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Vertex decomposability and regularity of very well-covered graphs

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ABSTRACT

A graph is called very well-covered if it is unmixed without isolated vertices such that the cardinality of each minimal vertex cover is half the number of vertices. We first prove that a very well-covered graph is Cohen–Macaulay if and only if it is vertex decomposable. Next, we show that the Castelnuovo–Mumford regularity of the quotient ring of the edge ideal of a very well-covered graph is equal to the maximum number of pairwise 3-disjoint edges.

1. Introduction

Let G be a simple graph with the vertex set $V(G) = \{x_1, x_2, \ldots, x_n\}$ and the edge set $E(G)$. By identifying the vertex x_i with the variable x_i in the polynomial ring $R = K[x_1, \ldots, x_n]$ over a field K, one can associate with G the square-free monomial ideal

$$I(G) = (x_i x_j \mid \{x_i, x_j\} \in E(G)).$$

The ideal $I(G)$ is called the *edge ideal* of G.

A graph G is called *unmixed* if all the minimal vertex covers have the same cardinality. In this case $R/I(G)$ is unmixed, i.e., all the associated primes of $I(G)$ have the same height. Assume that the graph G is unmixed without isolated vertices. In this case it is well-known that $2\mathrm{ht}\,(I(G)) \geq |V(G)|$. See, e.g., [8]. A graph G is called *very well-covered* if it is unmixed without isolated vertices and with $2\mathrm{ht}\,(I(G)) = |V(G)|$. The class of very well-covered graphs contains unmixed bipartite graphs without an isolated vertex and grafted graphs. See [6] for the grafted graph.

For an unmixed graph G the following hierarchy is known:

$$\text{vertex decomposability} \Longrightarrow \text{shellability} \Longrightarrow \text{Cohen–Macaulayness}.$$

It is known that the implications are strict. We are interested in a class of graphs such that the converses of the above implications hold for them. For unmixed bipartite graphs the converse of the right implication has been shown by Estrada and Villarreal [5]. The left converse has been shown by Van Tuyl [15].

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The first main result of this article generalizes these results to very well-covered graphs. Namely:

Theorem 1.1. *Let G be a very well-covered graph. Then the following conditions are equivalent:*

(1) *G is Cohen–Macaulay.*
(2) *G is shellable.*
(3) *G is vertex decomposable.*

The equivalence between (1) and (2) is already pointed out in [3, Theorem 4.1]. Theorem 1.1 has been proved independently by Constantinescu and Varbaro in [1, Theorem 2.3], which is a minor extension of their other joint paper on vertex cover algebras in [2, Theorem 4.7].

Our second topic is the (Castelnuovo–Mumford) regularity of the edge ideal of a very well-covered graph. The regularity is one of the most important invariants of a minimal free resolution of a graded ring. It is actively studied for edge ideals. See, e.g., [12,13,15,21].

Let G be a graph. A pair of edges $\{x, y\}$ and $\{u, v\}$ of G is called 3-*disjoint* if the induced subgraph of G on $\{x, y, u, v\}$ is disconnected. A set Γ of edges of G is called a *pairwise* 3-*disjoint set of edges* if any pair of edges of Γ is 3-disjoint. The maximum cardinality of all pairwise 3-disjoint sets of edges in G is denoted by $a(G)$.

Then the following lower bound of the regularity is known:

Theorem 1.2 (*[12], Lemma 2.2*)**.** *For a graph G, reg* $(R/I(G)) \geq a(G)$.

The following natural question arises: are there any families of graphs where this inequality is an equality? Zheng [21] proved the equality for trees. Francisco et al. [7] (resp. Van Tuyl [15]) proved that equality holds for Cohen–Macaulay bipartite graphs (resp. sequentially Cohen–Macaulay bipartite graphs). Note that a tree is a sequentially Cohen–Macaulay bipartite graph ([6]). Kummini [13] proved that equality holds also for unmixed bipartite graphs. We generalize Kummini's result to the class of very well-covered graphs. Namely:

Theorem 1.3. *Let G be a very well-covered graph. Then reg* $(R/I(G)) = a(G)$.

While Kummini has associated a directed graph with an unmixed graph, we associated a semidirected graph with a very well-covered graph to reduce to the Cohen–Macaulay case.

2. Basic definitions and notation

In this section we recall some definitions and properties that we will use in the paper.

A simplicial complex Δ on the vertex set $V = \{x_1, \ldots, x_n\}$ is a collection of subsets of V, with the following two properties: (1) $\{x_i\} \in \Delta$ for all i; (2) if $F \in \Delta$, then all subsets of F are also in Δ (including the empty set). An element of Δ is called a *face* of Δ and we define the dimension of F by $\dim F = |F| - 1$. The dimension of Δ is defined by $\dim \Delta = \max\{\dim F \mid F \in \Delta\}$. A maximal face of Δ with respect to inclusion is called a *facet* of Δ.

A simplicial complex Δ is called *shellable* if there is a linear order $F_1, \ldots, F_s$ of all the facets of Δ such that for all $1 \leq i < j \leq s$, there exists some $v \in F_j \setminus F_i$ and some $l \in \{1, \ldots, j-1\}$ with $F_j \setminus F_l = \{v\}$.

Let $F \in \Delta$ be a face of Δ. The *link* of F is the simplicial complex

$$\mathrm{lk}_\Delta(F) = \{F' \in \Delta | F' \cap F = \emptyset, F' \cup F \in \Delta\}$$

and the *deletion* of F is the simplicial complex

$$\mathrm{del}_\Delta(F) = \{F' \in \Delta \mid F' \cap F = \emptyset\}.$$

If Δ is the simplicial complex with the facets $F_1, \ldots, F_t$, then we write $\Delta = \langle F_1, \ldots, F_t \rangle$.

A simplicial complex Δ on the vertex set $V = \{x_1, \ldots, x_n\}$ is defined recursively to be *vertex decomposable* if one of the following conditions is satisfied:

(i) $n = 0$ and $\Delta = \{\emptyset\}$.
(ii) $n > 0$ and $\Delta = \langle \{x_1, \ldots, x_n\} \rangle$.
(iii) There exists some $x \in V$ such that $\mathrm{lk}_\Delta(\{x\})$ and $\mathrm{del}_\Delta(\{x\})$ are vertex decomposable, and every facet of $\mathrm{del}_\Delta(\{x\})$ is a facet of Δ.

A simplicial complex Δ on the vertex set $V = \{x_1, \ldots, x_n\}$ is defined recursively to be *semi-nonevasive* over a field K if one of the following conditions is satisfied:

(i) $n = 0$ and $\Delta = \{\emptyset\}$.
(ii) $n > 0$ and $\dim \Delta = 0$.
(iii) There exists some $x \in V$ such that $\mathrm{lk}_\Delta(\{x\})$ and $\mathrm{del}_\Delta(\{x\})$ are semi-nonevasive over K, and such that

$$\tilde{H}_i(\Delta; K) \cong \tilde{H}_i(\mathrm{del}_\Delta(\{x\}); K) \oplus \tilde{H}_{i-1}(\mathrm{lk}_\Delta(\{x\}); K)$$

for each i.

See [11] for detailed information.

Let $G = (V(G), E(G))$ be a graph. A subset C of $V(G)$ is called a *vertex cover* of G if we have $C \cap \{x, y\} \neq \emptyset$ for any $\{x, y\} \in E(G)$. A vertex cover C of G is called *minimal* if there is no proper subset of C which is a vertex cover. A subset F of $V(G)$ is called an *independent set* of G if any subsets of F with cardinality 2 do not belong to $E(G)$. The family of all independent sets of G is a simplicial complex on the vertex set $V(G)$, which is called the *independence complex* of G and is denoted by Δ_G. We call a graph G *vertex decomposable* (resp. *shellable*, *semi-nonevasive* etc.) if the independence complex Δ_G is vertex decomposable (resp. shellable, semi-nonevasive etc.).

It is known that a graph G is vertex decomposable if and only if its connected components are vertex decomposable. See [20, Lemma 20].

For $W \subseteq V(G)$ the subgraph $G \setminus W$ is defined to be

$$(V(G) \setminus W,\ \{\{x, y\} \in E(G) \mid \{x, y\} \cap W = \emptyset\}).$$

Moreover, for any $x \in V(G)$ we denote by $N_G(x)$ the *neighbor set* of x in G, i.e., $N_G(x) = \{y \in V(G) \mid \{x, y\} \in E(G)\}$. The following lemma will be crucial in the proof of our main results.

Lemma 2.1 (*[4], Lemma 4.2*)**.** *Let G be a graph and suppose that $x, y \in V(G)$ are two vertices such that $\{x\} \cup N_G(x) \subseteq \{y\} \cup N_G(y)$. If both $G \setminus \{y\}$ and $G \setminus (\{y\} \cup N_G(y))$ are vertex decomposable, then G is vertex decomposable.*

Let M be a noetherian graded R-module, and let

$$0 \to \bigoplus_j R(-j)^{\beta_{t,j}(M)} \to \bigoplus_j R(-j)^{\beta_{t-1,j}(M)} \to \cdots \to \bigoplus_j R(-j)^{\beta_{0,j}(M)} \to M \to 0$$

be a minimal graded free resolution of M over R.

The *(Castelnuovo–Mumford) regularity* of M, denoted by $\operatorname{reg}(M)$, is defined by

$$\operatorname{reg}(M) = \max\{j - i \mid \beta_{i,j}(M) \neq 0\}.$$

The *projective dimension* of M, denoted by $\operatorname{pd}(M)$ (sometimes $\operatorname{pd}_R(M)$), is defined by

$$\operatorname{pd}(M) = \max\{i \mid \beta_{i,j}(M) \neq 0 \text{ for some } j\}.$$

The *cover ideal* of G, denoted by $J(G)$, is defined to be the square-free monomial ideal

$$J(G) = (x_F \mid F \text{ is a (minimal) vertex cover of } G),$$

where $x_F = \prod_{x_i \in F} x_i$.

We require the following result.

Lemma 2.2 (*[14]*)**.** *For a graph G we have $\operatorname{reg}(R/I(G)) = \operatorname{pd}(J(G))$.*

3. The Cohen–Macaulay case

Throughout the article let $R = K[x_1, \ldots, x_h, y_1, \ldots, y_h]$ be a polynomial ring over a field K.

In this section we first prove that G is Cohen–Macaulay if and only if it is vertex decomposable, following the idea of Van Tuyl [15].

We next show that if G is Cohen–Macaulay, then the regularity of $R/I(G)$ is equal to the maximum number of pairwise 3-disjoint edges of G. We will use this result to prove the main theorem of the next section.

We reformulate a result in [3] for our purpose.

Lemma 3.1. *Let G be a very well-covered graph with $2h$ vertices. Then the following conditions are: equivalent:*

(1) *G is Cohen–Macaulay;*
(2) *There is a relabeling of vertices $V(G) = \{x_1, \ldots, x_h, y_1, \ldots, y_h\}$ such that the following five conditions hold:*
  (i) *$X = \{x_1, \ldots, x_h\}$ is a minimal vertex cover of G and $Y = \{y_1, \ldots, y_h\}$ is a maximal independent set of G;*
  (ii) *$\{x_1, y_1\}, \ldots, \{x_h, y_h\} \in E(G)$;*
  (iii) *if $\{z_i, x_j\}, \{y_j, x_k\} \in E(G)$, then $\{z_i, x_k\} \in E(G)$ for distinct i, j, k and for $z_i \in \{x_i, y_i\}$;*
  (iv) *if $\{x_i, y_j\} \in E(G)$, then $\{x_i, x_j\} \notin E(G)$;*
  (v) *if $\{x_i, y_j\} \in E(G)$, then $i \leq j$.*

Proof. (1) $\Rightarrow$ (2). Since G is very well-covered, G has a perfect matching (see [9, Remark 2.2]). Hence there is a relabeling of the vertices in G such that the conditions (i) and (ii) are satisfied. It is shown in [3, Lemma 3.5] that if G is Cohen–Macaulay, then there exists a suitable simultaneous change of labeling on both x_is and y_is (i.e., we relabel $(x_{i_1}, \ldots, x_{i_h})$ and $(y_{i_1}, \ldots, y_{i_h})$ as $(x_1, \ldots, x_h)$ and $(y_1, \ldots, y_h)$ at the same time) such that the condition (v) is satisfied. Hence under this relabeling the conditions (i), (ii) and (v) are fulfilled. Now the conditions (iii) and (iv) are also satisfied by [3, Theorem 3.6], since G is a Cohen–Macaulay very well-covered graph with the conditions (i), (ii) and (v).

(2) $\Rightarrow$ (1) is proved by [3, Theorem 3.6]. $\square$

Note that under the relabeling we have $\deg(y_1) = 1$ and $\{x_1, y_1\} \in E(G)$.

Theorem 3.2. *Let G be a very well-covered graph with $2h$ vertices. Then the following conditions are equivalent:*

(1) *G is Cohen–Macaulay.*
(2) *G is vertex decomposable.*

Proof. (2) $\Rightarrow$ (1) always holds for any unmixed graph G. So it suffices to prove (1) $\Rightarrow$ (2). We prove the assertion by induction on $h = \operatorname{ht} I(G)$. If $h = 1$, then G is just an edge and there is nothing to prove. So suppose $h > 1$. By Lemma 3.1 we may assume the conditions (i), (ii), ..., (v). We have $\{y_1\} \cup N_G(y_1) \subseteq \{x_1\} \cup N_G(x_1)$. By Lemma 2.1 it is enough to show that $G \setminus \{x_1\}$ and $G \setminus (\{x_1\} \cup N_G(x_1))$ are vertex decomposable. It is clear that $G \setminus \{x_1, y_1\}$ has an even number of vertices which are not isolated with $\operatorname{ht}(I(G \setminus \{x_1, y_1\})) = h - 1$. It follows from the above lemma that $G \setminus \{x_1, y_1\}$ is Cohen–Macaulay. Now the induction hypothesis implies that $G \setminus \{x_1, y_1\}$ is vertex decomposable. Since $\{y_1\}$ is isolated in $G \setminus \{x_1\}$, we know that $G \setminus \{x_1\}$ is vertex decomposable.
Now we show that $G \setminus (\{x_1\} \cup N_G(x_1))$ is vertex decomposable. We first prove the following claims:

Claim 1. *If $x_t \in N_G(x_1)$, then y_t is isolated in $G \setminus (\{x_1\} \cup N_G(x_1))$.*

Claim 2. *If $y_t \in N_G(x_1)$, then x_t is isolated in $G \setminus (\{x_1\} \cup N_G(x_1))$.*

Proof of Claim 1. Suppose $x_t \in N_G(x_1)$. If y_t is not isolated in $G \setminus (\{x_1\} \cup N_G(x_1))$, then there exists an integer k such that $\{x_k, y_t\} \in E(G \setminus (\{x_1\} \cup N_G(x_1)))$. From Lemma 3.1, we get that $\{x_1, x_k\} \in E(G)$ and hence $x_k \in N_G(x_1)$. This implies that $x_k \notin V(G \setminus (\{x_1\} \cup N_G(x_1)))$ but $\{x_k, y_t\} \in E(G \setminus (\{x_1\} \cup N_G(x_1)))$ which is impossible.

Proof of Claim 2. Suppose $y_t \in N_G(x_1)$ but x_t is not isolated in $G \setminus (\{x_1\} \cup N_G(x_1))$. If $\{x_k, x_t\} \in E(G \setminus (\{x_1\} \cup N_G(x_1)))$ for some k, then we get $\{x_1, x_k\} \in E(G)$ and so $x_k \in N_G(x_1)$, a contradiction.
If $\{x_t, y_k\} \in E(G \setminus (\{x_1\} \cup N_G(x_1)))$ for some k, then we must have $\{x_1, y_k\} \in E(G)$ and hence $y_k \in N_G(x_1)$. This shows that $y_k \notin V(G \setminus (\{x_1\} \cup N_G(x_1)))$ but $\{x_t, y_k\} \in E(G \setminus (\{x_1\} \cup N_G(x_1)))$ which is impossible.

The above statements show that

$$H = (G \setminus (\{x_1\} \cup N_G(x_1))) \setminus \{\text{isolated vertices of } G \setminus (\{x_1\} \cup N_G(x_1))\}$$

has an even number of vertices which are not isolated and its height is half of the number of vertices. It follows from the above lemma that H is Cohen–Macaulay and so it is vertex decomposable by induction. Therefore $G \setminus (\{x_1\} \cup N_G(x_1))$ is also vertex decomposable. □

Remark 3.3. Since a grafted graph is very well-covered and Cohen–Macaulay, we get that if G is a grafted graph, then it is vertex decomposable. Note that it is not very difficult to prove it directly. Also it is known that vertex decomposability implies the semi-nonevasive property, while shellability does not imply it, in general. See [11, Propositions 5.12 and 5.13]. But we know by Theorem 3.2 that if G is a shellable very well-covered graph, then G is semi-nonevasive.

Now we study the Castelnuovo–Mumford regularity of Cohen–Macaulay very well-covered graphs.

Lemma 3.4. *Let G be a very well-covered graph with $2h$ vertices. If G is Cohen–Macaulay, then $\operatorname{reg}(R/I(G)) = a(G)$.*

Proof. By Theorem 1.2 and Lemma 2.2 it is enough to show that $\operatorname{pd}(J(G)) \leq a(G)$, where $J(G)$ denotes the cover ideal of the graph G. We proceed by induction on the $\operatorname{ht} I(G) = h$. If $h = 1$, then G just has the single edge $\{x_1, y_1\}$ and $J(G) = (x_1, y_1)$. Therefore $\operatorname{pd}(J(G)) = 1 = a(G)$. Now suppose $h > 1$. By Lemma 3.1 we may assume that $\deg(y_1) = 1$, $N_G(y_1) = \{x_1\}$, and $N_G(x_1) = \{x_{i_1}, \ldots, x_{i_k}, y_1, y_{j_1}, \ldots, y_{j_s}\}$ with $\{i_1, \ldots, i_k\} \cap \{1, j_1, \ldots, j_s\} = \emptyset$. Note that there is no minimal vertex cover of G containing both x_1 and y_1 and that any minimal vertex cover of G not containing x_1 must contain $N_G(x_1)$. Set $G_1 = G \setminus (\{x_1\} \cup N_G(x_1))$ and $G_2 = G \setminus (\{y_1\} \cup N_G(y_1))$. Let $J(G_1)R$ and $J(G_2)R$ be ideals of $R = K[x_1, \ldots, x_h, y_1, \ldots, y_h]$ generated by the elements in $J(G_2)$ and in $J(G_2)$ respectively. Then using the same arguments as in [15, Theorem 3.3], we have:

(1) $J(G) = x_1 J(G_2)R + x_{i_1} \cdots x_{i_k} y_1 y_{j_1} \cdots y_{j_s} J(G_1)R$.
(2) $x_1 J(G_2)R \bigcap x_{i_1} \cdots x_{i_k} y_1 y_{j_1} \cdots y_{j_s} J(G_1)R = x_1 x_{i_1} \cdots x_{i_k} y_1 y_{j_1} \cdots y_{j_s} J(G_1)R$.

The above statements imply that there is an exact sequence

$$0 \longrightarrow x_1 x_{i_1} \cdots x_{i_k} y_1 y_{j_1} \cdots y_{j_s} J(G_1)R \longrightarrow x_1 J(G_2)R \oplus x_{i_1} \cdots x_{i_k} y_1 y_{j_1} \cdots y_{j_s} J(G_1)R \longrightarrow J(G) \longrightarrow 0.$$

The above exact sequence yields

$$\operatorname{pd}(J(G)) \leq \max\{\operatorname{pd}(x_1 x_{i_1} \cdots x_{i_k} y_1 y_{j_1} \cdots y_{j_s} J(G_1)R) + 1, \operatorname{pd}(x_1 J(G_2)R), \operatorname{pd}(x_{i_1} \cdots x_{i_k} y_1 y_{j_1} \cdots y_{j_s} J(G_1)R)\}.$$

Note that for any monomial ideal I and monomial f with property that $\operatorname{supp}(f) \cap \operatorname{supp}(g) = \emptyset$, for all $g \in \mathcal{G}(I)$ (the minimal generating set of I), we have $\operatorname{pd}(fI) = \operatorname{pd}(I)$. Therefore

$$\operatorname{pd}(J(G)) \leq \max\{\operatorname{pd}(J(G_1)R) + 1, \operatorname{pd}(J(G_2)R)\}.$$

As explained in the proof of Theorem 3.1, $G_1 \setminus \{\text{isolated vertices of } G_1\}$ and G_2 have an even number of vertices which are not isolated and their heights are half of the number of vertices. Since isolated vertices change neither value of $\operatorname{reg}(R/I(G))$ nor $a(G)$ for any graph G, our induction hypothesis implies that $\operatorname{pd}(J(G_1)R) + 1 = \operatorname{pd}_{R_1}(J(G_1)) + 1 \leq a(G_1) + 1$ and $\operatorname{pd}(J(G_2)R) = \operatorname{pd}_{R_2}(J(G_2)) \leq a(G_2)$, where $R_1 = K[x \mid x \in V(G_1)]$ and $R_2 = K[x \mid x \in V(G_2)]$. One can see that $a(G_2) \leq a(G)$ and $a(G_1) + 1 \leq a(G)$ (adding the edge $\{x_1, y_1\}$ to any pairwise 3-disjoint set of edges in G_1 is a set of pairwise 3-disjoint edges in G). Therefore $\operatorname{pd}(J(G)) \leq a(G)$. □

4. Regularity in the unmixed case

In this section we prove Theorem 1.3. First we quote some known results.

Lemma 4.1 (*[9], Remark 2.2*). *Let G be a very well-covered graph with $2h$ vertices. Then there is a relabeling of vertices $V(G) = \{x_1, \ldots, x_h, y_1, \ldots, y_h\}$ such that the following two conditions hold:*

(i) *$X = \{x_1, \ldots, x_h\}$ is a minimal vertex cover of G and $Y = \{y_1, \ldots, y_h\}$ is a maximal independent set of G;*
(ii) *$\{x_1, y_1\}, \ldots, \{x_h, y_h\} \in E(G)$.*

Lemma 4.2 (*[3], Proposition 2.3*). *Let G be a graph with $2h$ vertices, which are not isolated, and with $\operatorname{ht}(I(G)) = h$. We assume the conditions* (i) *and* (ii) *in Lemma 4.1. Then G is unmixed, i.e., G is very well-covered if and only if the following conditions hold:*

(iii) *if $\{z_i, x_j\}, \{y_j, x_k\} \in E(G)$, then $\{z_i, x_k\} \in E(G)$ for distinct i, j, k and for $z_i \in \{x_i, y_i\}$;*
(iv) *if $\{x_i, y_j\} \in E(G)$, then $\{x_i, x_j\} \notin E(G)$.*

To prove Theorem 1.3 we follow the basic idea of Kummini [13] for unmixed bipartite graphs, but to treat a very well-covered graph we need to introduce a notion of a semidirected graph.

Set $[h] = \{1, 2, \ldots, h\}$. Let $\binom{[h]}{2}$ be the family of all subsets of $[h]$ with cardinality 2. Let E_{u} be a subset of $\binom{[h]}{2}$. Let E_{d} be a subset of $[h] \times [h] \setminus D$, where $D = \{(i, i) \mid i \in [h]\}$. We have the canonical forgetful map $f : [h] \times [h] \setminus D \longrightarrow \binom{[h]}{2}((i, j) \longmapsto \{i, j\})$. We call $\mathfrak{d} = ([h], E_{\mathrm{u}}, E_{\mathrm{d}})$ a *semidirected graph* if $f(E_{\mathrm{d}}) \cap E_{\mathrm{u}} = \emptyset$. We call an element $\{i, j\}$ of E_{u} an *undirected edge* of $\mathfrak{d}$ and $E_{\mathrm{u}}(\mathfrak{d}) = E_{\mathrm{u}}$ the *undirected edge set* of $\mathfrak{d}$. We call an element (i, j) of E_{d} a *directed edge* from i to j of $\mathfrak{d}$ and $E_{\mathrm{d}}(\mathfrak{d}) = E_{\mathrm{d}}$ the *directed edge set* of $\mathfrak{d}$. We denote (i, j) by ij.

We say that a set $A \subseteq [h]$ is an *independent set* in $\mathfrak{d}$ if $\{i, j\} \notin E_{\mathrm{u}}(\mathfrak{d})$ and $ij, ji \notin E_{\mathrm{d}}(\mathfrak{d})$ for any $i, j \in A$. And $\Delta_{\mathfrak{d}}$ denotes the set of all independent sets in $\mathfrak{d}$, which is a simplicial complex on the vertex set $[h]$ called the *independence complex* of $\mathfrak{d}$.

We call a semidirected graph $\mathfrak{d}$ *acyclic* if there are no directed cycles in $\mathfrak{d}$, and *transitively closed* if the following two properties are satisfied for any distinct $i, j, k \in [h]$:

(1) if $ij, jk \in E_{\mathrm{d}}(\mathfrak{d})$, then $ik \in E_{\mathrm{d}}(\mathfrak{d})$;
(2) if $ij \in E_{\mathrm{d}}(\mathfrak{d})$ and $\{j, k\} \in E_{\mathrm{u}}(\mathfrak{d})$, then $\{i, k\} \in E_{\mathrm{u}}(\mathfrak{d})$.

Let G be a very well-covered graph with $2h$ vertices. Suppose G satisfies the conditions (i) and (ii) in Lemma 4.1. We define a semidirected graph $\mathfrak{d}_G = ([h], E_{\mathrm{u}}, E_{\mathrm{d}})$ associated with G as follows:

$$E_{\mathrm{u}} = \left\{ \{i, j\} \in \binom{[h]}{2} \mid \{x_i, x_j\} \in E(G) \right\}.$$
$$E_{\mathrm{d}} = \{ij \in [h] \times [h] \mid i \neq j, \{x_i, y_j\} \in E(G)\}.$$

Note that the condition $f(E_{\mathrm{d}}) \cap E_{\mathrm{u}} = \emptyset$ is satisfied by the condition (iv) in Lemma 4.2. And $\mathfrak{d}_G$ is transitively closed by the condition (iii) in Lemma 4.2. If G is also Cohen–Macaulay, then $\mathfrak{d}_G$ is acyclic. This fact can be checked by the condition (v) in Lemma 3.1.

Now we interpret $a(G)$ in terms of the semidirected graph $\mathfrak{d}_G$.

Lemma 4.3. *Let G be a very well-covered graph with $2h$ vertices. Suppose G satisfies the conditions* (i) *and* (ii) *in Lemma 4.1. Then*

$$a(G) = \dim \Delta_{\mathfrak{d}_G} + 1.$$

Proof. We first show that $a(G) \geq \dim \Delta_{\mathfrak{d}_G} + 1$. For $A \in \Delta_{\mathfrak{d}_G}$, set $B = \{\{x_i, y_i\} \mid i \in A\}$. Then it is easy to see that B is a set of pairwise 3-disjoint edges in G.

We next show that $a(G) \leq \dim \Delta_{\mathfrak{d}_G} + 1$. Let B be a set of pairwise 3-disjoint edges in G. Set

$$A = \{i \mid \{x_i, y_j\} \in B \text{ for some } j\} \cup \{i \mid \{x_i, x_k\} \in B \text{ for some } k > i\}.$$

We show that A is an independent set in $\mathfrak{d}_G$. Suppose to the contrary that A is not an independent set in $\mathfrak{d}_G$. If A contains an undirected edge $\{i, j\}$, this contradicts the fact that B is a set of pairwise 3-disjoint edges in G. Hence there is a directed edge ij with $i, j \in A$. We must consider the following three cases.

(1) Suppose $\{x_i, y_k\}, \{x_j, y_\ell\} \in B$ for some k, ℓ. We may assume that $\{x_i, y_j\} \in E(G)$. Since $\{x_i, y_j\}, \{x_j, y_\ell\} \in E(G)$ we have $\{x_i, y_\ell\} \in E(G)$ by the condition (iii) in Lemma 4.2. This contradicts the fact that B is a set of pairwise 3-disjoint edges in G.

(2) Suppose $\{x_i, x_k\}, \{x_j, y_\ell\} \in B$ for some k, ℓ. We first assume that $\{x_i, y_j\} \in E(G)$. Since $\{x_i, y_j\}, \{x_j, y_\ell\} \in E(G)$ we have $\{x_i, y_\ell\} \in E(G)$ by the condition (iii) in Lemma 4.2. This contradicts the fact that B is a set of pairwise 3-disjoint edges in G.

We next assume that $\{x_j, y_i\} \in E(G)$. Since $\{x_j, y_i\}, \{x_i, x_k\} \in E(G)$ we have $\{x_j, x_k\} \in E(G)$ by the condition (iii) in Lemma 4.2. This contradicts the fact that B is a set of pairwise 3-disjoint edges in G.

(3) Suppose $\{x_i, x_k\}, \{x_j, x_\ell\} \in B$ for some k, ℓ. We may assume that $\{x_i, y_j\} \in E(G)$. Since $\{x_i, y_j\}, \{x_j, x_\ell\} \in E(G)$ we have $\{x_i, x_\ell\} \in E(G)$ by the condition (iii) in Lemma 4.2. This contradicts the fact that B is a set of pairwise 3-disjoint edges in G.

Hence A is an independent set in $\mathfrak{d}_G$. This implies that $a(G) \leq \dim \Delta_{\mathfrak{d}_G} + 1$. □

Remark 4.4. Let G be a Cohen–Macaulay very well-covered graph with $2h$ vertices. Suppose G satisfies the conditions (i) and (ii) in Lemma 4.1. Then $\{x_1 - y_1, x_2 - y_2, \ldots, x_h - y_h\}$ is a regular sequence for $R/I(G)$ (see [3]). Set $T = R/(I(G) + (x_1 - y_1, x_2 - y_2, \ldots, x_h - y_h))$, which is an artinian ring. The *socle* $\operatorname{Soc} T$ of T is defined to be the ideal $(0 :_T T_+)$ of T, where T_+ is the irrelevant maximal ideal of the k-algebra T. See, e.g., [18, Lemma 4.3.3]. Then it is known that $\operatorname{reg} R/I(G) = \max\{k; (\operatorname{Soc} T)_k \neq 0\}$. See, e.g., [18, Lemma 4.3.3]. It is shown that $\max\{k; (\operatorname{Soc} T)_k \neq 0\} = \dim \Delta_{\mathfrak{d}_G} + 1$ in [3, Lemma 4.3]. Hence, taking this together with Lemma 4.3, we have $\operatorname{reg} R/I(G) = a(G)$. This is another proof of Lemma 3.4.

Let $\mathfrak{d}$ be a semidirected graph on the vertex set $[h]$. A pair $\{i, j\}$ of vertices in $\mathfrak{d}$ is called *strongly connected* if $ij, ji \in E_{\mathrm{d}}(\mathfrak{d})$. A *strong component* of $\mathfrak{d}$ is a maximal induced subgraph satisfying the property that every pair of vertices in it is strongly connected. The vertex sets of the strong components of $\mathfrak{d}$ give a partition of $[h]$.

We define a new semidirected graph $\widehat{\mathfrak{d}}_G$. Let G be a very well-covered graph with $2h$ vertices. Suppose G satisfies the conditions (i) and (ii) in Lemma 4.1. Let $\mathcal{Z}_1, \ldots, \mathcal{Z}_t$ be the vertex sets of the strong components of $\mathfrak{d}_G$. We define a semidirected graph $\widehat{\mathfrak{d}}_G = ([t], E_{\mathrm{u}}, E_{\mathrm{d}})$ as follows:

$$E_{\mathrm{u}} = \left\{ \{a, b\} \in \binom{[t]}{2} \mid \{i, j\} \in E_{\mathrm{u}}(\mathfrak{d}_G) \text{ for some } i \in \mathcal{Z}_a \text{ and for some } j \in \mathcal{Z}_b \right\}.$$
$$E_{\mathrm{d}} = \{ab \in [t] \times [t] \mid a \neq b,\ ij \in E_{\mathrm{d}}(\mathfrak{d}_G) \text{ for some } i \in \mathcal{Z}_a \text{ and for some } j \in \mathcal{Z}_b\}.$$

First we observe that $\{i, j\}$ (resp. ij) is an undirected (resp. a directed) edge in $\mathfrak{d}_G$ for all $i \in \mathcal{Z}_a$ and all $j \in \mathcal{Z}_b$ if $\{a, b\}$ (resp. ab) is an undirected (resp. a directed) edge in $\widehat{\mathfrak{d}}_G$.

Suppose $\{a, b\}$ (resp. ab) is an undirected (resp. a directed) edge in $\widehat{\mathfrak{d}}_G$. Take $i \in \mathcal{Z}_a$ and $j \in \mathcal{Z}_b$. There are $i' \in \mathcal{Z}_a$ and $j' \in \mathcal{Z}_b$ such that $\{i', j'\}$ (resp. $i'j'$) is an undirected (resp. a directed) edge in $\mathfrak{d}_G$. Since ii' is a directed edge in $\mathfrak{d}_G$, $\{i, j'\}$ (resp. ij') is an undirected (resp. a directed) edge in $\mathfrak{d}_G$ by the transitively closed property of $\mathfrak{d}_G$. Since jj' (resp. $j'j$) is a directed edge in $\mathfrak{d}_G$, we have that $\{j, i\}$ (resp. ij) is an undirected (resp. a directed) edge in $\mathfrak{d}_G$ by the transitively closed property of $\mathfrak{d}_G$ again.

Now we check the condition $f(E_{\mathrm{d}}) \cap E_{\mathrm{u}} = \emptyset$, where f is the canonical forgetful map from $[t] \times [t] \setminus \{(i, i) \mid i \in [t]\}$ to $\binom{[t]}{2}$.

Suppose to the contrary that there are both an undirected edge $\{a, b\}$ and a directed edge ab in $\widehat{\mathfrak{d}}_G$. By the above observation this means that for $i \in \mathcal{Z}_a$ and $j \in \mathcal{Z}_b$ the semidirected graph $\mathfrak{d}_G$ has both the undirected edge $\{i, j\}$ and the directed edge ij, which contradicts the definition of a semidirected graph.

It is easy to see that $\widehat{\mathfrak{d}}_G$ is also transitively closed by the above observation since $\mathfrak{d}_G$ is transitively closed.

We now check that $\widehat{\mathfrak{d}}_G$ is acyclic. Suppose to the contrary that there is a directed cycle $a_1a_2, a_2a_3, \ldots, a_{r-1}a_r, a_ra_1$ for $r \geq 2$ in $\widehat{\mathfrak{d}}_G$. Since $\widehat{\mathfrak{d}}_G$ is transitively closed, we can show that a_1a_r is a directed edge in $\widehat{\mathfrak{d}}_G$. Since both a_1a_r and a_ra_1 are directed edges, $\mathcal{Z}_{a_1} \cup \mathcal{Z}_{a_r}$ is a vertex set of a strong component in $\mathfrak{d}_G$, which contradicts the fact that $\mathcal{Z}_{a_1}$ is a vertex set of a maximal strong component in $\mathfrak{d}_G$.

We write $b \succ a$ if $ab \in E_{\mathrm{d}}(\widehat{\mathfrak{d}}_G)$. By $b \succcurlyeq a$ (and, equivalently, $a \preccurlyeq b$) we mean that $b \succ a$ or $b = a$. It is easy to see that $\preccurlyeq$ defines a partial order on $[t]$ since $\widehat{\mathfrak{d}}_G$ is transitively closed and acyclic.

For $A \subseteq [t]$, we write $b \succcurlyeq A$ if there exists $a \in A$ such that $b \succcurlyeq a$.

We define the *acyclic reduction* of G to be the graph $\widehat{G} = (V(\widehat{G}), E(\widehat{G}))$ where

$$V(\widehat{G}) = \{u_1, \ldots, u_t\} \cup \{v_1, \ldots, v_t\};$$
$$E(\widehat{G}) = \{\{u_a, v_a\} \mid 1 \leq a \leq t\} \cup \{\{u_a, u_b\} \mid \{a, b\} \in E_{\mathrm{u}}(\widehat{\mathfrak{d}}_G)\} \cup \{\{u_a, v_b\} \mid ab \in E_{\mathrm{d}}(\widehat{\mathfrak{d}}_G)\}.$$

Lemma 4.5. *Let G be a very well-covered graph. Suppose G satisfies the conditions* (i) *and* (ii) *in Lemma 4.1. Then $\widehat{G}$ is a Cohen–Macaulay very well-covered graph.*

Proof. By definition $\widehat{G}$ has an even number of vertices, which are not isolated, and with $2\operatorname{ht}(I(\widehat{G})) = |V(\widehat{G})|$. Clearly $\widehat{G}$ satisfies the conditions corresponding to (i) and (ii) in Lemma 4.1. Since $\widehat{\mathfrak{d}}_G$ is transitively closed, the condition (iii) in Lemma 4.2 is satisfied. Since $\widehat{\mathfrak{d}}_G$ is a semidirected graph, the condition (iv) in Lemma 4.2 is satisfied. Then $\widehat{G}$ is a very well-covered graph. Since $\widehat{\mathfrak{d}}_G$ is acyclic, its vertex set can be relabeled such that every directed edge of $\widehat{\mathfrak{d}}_G$ is of the form ij with $i < j$. Under this relabeling the condition (v) in Lemma 3.1 is fulfilled, while the conditions (i)–(iv) are preserved. Hence by Lemma 3.1, $\widehat{G}$ is Cohen–Macaulay. □

Remark 4.6. Let G be a very well-covered graph. Suppose G satisfies the conditions (i) and (ii) in Lemma 4.1. It is easy to see that $\widehat{\mathfrak{d}}_G = \mathfrak{d}_{\widehat{G}}$. Moreover, if G itself is Cohen–Macaulay, then $G = \widehat{G}$.

Next we show that $a(G) = a(\widehat{G})$.

Lemma 4.7. *Let G be a very well-covered graph. Suppose G satisfies the conditions* (i) *and* (ii) *in Lemma 4.1. Let $\widehat{G}$ be the acyclic reduction of G. Then*

$$\dim \Delta_{\mathfrak{d}_G} = \dim \Delta_{\mathfrak{d}_{\widehat{G}}}.$$

Proof. First we show that $\dim \Delta_{\mathfrak{d}_G} \leq \dim \Delta_{\mathfrak{d}_{\widehat{G}}}$. Take $A = \{i_1, \ldots, i_r\} \in \Delta_{\mathfrak{d}_G}$. Suppose $i_j \in \mathcal{Z}_{a_j}$ for all $1 \leq j \leq r$. Set $\widehat{A} = \{a_1, \ldots, a_r\}$. It is easy to see that $\widehat{A} \in \Delta_{\mathfrak{d}_{\widehat{G}}}$.

Next we show that $\dim \Delta_{\mathfrak{d}_G} \geq \dim \Delta_{\mathfrak{d}_{\widehat{G}}}$. Take $\widehat{A} = \{a_1, \ldots, a_r\} \in \Delta_{\mathfrak{d}_{\widehat{G}}}$. Take $i_j \in \mathcal{Z}_{a_j}$ for all $1 \leq j \leq r$ and set $A = \{i_1, \ldots, i_r\}$. Then A is an independent set in $\mathfrak{d}_G$. □

In the next lemma, $\mathrm{Ass}(R/I)$ denotes the set of all associated prime ideals of I.

Lemma 4.8. *Let G be a very well-covered graph with $2h$ vertices. Suppose G satisfies the conditions* (i) *and* (ii) *in Lemma 4.1. For all $\mathfrak{p} \in Ass(R/I(G))$, if $y_i \in \mathfrak{p}$ and ij is a direct edge in $\mathfrak{d}_G$, then $y_j \in \mathfrak{p}$.*

Proof. Set $I = I(G)$. Now let $k \in [h]$. Since $x_k y_k \in I \subseteq \mathfrak{p}$, $x_k \in \mathfrak{p}$ or $y_k \in \mathfrak{p}$, but since $\mathrm{ht}(\mathfrak{p}) = h$, we have that $x_k \in \mathfrak{p}$ if and only if $y_k \notin \mathfrak{p}$. Now $y_i \in \mathfrak{p}$ implies that $x_i \notin \mathfrak{p}$, which together with $x_i y_j \in \mathfrak{p}$ shows that $y_j \in \mathfrak{p}$. □

Let G be a very well-covered graph with $2h$ vertices. Suppose G satisfies the conditions (i) and (ii) in Lemma 4.1. For an independent set $\widehat{A} \neq \emptyset$ of $\widehat{\mathfrak{d}}_G$ we define

$$\Omega_{\widehat{A}} = \cup_{b \succcurlyeq \widehat{A}} \mathcal{Z}_b.$$

For $\emptyset$ we define $\Omega_{\emptyset} = \emptyset$.

Lemma 4.9. *Let G be a very well-covered graph. Suppose G satisfies the conditions* (i) *and* (ii) *in Lemma 4.1. Then $\Omega_{\widehat{A}}$ does not contain any undirected edge in $\mathfrak{d}_G$ for any independent set $\widehat{A}$ of $\widehat{\mathfrak{d}}_G$.*

Proof. Suppose to the contrary that $\{i, j\} \subseteq \Omega_{\widehat{A}}$ is an undirected edge in $\mathfrak{d}_G$. Suppose $i \in \mathcal{Z}_a$ and $j \in \mathcal{Z}_b$. Then $\{a, b\}$ is an undirected edge in $\widehat{\mathfrak{d}}_G$. By the definition of $\Omega_{\widehat{A}}$ there are elements $a', b' \in \widehat{A}$ such that $a \succcurlyeq a'$ and $b \succcurlyeq b'$. We claim that $\{a', b'\}$ is an undirected edge in $\widehat{\mathfrak{d}}_G$, which is a contradiction to the fact that $\widehat{A}$ is an independent set. Now we prove the claim. Since $a'a \in E_{\mathrm{d}}(\widehat{\mathfrak{d}}_G)$ and $\{a, b\} \in E_{\mathrm{u}}(\widehat{\mathfrak{d}}_G)$, we have $\{a', b\} \in E_{\mathrm{u}}(\widehat{\mathfrak{d}}_G)$. Since $b'b \in E_{\mathrm{d}}(\widehat{\mathfrak{d}}_G)$ and $\{b, a'\} \in E_{\mathrm{u}}(\widehat{\mathfrak{d}}_G)$, we have $\{b', a'\} \in E_{\mathrm{u}}(\widehat{\mathfrak{d}}_G)$. □

Lemma 4.10. *Let G be a very well-covered graph with $2h$ vertices. Suppose G satisfies the conditions* (i) *and* (ii) *in Lemma 4.1. Then*

$$Ass(R/I(G)) = \{(x_i \mid i \notin \Omega_{\widehat{A}}) + (y_i \mid i \in \Omega_{\widehat{A}}) \mid \widehat{A} \in \Delta_{\widehat{\mathfrak{d}}_G}\}.$$

Proof. Set $I = I(G)$. Let $\mathfrak{p} \in \mathrm{Ass}(R/I)$. Since I is unmixed, just one of x_i and y_i belongs to $\mathfrak{p}$ for all $i = 1, \ldots, h$. Let $U = \{b \mid y_j \in \mathfrak{p} \text{ for some } j \in \mathcal{Z}_b\}$. Since $\mathcal{Z}_1, \ldots, \mathcal{Z}_t$ are the vertex sets of the strong components of $\mathfrak{d}_G$, from Lemma 4.8 it follows that $y_j \in \mathfrak{p}$ for all $j \in \cup_{b \in U} \mathcal{Z}_b$, and that if $b' \succ b$ and $b \in U$, then $b' \in U$. Suppose $\widehat{A}$ is the set of the minimal elements of U with respect to $\succ$. One can see that $\widehat{A}$ does not contain any directed edges in $\widehat{\mathfrak{d}}_G$, $U = \{b \mid b \succcurlyeq \widehat{A}\}$, and $\Omega_{\widehat{A}} = \cup_{b \in U} \mathcal{Z}_b = \{j \mid y_j \in \mathfrak{p}\}$. Now we show that $\widehat{A}$ does not contain any undirected edges of $\widehat{\mathfrak{d}}_G$. Suppose to the contrary that $\{a, b\} \subseteq \widehat{A}$ is an undirected edge in $\widehat{\mathfrak{d}}_G$. Take $i \in \mathcal{Z}_a$ and $j \in \mathcal{Z}_b$. Thus $x_i x_j \in I \subseteq \mathfrak{p}$ and hence we may assume that $x_i \in \mathfrak{p}$. Therefore $y_i \notin \mathfrak{p}$ by the form of the associated primes of $\mathrm{Ass}(R/I)$. On the other hand, since $i \in \mathcal{Z}_a$ and $a \in \widehat{A}$, we have $i \in \Omega_{\widehat{A}}$. This means that $y_i \in \mathfrak{p}$, which is a contradiction. Hence $\mathrm{Ass}(R/I) \subseteq \{(x_i \mid i \notin \Omega_{\widehat{A}}) + (y_i \mid i \in \Omega_{\widehat{A}}) \mid \widehat{A} \in \Delta_{\widehat{\mathfrak{d}}_G}\}$.

Conversely, take $\widehat{A} \in \Delta_{\widehat{\mathfrak{d}}_G}$ and set $\mathfrak{p} = (x_i \mid i \notin \Omega_{\widehat{A}}) + (y_i \mid i \in \Omega_{\widehat{A}})$.

Therefore $\mathrm{ht}(\mathfrak{p}) = \mathrm{ht}(I) = h$. Since I is unmixed, it suffices to prove that $I \subseteq \mathfrak{p}$. I is generated by monomials of the forms $x_i y_i$ $(i = 1, \ldots, h)$, $x_i y_j$, and $x_i x_j$ for some $1 \leq i \neq j \leq h$. It is clear that $x_i y_i \in \mathfrak{p}$ for all $i = 1, \ldots, h$. So assume that $i \neq j$. First let $x_i y_j \in I$. If $i \notin \Omega_{\widehat{A}}$, we have $x_i \in \mathfrak{p}$ and $x_i y_j \in \mathfrak{p}$. If $i \in \Omega_{\widehat{A}}$, then there exists a, b, b' such that $a \in \widehat{A}$, $b \succcurlyeq a$, $i \in \mathcal{Z}_b$, and $j \in \mathcal{Z}_{b'}$. Since ij is a directed edge in $\mathfrak{d}_G$, we get that $b' \succcurlyeq b$ in $\widehat{\mathfrak{d}}_G$. Hence $b' \succcurlyeq a$, and $j \in \Omega_{\widehat{A}}$, which shows that $y_j \in \mathfrak{p}$ and so $x_i y_j \in \mathfrak{p}$. Now let $x_i x_j \in I$. Since $\widehat{A}$ does not contain any undirected edges of $\widehat{\mathfrak{d}}_G$, $\Omega_{\widehat{A}}$ does not contain any undirected edges of $\mathfrak{d}_G$ by Lemma 4.9. Then we have $\{i, j\} \nsubseteq \Omega_{\widehat{A}}$. Therefore $i \notin \Omega_{\widehat{A}}$ or $j \notin \Omega_{\widehat{A}}$ which shows that $x_i \in \mathfrak{p}$ or $x_j \in \mathfrak{p}$. Hence $x_i x_j \in \mathfrak{p}$. □

Now we consider the polynomial ring $S = K[u_1, \ldots, u_t, v_1, \ldots, v_t]$. Let $I(\widehat{G})$ be the edge ideal of the acyclic reduction $\widehat{G}$ of G in S. Thanks to Lemma 4.10, the next result is a generalization of [13, Remark 3.3]. Since the proof follows by arguments similar to those in [13, Proposition 3.2 and Remark 3.3], we omit it.

Proposition 4.11. *Let G be a very well-covered graph. Suppose G satisfies the conditions* (i) *and* (ii) *in Lemma 4.1. Let $\widehat{G}$ be the acyclic reduction of G with edge ideal $I(\widehat{G}) \subseteq S$. Then $reg\,(R/I(G)) = reg\,(S/I(\widehat{G}))$.*

Now we have all the ingredients that we need and we combine them to prove our second main result.

Theorem 4.12. *Let G be a very well-covered graph. Suppose G satisfies the conditions* (i) *and* (ii) *in Lemma 4.1. Then*

$$reg\,(R/I(G)) = \dim \Delta_{\mathfrak{d}_G} + 1 = a(G).$$

Proof. Let $\widehat{G}$ be the acyclic reduction of G on the vertex set $\{u_1, \ldots, u_t\} \cup \{v_1, \ldots, v_t\}$ with edge ideal $I(\widehat{G}) \subseteq S$. Then $\widehat{G}$ is a Cohen–Macaulay very well-covered graph by Lemma 4.5. Hence by Proposition 4.11, Lemmas 3.4 and 4.7, we have

$$\operatorname{reg}(R/I(G)) = \operatorname{reg}(S/I(\widehat{G})) = a(\widehat{G}) = \dim \Delta_{\mathfrak{d}_{\widehat{G}}} + 1 = \dim \Delta_{\mathfrak{d}_G} + 1 = a(G). \quad \square$$

It was suggested by Villarreal that if G is a Cohen–Macaulay graph, then $G \setminus \{v\}$ is Cohen–Macaulay for some vertex v in G; see [17]. Estrada and Villarreal proved this for Cohen–Macaulay bipartite graphs by showing that there is a vertex $v \in V(G)$ such that $\deg(v) = 1$ ([5, Theorem 2.4]). Van Tuyl and Villarreal proved the same result for sequentially Cohen–Macaulay bipartite graphs in [16, Lemma 3.9]. Using the above fact, Van Tuyl in [15] showed that if G is bipartite, then:

- G is sequentially Cohen–Macaulay if and only if it is vertex decomposable.
- If G is sequentially Cohen–Macaulay, then $\operatorname{reg}(R/I(G)) = a(G)$.

So it is natural to ask the following question:

Question 4.13. *Let G be a sequentially Cohen–Macaulay graph with $2h$ vertices which are not isolated and with $ht\,(I(G)) = h$. Then do we have the following statements?*

(1) *G is vertex decomposable.*
(2) *$reg\,(R/I(G)) = a(G)$.*

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