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ARTIN–SCHREIER THEORY FOR COMMUTATIVE REGULAR RINGS

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Introduction

In a famous paper [14] of 1926, E. Artin and O. Schreier introduced the notion of "real field" and showed how the condition of reality is connected with the existence of orderings on a field. In a subsequent paper [15] Artin used these results to solve positively Hilbert's 17th problem, whether every positive definite rational function over $\mathbf{Q}$ is a sum of squares of rational functions over $\mathbf{Q}$. In this paper we show that many of the results of Artin and Schreier for fields carry over to commutative (von Neumann) regular rings. The analogue of the notion of "ordering on a field" is called here "good preorder on a regular ring" (Section 3). It turns out that the good preordered regular rings are known in the literature as regular f-rings (see [9, 12]).

In Section 3 we prove the basic quantifier elimination result, using a method of Lipshitz and Saracino [3] (in fact their work convinced me of the possibility of a generalization of the Artin–Schreier theory). The existence of a quantifier elimination opens the way to algebraic applications as A. Tarski, A. Robinson, P. J. Cohen, and S. Kochen have shown in several instances [16, 5, 17, 18]. We use it in Section 4 to solve Hilbert's 17th problem for regular f-rings explicitly, in the following sense.

Let $A = (A_1, \ldots, A_m)$, $X = (X_1, \ldots, X_n)$; for given $\tau(A, X) \in \mathbf{Q}[A, X]$ there exist finitely many $\alpha_i(A)$ and $\tau_i(A, X)$, such that $\tau(A, X) = \Sigma\alpha_i(A)\cdot\tau_i^2(A, X)$, and such that for any regular f-ring R and $a \in R^m$, $\tau(a, X)$ is positive definite over R iff $0 \leq \alpha_i(a)$ for all i.

Here $\alpha_i(A) \in \mathbf{Q}|A|$ and $\tau_i(A, X) \in \mathbf{Q}|A|\langle X\rangle$; these ring constructions $R|X|$ and $R\langle X\rangle$ are introduced in Section 4. $R\langle X\rangle$ looks like $R(X)$ if R is a field, in fact $R(X)$ is a quotient of $R\langle X\rangle$. $R|X|$ is generated over R by $X_1, \ldots, X_n$, using ring operations and an absolute value function $|\ |$.

This result generalizes and gives an elegant formulation of a theorem of L. Henkin (see [19]) who proved the existence of a finite number of possible decompositions of a positive definite $f(a, X) \in (\mathbf{Q}(a))(X)$ as a finite sum $\Sigma\alpha_i(a)f_i^2(a, X)$, $(0 \leq \alpha_i(a))$, for any given $f(A, X) \in \mathbf{Q}[A, X]$.

In Section 5 we introduce sheaves of ordered fields and make our result of Section 4 a little more concrete (following a suggestion of G. Cherlin). In Section 6 two notions of "real closure" of a regular f-ring are defined, and existence and uniqueness is proved for both. For the special case of an ordered field F the *invariant real closure* of F is nothing else than the classical real closure $\bar{F}$, but the *atomless real closure* of F is the ring of locally constant functions defined on the Cantorspace $\mathscr{C}$ and with values in $\bar{F}$. In Section 7, decidability and related properties are discussed for real closed regular f-rings. With respect to Section 3: I was preceded by A. Macintyre and V. Weispfenning [9, 20], in the proof of the existence of a quantifier-elimination. Furthermore, Leonard Lipshitz has obtained Theorems 6.11 and 6.14 independently, see [25].

I wish to thank Greg Cherlin and Jan Treur for stimulating discussions and pointing out errors in an earlier version.

A last remark on notation: the inclusion symbol $\subset$ is also used for the substructure relation: if A and B are rings (with unit), then $A \subset B$ means that A is a subring of B (with the same unit). In general we use the model theoretic notions and notations of [21, 22].

0. Some elementary facts about rings

Conventions. *Rings* are always assumed to be commutative with 1. The *language of rings* contains the binary function symbols + and ·, the unary function symbol −, and the constants 0,1. A *multiplicatively closed subset of a ring* contains 1 but does not contain 0. A *domain* is a ring without zero divisors and with $0 \neq 1$. A *prime ideal* p of a ring R is an ideal such that R/p is a domain (hence $p \neq R$).

It is well known that every multiplicatively closed set contains a prime ideal in its complement, and conversely, that the complement of a prime ideal is multiplicatively closed. From this follow:

Fact 0.1. The minimal prime ideals are exactly the complements of the maximal multiplicatively closed sets. Every prime ideal contains a minimal prime ideal.

Fact 0.2. Suppose S is a maximal multiplicatively closed set in the ring R, and $p = R - S$; then for each $a \in p$: $a^n s = 0$ for some $n \in \mathbf{N}$ and $s \in S$ (otherwise a and S would generate a multiplicatively closed set strictly containing S).

Fact 0.3. $\{x \in R \mid \exists n \in \mathbf{N}\ x^n = 0\}$ is the intersection of all (minimal) prime ideals in R (for if $x^n \neq 0\ \forall n \in \mathbf{N}$, then $\{x^n \mid n \in \mathbf{N} \cup \{0\}\}$ is a multiplicatively closed set, hence is contained in the complement of a prime ideal).

Fact 0.4. If R is a reduced ring (i.e. has no nilpotents other than 0), then the canonical mapping $R \to \Pi_p R/p$, where p varies over the minimal prime ideals of R, is an embedding of R in a product of domains.

Fact 0.5. Conversely, if a ring is embeddable in a direct product of domains, then the ring is reduced.

1. Real rings

Let K be a class of rings such that $A \simeq B,\ B \in K \Rightarrow A \in K$.

Definition 1.1. Let R be a *ring*, I an *ideal* in R.
(a) I is called a *K-prime* if $R/I \in K$,
(b) I is called a *K-radical* if I is an intersection of K-prime ideals,
(c) $K\text{-rad}(I) = \bigcap\{p \mid p \text{ is } K\text{-prime},\ p \supset I\}$,
(d) $K\text{-spec}(R) = \{p \mid p \text{ is } K\text{-prime}\}$.

If K is the class of all domains, these concepts coincide with "prime ideal", "radical ideal", "the radical of an ideal", and "the spectrum of a ring" respectively. These concepts, and the following application, were inspired by a study of [1].

Definition 1.2. A ring R is called *real* iff

$$\forall n \in \mathbf{N}\ \forall x_1 \dots \forall x_n \left(\sum_{i=1}^{n} x_i^2 = 0 \Rightarrow x_1 = \dots = x_n = 0 \right)$$

For fields this is in accordance with the usual definition; a domain is real iff its quotient field is real.

Let K be the class of real domains. Instead of K-prime, K-radical, K-rad, K-Spec, we'll use the terms real prime, real radical, realrad, Realspec. Hence an ideal I of a ring R is real prime iff it is prime and

$$\forall n \in \mathbf{N}\ \forall x_1 \dots \forall x_n \left(\sum_{i=1}^{n} x_i^2 \in I \Rightarrow x_1 \in I \text{ and } \dots \text{ and } x_n \in I \right)$$

Lemma 1.3. *Let R be a real ring. Then R is reduced and all minimal prime ideals are real prime ideals.*

Proof. Suppose $r^n = 0$; we may assume $n = 2^k\ (k \in \mathbf{N})$, and we get

$$r^{2^k} = 0 \Rightarrow (r^{2^{k-1}})^2 = 0 \Rightarrow r^{2^{k-1}} = 0 \Rightarrow \dots \Rightarrow r^2 = 0 \Rightarrow r = 0.$$

Let S be a multiplicatively closed subset of R. Then there is a multiplicatively closed $S' \supset S$ such that $n \in \mathbf{N},\ x_1, \dots, x_n \in S \Rightarrow \sum_{i=1}^{n} x_i^2 \in S'$, namely take

$$S' = S \cup S \cdot \left\{ \sum_{i=1}^{n} x_i^2 \,\middle|\, x_1, \dots, x_n \in S,\ n \in \mathbf{N} \right\}$$

Suppose $0 \in S'$; then

$$0 = s \sum_{i=1}^{n} x_i^2 \quad (s \in S, \quad x_i \in S, \quad n \in \mathbf{N}),$$

hence $0 = \sum_{i=1}^{n} (sx_i)^2$, implying $sx_i = 0$, but also $sx_i \in S$, contradiction! So S' is a multiplicatively closed set. From this it follows that the maximal multiplicatively closed sets S satisfy

$$\left.\begin{array}{l} x_1, \ldots, x_n \in S \\ n \in \mathbf{N} \end{array}\right\} \Rightarrow \sum_{i=1}^{n} x_i^2 \in S.$$

Hence their complements, the minimal prime ideals are real prime ideals. □

This generalizes a result of [2].

Theorem 1.4. *Let R be a ring. Then the following are equivalent:*

(a) *R is a real ring,*

(b) *R is embeddable in a direct product of real fields,*

(c) *R is embeddable in a function ring L^X with L a real field and X a set.*

Proof.

(a) ⇒ (b) follows immediately from Fact 0.4 and Lemma 1.3 (and the fact that the quotient field of a real domain is a real field).

(b) ⇒ (c) follows from the fact that for every set of real fields, there is a real field in which all are embeddable.

(c) ⇒ (a) If L is a real field, then L^X is a real ring and also every subring of L^X is a real ring. □

Theorem 1.5. *Let R be a ring, I an ideal of R. Then*

(i) *I is real radical $\Leftrightarrow \forall n \in \mathbf{N} \forall x_1 \ldots \forall x_n (\sum_{i=1}^{n} x_i^2 \in I \Rightarrow x_1 \in I$ and ... and $x_n \in I)$,*

(ii) *if I is a real radical ideal, then I is a radical ideal, all its minimal prime ideals are real prime ideals and their intersection is I.*

Proof. Ad (i): if I is real radical, then I is intersection of ideals which satisfy the right-hand side of (i), hence I itself satisfies the right-hand side of (i). Conversely if the right-hand side of (i) holds, then R/I is a real ring, and by using the lemma and the well known 1–1 correspondence between the ideals of R/I and the ideals of R containing I, we get the left-hand side of (i). In the same way we prove (ii). □

Theorem 1.6. *Let R be a ring, I an ideal in R. Then*

$$\text{realrad}(I) = \{x \in R \mid \exists k, l \in \mathbf{N} \exists y_1, \ldots, y_l \in R: x^{2k} + y_1^2 + \cdots + y_l^2 \in I\}.$$

Proof. If $x^{2k} + y_1^2 + \cdots + y_l^2 \in I$, then $(x^k)^2 + y_1^2 + \cdots + y_l^2 \in \text{realrad}(I)$, hence $x^k \in \text{realrad}(I)$, so $x \in \text{realrad}(I)$ (use Theorem 1.5). Of course realrad(I) is the smallest real radical ideal containing I, so it suffices to prove that the right-hand side of the equality is a real radical ideal

(a) Suppose $x^{2k}+y_1^2+\cdots+y_l^2\in I$ then for all $r\in R$ $(rx)^{2k}+(r^k y_1)^2+\cdots+(r^k y_l)^2\in I$

(b) Suppose $x^{2k}+y_1^2+\cdots+y_l^2\in I$, $u^{2m}+v_1^2+\cdots+v_n^2\in I$, we may assume $m=k$, then

$$(x+u)^{4k}+(x-u)^{4k}=x^{2k}\cdot S_1+u^{2k}\cdot S_2,$$

where S_1 and S_2 are sums of squares, but $x^{2k}\cdot S_1+\sum y_i^2\cdot S_1\in I$ and $u^{2k}\cdot S_2+(\sum v_j^2)\cdot S_2\in I$, hence

$$(x+u)^{4k}+(x-u)^{4k}+\left(\sum y_i^2\right)\cdot S_1+\left(\sum v_j^2\right)\cdot S_2\in I$$

From (a) and (b) it follows that the right-hand side is an ideal, the easy proof that this ideal is real radical is left to the reader □

In the following Realspec(R) will be endowed with the Zariski topology, i.e. the closed sets are the

$$V(X)=\{p\in \text{Realspec}(R)\mid X\subset p\}(X\subset R)$$

Corollary 1.7. *Let R be a ring. Then Realspec(R) is compact*

Proof. Let $V(X_i)(i\in I)$ be a family of closed sets every finite subfamily of which has a nonvoid intersection. It suffices to prove that the real radical of the ideal generated by $\bigcup_{i\in I}X_i$ is a proper ideal. Suppose 1 is an element of this radical, then, by Theorem 1.6, $1+\sum_{i=1}^{l}y_i^2=\sum_{k=1}^{n}a_kb_k$ with $b_k\in\bigcup_{i\in I}X_i$, but then 1 is an element of the real radical of $Rb_1+\cdots+Rb_n$, hence $V\{b_1,\ldots,b_n\}=\emptyset$, contradiction! □

The Corollary and Theorem 1.4 will be applied in proving that the theory of real rings has a model companion, thus doing for real rings what Lipshitz and Saracino [3], and Carson [4], have done for reduced rings.

2. The model companion of the theory of real rings

We begin with some useful facts on idempotents and regular rings

Definition 2.1. An element x of a ring R is called *idempotent* if $x^2=x$. If p is a prime ideal of R and $x\in R$ is idempotent then either $x/p=1$ or $x/p=0$ in R/p

The set of idempotents of a ring R will be denoted by $B(R)$ and is made a Boolean algebra by defining

$$x \vee y = x + y - xy$$

$$x \wedge y = x \cdot y$$

$$\bar{x} = 1 - x$$

0 is the smallest, 1 the largest element of $B(R)$

There is often a very convenient "geometrical" interpretation of this Boolean algebra: suppose there is given a family $(p_i)_{i \in I}$ of prime ideals of R, such that $\bigcap_{i \in I} p_i = \{0\}$. Then R is canonically embedded into $\prod_{i \in I} R/p_i$, and $B(R)$ is 1–1 mapped into the Boolean algebra $\mathcal{P}I$ by $e \mapsto \{i \in I \mid e/p_i = 1\}$, *and this is even an embedding of Boolean algebras*

Summarized: *if we look at the elements of R as functions defined on I, then the idempotents are the characteristic functions*

Definition 2.2. A ring R is called (Von Neumann) *regular* if $\forall x \exists y (x^2 y = x)$

A regular domain is a field, hence all prime ideals in a regular ring are maximal (hence are also minimal prime ideals). A regular ring is a reduced ring. Hence, if R is regular, then $R \to \prod_{p \in \operatorname{Spec} R} R/p$ is an embedding of R in a product of fields

Lemma 2.3. *Let R be a subring of S. For each minimal prime ideal p of R there exists a minimal prime ideal q of S such that $q \cap R = p$*

Corollary 2.4. *Let the regular ring R be a subring of the regular ring S. Then the map $q \mapsto q \cap R$ $(\operatorname{Spec}(S) \to \operatorname{Spec}(R))$ is onto*

Proof of the lemma. Extend $R \setminus p$ to a maximal multiplicatively closed subset T of S, and put $q = S \setminus T$. Then $q \cap S \subset p$, and $q \cap R$ is prime, hence $q \cap R = p$ □

Let T be the first order theory of real rings. We'll prove that T has a model companion $\bar{T}$ (i.e. $\bar{T}$ is an extension of T in the same language as T, each model of T can be embedded in a model of $\bar{T}$ and conversely, and $\bar{T}$ is model complete, A Robinson [5] has proved that a theory has at most one model companion)

The axioms of $\bar{T}$ are

(i) the axioms of T,

(ii) regularity, i.e. $\forall x \exists y (x^2 y = x)$,

(iii) there are no minimal idempotents, i.e.

$$\forall e (e^2 = e \neq 0 \to \exists f (f \neq e \wedge f \neq 0 \wedge f^2 = f = ef)),$$

(iv) every monic polynomial of odd degree has a root,

(v) $\forall x \exists y (x^2 = y^4)$

Theorem 2.5. *$\bar{T}$ is the model companion of* T

Proof. Let R be a model of $\bar{\mathrm{T}}$, and p be a prime ideal of R. Then R/p is a real closed field: that R/p is a real field follows from the Lemma 1.3, that every monic polynomial of odd degree has a root follows from the axioms (iv), from axiom (v) it follows that every $x \in R/p$ is of the form y^2 or $-y^2$. The rest of the proof follows the lines of Lipshitz and Saracino [3], I only remark that the above corollary simplifies some arguments. □

Convention. If the contrary is not explicitly stated all rings are assumed to be non-trivial, so the theory T of real rings will include the axiom $0 \neq 1$.

Theorem 2.6. *For every two real rings R and S there is a real ring in which both can be embedded (in other words* Mod(T) *has* JEP = *the joint embedding property).*

Proof. (following Lipshitz and Saracino) Let K be a real field, and X, Y nonempty sets and $f: R \to K^X$, $g: S \to K^Y$ embeddings (these exist by Theorem 1.4 (c) and the fact that the class of real fields has JEP). By means of diagonal maps K^X is embedded in $(K^X)^Y \simeq K^{X \times Y}$ and K^Y in $(K^Y)^X \simeq K^{X \times Y}$ hence R and S can both be embedded into $K^{X \times Y}$. □

Using a well known result of A. Robinson [6], we get

Corollary 2.7. *$\bar{T}$ is a complete theory.*

Remark 1. This corollary also follows from the fact that $\bar{\mathrm{T}}$ has a (necessarily unique) prime model, i.e. a model of $\bar{\mathrm{T}}$ which can be embedded in every model of $\bar{\mathrm{T}}$.

Let $\bar{\mathbf{Q}}$ be the real closure of $\mathbf{Q}$, let $C^0(\mathscr{C}, \bar{\mathbf{Q}})$ be the ring of locally constant functions defined on the Cantor space $\mathscr{C}$ and with values in $\bar{\mathbf{Q}}$. Then $C^0(\mathscr{C}, \bar{\mathbf{Q}})$ is the prime model of $\bar{\mathrm{T}}$. $C^0(\mathscr{C}, \bar{\mathbf{Q}})$ is not a minimal model of $\bar{\mathrm{T}}$. This will be proved in Section 6.

Remark 2. Regular rings R without minimal idempotents have several peculiar properties.

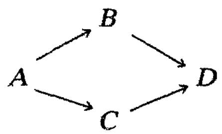
(a) R is neither noetherian nor artinian.

Proof. There exist a strictly descending sequence of idempotents $(e_n)_{n \in \mathbf{N}}$ and a strictly ascending sequence of idempotents $(f_n)_{n \in \mathbf{N}}$ and these give rise to a strictly descending sequence of ideals $(e_n R)_{n \in \mathbf{N}}$ and to a strictly ascending sequence of ideals $(f_n R)_{n \in \mathbf{N}}$.

(b) If $f \in R[X]$ has two distinct roots in R, then it has infinitely many roots in R.

Proof. Let $\alpha_1 \neq \alpha_2$ be roots of f, then $e\alpha_1 + (1-e)\alpha_2$ is also a root for every idempotent e, and in this way we get infinitely many roots.

Remark 3. Mod(T) does not have AP (the Amalgamation Property). Lipshitz and Saracino [3], give an example of three real rings A, B, C such that $A \subset B$, $A \subset C$, and such that there are no reduced ring D and embeddings $B \to D$, $C \to D$ such that



commutes.

3. Preordered regular rings

Definition 3.1. A preorder on a ring R is a *subset* O of R such that

(i) $O + O \subset O$,
(ii) $O \cdot O \subset O$,
(iii) $O \cap (-O) = \{0\}$,
(iv) $\forall a \in R \; a^2 \in O$.

This terminology is taken from [7].

A preorder O on a ring R defines a partial ordering $\leqslant$ by $a \leqslant b \Leftrightarrow b - a \in O$, which satisfies $a \leqslant b \Rightarrow a + c \leqslant b + c$, $a \leqslant b$ and $0 \leqslant c \Rightarrow ac \leqslant bc$ (note that $a \in O \Leftrightarrow 0 \leqslant a$).

Example 1. The set of sums of squares in a real ring is a pre-order.

Example 2. Suppose R is a field. If O is a preorder and $-x \notin O$, then $O + Ox$ is again a preorder on R.

This implies (via Zorn's lemma) that the maximal preorders on R are precisely the orderings on R (where an ordering is identified with the set of its nonnegative elements), and also that $O \subset R$ is a preorder if and only if O is the intersection of a nonempty collection of orderings.

Remark. A preordered reduced ring is a real ring.

Now the fundamental lemma.

Lemma 3.2. *Let (R, O) be a preordered ring, let $\underline{p}$ be a minimal prime ideal of R. Then $O/\underline{p} = \{a/\underline{p} \mid a \in O\}$ is a preorder on $R/\underline{p}$.*

Proof. It suffices to check that

$$\left.\begin{array}{l} a+b \in \underline{p} \\ a, b \in O \end{array}\right\} \Rightarrow a \in \underline{p}$$

So let $a+b \in \underline{p}$, $a, b \in O$ Fact 0 2 implies that there is $n \in \mathbb{N}$ and $x \in R$ $\underline{p}$ such that $(a+b)^n x = 0$ Let

$$a_1 = a^n, \qquad b_1 = \sum_{i=1}^{n} \binom{n}{i} a^{n-i} b^i,$$

then

$$a_1 + b_1 \in \underline{p},\ a_1,\ b_1 \in O, \qquad (a_1 + b_1)x = 0$$

Hence $a_1 x = -b_1 x$, so $a_1 x^2 = -b_1 x^2$, but also $a_1 x^2$, $b_1 x^2 \in O$, implying $a_1 x^2 = 0$, hence $a_1 \in \underline{p}$, i e $a^n \in \underline{p}$, so $a \in \underline{p}$ □

Theorem 3.4. *Every model R of $\bar{T}$ has a unique preorder O, namely O is the set of squares of R*

Proof. $O/\underline{p}$ is for each prime ideal $\underline{p}$ a preorder on the real closed field $R/\underline{p}$, hence consists exactly of the squares of $R/\underline{p}$ Hence every element $a \in O$ is "locally" a square, and a compactness argument proves that a is a square That the set of squares is indeed a preorder is proved in the same way □

We have seen that for preordered regular rings (R, O) with $R \models \bar{T}$ the following holds $\forall \underline{p} \in \mathrm{Spec}(R)$ $(R/\underline{p}, O/\underline{p})$ is an ordered field

We'll now characterize those preordered regular rings (R, O) which have this property

Definition 3.5. A good preorder on a regular ring R is a *preorder* O on R such that $\forall a \in R\ \exists e \in B(R)\ (ea \in O$ and $-(1-e)a \in O)$

First a small

Lemma 3.6. *Let (R, O) be a preordered regular ring Then for all $a \in R$ $a \in O \Leftrightarrow \forall \underline{p} \in \mathrm{Spec}(R)$ $a/\underline{p} \in O/\underline{p}$ Moreover, if $a/\underline{p} \in O/\underline{p}$ $(a \in R,\ \underline{p} \in \mathrm{Spec}(R))$, then there is an idempotent e with $e/\underline{p} = 1$ and $ea \in O$*

Proof. We first prove the second statement $a/\underline{p} \in O/\underline{p}$ implies that there is $a^1 \in O$ s t $a/\underline{p} = a^1/\underline{p}$, let e be the idempotent on which a and a^1 are equal (see the lemma of Lipshitz and Saracino [3], for the meaning of this) Then $e/\underline{p} = 1$ and $ea = ea^1 \in O$ The implication $\Rightarrow$ of the first statement holds by definition, and $\Leftarrow$ follows by a compactness argument from the second statement

Theorem 3.7. *Let (R, O) be a preordered regular ring Then we have O is a good preorder $\Leftrightarrow \forall \underline{p} \in \mathrm{Spec}(R)$ $(R/\underline{p}, O/\underline{p})$ is an ordered field*

Proof. $\Rightarrow$ let $a \in R$, $p \in \mathrm{Spec}(R)$, we have to prove that $a/p \in O/p$ or $-a/p \in O/p$ Choose $e \in B(R)$ s t $ea \in O$ and $-(1-e)a \in O$ If $e \notin p$, then $a/p = ea/p \in O/p$, if $e \in p$ then $1-e \notin p$ and $-a/p = -(1-e)a/p \in O/p$

$\Leftarrow$ let $a \in R$, choose for each $p \in \mathrm{Spec}(R)$ an idempotent e_p s t $e_p/p = 1$, and $e_p \quad a \in O$ if $a/p \notin O/p$, $-e_p \quad a \in O$ if $-a/p \in O/p$ (such an e_p exists by the second statement of the lemma)

Finitely many of these idempotents $e_1, \quad , e_n, e_{n+1}, \quad , e_{n+m}$ cover $\mathrm{Spec}(R)$, (note that we often identify $e \in B(R)$ with $\{p \in \mathrm{Spec}(R) \mid e \notin p\}$), where for $1 \leq i \leq n$ $e_i a \in O$, for $n+1 \leq i \leq n+m$ $-e_i a \in O$, and we may also assume that they are pairwise disjoint (for if necessary, we replace them by smaller idempotents), then the following holds for

$$e = \sum_{i=1}^{n} e_i \quad e \in B(R), \quad ea \in O, \qquad 1-e = \sum_{i=n+1}^{n+m} e_i,$$

hence $-(1-e)a \in O$ □

Corollary 3.8. *If $R \models \tilde{T}$, then the unique preorder on R is good*

Definition 3.9. A preorder O on a ring R is called archimedean iff $\forall r \in R\ \exists n \in \mathbf{N}$ $r \leq n$

Theorem 3.10. *Let (R, O) be a good preordered regular ring Then the following are equivalent*

(i) *O is archimedean,*

(ii) *$\forall p \in \mathrm{Spec}(R)$ $(R/p, O/p)$ is an archimedean ordered field,*

(iii) *(R, O) is embeddable in the ring of bounded functions $f \quad X \to \mathbf{R}$ (preordered by $f \geq 0 \Leftrightarrow \forall x \in X\ f(x) \geq 0$) for some nonempty X*

Proof. (i) $\Rightarrow$ (ii) is trivial Assume (ii), from the Lemma 3 6 follows that $(R, O) \to \prod_{p \in \mathrm{Spec}(R)} (R/p, O/p)$ is an embedding but $(R/p, O/p)$ is (uniquely) embedded in $\mathbf{R}$, hence we have a canonical embedding $(R, O) \to \mathbf{R}^{\mathrm{Spec}(R)}$, where $\mathbf{R}^{\mathrm{Spec}(R)}$ is preordered by $f \geq 0 \Leftrightarrow \forall x \in \mathrm{Spec}(R)\ f(x) \geq 0$ Moreover, if $r \in R$, $p \in \mathrm{Spec}(R)$, then there is $n_p \in \mathbf{N}$, with $r/p \leq n_p/p$, and with the lemma we can find $e_p \in B(R)$, $e/p = 1$, s t $re_p \leq n_p$ Finitely many of these e_p's cover $\mathrm{Spec}(R)$, and taking n to be the maximum of the corresponding n_p's, we get $r \leq n$, and this holds of course also for their images in $\mathbf{R}^{\mathrm{Spec}(R)}$ We have proved (iii)

(iii) $\Rightarrow$ (i) is trivial □

Theorem 3.11. *The class of good preordered regular rings has the amalgamation property*

Proof. Let (R, O), (S, P), (T, Q) be good preordered regular rings with $(R, O) \subset (S, P)$, $(R, O) \subset (T, Q)$ By the Corollary 2 4, it is easy to see that we can find an index set I and families of prime ideals $(\mathfrak{p}_i)_{i\in I}$ of S, $(\mathfrak{q}_i)_{i\in I}$ of T, $(\mathfrak{m}_i)_{i\in I}$ of R such that $\mathrm{Spec}(S)=\{\mathfrak{p}_i \mid i\in I\}$, $\mathrm{Spec}(T)=\{\mathfrak{q}_i \mid i\in I\}$, $\mathrm{Spec}(R)=\{\mathfrak{m}_i \mid i\in I\}$, and for all $i\in I$ $\mathfrak{p}_i\cap R=\mathfrak{q}_i\cap R=\mathfrak{m}_i$

Hence we have for each $i\in I$ embeddings $(R/\mathfrak{m}_i, O/\mathfrak{m}_i)\to(S/\mathfrak{p}_i, P/\mathfrak{p}_i)$ and $(R/\mathfrak{m}_i, O/\mathfrak{m}_i)\to(T/\mathfrak{q}_i, Q/\mathfrak{q}_i)$ of ordered fields, hence for each $i\in I$ we can find an ordered field (L_i, O_i) and embeddings such that

$$\begin{array}{ccccc} & & (S/\mathfrak{p}_i, P/\mathfrak{p}_i) & & \\ & \nearrow & & \searrow & \\ (R/\mathfrak{m}_i, O/\mathfrak{m}_i) & & & & (L_i, O_i) \\ & \searrow & & \nearrow & \\ & & (T/\mathfrak{q}_i, Q/\mathfrak{q}_i) & & \end{array}$$

commutes Putting together all these commutative diagrams we get a commutative diagram

$$\begin{array}{ccccccc} & & (S,P) & \longrightarrow & \prod_{i\in I}(S/\mathfrak{p}_i, P/\mathfrak{p}_i) & & \\ & \nearrow & & \nearrow & & \searrow & \\ (R,O) & \longrightarrow & \prod_{i\in I}(R/\mathfrak{m}_i, O/\mathfrak{m}_i) & & & & \prod_{i\in I}(L_i, O_i) \\ & \searrow & & \searrow & & \nearrow & \\ & & (T,Q) & \longrightarrow & \prod_{i\in I}(T/\mathfrak{q}_i, Q/\mathfrak{q}_i) & & \end{array}$$

hence an amalgamation diagram

$$\begin{array}{ccccc} & & (S,P) & & \\ & \nearrow & & \searrow & \\ (R,O) & & & & \prod_{i\in I}(L_i, O_i) \\ & \searrow & & \nearrow & \\ & & (T,Q) & & \end{array} \qquad \square$$

Remark 1. Note that we used that ordered fields are good preordered regular rings, and that the class of good preordered regular rings is closed under direct products (both facts are easy)

Remark 2. This method works also for the class of regular rings, for which Lipshitz and Saracino state the amalgamation property However the reference in their proof to a result of P M Cohn is in my opinion not correct, because Cohn proves amalgamation for a wider class of (not necessarily commutative) rings

The language $L(O)$ of preordered rings is the language L of rings, augmented by one unary predicate symbol Q However, to get better model theoretic results, we have to change the language (just as Lipshitz and Saracino in [3] do) We introduce two new unary function symbols $^{-1}$, $|\ \ |$, which we define in the theory $T(Q)$ of good preordered regular rings by the defining axioms

$$\begin{cases} x^2 \cdot x^{-1} = x \\ x \cdot (x^{-1})^2 = x^{-1} \end{cases} \qquad \begin{cases} |x|^2 = x^2 \\ Q(|x|) \end{cases}$$

These functions are indeed uniquely defined in every good preordered regular ring by these axioms. Let $T(Q, ^{-1}, |\ |)$ be the extension of definitions of $T(Q)$ just described. But we can also go the other direction: we can axiomatize the theory of good preordered regular rings in the language $L(^{-1}, |\ |)$ (this axiomatization is denoted by $T(^{-1}, |\ |)$) in such a way that if we define the unary predicate symbol Q by $Q(x) \leftrightarrow x = |x|$, then the corresponding extension by definitions $T(^{-1}, |\ |)(Q)$ is equivalent with $T(Q, ^{-1}, |\ |)$. The main advantage of this is that $T(^{-1}, |\ |)$ is a universal theory (the reader can easily provide a set of universal axioms). Note however that the notions of embedding and homomorphism (between two good preordered regular rings) do not change. Let $\bar{T}(^{-1}, |\ |)$ be the corresponding extension by definitions of $\bar{T}$. Now $\bar{T}(^{-1}, |\ |)$ is the model companion of the universal theory $T(^{-1}, |\ |)$, which has the amalgamation property, and by results of A. Robinson [5] and Eklof and Sabbagh [8] we get

Corollary 3.12. *$\bar{T}(^{-1}, |\ |)$ is the model completion of $T(^{-1}, |\ |)$, and admits elimination of quantifiers.*

Remark. After obtaining these results, I read A. Macintyre's 'Model completeness for sheaves of structures' [9], where weaker but more general results are proved. The discussion on pages 86, 87 and 88 of his paper establishes essentially that $\bar{T}(^{-1}, |\ |)$ is complete and the model companion of $T(^{-1}, |\ |)$, but a slightly different terminology is used, which I'll now explain. His notion of regular f-ring is equivalent with the notion of good preordered regular ring in the following sense:

If $(R, \leq, \vee, \wedge)$ is a regular f-ring, then $(R, ^{-1}, |\ |)$ is a good preordered regular ring (where $^{-1}$ and $|\ |$ are defined by: $x^2 x^{-1} = x$, $x(x^{-1})^2 = x^{-1}$, $|x| = (x \vee 0) + (-x \vee 0)$) *and conversely: the partial ordering $\leq$ of a good preordered regular ring $(R, ^{-1}, |\ |)$ defines a lattice structure $(\leq, \vee, \wedge)$ on R such that $(R, \leq, \vee, \wedge)$ is a regular f-ring.*

As his terminology is more standard we'll adopt the

Convention. In the following good preordered regular rings will be called *regular f-rings*.

Added in proof: recently Weispfenning's paper [20] appeared where the results of [9] are generalized still further. It contains also a general version of Theorem 3.11.

4. Positive definite functions over regular f-rings

Up till now we excluded the trivial ring $\{0\}$, for the sake of making $\bar{T}$ complete. However, it will be convenient in the following to include $\{0\}$ in our considerations, so $\{0\}$ will count as a real ring, regular ring, regular f-ring and even as a model of $\bar{T}(^{-1}, |\ |)$. This has the following effect.

Lemma 4.1. (I) $\operatorname{Mod} T(^{-1})$, *the class of real regular rings, is an equational class.*
(II) $\operatorname{Mod} T(^{-1}, |\ |)$, *the class of regular f-rings, is an equational class.*

Proof. Ad (I). We can axiomatize $T(^{-1})$ by
(1) the axioms for rings (which can be expressed by equations),
(2) $x^2 x^{-1} = x$, $x(x^{-1})^2 = x^{-1}$,
(3) $(x^2+\sum_{i=1}^{k} y_i^2)(x^2+\sum_{i=1}^{k} y_i^2)^{-1} x = x$ (for each $k \in \omega$).
For suppose a ring $(R, ^{-1})$ satisfies (1), (2) and (3). If $a^2 + \sum_{i=1}^{k} b_i^2 = 0$, then (3) implies that $a = 0$, hence $(R, ^{-1})$ is indeed a real regular ring. Conversely, let $(R, ^{-1})$ be a real ring, then for any prime ideal $\underline{p}$ (and $x, y_1, \ldots, y_k \in R$) either $x \equiv 0 (\operatorname{mod} \underline{p})$ and hence

$$\left(x^2+\sum_{i=1}^{k} y_i^2\right)\left(x^2+\sum_{i=1}^{k} y_i^2\right)^{-1} x \equiv x (\operatorname{mod} \underline{p})$$

or $x \not\equiv 0 (\operatorname{mod} \underline{p})$ implying $x^2 + \sum_{i=1}^{k} y_i^2 \not\equiv 0 (\operatorname{mod} \underline{p})$ and hence

$$\left(x^2+\sum_{i=1}^{k} y_i^2\right)\left(x^2+\sum_{i=1}^{k} y_i^2\right)^{-1} x \equiv x (\operatorname{mod} \underline{p}).$$

We have proved that the equations (3) hold locally, hence they hold.
Ad (II). We can axiomatize $T(^{-1}, |\ |)$ by
(1) The ring axioms,
(2) $x^2 x^{-1} = x$, $x(x^{-1})^2 = x^{-1}$,
(3) $|x| + |y| = ||x| + |y||$, $|x \cdot y| = |x| \cdot |y|$, $x^2 = |x|^2$, $(|x| + |y|)$
$(|x| + |y|)^{-1} x = x$.
In the same way as in case (I) we prove that a regular f-ring satisfies (1), (2) and (3). Conversely, if $(R, ^{-1}, |\ |)$ satisfies (1), (2) and (3), then $O = \{|x| : x \in R\}$ is a preorder [by (3)], and even a good preorder (by the equation $x^2 = |x|^2$) and $|\ |$ is just the absolute value function induced by O; hence $(R, ^{-1}, |\ |)$ is an f-regular ring. □

It is well known that one can define the polynomial ring $R[X_1, \ldots, X_n]$ in n variables up to isomorphism over the ring R as follows:
$R[X_1, \ldots, X_n]$ is a ring extension of R generated over R by n distinguished elements $X_1, \ldots, X_n$ such that for each ring morphism $\phi: R \to S$ and each n-tuple $(a_1, \ldots, a_n) \in S^n$, there is a unique extension $\psi: R[X_1, \ldots, X_n] \to S$ of ϕ with $\psi(X_i) = a_i$ $(1 \leq i \leq n)$.
The existence and uniqueness of such an extension follows from the fact that the class of rings is an equational class. From the same universal-algebraic arguments it follows:
(I) *Each real regular ring R has a real regular extension $R\langle X_1, \ldots, X_n \rangle$ generated*

over R by n distinguished elements $X_1, \ldots, X_n$ *such that for each homomorphism* $\phi: R \to S$ *with S real regular and each n-tuple* $(a_1, \ldots, a_n) \in S^n$ *there is a unique extension* $\psi: R\langle X_1, \ldots, X_n\rangle \to S$ *of* ϕ *with* $\psi(X_i) = a_i$ $(1 \leq i \leq n)$. (Note that $R\langle X_1, \ldots, X_n\rangle$ is generated over R by $X_1, \ldots, X_n$ using the ring operations *and the real regular ring operation* $^{-1}$.)

(II) *Each regular f-ring R has a regular f-extension* $R|X_1, \ldots, X_n|$ *generated over R by n distinguished elements* $X_1, \ldots, X_n$ *such that for each homomorphism* $\phi: R \to S$ *with S a regular f-ring and each n-tuple* $(a_1, \ldots, a_n) \in S^n$ *there is a unique extension* $\psi: R|X_1, \ldots, X_n| \to S$ *of* ϕ *with* $\psi(X_i) = a_i$ $(1 \leq i \leq n)$. (Note that $R|X_1, \ldots, X_n|$ is generated over R by $X_1, \ldots, X_n$ using the ring operations *and the regular f-ring operations* $^{-1}$ *and* $|\ |$.)

In case (I) as well as in case (II) the uniqueness of the extension (up to R-isomorphism) follows easily from the existence.

Now we have to make a number of essentially trivial remarks.

Note that a term $\tau(X_1, \ldots, X_n)$ in the language $L(^{-1}, \underline{a})_{a \in R}$ (with a constant $\underline{a}$ for each $a \in R$, $R \models T(^{-1})$) denotes an element of $R\langle X_1, \ldots, X_n\rangle$. Similarly for terms in the language $L(^{-1}, |\ |, \underline{a})_{a \in R}$ and $R \models T(^{-1}, |\ |)$.

It is well known that a polynomial $\tau \in R[X_1, \ldots, X_n]$ defines a polynomial function $S^n \to S$ for each ring extension S of R; the image of $a \in S^n$ is denoted by $\tau(a)$.

In the same way we have:

(1) If $R \subset S$, R, S real regular rings, $\tau \in R\langle X_1, \ldots, X_n\rangle$, then for each $a \in S^n$, the image of τ under the homomorphism $R\langle X_1, \ldots, X_n\rangle \to S$ which fixes R and maps X_i onto a_i, is denoted by $\tau(a)$. So $\tau \in R\langle X_1, \ldots, X_n\rangle$ defines a map $a \mapsto \tau(a)$ $(S^n \to S)$.

(2) Similarly: If $R \subset S$, R, S regular f-rings, then $\tau \in R|X_1, \ldots, X_n|$ defines a map $S^n \to S$, and the image of $a \in S^n$ under this map is denoted by $\tau(a)$.

If R is an infinite field, then $f \in R[X_1, \ldots, X_n]$ *is uniquely determined by its corresponding polynomial function* $R^n \to R$, *however this is not true in the case of real regular rings*: $\tau = (X^2-2)(X^2-2)^{-1} \in \mathbf{Q}\langle X\rangle$ defines the map $r \mapsto 1$ $(\mathbf{Q} \to \mathbf{Q})$ as does $1 \in \mathbf{Q}\langle X\rangle$, but $\tau \neq 1$, because τ does not define a constant map from $\mathbf{Q}(\sqrt{2})$ to $\mathbf{Q}(\sqrt{2})$. Still we have:

Lemma 4.2. *Let R be a regular f-ring,* $R \subset S \models \bar{T}(^{-1}, |\ |)$. *Then for* $\tau \in R|X_1, \ldots, X_n|$ *we have*:

$$\tau = 0 \Leftrightarrow \forall a \in S^n: \tau(a) = 0.$$

Proof. $\Rightarrow$ is trivial.

$\Leftarrow$: regular f-rings have amalgamation, so we may assume even that $R|X_1, \ldots, X_n| \subset S$; then the n-tuple $(X_1, \ldots, X_n)$ is an element of S, so $\tau(X_1, \ldots, X_n) = 0$, but $\tau(X_1, \ldots, X_n) = \tau$, (this can be proved by induction), and we have proved the statement for $\tau \in R|X_1, \ldots, X_n|$. □

Lemma 4.3. (1) *If R is a real ring, then $R[X_1, \ldots, X_n]$ is a real ring (Hence $R[X_1, \ldots, X_n]$ can be embedded in a real regular ring.)*

(2) *Let (R, O) be a preordered regular ring. Then (R, O) can be embedded in a (good) preordered regular ring (S, P) with $S \models \bar{\mathrm{T}}$.*

(3) *Let R be a regular f-ring. Then*

$$O \overset{\text{def}}{=} \left\{ \sum_{i=1}^{k} a_i \tau_i^2 \,\middle|\, k \in \omega,\ 0 \leq a_i \in R,\ \tau_i \in R[X_1, \ldots, X_n] \right\}$$

is a preorder on $R[X_1, \ldots, X_n]$ which extends the (good) preorder of R.

Proof. (1) It will suffice to prove this for $n = 1$. If $\sum_{i=1}^{k} f_i^2 = 0$ ($f_i \in R[X]$) and not all f_i are zero, then let $a_i \in R$ be the coefficient of X^n in f_i and take n maximal with the property that some $a_i \neq 0$, then we have $\sum_{i=1}^{k} a_i^2 = 0$, contradiction!

(2) $(R, O) \to \prod_{p \in \mathrm{Spec}(R)} (R/p, O/p)$ is an embedding by the lemma of Theorem 3.7. But O/p is an intersection of a family of orderings $(O_{p,i})_{i \in I_p}$, $I_p \neq \emptyset$ (see Example 2 of Section 3), hence for each p $(R/p, O/p) \to \prod_{i \in I_p} (R/p, O_{p,i})$ is an embedding of $(R/p, O/p)$ in a product of ordered fields.

Let for each (p, i) with $i \in I_p$ $(R_{p,i}, P_{p,i})$ be an extension of $(R/p, O_{p,i})$ with $R_{p,i} \models \bar{\mathrm{T}}$. Then $(R/p, O/p)$ is naturally embedded in $\prod_{i \in I_p} (R_{p,i}, P_{p,i})$ and finally (R, O) is naturally embedded in

$$\prod_{p \in \mathrm{Spec}(R)} \prod_{i \in I_p} (R_{p,i}, P_{p,i}) \overset{\text{def}}{=} (S, P)$$

and $S \models \bar{\mathrm{T}}$. □

(3) It will suffice to prove that

$$\sum_{i=1}^{k} a_i \tau_i^2 = 0 \Rightarrow a_i \tau_i^2 = 0 \quad \forall 1 \leq i \leq k,$$

so suppose $\sum_{i=1}^{k} a_i \tau_i^2 = 0$, take a regular f-extension S of R with $S \models \bar{\mathrm{T}}$. Then $\sum_{i=1}^{k} a_i \tau_i^2(s) = 0\ \forall s \in S^n$, hence $a_i \tau_i^2(s) = 0\ \forall s \in S^n$ (note that we have assumed all $a_i \geq 0$), which by Lemma 4.2 implies that $a_i \tau_i^2 = 0$. $O \cap R$ is a preorder on R, which contains the given good preorder on R, and as a good preorder is a maximal preorder, $O \cap R$ is equal to the given preorder on R. □

Before coming to our main topic we indicate which inclusion relations hold between the rings introduced so far.

Theorem 4.4. (A) *If R and S are real regular rings and $R \subset S$ then the following relations hold:*

(1) $R[X_1, \ldots, X_n] \subset R\langle X_1, \ldots, X_n\rangle$.

(2) *Let R be $\mathbf{Q}(\sqrt{2})$, S be $\mathbf{Q}(\sqrt[4]{2})$, then $R\langle X\rangle \not\subset S\langle X\rangle$.*

(B) *If R and S are regular f-rings and $R \subset S$, then*

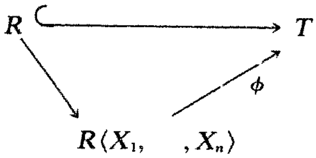
(1) $R[X_1, \ldots, X_n] \subset R|X_1, \ldots, X_n|$

(2) $R[X_1, \ldots, X_n] \subset S[X_1, \ldots, X_n]$

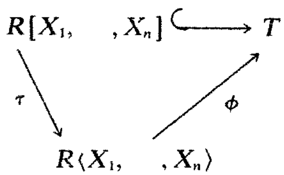
(3) *Let R be $\mathbf{Q}(\sqrt{2})$, ordered by $\sqrt{2} > 0$, then $R\langle X\rangle \not\subset R[X]$.*

(The inclusions are all supposed to be induced by canonical homomorphisms)

Proof. (A) (1) Let ϕ $R[X_1, \ldots, X_n] \to R\langle X_1, \ldots, X_n\rangle$ be the canonical mapping fixing R and the X_i, $R[X_1, \ldots, X_n]$ is real (Lemma 4 3), hence is a subring of a real regular ring T Now there is a unique homomorphism ψ $R\langle X_1, \ldots, X_n\rangle \to T$ fixing the X_i such that



commutes But then



also commutes because $\phi \circ \tau$ fixes R and the X_i Hence τ is 1–1

(2) $(X^2+\sqrt{2})(X^2+\sqrt{2})^{-1}$ considered as an element of $S\langle X\rangle$ is equal to the identity 1, but $(X^2+\sqrt{2})(X^2+\sqrt{2})^{-1}$ considered as an element of $R\langle X\rangle$ is not equal to 1

R is a subfield of the real field $\mathbf{Q}(\sqrt{-\sqrt{2}})$ and

$$((X^2+\sqrt{2})(X^2+\sqrt{2})^{-1})(\sqrt{-\sqrt{2}}) = 0$$

Hence the canonical map $R\langle X\rangle \to S\langle X\rangle$ is not 1–1

(B) (1) As in Lemma 4 3 (1) one proves that

$$O = \left\{ \sum_{i=1}^{k} a_i \tau_i^2 \,\middle|\, 0 \leq a_i \in R,\ \tau_i \in R[X_1, \ldots, X_n] \right\}$$

is a preorder on $R[X_1, \ldots, X_n]$ (although one has to be a little more careful), hence for each minimal prime ideal $\underline{p}$ of $R[X_1, \ldots, X_n]$, $O/\underline{p}$ is a preorder on $R[X_1, \ldots, X_n]/\underline{p}$

Now we use the following fact which is easy to prove

If (D, P) is a preordered domain, then $(Qt(D), Qt(P))$ is a preordered field and $P \subset Qt(P)$, where $Qt(D)$ is the quotient field of D, and

$$Qt(P) = \{a/b \mid a \in P,\ b \in P \setminus \{0\}\}$$

In fact $Qt(P)$ is the smallest preorder on $Qt(D)$ containing P

So let $T_p = (Qt(R[X_1, \ldots, X_n]/p), Qt(O/p))$ (p a minimal prime), then T_p is a preordered regular ring, as is $\Pi_p T_p$, using Lemma 4.3 (2) $\Pi_p T_p$ is embedded in a regular f-ring T, and it is easy to see that $R \to T$ (the composition of $R \to R[X_1, \ldots, X_n] \to \Pi_p R[X_1, \ldots, X_n]/p \to \Pi_p T_p \to T$) is an embedding of regular f-rings. Now the proof proceeds as in (A) (1).

(2) is easy, using Lemma 4.2.

(3) As in (A) (2) we prove that $(X^2 + \sqrt{2})(X^2 + \sqrt{2})^{-1}$, considered as an element of $R|X|$, is the identity, but $(X^2 + \sqrt{2})(X^2 + \sqrt{2})^{-1}$ considered as an element of $R\langle X\rangle$ is not the identity. □

We have the following necessary and sufficient condition for

$$R\langle X_1, \ldots, X_n\rangle \subset R|X_1, \ldots, X_n|$$

Proposition 4.5. *Let R be a regular f-ring.*

(a) *If $R\langle X\rangle \subset R|X|$ then the preorder $\{\sum_{i=1}^k a_i^2 \mid k \in \omega, a_i \in R\}$ equals the given good preorder of R.*

(b) *If $\{\sum_{i=1}^k a_i^2 \mid k \in \omega, a_i \in R\}$ equals the given good preorder of R, then $R\langle X_1, \ldots, X_n\rangle \subset R|X_1, \ldots, X_n|$ $(\forall n \in \omega)$.*

Proof. (a) Suppose $0 \leq r \in R$, but $r \notin O =^{\text{def}} \{\sum_{i=1}^k a_i^2 \mid k \in \omega,\ a_i \in R\}$. Then for some prime p of R, $r/p \notin O/p$. We easily see that one can change r in such a way that r becomes a unit in R without changing the value of r/p. Now there exists an ordering O_p on R/p such that $-r/p \in O_p$, let S be the real extension field $(R/p)(\sqrt{-r/p})$ of R/p, we see that $(X^2 + r)(X^2 + r)^{-1}$ is mapped onto zero by the map $R\langle X\rangle \to S$ which extends $R \to R/p$ and maps X onto $\sqrt{-r/p}$, hence $(X^2 + r)(X^2 + r)^{-1}$, as an element of $R\langle X\rangle$, is not the identity, but $(X^2 + r)(X^2 + r)^{-1}$ is the identity of $R|X|$.

(b) Under the stated condition $O = \{\sum_{i=1}^k \tau_i^2 \mid k \in \omega,\ \tau_i \in R\langle X_1, \ldots, X_n\rangle\}$ is a preorder on $R\langle X_1, \ldots, X_n\rangle$ extending the given preorder on R. Hence by Lemma 4.2 there is an extension (S, P) of $(R\langle X_1, \ldots, X_n\rangle, O)$, with $S \models \bar{\mathrm{T}}$, consequently (S, P) is also a regular f-extension of R. Now we have a unique regular f-ring morphism $\psi: R|X_1, \ldots, X_n| \to (S, P)$ fixing R and the X_i. But this implies that $\psi \circ \iota$ coincides with the embedding $R\langle X_1, \ldots, X_n\rangle \to S$ (where ι is the canonical map $R\langle X_1, \ldots, X_n\rangle \to R|X_1, \ldots, X_n|$) and this implies that ι is 1–1. □

Now we are going to discuss positive definiteness.

Definition 4.6. Let R be a regular f-ring, S a regular f-extension of R with $S \models \bar{\mathrm{T}}$. Then $\tau \in R|X_1, \ldots, X_n|$ is positive definite $\Leftrightarrow^{\text{def}} \forall s \in S^n\ \tau(s) \geq 0$.

Remark. Using the fact that $\bar{\mathrm{T}}(^{-1}, |\ |)$ is the model completion of $\mathrm{T}(^{-1}, |\ |)$ it doesn't matter which S we take, in a later section I will prove that every regular f-ring R

has a unique prime model extension (in the sense of "Saturated Model Theory" of G. Sacks), so we could have taken S as this prime model extension of R. Still another alternative is to define $\tau \in R|X_1, \ldots, X_n|$ to be positive definite iff for each regular f-extension S of R we have $\forall s \in S^n\ \tau(s) \geq 0$.

A rather trivial fact: $\tau \in R|X_1, \ldots, X_n|$ is positive definite iff $\tau = |\tau|$, this follows from $\tau = \tau(X_1, \ldots, X_n)$ for each $\tau \in R|X_1, \ldots, X_n|$. Now $\tau \in R|X_1, \ldots, X_n|$ may involve the absolute value operation. Those which do not, form the regular subring $R \lessdot X_1, \ldots, X_n \gtrdot$, more precisely:

Definition 4.7. Let R be a regular *f-ring*, and let $R\langle X_1, \ldots, X_n\rangle \to R|X_1, \ldots, X_n|$ be the real *regular ring* homomorphism fixing R and the X_i. The image of this map is by definition $R \lessdot X_1, \ldots, X_n \gtrdot$.

Theorem 4.8. *Let R be a regular f-ring, $\tau \in R \lessdot X_1, \ldots, X_n \gtrdot$. Then τ is positive define $\Leftrightarrow \tau = \sum_{i=1}^{k} a_i\tau_i^2$ for some $k \in \omega$, $0 \leq a_i \in R$, $\tau_i \in R \lessdot X_1, \ldots, X_n \gtrdot$.*

Proof. $\Leftarrow$ is trivial. So let τ be positive definite.

$$O = \left\{ \sum_{i=1}^{k} a_i\tau_i^2 \,\middle|\, k \in \omega, \quad \tau_i \in R \lessdot X_1, \ldots, X_n \gtrdot \right\}$$

is a preorder on $R \lessdot X_1, \ldots, X_n \gtrdot$ extending the given preorder of R. By (2) and (3) of Lemma 4.3 there is a preordered extension (S, P) of $(R \lessdot X_1, \ldots, X_n \gtrdot, O)$ with $S \models \bar{T}$. Then (S, P) is a regular f-extension of R. Hence

$$\tau = \tau(X_1, \ldots, X_n) \in P \cap (R \lessdot X_1, \ldots, X_n \gtrdot) = O. \quad \square$$

I do not know whether the theorem can also be proved for "positive definite" $\tau \in R\langle X_1, \ldots, X_n\rangle$ (R a regular f-ring). At least the corresponding lemma does not hold:

let $R = \mathbf{Q}(\sqrt{2})$, ordered such that $\sqrt{2} > 0$, then

$$O = \left\{ \sum_{i=1}^{k} a_i\tau_i^2 \,\middle|\, 0 \leq a_i \in R, \quad \tau_i \in R\langle X\rangle \right\}$$

is not a preorder on $R\langle X\rangle$, for let $e = (X^2 + \sqrt{2})(X^2 + \sqrt{2})^{-1}$, then $(1-e)(X^2 + \sqrt{2})$ is the zero-element of $R\langle X\rangle$, but $(1-e)X^2$ is an element of O and $\neq 0$, and $(1-e)\sqrt{2}$ is an element of O and $\neq 0$ (note that $1-e$ takes the value 1 on substituting $X \mapsto \sqrt{-\sqrt{2}}$ in the real extension $\mathbf{Q}(\sqrt{-\sqrt{2}})$ of R).

We are now going to strengthen this theorem by showing that there exists a "uniform" decomposition of a positive definite $\tau \in R \lessdot X_1, \ldots, X_m \gtrdot$ as a sum of squares with positive coefficients.

Lemma 4.9. *Let for each $\mu \in M$ S_μ be a regular f-ring, let $S = \prod_{\mu \in M} S_\mu$. Each of the projections $S \to S_\mu$ can be extended to a regular f-ring homomorphism $\pi_\mu: S|X_1, \ldots, X_n| \to S_\mu|X_1, \ldots, X_n|$ by putting $\pi_\mu(X_i) = X_i$. Then*

$\tau \in S|X_1, \ldots, X_n|$ *is positive definite* $\Leftrightarrow \forall_{\mu \in M} \pi_\mu(\tau) \in S_\mu|X_1, \ldots, X_n|$ *is positive definite*

Proof. Let $\bar{S}_\mu$ be any regular f-extension of S_μ with $\bar{S}_\mu \models \bar{T}$. Then $\bar{S} =^{\text{def}} \prod \bar{S}_\mu$ is a regular f-extension of S with $\bar{S} \models \bar{T}$. Now it is easily seen that for $\tau \in S|X_1, \ldots, X_n|$ and $\alpha = (\alpha^1, \ldots, \alpha^n) \in (\bar{S})^n$ we have $(\tau(\alpha))_\mu = (\pi_\mu(\tau))(\alpha^1_\mu, \ldots, \alpha^n_\mu)$ for all $\mu \in M$. Hence

$$\tau(\alpha) \geq 0 \Leftrightarrow \forall_{\mu \in M} (\pi_\mu(\tau))(\alpha^1_\mu, \ldots, \alpha^n_\mu) \geq 0$$

From this the conclusion follows. □

Theorem 4.10. *Let R be a regular f-ring, $\tau(A, X) \in (R|A_1, \ldots, A_m|)\langle X_1, \ldots, X_n \rangle$. Then there exist $k \in \omega$, $\alpha_i(A) \in R|A_1, \ldots, A_m|$,*

$$\tau_i(A, X) \in (R|A_1, \ldots, A_m|)\langle X_1, \ldots, X_n\rangle \; (1 \leq i \leq k)$$

such that for each regular f-ring extension S of R and each $a \in S^m$ with positive definite $\tau(a, X) \in S\langle X_1, \ldots, X_n\rangle$ we have

$$\tau(a, X) = \sum_{i=1}^{k} \alpha_i(a) \cdot \tau_i^2(a, X),$$

and $\alpha_i(a) \geq 0$ for all $1 \leq i \leq k$ (A, X denote the tuples $(A_1, \ldots, A_m)$, $(X_1, \ldots, X_n)$).

Proof. Suppose the theorem does not hold. Using a Cantor diagonal argument we'll derive a contradiction. The negation of the theorem is as follows: for each finite sequence $\mu = (\alpha_1, \tau_1), \ldots, (\alpha_k, \tau_k)$ with

$$\alpha_i \in R|A_1, \ldots, A_m|, \qquad \tau_i \in (R|A_1, \ldots, A_m|)\langle X_1, \ldots, X_n\rangle$$

there is a regular f-ring extension S_μ of R and an m-tuple $a_\mu = ((a_\mu)_1, \ldots, (a_\mu)_m) \in (S_\mu)^m$, such that $\tau(a_\mu, X) \in S_\mu\langle X_1, \ldots, X_n\rangle$ is positive definite and either

$$\tau(a_\mu, X) \neq \sum_{i=1}^{k} \alpha_i(a_\mu)\tau_i(a_\mu, X)$$

or (*)

$$\alpha_i(a_\mu) \not\geq 0 \quad \text{for some } 1 \leq i \leq k$$

Now we form the direct product $S = \prod_\mu S_\mu$. Note that R is embedded in S by the diagonal map. We define the m-tuple $a = (a_1, \ldots, a_m) \in S^m$ by $(a_j)_\mu = (a_\mu)_j$ for $1 \leq j \leq m$. Then by the lemma $\tau(a, X) \in S\langle X_1, \ldots, X_n\rangle$ is positive definite. Let T be the regular f-subring of S generated over R by $a_1, \ldots, a_m$, in other words $T = \{\alpha(a) \mid \alpha \in R|A_1, \ldots, A_m|\}$. Then $\tau(a, X)$ is also a positive definite element of $T\langle X_1, \ldots, X_n\rangle$ (making a trivial identification provided by Theorem 4.4). Hence, by Theorem 4.8 there exist $\alpha_1(a), \ldots, \alpha_k(a) \in T$,

$\tau_1(a, X), \quad , \tau_k(a, X) \in T\langle X_1, \quad , X_n\rangle$

(where $\alpha_i = \alpha_i(A) \in R|A_1, \quad , A_m|$, $\tau_i(A, X) \in (R|A_1, \quad , A_m|)\langle X_1, \quad , X_n\rangle$)

such that $\tau(a, X) = \sum_{i=1}^k \alpha_i(a) \ \tau_i^2(a, X)$ and $\alpha_i(a) \geq 0$ for all $1 \leq i \leq k$ (in the ring $T\langle X_1, \quad , X_n\rangle$, hence also in $S\langle X_1, \quad , X_n\rangle$) Let μ be this sequence $(\alpha_1, \tau_1), \quad , (\alpha_k, \tau_k)$, then we have (by applying π_μ)

$$\tau(a_\mu \ X) = \sum_{i=1}^k \alpha_i(a_\mu) \ \tau_i^2(a_\mu, X),$$

and $\alpha_i(a_\mu) \geq 0$ for all $1 \leq i \leq k$ This contradicts (*)! □

Perhaps a more attractive formulation of the theorem is the following

Corollary 4.11. *Let $\tau(A, X) \in \mathbf{Q}[A, X]$ There exist $k \in \omega$, $\alpha_i(A) \in \mathbf{Q}|A|$, $\tau_i(A, X) \in \mathbf{Q}|A|\langle X\rangle (1 \leq i \leq k)$, such that for each ordered field R and each $a \in R^m$ with positive definite $\tau(a, X) \in R[X]$ we have $\tau(a, X) = \sum_{i=1}^k \alpha_i(a)\tau_i^2(a, X)$ and $\alpha_i(a) \geq 0$ for all $1 \leq i \leq k$ ($A = (A_1, \quad , A_m)$, $X = (X_1, \quad , X_n)$)*

Proof. It suffices to note that $\mathbf{Q}[A \ X]$ is naturally embedded as a ring in $(\mathbf{Q}|A|)\langle X\rangle$, and that for an ordered field R and $f(X) \in R[X]$ we have f is positive definite as an element of $R(X)$ iff f i positive definite as an element of $R|X|$ This equivalence follows by considering R as embedded in $C^\circ(\mathscr{C} \to \bar{R}) \models \bar{T}$ where $\mathscr{C}$ is Cantor space, $\bar{R}$ the real closure of R and $C^\circ(\mathscr{C} \to \bar{R})$ the ring of locally constant functions with domain $\mathscr{C}$ and codomain $\bar{R}$ □

A remark of G Cherlin suggested to me that these results can be strengthened still further

Theorem 4.12. *Let $\tau(A, X) \in (\mathbf{Q}|A|)\langle X\rangle$. There exist $K \in \omega$, $c_i(A) \in \mathbf{Q}|A|$, $\tau_i(A, X) \in (\mathbf{Q}|A|)\langle X\rangle$ $(1 \leq i \leq K)$ such that*

$$\tau(A, X) = \sum_{i=1}^K c_i(A) \ \tau_i^2(A, X)$$

and such that for every regular f-ring R and m-tuple $a = (a_1, \quad , a_m) \in R^m$ (the following holds $\tau(a, X) \in R\langle X\rangle$ is positive definite $\Leftrightarrow 0 \leq c_i(a)$ for all $1 \leq i \leq K$)

We first collect the necessary tools in a lemma

Lemma 4.13. (1) *For each open formula $\phi(A) = \phi(A_1, \quad , A_m)$ of $L(^{-1}, | \ |)$ there exists $\tau_\phi \in \mathbf{Q}|A|$, such that for each regular f-ring R, each m-tuple $(a_1, \quad , a_m) \in R^m$ and each $\underline{m} \in \mathrm{Spec}(R)$ we have*

$$R/\underline{m} \models (\phi(a_{1/\underline{m}}, \quad , a_{m/\underline{m}}) \Leftrightarrow \tau_\phi(a_{1/\underline{m}}, \quad , a_{m/\underline{m}}) = 1,$$

$$R/\underline{m} \models \neg\phi(a_{1/\underline{m}}, \quad , a_{m/\underline{m}}) \Leftrightarrow \tau_\phi(a_{1/\underline{m}}, \quad , a_{m/\underline{m}}) = 0$$

(2) *For each* $\tau(A,X) \in (\mathbf{Q}\ A|)\langle X\rangle$ *and each regular f-ring R and each m-tuple* $a = (a_1, \ldots, a_m) \in R^m$ *we have*

$$\tau(a,X) = 0 \text{ (in } R\langle X\rangle) \Leftrightarrow \forall \underline{m} \in \mathrm{Spec}(R)\,\tau(a/\underline{m}, X) = 0 \text{ in } R/\underline{m}\langle X\rangle \text{ and}$$

$\tau(a,X) \in R\langle X\rangle$ *is positive definite* $\Leftrightarrow \forall \underline{m} \in \mathrm{Spec}(R)\,\tau(a/\underline{m},X) \in R/\underline{m}\langle X\rangle$ *is positive definite* ($a/\underline{m}$ *denotes the m-tuple* $(a_{1/\underline{m}}, \ldots, a_{m/\underline{m}})$).

Proof. (1) with induction we put $\tau_{\neg\phi} = 1 - \tau_\phi$, $\tau_{\phi\wedge\psi} = \tau_\phi \cdot \tau_\psi$ and for an atomic formula $\phi = \sigma(A_1, \ldots, A_m) = 0$ we put $\tau_\phi = 1 - (\sigma^{-1} \cdot \sigma)$.

(2) both equivalences are proved using the technique of the preceding lemma (note that $R \to \prod_{\underline{m}\in\mathrm{Spec}(R)} R/\underline{m}$ is an embedding of regular f-rings). □

Proof of the theorem. $\bar{\mathrm{T}}(^{-1},|\ |)$ has a quantifier elimination, hence there is an open formula Pos(A) of L($^{-1}$,| |) such that for every regular f-ring R and m tuple $a = (a_1, \ldots, a_m) \in R^m$ we have $\tau(a,X) \in R\langle X\rangle$ is positive definite $\Leftrightarrow$ $R \models \mathrm{Pos}(a_1, \ldots, a_m)$. Let $P(A) \in Q|A|$ correspond to the formula Pos(A) as described in the lemma, part (1). Then we have, using part (2) of the lemma

$$\tau(A,X) = \sum_{i=1}^{k} (P(A) \cdot \alpha_i(A)) \cdot \tau_i^2(A,X)$$
$$+ (1 - P(A))(\tfrac{1}{2} + \tfrac{1}{2}\tau(A,X))^2$$
$$+ (P(A) - 1)(\tfrac{1}{2} - \tfrac{1}{2}\tau(A,X))^2,$$

(where $\alpha_i(A)$ and $\tau_i(A,X)$ are chosen as in Theorem 4.10). Put $K = k + 2$,

$$c_i(A) = P(A) \cdot \alpha_i(A) \qquad (1 \le i \le k),$$
$$c_{k+1}(A) = 1 - P(A), \qquad \tau_{k+1}(A,X) = \tfrac{1}{2} + \tfrac{1}{2}\tau(A,X),$$
$$c_{k+2}(A) = P(A) - 1, \qquad \tau_{k+2}(A,X) = \tfrac{1}{2} - \tfrac{1}{2}\tau(A,X).$$

Then again part (2) of the lemma shows that $\tau(a,X)$ is positive definite $\Leftrightarrow \forall 1 \le i \le K\ c_i(a) \ge 0$. □

Finally we give some easy examples of decompositions as described in the theorem.

Example 1.

$$X^2 + A_1X + A_2 = 1 \cdot (X + \tfrac{1}{2}A_1)^2 + (A_2 - \tfrac{1}{4}A_1^2) \cdot 1^2$$

and $X^2 + a_1X + a_2$ is positive definite iff $a_2 - \frac{1}{4}a_1^2 \ge 0$.

Example 2.

$$X^4 + A_1X^2 + A_2 = 1 \cdot (X^2 + \tfrac{1}{2}(A_1 \wedge 0))^2 + (A_1 \vee 0) \cdot X^2 + (A_2 - (\tfrac{1}{2}(A_1 \wedge 0))^2) \cdot 1^2$$

and $X^4 + a_1X^2 + a_2$ is positive definite iff $a_2 - (\frac{1}{2}(A_1 \wedge 0))^2 \ge 0$.

(In both examples a_1 and a_2 are elements of some regular f-ring, and "$\vee$" and "$\wedge$" denote the lattice operations.)

5. Sheaves of ordered fields

R. Wiegand has constructed the "regular hull" $\hat{R}$ of a ring R ([10]). We'll generalize his construction. Let T be a set of universal sentences in the language of rings.

Let us call a ring R T-regular if R is regular and embeddable in a direct product of T-fields (where a T-field is by definition a field which is a model of T).

Proposition 5.1. (1) *A ring R is T-regular iff R is regular and $\forall \underline{m} \in \mathrm{Spec}(R)$: $R/\underline{m} \models T$.*

(2) *The class of T-regular rings is a variety (or equational class) using the language $L(^{-1})$.*

Proof. If R is regular and $R/\underline{m} \models T$ for all $\underline{m} \in \mathrm{Spec}(R)$, then R is T-regular because $R \to \prod_{\underline{m} \in \mathrm{Spec}(R)} R/\underline{m}$ is an embedding of R in a direct product of T-fields. Conversely, let R be a regular subring of the direct product $\prod_{i \in I} F_i$ with F_i a T-field $\forall i \in I$. Let $\underline{m} \in \mathrm{Spec}(R)$. By the corollary of Section 2 there is a maximal ideal M of $\prod F_i$ such that $M \cap R = \underline{m}$. Then $R/\underline{m}$ is a subfield of $(\prod F_i)/M$, but $(\prod F_i)/M$ is in fact an ultraproduct of the fields F_i, hence $(\prod F_i)/M \models T$, implying $R/\underline{m} \models T$. Now (1) is proved. Using (1) it is easily shown that the class of regular T-rings is closed under homomorphic images, regular subrings and direct products, hence (2) holds. □

Definition 5.2. Let R be a ring; a pair $(i, \hat{R})$ where i is a ringmorphism $i: R \to \hat{R}$ and $\hat{R}$ is T-regular, is called a *T-regular hull* of R iff for each morphism $j: R \to S$ with S T-regular, there is a unique morphism $\Theta: \hat{R} \to S$ such that

$$\begin{array}{ccc} R & \xrightarrow{\;i\;} & \hat{R} \\ {\scriptstyle j}\downarrow & \swarrow{\scriptstyle \Theta} & \\ S & & \end{array}$$

commutes. For $T = \emptyset$ the following has been proved by R. Wiegand.

Theorem 5.3. *Every ring R has an (up to R-isomorphism) unique T-regular hull $(i, \hat{R})$; $i: R \to \hat{R}$ is 1–1 iff $\hat{R}$ is embeddable in a product of T-fields. Every element of $\hat{R}$ is a finite sum of elements $(ia)\cdot(ib)^{-1}$ $(a, b \in R)$.*

(The proof of this theorem will contain more information on $(i, \hat{R})$.)

Proof. Let us define for any ideal I of R: I is T-prime iff R/I is embeddable in a T-field. Let $\mathrm{Spec}_T(R)$ be the set of T-primes, let for $a \in T$

$$D_T(a) = \{\underline{p} \in \mathrm{Spec}_T(R) \mid a \notin \underline{p}\},$$

$$V_T(a) = \{\underline{p} \in \mathrm{Spec}_T(R) \mid a \in \underline{p}\}.$$

Then the $D_T(a)$ and $V_T(a)$ form a clopen subbase for a Boolean topology on $\mathrm{Spec}_T(R)$. This is proved by a nice model theoretic argument (this is one of the points where the proof differs from Wiegand's for $T = \emptyset$).

Let FL be the theory of fields and D^+F be the positive diagram of R. Then a T-field which extends $R/\underline{p}$ ($\underline{p} \in \mathrm{Spec}_T(R)$) is essentially a model of $T \cup \mathrm{FL} \cup D^+R$ in which $\{\underline{a} = 0 \mid a \in \underline{p}\} \cup \{\underline{a} \neq 0 \mid a \notin \underline{p}\}$ is satisfied, more precisely: let $\mathrm{L}(R)$ be the language of rings augmented by a constant $\underline{a}$ for each $a \in R$, let B be the Boolean algebra of open $\mathrm{L}(R)$-sentences modulo equivalence by $T \cup \mathrm{FL} \cup D^+R$. Let $S(B)$ be the Boolean space of ultrafilters of B, then $\underline{p} \mapsto \{\phi \in B \mid R/\underline{p} \models \phi\}$ is a bijection of $\mathrm{Spec}_T(R)$ onto $S(B)$. The inverse map is given by $\mathscr{F} \mapsto \{a \in R \mid$ the formula $\underline{a} = 0$ belongs to $\mathscr{F}\}$. Now the Boolean space $S(B)$ has as a clopen subbase the collection of sets

$$d(a) = \{\mathscr{F} \in S(B) \mid \underline{a} = 0 \text{ belongs to } \mathscr{F}\},$$

$$v(a) = \{\mathscr{F} \in S(B) \mid \underline{a} \neq 0 \text{ belongs to } \mathscr{F}\}.$$

If we transfer this subbase to $\mathrm{Spec}_T(R)$ by the above bijection we obtain the clopen subbase consisting of the $D_T(a)$ and $V_T(a)$.

Let X be the Boolean space $\mathrm{Spec}_T R$ with the indicated topology. Put $K_x = R/x$ ($x \in X$) and let $\mathscr{R}$ be the disjoint union $\bigcup_{x \in X} K_x$, for $a, b \in R$ we define

$$[a, b]: X \to \mathscr{R} \text{ by } [a, b](x) = a_x/b_x \in K_x \quad \text{if } b_x \neq 0,$$
$$= 0 \in K_x \quad \text{if } b_x = 0.$$

We topologize $\mathscr{R}$ by the strongest topology which makes all maps $[a, b]: X \to \mathscr{R}$ continuous, i.e. $O \subset \mathscr{R}$ is open iff $\forall a, b \in R\, [a, b]^{-1}(O)$ is open in X. If we define $\pi: \mathscr{R} \to X$ by $\pi^{-1}\{x\} = K_x$, then a long but tedious argument shows that $(\mathscr{R}, \pi, X)$ is a ringed space. All stalks K_x are T-fields, hence, using a result of R.S. Pierce (Theorem 10.3 in [11]) we have $\hat{R} =^{\mathrm{def}} \Gamma(X, \mathscr{R})$ is a regular ring, $x \mapsto \hat{x} =^{\mathrm{def}} \{\sigma \in \hat{R} \mid \sigma(x) = 0\}$ is a homeomorphism of X onto $\mathrm{Spec}(\hat{R})$ and $[a, b](x) \mapsto [a, b]/\hat{x}$ is an isomorphism of K_x onto $\hat{R}/\hat{x}$ ($\forall x \in X$). It is easily seen that $[a, b] \in \hat{R}$, and that $a \mapsto [a, 1]$ is a ringmorphism $\iota: R \to \hat{R}$, and $[a, b] = (\iota a)\,(\iota b)^{-1}$. We have $[a, 1] = 0 \Leftrightarrow a \in \bigcap_{x \in X} x$, hence $\iota: R \to \hat{R}$ is 1–1 iff R is embeddable in a product of T-fields.

We now prove that each $\sigma \in \hat{R}$ is a finite sum $\sum [a_i, b_i]$ (this proof is more elementary than Wiegand's). First of all $\sigma(x) = [a, b](x)$ for some $a, b \in R$, depending on x, hence, using that X is Boolean, $\sigma = \sum_{i=1}^{k} e_i [a_i, b_i]$ for idempotents $e_i \in \hat{R}$, $a_i, b_i \in R$. Every idempotent $e \in \hat{R}$ is a boolean combination of idempotents which are characteristic functions of a set $D_T(a) \subset X$. Using the following (α) $[a, a]$ = characteristic function of $D_T(a)$,

(β) $[a, b][c, d] = [ac, bd]$, $-[a, b] = [-a, b]$, $[a, b]^{-1} = [b, a]$, we see by induction that an idempotent $e \in \hat{R}$ is a finite sum of elements $[a, b]$, substituting these sums in $\sigma = \sum_{i=1}^{k} e_i [a_i, b_i]$, and again using (β) we arrive at the conclusion that σ is a finite sum of elements $[a, b]$.

Let $j: R \to S$ be any ringmorphism with S a T-regular ring. We want $\Theta: \hat{R} \to S$ with $\Theta\iota = j$. Let $y \in \mathrm{Spec}_T S$, then $j^{-1}y \in \mathrm{Spec}_T R = X$, and $R/j^{-1}y$ is embedded in S/y. We put $(\Theta(\sigma))(y) = \sigma(j^{-1}y)$ for $\sigma \in \hat{R}$. Again it is a tedious exercise to check that $\Theta(\sigma): \mathrm{Spec}_T S \to \bigcup_{y \in \mathrm{Spec}_T S} S/y$ is a global section of the ringed space belonging to S as defined in [11], and that Θ is a ringmorphism of $\hat{R}$ into the ring of global sections, which is S itself after an identification, and that $\Theta\iota = j$. Uniqueness of Θ follows by $\Theta(\sum [a_i, b_i]) = \sum \Theta(a_i) \cdot (\Theta(b_i))^{-1}$. □

Remark 1. Θ is onto $\Leftrightarrow$ $j(R)$ generates S as a regular ring.

Remark 2. Let us take for T the set of axioms

$$\sum_{i=1}^{n} x_i^2 = 0 \Rightarrow x_1 = \cdots = x_n = 0.$$

Then we have that every real ring R has a unique real regular hull $\hat{R}$, i.e. $\hat{R}$ is a real regular ring containing R as a subring such that each morphism of R into a real regular ring S can be uniquely extended to a morphism of $\hat{R}$ into S, moreover every element of $\hat{R}$ is a finite sum $\sum a_i b_i^{-1}$ $(a_i, b_i \in R)$. Hence the following corollary is immediate.

Corollary 5.4. *Let R be a real regular ring. Then $R\langle X_1, \ldots, X_n\rangle = (R[X_1, \ldots, X_n])^\wedge$ and every element of $R\langle X_1, \ldots, X_n\rangle$ (and also of $R\{X_1, \ldots, X_n\}$ if R is a regular f-ring) is a finite sum $\sum f_i \cdot g_i^{-1}$ with $f_i, g_i \in R[X_1, \ldots, X_n]$.*

This makes our results in the preceding section more concrete. We shall also give a more concrete description of $R|X_1, \ldots, X_n|$ by this method.

We adopt the definition of "sheave of structures" given in [9]. However, the 4-tuple (S, X, π, μ) will be abbreviated here as $(\bigcup_{x \in X} \pi^{-1}\{x\}, \pi, X)$, or even as $(\bigcup_{x \in X} \pi^{-1}\{x\}, X)$. In the following all structures are $L(Q)$-structures, and Q is used mostly for preorders on a ring. It will be clear now what is meant by a sheaf of ordered fields. Let (R, O) be a regular f-ring. As a regular ring $R \simeq \Gamma(X, \mathcal{S}_R)$ where $X = \mathrm{Spec}(R)$, $\mathcal{S}_R = (\bigcup_{x \in X} R/x, \pi, X)$ and $\pi: \bigcup_{x \in X} R/x \to X$ is defined by $\pi(a/x) = x$ $(a \in R)$ (see [11]).

We make $\mathcal{S}_R$ a sheaf of ordered fields $\mathcal{S}_{(R,O)} = (\bigcup_{x \in X} (R/x, O/x), \pi, X)$, then the isomorphism is even an isomorphism between (R, O) and $\Gamma(X, \mathcal{S}_{(R,O)})$, this is essentially Theorem 3.7 and the lemma which precedes it. Conversely, if $\mathcal{S}$ is a sheaf of ordered fields on a boolean space X, then $(R, O) =^{\mathrm{def}} \Gamma(X, \mathcal{S})$ is a regular f-ring and X is homeomorphic with $\mathrm{Spec}(R)$ via $x \mapsto \{\sigma \in \Gamma(X, \mathcal{S}) \mid \sigma(x) = 0\}$.

To derive an analogue of Theorem 5.3 for regular f-rings we need an analogue of "T-prime ideal".

Definition 5.5. Let R be a ring. $O \subset R$ is called a linear ordering ideal of R if $O + O \subset O$, $O \times O \subset O$, $O \cup -O = R$.

If O is a linear ordering ideal then $I = O \cap -O$ is an ideal of R and $O/I = \{a/I \mid a \in R\}$ is a linear ordering on the ring R/I. Conversely, if $\phi : R \to S$ is a ringmorphism and P is a linear ordering on S, then $O = \phi^{-1}(P)$ is a linear ordering ideal of R and $\ker \phi = O \cap (-O)$ and $(R/\ker \phi, O/\ker \phi) \to (S, P)$ is an embedding.

Let (R, P) be any ring with a subset P. We let X be the space of linear ordering ideals O with $O \cap -O$ prime and $O \supset P$. We define

$$V(a) = \{O \in X \mid a \notin O\}, \qquad D(a) = \{O \in X \mid a \in O\} (a \in R),$$

and again the $V(a)$ and $D(a)$ form a clopen subbase for a boolean topology on X. This is shown by an argument similar to that in the proof of Theorem 5.3: let B be the boolean algebra of open sentences of $L(\underset{\sim}{O}, \underset{\sim}{a})_{a \in R}$, modulo equivalence by $\mathrm{OFL} \cup D^{+}(R, P)$ where OFL is the theory of ordered fields in the language $L(\underset{\sim}{O})$ and $D^{+}(R, P)$ is the positive diagram of (R, P). Then $\mathscr{F} \to \{a \in R \mid \underset{\sim}{O}(\underset{\sim}{a})$ belongs to $\mathscr{F}\}$ is a bijection of the Stonespace $S(B)$ of B onto X, and using this bijection we transfer the topology on $S(B)$ to X (note that an atomic formula $\underset{\sim}{a} = 0$, is equivalent to $\underset{\sim}{O}(\underset{\sim}{a}) \wedge \underset{\sim}{O}(-\underset{\sim}{a})$). For $x = O \in X$ we put K_x = quotientfield of $R/O \cap -O$, ordered by the ordering which extends $O/O \cap -O$, we define the maps $[a, b] : X \to \bigcup_{x \in X} K_x$ and the sheaf $\mathscr{R}$ as in the proof of Theorem 5.3. Then $\mathscr{R}$ is a sheaf of ordered fields and for $(\hat{R}, \hat{P}) =^{\mathrm{def}} \Gamma(X, \mathscr{R})$ we have

Theorem 5.6. (1) *$(\hat{R}, \hat{P})$ is a regular f-ring and the map $\iota : (R, P) \to (\hat{R}, \hat{P})$ is a morphism such that for each morphism $j : (R, P) \to (S, Q)$ with (S, Q) a regular f-ring, there is a unique morphism $\Theta : (\hat{R}, \hat{P}) \to (S, Q)$ such that*

$$\begin{array}{ccc} (R, P) & \xrightarrow{\ \iota\ } & (\hat{R}, \hat{P}) \\ \downarrow & \swarrow_{\Theta} & \\ (S, Q) & & \end{array}$$

commutes.

(2) *ι is an embedding iff (R, P) is embeddable in a direct product of ordered fields.*

(3) *Every element of $\hat{R}$ is a finite sum of elements $e \cdot (\iota a) \cdot (\iota b)^{-1}$ with $a, b \in R$, e an idempotent of R.*

Proof. Similar to the proof of Theorem 5.3. We indicate only the differences. The characteristic function of $D(a)$ is $[a, 1]^{-}$, $([a, 1]^{-})^{-1}$ where $x_{-} =^{\mathrm{def}} x \wedge 0$ for x in an f-ring.

Hence. *the idempotent e in* (3) *can be chosen as a boolean combination of elements of the form* $[a,1]^-$ $([a,1]^-)^{-1}$ $(a \in R)$. This fact can be used to prove the uniqueness of Θ To check that the construction of Θ goes through one needs that for (S,Q) as in (1) $O \mapsto O \cap -O$ defines a homeomorphism of the space of prime linear ordering ideals over Q onto Spec(S), this is easily proved using the compactness of the spaces □

Remark 1. Θ is onto $\Leftrightarrow$ $j(R)$ generates (S,Q) as a regular f-ring

Remark 2. Let R be a regular f-ring, $P = \{x \in R \mid x \geq 0\}$ Then, with the notations of Theorem 5 6 , we have

$$R|X_1, \quad, X_n| = (R[X_1, \quad, X_n], \hat{P}),$$

hence every element of $R|X_1, \quad, X_n|$ is a finite sum of elements $e, f\ g^{-1}$ with $f, g \in R[X_1, \quad, X_n]$ and e an idempotent

Although it will not be needed it seems appropriate here to give an elementary characterization of those (R,P) which are embeddable in a direct product of ordered fields

Definition 5.7. For a subset P of a ring R and $a_1, \quad, a_n \in R$ we define $P[a_1, \quad, a_n]$ as the *smallest subset* of R containing P and $a_1, \quad, a_n$ and which is closed under addition and multiplication

If P contains all squares of R and is closed under addition and multiplication, then clearly

$$P[a_1, \quad, a_n] = \{f(a_1, \quad, a_n) \mid f \in R[X_1, \quad, X_n]$$

has all coefficients in P and every monomial has degree at most 1 in each $X_i\}$

Theorem 5.8. *Let R be a ring P ⊂ R then the following holds* (R,P) *is embeddable in a direct product of ordered fields* $\Leftrightarrow$

(1) P *is a preorder on* R,

(2) $\forall n \in \mathbb{N}\ \forall a_1 \quad \forall a_n \in R\ \bigcap_{\varepsilon_i = \pm 1} P[\varepsilon_1 a_1, \quad, \varepsilon_n a_n] = P$,

(3) $\forall a \in R (a^3 \in P \Rightarrow a \in P)$

Remark. In [12] it is proved that the conjunction of (1) and (2) is equivalent to R, P is embeddable in a direct product of linearly ordered rings.

Proof. $\Rightarrow$ is straightforward.

$\Leftarrow$. assume that (1), (2) and (3) hold. Let $a \notin P$. If we put $O := P$, then the following conditions on O hold

(a) $P \subset O$,

(b) O is closed under addition and multiplication

(c) $\forall a \, \forall a_1 \ldots \forall a_k \, \forall n \in \mathbf{N} \; a^{2n+1} \notin \bigcap_{\varepsilon_i = \pm 1} O[\varepsilon_1 a_1, \ldots, \varepsilon_k a_k]$

For (a) and (b) this is clear, if

$$a^{2n+1} \in \bigcap_{\varepsilon_i = \pm 1} P[\varepsilon_1 a_1, \ldots, \varepsilon_n a_n] = P,$$

then $a^{3m} \in P$ for some $m \in \mathbf{N}$ (by multiplication with an even power of a), and by induction on m this gives $a \in P$, contradiction!, so (c) also holds for P Let O_a be a maximal subset of R which satisfies (a), (b) and (c)

Then O_a is a linear ordering ideal of R for suppose $c \notin O_a$, $-c \notin O_a$, then $O_a[c]$ and $O_a[-c]$ are proper extensions of O_a which satisfy (a) and (b), hence there are $n \in \mathbf{N}$, $a_1, \ldots, a_k$, $b_1, \ldots, b_l \in R$ with

$$a^{2n+1} \in \bigcap_{\varepsilon_i = \pm 1} (O_a[c])[\varepsilon_1 a_1, \ldots, \varepsilon_k a_k],$$

$$a^{2n+1} \in \bigcap_{\delta_j = \pm 1} (O_a[-c])[\delta_1 b_1, \ldots, \delta_l b_l]$$

hence

$$a^{2n+1} \in \bigcap_{\substack{\varepsilon_i = \pm 1 \\ \delta_j = \pm 1}} O_a[\varepsilon_0 c, \varepsilon_1 a_1, \ldots, \varepsilon_k a_k, \delta_1 b_1, \ldots, \delta_l b_l]$$

which contradicts (c) Put $I_a = O_a \cap -O_a$ Then $(\bar{R}_a, \bar{O}_a) = (R/I_a, O_a/I_a)$ is a linearly ordered ring From (c) we know that $a^{2n+1} \notin O_a$, $(\forall n \in \mathbf{N})$, hence $\boldsymbol{a} = a/I_a$ is not nilpotent in $\bar{R}_a$, and consequently there is a minimal prime ideal $\underline{p}_a$ of $\bar{R}_a$ such that for $S_a = (\bar{R}_a/\underline{p}_a, \bar{O}_a/\underline{p}_a)$ we have $\boldsymbol{a}/\underline{p}_a < 0$ in S_a and S_a is a linearly ordered domain Summarized for an arbitrary $a \notin P$ we have found a linearly ordered domain S_a with a homomorphism $(R, P) \to S_a$ such that the image of a in S_a is strictly negative in S_a Then $(R, P) \to \prod_a S_a$ where a runs over all elements $\notin P$, is an embedding of (R, P) in a product of linarly ordered domains □

As a final application of sheaves we give an example of a real regular ring R, which has no good preorder This contrasts with the situation for fields where it is an old result of Artin and Schreier that every real field has an ordering

Example. Let $X = \mathbf{N} \cup \{\infty\}$ be the one point compactification of the discrete space $\mathbf{N}$ In [13], page 250, A B Carson constructs a sheaf $K = (\bigcup_{x \in X} K_x, X)$ of fields on X with $K_n = \mathbf{R}$ for $n \in \mathbf{N}$, $K_\infty = \mathbf{Q}(\sqrt{2})$, such that τ defined by

$$\tau(n) = (-1)^n \sqrt{2} \in K_n,$$

$$\tau(\infty) = \sqrt{2} \in K_\infty,$$

is an element of $\Gamma(X, K)$ As X is boolean and all K_x are real fields $R \stackrel{\text{def}}{=} \Gamma(X, K)$ is a real regular ring Suppose R has a good preorder O, this O induces an ordering on each stalk K_x, suppose it induces on K_∞ the ordering with $\tau(\infty) = \sqrt{2} > 0$, but this would imply that $\tau(n) = (-1)^n \sqrt{2} > 0$ in $K_n = \mathbf{R}$ for all sufficiently large $n \in \mathbf{N}$, which is impossible; similarly we reach a contradiction if the induced ordering on K_α satisfies $\sqrt{2} \leq 0$. Hence R has no good preorder.

6. Real closures of regular f-rings

We introduce two notions of "real closure of a regular f-ring R", and prove existence and uniqueness for both. In general the two real closures of a regular f-ring R do not coincide; they coincide if and only if R has no minimal idempotents. Lemma 6.1 is basic for all the following. It was inspired by a sheaf construction of A. Carson (Lemma 2.2 of [13]). Due to the fact that we can distinguish the distinct roots of a polynomial over an ordered field, we can prove stronger statements than in the situation considered by Carson.

Let R be in the following a regular f-ring, $K = (\bigcup_{x \in X} K_x, X)$ be the corresponding sheaf of ordered fields on $X = \mathrm{Spec}(R)$ as defined before. We identify R with $\Gamma(X, K)$. Let $f(Z) = \sum_{i=0}^{n+1} f_i Z^i \in R[Z]$ be a monic polynomial ($f_{n+1} = 1$) of degree $n+1$, $n \geq 1$, where either $n+1$ is odd, or $n+1 = 2$ and $f(Z) = Z^2 - r$ for some $0 \leq r \in R$. For $x \in X$ we put $f_x(Z) = \sum_{i=0}^{n+1} f_i(x) Z^i \in K_x[Z]$.

Lemma 6.1. *There exists an up to K-isomorphism unique sheaf L such that*

(a) *L is a sheaf of ordered fields $(\bigcup_{x \in X} L_x, X)$ over X with K a subsheaf of L.*

(b) *$\forall x \in X$ $L_x = K_x(\lambda_x)$ where λ_x is the largest zero of $f_x(Z)$ in the real closure of K_x.*

Proof. We define λ_x as the largest zero of $f_x(Z)$ in the real closure of K_x and order $K_x(\lambda_x)$ by the ordering induced by this real closure. Put $L_x = K_x(\lambda_x)$ and define $\sigma: X \to \bigcup_{x \in X} L_x$ by $\sigma(x) = \lambda_x$ $(x \in X)$.

Let $\mathcal{F}$ be the collection of all sets $\{\sum_{i=0}^{n} a_i(x)(\sigma(x))^i \mid x \in N\}$ where $(a_0, \ldots, a_n)$ runs over R^{n+1} and N over the clopen subsets of X. $\mathcal{F}$ is an open basis for a topology on $\bigcup_{x \in X} L_x$, and with this topology $L = (\bigcup_{x \in X} L_x, X)$ is a sheaf satisfying (a) and (b). For this last statement to be true it suffices to check the following:

(1) $\forall x \in X$ $L_x = \left\{ \sum_{i=0}^{n} a_i(x)(\sigma(x))^i \mid a_0, \ldots, a_n \in R \right\}$.

(2) $\forall (a_0, \ldots, a_n) \in R^{n+1}$ $\forall (b_0, \ldots, b_n) \in R^{n+1}$ $\exists (c_0, \ldots, c_n) \in R^{n+1}$

$$\sum_{i=0}^{n} a_i \sigma^i \cdot \sum_{i=0}^{n} b_i \sigma^i = \sum_{i=0}^{n} c_i \sigma^i,$$

(3) $\forall (a_0, \ldots, a_n) \in R^{n+1}$ $\forall x \in X \left(\sum_{i=0}^{n} a_i(x) \cdot (\sigma(x))^i = 0 \Rightarrow \right.$

$$\left. \exists \text{ open } N \ni x \ \forall y \in N \sum_{i=0}^{n} a_i(y) \cdot (\sigma y)^i = 0 \right),$$

(4) $\forall(a_0, \ldots, a_n) \in R^{n+1}\ \forall x \in X\left(\sum_{i=0}^{n} a_i(x)\ (\sigma x)^i \geq 0 \Rightarrow\right.$

$$\left.\exists \text{ open } N \ni x\ \forall y \in N\left(\sum_{i=0}^{n} a_i(y)\ (\sigma y)^i \geq 0\right)\right)$$

(1) follows because X is boolean (hence $K_x = \{a(x) \mid a \in R\}$) and because $\sigma(x)$ is a root of a monic polynomial of $(n+1)$th degree.

(2) $\sigma^{n+1} = -\sum_{i=0}^{n} f_i\sigma^i$, and with induction it is shown that every power $\sigma^K (K \in \mathbf{N})$ is of the form $\sum_{i=0}^{n} c_i\sigma^i (c_i \in R)$.

(3) and (4) the theory RCF of real closed fields has a quantifier elimination so there exist open formulas $\phi_1(\boldsymbol{a}, \boldsymbol{f})$ and $\phi_2(\boldsymbol{a}, \boldsymbol{f})$, ($\boldsymbol{a} = (a_0, \ldots, a_n)$, $\boldsymbol{f} = (f_0, \ldots, f_n)$ are taken as n-tuples of variables here) such that RCF$\models \phi_1(\boldsymbol{a}, \boldsymbol{f}) \leftrightarrow \forall s$ (s is the largest zero of $f_0 + f_1Z + \cdots + f_nZ^n + Z^{n+1} \rightarrow a_0 + a_1s + \cdots + a_ns^n = 0$) and RCF$\models \phi_2(\boldsymbol{a}, \boldsymbol{f}) \leftrightarrow \forall s$ (s is the largest zero of $f_0 + f_1Z + \cdots + f_nZ^n + Z^{n+1} \rightarrow a_0 + a_1s + \cdots + a_ns^n \geq 0$).

Because $\phi_i(\boldsymbol{a}, \boldsymbol{f})$ is an open formula and $\bigcup_{x \in X} K_x$ is Hausdorff, we can use Lemma 3 of [9] to get: if $K_x \models \phi_i(\boldsymbol{a}(x), \boldsymbol{f}(x))$, then there exists an open $O \ni x$ with $K_y \models \phi_i(\boldsymbol{a}(y), \boldsymbol{f}(y))$ for all $y \in O$ ($\boldsymbol{a} \in R^{n+1}, \boldsymbol{f} \in R^{n+1}$). Now (3) and (4) follow easily.

Uniqueness. Let $L = (\bigcup_{x \in X} L_x, X)$ be a sheaf as described in the lemma. We define $\sigma: X \rightarrow \bigcup_{x \in X} L_x$ by $\sigma(x) = \lambda_x (\forall x \in X)$, and we prove that σ is continuous: let $x \in X$, and suppose $\sigma(x) = a(x)$ with an $a \in \Gamma(X, L)$; there is an open formula $\phi(s, f_0, \ldots, f_n)$ with

$$\text{RCF} \models \phi(s, f_0, \ldots, f_n) \leftrightarrow s$$

is the largest zero of

$$f_0 + f_1Z + \cdots + f_nZ^n + Z^{n+1},$$

hence $L_x \models \phi(a(x), f_0(x), \ldots, f_n(x))$, and using Lemma 3 of [9] there is an open set $O \ni x$ with

$$L_y \models \phi(a(y), f_0(y), \ldots, f_n(y)) \quad \forall y \in O,$$

hence $\sigma(y) = a(y)$ for all $y \in O$. From $\sigma \in \Gamma(X, L)$ it follows that $\mathcal{F}$ is a basis for the given topology on $\bigcup_{x \in X} L_x$, so this topology is uniquely determined. □

Remark. Note that for $S = \Gamma(X, L)$ we have: R is a regular f-subring of S, $B(R) = B(S)$, $\sigma \in S$, $\sum_{i=0}^{n+1} f_i\sigma^i = 0$ and consequently $S = R + R \cdot \sigma + \cdots + R \cdot \sigma^n$.

$S = R[\sigma]$ is in a certain sense a universal construction.

Lemma 6.2. *Let T be a regular f-extension of R, $L = (\bigcup_{y \in Y} L_y, Y)$ its corresponding sheaf of ordered fields over $Y = \text{Spec}(T)$, suppose $s \in T$ is a zero of $f(Z)$, such that for each $y \in Y$ $s(y)$ is the largest zero of $f_y(Z) \in L_y[Z]$ in the real closure of L_y. Then there exists a unique R-morphism $\Phi: S = R[\sigma] \rightarrow T$, moreover Φ is an embedding with $\Phi(\sigma) = s$.*

Proof. *Existence* define Φ by:

$$a_0 + a_1\sigma + \quad + a_n\sigma^n \mapsto a_0 + a_1 s + \quad + a_n s^n,$$

where $a_i \in R$ We have to show the following

(1) Φ is welldefined, i e if $a_0 + a_1\sigma + \quad + a_n\sigma^n = 0$, then

$$a_0 + a_1 s + \quad + a_n s^n = 0,$$

(2) Φ is 1–1, i e if $a_0 + a_1 s + \quad + a_n s^n = 0$, then $a_0 + a_1\sigma + \quad + a_n\sigma^n = 0$,

(3) Φ is a regular f-ring morphism

Ad (1). let $a_0 + a_1\sigma + \quad + a_n\sigma^n = 0$, and take any $y \in Y$, for the image $x = y \cap R$ of y in X we have K_x is an ordered subfield of K_y and $K_x \models \phi_1(a(x), f(x))$ where ϕ_1 is the open formula defined in the proof of Lemma 6 1, this gives $K_y \models \phi_1(a(y), f(y))$, implying $a_0(y) + a_1(y)s(y) + \quad + a_n(y)(s(y))^n = 0$, as this holds for any $y \in Y$, we get $a_0 + a_1 s + \quad + a_n s^n = 0$

Ad(2). use the first corollary of Section 2 and reverse the proof of (1)

Ad(3) that Φ is a ring morphism is trivial by (1)

The equivalence

$$a_0 + a_1\sigma + \quad + a_n\sigma^n \geq 0 \Leftrightarrow a_0 + a_1 s + \quad + a_n s^n \geq 0$$

is proved as in (1) (2) using the open formula $\phi_2(a, f)$ instead of $\phi_1(a, f)$

Uniqueness Suppose Φ $R[\sigma] \to T$ is an R-morphism, let $\phi(s, f)$ be the open formula defined in the last part of the proof of Lemma 6 1, take any $y \in Y$ and let $x = \Phi^{-1}y$, Φ induces a morphism $R[\sigma]/x \to L_y$ of ordered fields, which is necessarily an embedding, by the meaning of $\phi(s, f)$ we have $R[\sigma]/x \models \phi(\sigma(x), f_0(x), \quad, f_n(x))$, hence $L_y \models \phi(\Phi(\sigma)(y), f_0(y), \quad, f_n(y))$, and this means that $\Phi(\sigma)(y)$ is the largest zero of $f_y(Z)$ in the real closure of L_y This holds for any $y \in Y$, so $s = \Phi(\sigma)$ □

Let R, S, T denote n the following regular f-rings

Definition 6.3. S is called an (idempotent) *invariant* R*-extension*, if S is an extension of R with $B(R) = B(S)$

Definition 6.4. S is called an *integral* R*-extension* (or integral over R), if every $s \in S$ is a zero of a monic polynomial over R

A standard argument shows if S is an invariant R-extension then S is integral over R iff ($\forall \underline{m} \in \mathrm{Spec}(S)$ $S/\underline{m}$ is algebraic over $R/\underline{m} \cap R$)

Definition 6.5. S is called *real closed* if every $s \geq 0$ in S is a square and every monic polynomial of odd degree over S has a root in S, equivalently $S/\underline{m}$ is a real closed field for all $\underline{m} \in \mathrm{Spec}(S)$

Definition 6.6. S is called an *invariant real closure* of R if S is an invariant integral R extension which is real closed

Lemma 6.7. *Let T be a real closed extension of R Then there exists an S with $R \subset S \subset T$ such that S is a real closed invariant R-extension*

Proof. Take for S a regular f-subring of T including R, which is maximal with respect to the property of being an invariant R-extension We show that S is real closed, let $f(Z) \in S[Z]$ be monic where either $f(Z) = Z^2 - r$ for some $0 \leq r \in S$, or f is of odd degree,

$f(Z)$ has a zero t in T such that for all $y \in Y = \mathrm{Spec}(T)$, $t(y)$ is the largest zero of $f_y(Z) \in L_y[Z]$ in L_y, (where $(\bigcup_{y \in Y} L_y, Y)$ is is the sheaf corresponding to T)

This statement is proved using the argument in the uniquenesspart of Lemma 6 1 to show that σ is a global section From Lemma 6 2 and the remark following Lemma 1 it follows that $S[t] \subset T$ is an invariant extension of S, and by the maximality of S, this yields $t \in S$ Hence $f(Z) \in S[Z]$ has a zero in S □

Lemma 6.8. *With the same hypothesis as in Lemma 6 7, there is an S with $R \subset S \subset T$ such that S is a real closed integral R-extension*

Proof. Take for S a regular f-subring of T including R which is maximal with respect to the property of being integral over R We show that S is real closed let $f(Z) \in S[Z]$ be monic, where either $f(Z) = Z^2 - r$ with $0 \leq r \in S$, or $f(Z)$ is of odd degree As in the proof of Lemma 6 7 we see that $f(Z)$ has a root $t \in T$ such that $S[t]$ is a regular f-subring of T Because S is integral over R and t is integral over S, we get $S[t]$ is integral over R, hence by the maximality of S $t \in S$ □

By first applying Lemma 6 7 and then Lemma 6 8 we get Lemma 6 9

Lemma 6.9. *If T is a real closed extension of R, then there is S with $R \subset S \subset T$ such that S is an invariant real closure of R*

Lemma 6.10. *If S is an invariant real closure of R, then there is no S' with $R \subset S' \subsetneq S$ with S' real closed*

Proof. Let $\underline{m} \in \mathrm{Spec}(S)$ Then $R/\underline{m} \cap R \subset S'/\underline{m} \cap S' \subset S/\underline{m}$ and $S/\underline{m}$ is the real closure of $R/(\underline{m} \cap R)$, hence, if S' were real closed, then $S'/(\underline{m} \cap S') = S/\underline{m}$ for each $\underline{m} \in \mathrm{Spec}(S)$, and from this it follows that $S' = S$ (because S is an invariant S'-extension) □

Theorem 6.11. *R has an invariant real closure $\bar{R}$, $\bar{R}$ is unique up to R-isomorphism For any real closed extension T of R, there is a unique R-morphism of $\bar{R}$ into T, this morphism is an embedding*

Proof. R has a real closed extension, hence by Lemma 6 9, R has an invariant real closure $\bar{R}$. Uniqueness of $\bar{R}$ follows in the usual way from the last statement of the theorem, so we prove this statement first Let T be any real closed extension of R We consider the set of all pairs (S, ϕ) with $R \subset S \subset \bar{R}$ and ϕ an R-embedding of S in T, and we partially order this set by $(S, \phi) \leqslant (S', \phi') \Leftrightarrow^{\text{def}} S \subset S'$ and ϕ is a restriction of ϕ' Let K be the sheaf of $\bar{R}$ and L the sheaf of T

By Zorn's lemma this set has a maximal member, say (S, ϕ); suppose $S \neq \bar{R}$, then by Lemma 6 10 S is not real closed, hence there exists a monic polynomial $f(Z) \in S[Z]$, either of the form $Z^2 - r$, $0 \leqslant r \in S$, or of odd degree, which has no root in S But $f(Z)$ has a root a in R, and a root b in T, such that $\forall x \in \operatorname{Spec}(R)$ $a(x)$ is the largest root of $f_x(Z)$ in K_x, and such that $\forall y \in \operatorname{Spec}(T)$ $b(y)$ is the largest root of $f_y(Z)$ in L_y Lemma's 6 1 and 6 2 imply that $S[a]$ is a regular f-subring of $\bar{R}$ and that we can extend ϕ to an embedding $S[a] \to T$ by $a \mapsto b$ This contradicts the maximality of (S, ϕ) Hence we have proved the existence of an R-embedding ϕ of $\bar{R}$ into T

Suppose ψ is any R-morphism of $\bar{R}$ into T We look at the set of all S with $R \subset S \subset \bar{R}$ such that $\psi \mid S = \phi \mid S$ By Zorn's lemma there exists a maximal S in this set, suppose $S \neq \bar{R}$, then we use the same argument as before and the uniqueness-part of Lemma 6 2, to deduce a contradiction □

Example 1 For ordered fields the concept of "invariant real closure" coincides with the usual concept of "real closure"

Example 2. If X is a boolean space, F an ordered field $\bar{F}$ its real closure, then $C^0(X, \bar{F})$ is the invariant real closure of $C^0(X, F)$

Here $C^0(X, F)$ is the regular f-ring of locally constant functions with domain X, and with values in F, and with $f \geqslant 0 \Leftrightarrow \forall x \in X(f(x) \geqslant 0)$, for $f \in C^0(X, F)$ $C^0(X, F)$ will also be used in this sense if F is not ordered, of course $C^0(X, F)$ is then only a regular ring

Up till now we only considered invariant extensions, i e extensions in which no new idempotents occur Now we are going to study extensions which are generated over a given ring by idempotents

Definition 6.12. S is called *atomless* if $B(S)$ is an atomless boolean algebra

(Of course in this definition R and S can be arbitrary rings)

Definition 6.13. S is called an *atomless real closure of* R if S is a real closed atomless extension of R such that for any real closed atomless extension T of R there exists an R-embedding of S into T the diagram

$$\begin{array}{ccc} R & \longrightarrow & S \\ \downarrow & \swarrow \Theta & \\ T & & \end{array}$$

commutes for a suitable embedding Θ

Of course the real closed atomless extensions are exactly the extensions which are models of $\bar{T}(^{-1}, |\ |)$, and those who are familiar with "Saturated Model Theory" of G Sacks will see that the notion of "atomless real closure' coincides with the notion of "primemodel extension with respect to the theory $\bar{T}(^{-1}, |\ |)$" The following will be proved

Theorem 6.14. *Every regular f-ring R has an atomless real closure $\tilde{R}$ $\tilde{R}$ is unique up to R-isomorphism and is integral over R*

First some lemma's

Lemma 6.15. *Let F be an ordered field (considered as a regular f-ring) Then $C^0(\mathscr{C}, F)$ is a prime atomless extension of F i e $C^0(\mathscr{C}, F)$ is an atomless extension which can be F-embedded into any atomless extension of F, moreover $C^0(\mathscr{C}, F)$ is up to F-isomorphism the only prime atomless extension of F*

Proof. Let R be any atomless extension of F $B(R)$ is then an atomless boolean algebra, hence includes a countable atomless boolean algebra Applying the (contravariant) Stonespace functor we get a continuous map h of the Stonespace of $B(R)$ onto $\mathscr{C}$ (which is the Stonespace of any countable atomless boolean algebra) But the Stonespace of $B(R)$ is naturally homeomorphic with $X = \mathrm{Spec}(R)$ (see [13], Theorem 1 5), so we may consider h as a continuous map of X onto $\mathscr{C}$ Let $K = (\bigcup_{x \in X} K_x, X)$ be the sheaf of ordered fields associated with R, F is naturally embedded in each $K_x = R/x$ (because $x \cap F = \{0\}$), so we have $C^0(X, F) \subset \Gamma(X, K) = R$ Let ι be the map $\sigma \mapsto \sigma \circ h$ of $C^0(\mathscr{C}, F)$ into $C^0(X, F)$, then ι is an F-embedding of $C^0(\mathscr{C}, F)$ into R
Uniqueness It suffices to prove the following let $F \subset R \subset C^0(\mathscr{C}, F)$ and suppose R is atomless, then R is F-isomorphic with $C^0(\mathscr{C}, F)$

Here follows the proof $B(R) \subset B(C^0(\mathscr{C}, F))$ and $B(C^0(\mathscr{C}, F))$ is countable hence $B(R)$ is a countable atomless boolean algebra, by applying the Stonespace functor this gives us that $X = \mathrm{Spec}(R)$ and $\mathscr{C}$ are homeomorphic Let $K = (\bigcup_{x \in X} K_x, X)$ be the sheaf associated with R, then $K_x = F$ for all $x \in X$ (here we need both inclusions $F \subset R$ and $R \subset C^0(\mathscr{C}, F)$), hence

$$R = \Gamma(X, K) \simeq C^0(X, F) \simeq C^0(\mathscr{C}, F) \quad \square$$

We need the following notations Let R be a regular f-ring and e an idempotent of R, then we have a canonical decomposition $R = (R \mid e) \times (R \mid 1 - e)$ of R as a direct product of two regular f-rings, here $(R \mid e)$ is taken as the ideal eR which we make a regular f-ring by taking e as the identity, and defining the other operations and relations as restrictions of the operations and relations on R, note that e is an atom iff $R\,|\,e$ is a field, let $\mathrm{At}(R)$ be the set of atoms of $B(R)$

Proposition 6.16. *Every regular f-ring R has a prime atomless extension S, i.e. an atomless extension which can be R-embedded in each atomless extension of R. Moreover S is unique up to R-isomorphism.*

Proof. Let $\kappa = \mathrm{card}(\mathrm{At}(R))$ and let $(e_\lambda)_{\lambda<\kappa}$ be a 1–1 enumeration of $\mathrm{At}(R)$. We define an ascending chain of regular f-rings $(R_\lambda)_{\lambda\leq\kappa}$, beginning with $R_0 = R$ such that

(A) $\qquad \mathrm{At}(R_\lambda) = \{e_v \mid \lambda \leq v < \kappa\}$

Suppose R_λ is already defined such that (A) holds, and $\lambda < \lambda + 1 \leq \kappa$, then $R_\lambda = (R_\lambda\,|\,1-e_\lambda)\times(R_\lambda\,|\,e_\lambda)$ and $R_\lambda\,|\,e_\lambda$ is an ordered field, we "replace" it by its prime atomless extension $C^0(\mathscr{C}, R_\lambda\,|\,e_\lambda)$, i.e. $R_{\lambda+1}$ is the extension $(R_\lambda\,|\,1-e_\lambda)\times C^0(\mathscr{C}, R_\lambda\,|\,e_\lambda)$ of R_λ, it is easily shown that $\mathrm{At}(R_{\lambda+1}) = \mathrm{At}(R_\lambda)\setminus\{e_\lambda\}$ and $R_{\lambda+1}\,|\,1-e_\lambda = R_\lambda\,|\,1-e_\lambda$

Let $0<\mu\leq\kappa$ be a limit ordinal and let (A) hold for all $\lambda<\mu$, we put $R_\mu = \bigcup_{\lambda<\mu} R_\lambda$ and we see that (A) also holds for $\lambda = \mu$. This construction shows also that $R_\lambda\,|\,e_\lambda = R\,|\,e_\lambda$ for all $\lambda \leq \lambda' \leq \kappa$

We claim that R_κ is a prime atomless extension of R, by (A) R_κ is an atomless extension of R, let T be any atomless extension of R, with induction we construct a sequence of embeddings $\iota_\lambda : R_\lambda \to T$ $(0\leq\lambda\leq\kappa)$ such that ι_λ is a restriction of ι_μ for $\lambda<\mu\leq\kappa$. ι_0 is the inclusion map $R\to T$, suppose ι_λ is defined $(\lambda<\lambda+1\leq\kappa)$, ι_λ embeds the direct factor $R_\lambda\,|\,e_\lambda$ of R_λ into $T\,|\,e_\lambda$, and $R_\lambda\,|\,e_\lambda$ is an ordered field and $T\,|\,e_\lambda$ is atomless (because T is), hence by Lemma 6.15 ι_λ can be extended to an embedding $\iota_{\lambda+1}$ $(R_\lambda\,|\,1-e_\lambda)\times C^0(\mathscr{C}, R\,|\,e_\lambda) = R_{\lambda+1}\to T$

For μ a limit ordinal $\leq\kappa$ we put $\iota_\mu = \bigcup_{\lambda<\mu}\iota_\lambda$. Hence we have constructed the sequence $(\iota_\lambda)_{\lambda\leq\kappa}$, whose last member ι_κ gives the desired R-embedding of R_κ into T

Uniqueness. As in Lemma 6.15 it suffices to prove the following: let $R\subset T\subset R_\kappa$ and suppose T is atomless, then T and R_κ are R-isomorphic. Proof of this fact. First note that from $(R\,|\,e_\lambda)\subset(T\cap R_\lambda)\,|\,e_\lambda\subset(R_\lambda\,|\,e_\lambda) = (R\,|\,e_\lambda)$ it follows that these inclusions are in fact equalities. We construct a sequence of R-embeddings ι_λ $R_\lambda\to T$ $(0\leq\lambda\leq\kappa)$ such that for $\lambda<\mu\leq\kappa$ ι_λ is a restriction of ι_μ and such that for all $0\leq\lambda\leq\kappa$ $\iota_\lambda(R_\lambda) = T\cap R_\lambda$, the construction is like the preceding one except for the following essential detail: suppose that for $\lambda<\lambda+1\leq\kappa$ we have already constructed ι_λ mapping R_λ isomorphically onto $T\cap R_\lambda$ and fixing R, restrictions i and j of ι_λ map

$R_\lambda\,|\,(1-e_\lambda)$ isomorphically onto $(T\cap R_\lambda)\,|\,(1-e_\lambda)$ and $R_\lambda\,|\,e_\lambda$ isomorphically onto $(T\cap R_\lambda)\,|\,e_\lambda$ respectively

First of all

$$(1-e_\lambda)(T\cap R_{\lambda+1}) = T\cap(1-e_\lambda)R_{\lambda+1}$$
$$= T\cap(1-e_\lambda)R_\lambda = (1-e_\lambda)(T\cap R_\lambda),$$

implying

$$(T\cap R_{\lambda+1})\,|\,(1-e_\lambda) = (T\cap R_\lambda)\,|\,(1-e_\lambda)$$

Secondly $(T\cap R_{\lambda+1})\,|\,e_\lambda$ is atomless (because T is atomless and $R_{\lambda+1}$ contains all idempotents of R_κ smaller than e_λ) and

$$R\,|\,e_\lambda = R_\lambda\,|\,e_\lambda = (T\cap R_\lambda)\,|\,e_\lambda \subset (T\cap R_{\lambda+1})\,|\,e_\lambda \subset R_{\lambda+1}\,|\,e_\lambda$$
$$= C^0(\mathscr{C}, R_\lambda\,|\,e_\lambda),$$

hence, by the uniquenesspart of Lemma 6 15 we can extend j to an isomorphism L of $R_{\lambda+1}\,|\,e_\lambda$ onto $(T\cap R_{\lambda+1})\,|\,e_\lambda$. using (*) we get an isomorphism

$$\iota_{\lambda+1} = j_\lambda \times L \text{ of } R_{\lambda+1} = (R_\lambda\,|\,1-e_\lambda)\times(R_{\lambda+1}\,|\,e_\lambda)$$

onto

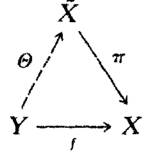
$$T\cap R_{\lambda+1} = (T\cap R_\lambda)\,|\,(1-e_\lambda)\times(T\cap R_{\lambda+1})\,|\,e_\lambda$$

The sequence (ι_λ) being constructed ι_κ is the desired R-isomorphism of R_κ onto $T = T\cap R_\kappa$ □

Remark 1. Beginning with Lemma 6 15 everything remains valid if we omit everywhere the predicate "ordered" in "ordered field" and the prefix "f" in "f-ring". I think this is of independent interest For instance if we consider boolean rings, it shows that a boolean algebra has a prime atomless extension (probably this is known but I have not seen it in the literature) By Stone's representation theory of boolean algebras this implies the following

let X be a boolean space, we consider pairs (Y, f) with Y a boolean space without isolated points and f $Y \to X$ continuous and onto, there exists a pair $(\tilde{X}, \pi)$ such that for every pair (Y, f) we can complete the diagram

Θ continuous and onto



Moreover $(\tilde{X}, \pi)$ is determined up to X-isomorphism by this property

Remark 2. Note that in Lemma 6 15 $C^0(\mathscr{C}, F)$ is generated as a ring over F by idempotents, going through the construction of the prime atomless extension $S = R_\kappa$ of R we see that also S is generated over R by idempotents, hence S is integral over R, moreover, if R is real closed S is also real closed

Now we can finally prove Theorem 6 14 Take for $\tilde{R}$ the prime atomless extension of the invariant real closure $\bar{R}$ of R By Proposition 6 15 and the preceding remark $\tilde{R}$ is an atomless real closed integral extension of R If T is any atomless real closed extension of $\bar{R}$, we first R-embed $\bar{R}$ into T (using Theorem 6 11) and then extend this embedding to an embedding of $\tilde{R}$ into T

Uniqueness Suppose $R \subset S \subset \tilde{R}$ and S atomless and real closed, then $\bar{R} \subset S$ by Theorem 6 11, and we have the situation $\bar{R} \subset S \subset \tilde{R}$ and S atomless, using the proof of the uniquenesspart of Proposition 6 16, this yields S is R-isomorphic with $\tilde{R}$ □

We make one further remark on these matters An unpublished result of S Shelah says that for each complete theory A in a countable language which admits quantifier elimination the following holds

if A is stable or quasi-totally transcendental, then each substructure of a model of A has a unique primemodel extension to a model of A (See [21] and [22] for these concepts)

We'll show that this result cannot be used in our situation to get Theorem 6 14 Note that $\bar{T}(^{-1},|\ |)$ is complete, admits quantifierelimination and that the regular f-rings are exactly the substructures of models of $\bar{T}(^{-1},|\ |)$

Proposition 6.17. *The theory $\bar{T}(^{-1},|\ |)$ is not stable and not quasi-totally transcendental*

Proof. $\mathbf{Z}$ is an infinite subset of each model of $\bar{T}(^{-1},|\ |)$ which is linearly ordered by the (definable) ordering of the model Hence by Theorem 7 1 33 of [21] $\bar{T}(^{-1},|\ |)$ is not stable For any regular f-ring R, let $S(R)$ be the space of 1-types over R as defined in Section 27 of [22], and let $S_B(R) = \{p \in S(R) \mid \text{the formula } x^2 = x$ belongs to $p\}$ Hence $S_B(R)$ is a clopen subset of $S(R)$

In particular $S_B(\mathbf{Q}) = \{p_0, p_1, p_2\}$ where p_0 is the principal type generated by $x = 0$, p_1 is generated by $x = 1$ and p_2 is generated by $x^2 = x \wedge 0 < x < 1$

Let $R = C^0(\mathscr{C}, \mathbf{Q})$, so $B(R)$ is a countable atomless boolean algebra, hence $B(R)$ contains a subset which is a dense linear ordering without endpoints and this implies by a wellknown argument that $S_B(R)$ is uncountable Hence $D^\alpha S(R) \cap S_B(R) \neq \emptyset$ for all ordinals α (for otherwise we can find for each $p \in S_B(R)$ a clopen $N_p \subset S(R)$ with $\{p\} = D^\alpha S(R) \cap S_B(R) \cap N_p$ for some α, and this implies that $N_p \neq N_q$ if $p \neq q$, but there are only countable many clopen subsets of $S(R)$, hence $S_B(R)$ is countable, contradiction!) Further the natural embedding $\mathbf{Q} \longrightarrow R$ satisfies

$$S_1(D^\alpha S(R) \cap S_B(R)) \subset D^\alpha S(\mathbf{Q}) \cap S_B(\mathbf{Q}),$$

hence $D^\alpha S(\mathbf{Q}) \cap S_B(\mathbf{Q}) \neq \emptyset$ for all α, hence $p_2 \in D^\alpha S(\mathbf{Q}) \cap S_B(\mathbf{Q})$ for all α, but p_2 is an isolated point of $S(\mathbf{Q})$, hence the ranked points of $S(\mathbf{Q})$ are not dense in $S(\mathbf{Q})$, so $\bar{T}(^{-1},|\ |)$ is not quasi-totally trascendental □

7. Decidability

Theorem 7.1. *If R and S are real closed regular f-rings then* $R \equiv S \Leftrightarrow B(R) \equiv B(S)$

Proof. $\Rightarrow$ is trivial Assume that $B(R) \equiv B(S)$ By the ultrapower theorem of Shelah-Keisler we may even assume that $B(R) = B(S)$ Then $R \equiv S$ follows from Theorem 1 2 of [23] □

Theorem 7.2. *The theory of real closed regular f-rings is decidable*

Proof. It is easy to see that this falls under the scope of Theorem 1 5 of [23] □

Originally I proved Theorem 7 1 and Theorem 7 2 by a method due to A B Carson [24] This method gives some additional results which may be useful to mention

Theorem 7.3. *Let R, S be real closed regular f-rings with* $R \subset S$ *Then we have*

(1) $T \subset_\vee S$ *iff every atom of* $B(R)$ *is an atom of* $B(S)$

(2) $R \prec S$ *iff* $B(R) \prec B(S)$

Proof. Replace in Lemma 2 2 and Proposition 2 4 of [24] the theory Σ_1 by the theory of real closed regular f-rings, and note that the proofs go through □

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