Comp. & Maths. with Appls. Vol. 1, pp. 377–380. Pergamon Press, 1975. Printed in Great Britain.

LANCZOS' SPLITTING OF THE RIEMANN TENSOR*

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Communicated by E. Y. Rodin

(*Received August*, 1975)

1. INTRODUCTION

Lanczos had a deep interest in the General Theory of Relativity and its study by means of variational principles. He explored many modifications of the Einstein field equations based on Lagrangian functions which were quadratic in the components of the Riemann curvature tensor and whose Euler equations were therefore different from the Einstein ones since the latter may be obtained by using the scalar curvature alone.

In his 1962 paper[1] entitled "The Splitting of the Riemann Tensor" he returned to the study of the algebraically independent parts of the Riemann tensor and to the study of a quadratic Lagrangian whose variation vanishes identically. He paid particular attention to a part of this tensor which he called self-dual and which satisfied other conditions. This part is the conformal or Weyl tensor but Lanczos did not note or use this fact. He was mainly concerned with finding a "generating function" for the conformal tensor. He used the discussion of the variational principle mentioned above to express the conformal tensor in terms of the derivatives of a tensor $H_{\lambda\mu\nu}$. He also discussed the equations this tensor has to satisfy in case the conformal tensor vanished.

It is the purpose of this note to use the two component spinor formalism to describe Lanczos' results. This formalism involves a set of four matrices

$$q_\sigma = \|\gamma_{\sigma\dot{A}B}\| \qquad \sigma = 1,2,3,4.$$
$$A,B = 1,2.$$

such that

$$\|\gamma_{\sigma\dot{A}B}\| = \bar{\epsilon q}_\sigma = \|\bar{\gamma}_{\sigma\dot{B}A}\|$$

where

$$\epsilon = \left\|\begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix}\right\|.$$

$$\bar{q}_\sigma q_\tau + \bar{q}_\tau q_\sigma = 2g_{\sigma\tau} 1_2.$$

and $g_{\sigma\tau}$ is the metric tensor of space-time.

A spin connection may be defined by the requirement that the covariant derivative of q_σ and of ϵ_{AB} vanishes. We also define the matrices

$$p^{\sigma\tau} = \|p^{\sigma\tau}_{\dot{A}\dot{B}}\| = \frac{1}{2}(\bar{q}^\sigma q^\tau - \bar{q}^\tau q^\sigma).$$

It may be shown that

$$p^{\sigma\tau} = -p^{\tau\sigma} = \frac{1}{2}E^{\sigma\tau\mu\nu}p_{\mu\nu} = p^{\upsilon\sigma\tau}.$$

*Work on this paper was supported by the National Science Foundation under grant number GP-31358. It was carried out while the author was in residence at the Mathematical Institute, University of Oxford.

where

$$E^{\sigma\tau\mu\nu} = g^{-1/2}\epsilon^{\sigma\tau\mu\nu}$$

$$E_{\sigma\tau\mu\nu} = g^{1/2}\epsilon_{\sigma\tau\mu\nu}.$$

The ϵ's are the Levi–Civita alternating tensor densities. The E's are pure imaginary quantities since the determinant of the metric tensor, g, is negative.

We shall need the following formulas in the sequel

$$\gamma^{\sigma}_{\dot{A}B}\gamma^{\dot{A}B}_{\tau} = -2\delta^{\sigma}_{\tau},$$

$$\gamma^{\sigma}_{\dot{A}B}\gamma^{\dot{C}D}_{\sigma} = -2\delta^{C}_{A}\delta^{D}_{B},$$

$$\begin{aligned}
\text{trace}\, p_{\lambda\mu}p_{\sigma\tau} &= -p_{\lambda\mu AB}p^{AB}_{\sigma\tau} \\
&= -2(E_{\lambda\mu\sigma\tau} + g_{\lambda\sigma}g_{\mu} - g_{\lambda\tau}g_{\mu\sigma}) \\
p^{\sigma\tau}_{CD}p^{AB}_{\sigma\tau} &= 4(\delta^{A}_{C}\delta^{B}_{D} + \delta^{A}_{D}\delta^{B}_{C}) \\
\gamma^{\sigma\dot{C}D}p_{\sigma\tau AB} &= -(\gamma^{\dot{C}}_{\tau A}\delta^{D}_{B} + \gamma^{\dot{C}}_{\tau B}\delta^{D}_{A}).
\end{aligned}$$

2. THE SPINOR FORM $H_{\lambda\mu\nu}$

Lanczos requires that the tensor $H_{\lambda\mu\nu}$ satisfy the following algebraic conditions

$$H_{\lambda\mu\nu} = -H_{\mu\lambda\nu}$$

$$H_{\lambda\mu\nu}g^{\lambda\nu} = 0 \tag{2.1}$$

$$\frac{1}{2}H_{\lambda\mu\nu}E^{\lambda\mu\nu\rho} = H^{\nu\nu\rho}_{\nu} = 0.$$

Thus there are 16 linearly independent components to this tensor. It is also required to satisfy the six differential equations

$$H^{\alpha}_{\mu\nu;\alpha} = 0, \tag{2.2}$$

where the semi-colon denotes the covariant derivative.

We define

$$H_{ABC\dot{D}} = H_{\lambda\mu\nu}p^{\lambda\mu}_{AB}\gamma^{\nu}_{\dot{D}C} \tag{2.3}$$

and note that as a result of the formulas given at the end of Section 1, that

$$H_{\sigma\tau\rho} + H^{\upsilon}_{\sigma\tau\rho} = -\frac{1}{8}(H_{ABC\dot{D}}\gamma^{\dot{D}C}_{\rho}p^{AB}_{\sigma\tau}. \tag{2.4}$$

These equations are the inverses to eqns (2.3) and $H_{\sigma\tau\rho}$ may be obtained by taking the real part of the right hand side of these equations. It is a consequence of the second and third of eqns (2.1) and of the results given in the introduction that

$$H^{A}_{BA\dot{D}} = \epsilon^{AC}H_{CBA\dot{D}} = 0.$$

That is, the spinor $H_{ABC\dot{D}}$ is symmetric in the first three indices. In other words

$$H_{ABC\dot{D}} = H_{BAC\dot{D}} = H_{ACB\dot{D}} \quad \text{etc.}$$

This implies that there are eight complex independent components to this spinor.

Equations (2.2) are equivalent to

$$(H^{\alpha}_{\mu\nu} + H^{\upsilon\alpha}_{\mu\nu});_{\alpha} = 0.$$

In view of eqns (2.4) we may write these as

$$H_{ABC\dot{D};\rho}\gamma^{\rho\dot{D}C}p^{\mu\nu}_{AB} = 0.$$

On using the results from the introduction we see that these are in turn equivalent to

$$H_{ABC\dot{D};\rho}\gamma^{\rho\dot{D}C} = 0. \tag{2.5}$$

These equations are three complex first order differential equations. They are the necessary and sufficient conditions that the spinor

$$K_{ABCD} = H_{ABC\dot{E};\rho}\gamma^{\rho\dot{E}}_{D} = K_{ABDC}, \tag{2.6}$$

and hence be symmetric in all its indices.

3. THE CONFORMAL SPINOR

Lanczos has shown in [1] that the conformal tensor may be expressed in terms of the tensor $H_{\lambda\mu\nu}$ as follows

$$\begin{aligned} C_{\lambda\mu\sigma\tau} = {} & H_{\lambda\mu\alpha;\beta}\delta^{\alpha\beta}_{\sigma\tau} + H_{\sigma\tau\alpha;\beta}\delta^{\alpha\beta}_{\lambda\mu} \\ & + H^{v}_{\lambda\mu\alpha;\beta}E^{\alpha\beta}_{\sigma\tau} + H^{v}_{\sigma\tau\alpha;\beta}E^{\alpha\beta}_{\lambda\mu}. \end{aligned} \tag{3.1}$$

The conformal spinor is defined as

$$C_{ABCD} = \frac{1}{4}p^{\lambda\mu}_{AB}C_{\lambda\mu\sigma\tau}p^{\sigma\tau}_{CD}, \tag{3.2}$$

and may be shown to be symmetric in all its indices. Thus it has five independent complex components.

When eqns (3.2) are used to substitute for the conformal tensor in eqns (3.1) and the equations and definitions given above are applied we obtain

$$C_{ABCD} = \frac{1}{2}[\gamma^{\beta\dot{F}}_{D}H_{ABC\dot{F};\beta} + \gamma^{\beta\dot{F}}_{C}H_{ABD\dot{F};\beta} + \gamma^{\beta\dot{F}}_{B}H_{CDA\dot{F};\beta} + \gamma^{\beta\dot{F}}_{A}H_{CDB\dot{F};\beta}].$$

If we now make use of eqn (2.5) in the form of eqn (2.6) we obtain

$$C_{ABCD} = 2\gamma^{\rho\dot{F}}_{D}H_{ABC\dot{F};\rho}. \tag{3.3}$$

This is the spinor form of Lanczos' equation relating the conformal tensor to his generating tensor $H_{\lambda\mu\nu}$ satisfying eqns (2.1) and (2.2).

This equation is similar to various spinor equations that occur in relativity. Thus the neutrino equation is

$$\gamma^{\rho\dot{A}}_{B}\psi^{B}_{;\rho} = 0.$$

Maxwell's vacuum equations may be written as

$$\gamma^{F\dot{C}}_{A}F^{AB}_{;\rho} = 0$$

where

$$F^{AB} = F^{\mu\nu}p^{\mu\nu AB},$$

and $F^{\mu\nu}$ is the Maxwell tensor. The Bionchi identities for a vacuum space-time take the form

$$\gamma^{\rho\dot{A}}_{B}C^{BCDE}_{;\rho} = 0.$$

Thus the differential operator

$$\gamma_B^{\rho\dot{A}}(\quad)_{;\rho},$$

plays a role in all three of these equations. A similar differential operator plays a role in relating the generating spinor $H_{ABC\dot{D}}$ to the conformal tensor. However, the fact that the operator which appears in the neutrino (and Dirac) equation also appears in the discussion of the conformal tensor and its generating function is now regarded as an alternative way of describing tensorial equations. Some quantum effects require spinorial quantities whose tensorial representation is quite different from the tensors discussed above. Thus the mere fact that the conformal tensor and its generating function are related by a simple spinorial equation is not in itself an indication that quantum effects will be describable by the quantities dealt with by Lanczos.

REFERENCES

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