ACADEMIC PRESS

J. Math. Anal. Appl. 270 (2002) 150–164

Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS

www.academicpress.com

Existence of solutions for wave-type hemivariational inequalities with noncoercive viscosity damping

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Received 31 May 2001

Submitted by H.R. Thieme

Abstract

In this paper we prove the existence of solutions for a hyperbolic hemivariational inequality of the form

$$u'' + Au' + Bu + \partial j(u) \ni f,$$

where B is a linear elliptic operator and A is linear and nonnegative (not necessarily coercive). © 2002 Elsevier Science (USA). All rights reserved.

Keywords: Hemivariational inequalities; Clarke subdifferential; Viscosity damping

1. Introduction

The theory of variational inequalities provides us with an appropriate mathematical model to describe many physical problems (cf. Duvaut and Lions [8]). It was started in 60-ties with the pioneer works of G. Fichera, J.L. Lions, and G. Stampacchia. All the inequality problems studied by the use of these methods were related to convex energy functionals and therefore were closely connected with the notion of monotonicity. In the 80-ties, Panagiotopoulos intro-

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PII: S0022-247X(02)00057-4

duced the notion of nonconvex superpotential by the use of the general gradient of Clarke [7]. Due to the lack of convexity new types of variational expressions were obtained. These are so called *hemivariational inequalities* and they are no longer connected with monotonicity.

For a comprehensive treatment of the hemivariational inequality problems as well as for many applications, we refer to the monographs of Panagiotopoulos [23, 25], Motreanu and Panagiotopoulos [19], Naniewicz and Panagiotopoulos [20].

In this paper we study the following hyperbolic hemivariational inequality

$$\begin{cases} u'' + Au' + Bu + \chi = f, \\ u(0) = \psi_0, \quad u'(0) = \psi_1 \quad \text{in } \Omega, \\ \chi(t,x) \in \partial j(u(t,x)) \quad \text{a.e. in } (0,T) \times \Omega, \end{cases} \tag{1}$$

where $A \in \mathcal{L}(H, V')$ is an operator (not necessarily coercive), $B \in \mathcal{L}(V, V')$ is a coercive operator, $j : \mathbb{R} \to \mathbb{R}$ is a locally Lipschitz function, $\psi_0, \psi_1 : \Omega \to \mathbb{R}$ and $f : (0,T) \to V'$ are given functions.

The model for our problem is the following second-order nonlinear evolution equation called *sine-Gordon equation*:

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} + \alpha \frac{\partial u}{\partial t} - \Delta u + \gamma \sin u = f, \\ u(0,x) = \psi_0(x), \qquad \frac{\partial u}{\partial t}(0,x) = \psi_1(x), \end{cases}$$

which is of great importance because of its physical applications (cf. Temam [31, Chapter IV.2, p. 188]). Our work allows to introduce nonmonotone multivalued constitutive laws into this model. Moreover, in our framework we can consider damping terms more general than simple multiplication by a positive number (see Section 4).

We prove the existence of solutions of (1) using a method similar to the *parabolic regularization* method from the book of Lions and Magenes [15]; namely we approximate the solution of our problem by a sequence of solutions of some modified problems containing a coercive damping term. For the modified problem we apply the result of Gasiński [12].

2. Preliminaries

Let X be a Banach space with a norm $\|\cdot\|_X$ and X' its topological dual. By $\langle\cdot,\cdot\rangle_{X' \times X}$ we shall denote the duality brackets for the pair (X, X'). If X is in addition a Hilbert space, then by $(\cdot,\cdot)_X$ we shall denote the scalar product in X.

In the formulation of our hemivariational inequality the crucial role will be played by the notion of Clarke subdifferential of a locally Lipschitz function. A function $j : X \to \mathbb{R}$ is said to be *locally Lipschitz* if for every $x \in X$ there exists a neighbourhood U of x and a constant $k_x > 0$ depending on U such that

$$|j(z) - j(y)| \leqslant k_x \|z - y\|_X$$

for all $z, y \in U$. In analogy with the directional derivative of a convex function, we define *the generalized directional derivative* of a locally Lipschitz function j at $x \in X$ in the direction $h \in X$ by

$$j^0(x;h) \stackrel{\text{def}}{=} \limsup_{\substack{x' \to 0 \\ t \searrow 0}} \frac{j(x+x'+th) - j(x+x')}{t}.$$

It is easy to check that the function $X \ni h \mapsto j^0(x;h) \in \mathbb{R}$ is sublinear and continuous and that $|j^0(x;h)| \leqslant k_x \|h\|_X$. Hence by the Hahn–Banach theorem $j^0(x;\cdot)$ is the support function of nonempty, convex and w^*-compact set

$$\partial j(x) \stackrel{\text{def}}{=} \bigl\{x^* \in X' : \langle x^*, h\rangle_{X' \times X} \leqslant j^0(x;h) \text{ for all } h \in X\bigr\},$$

known as *the Clarke subdifferential* of j at x. Note that for every $x^* \in \partial j(x)$ we have $\|x^*\|_{X'} \leqslant k_x$. We have also that if $j, g : X \to \mathbb{R}$ are locally Lipschitz functions, then $\partial(j+g)(x) \subset \partial j(x) + \partial g(x)$ and $\partial(tj)(x) = t\partial j(x)$ for all $t \in \mathbb{R}$. Moreover, if $j : X \to \mathbb{R}$ is also convex, then the subdifferential of j in the sense of convex analysis coincides with the generalized subdifferential introduced above. Finally, if j is strictly differentiable at x (in particular if j is continuously Gateaux differentiable at x), then $\partial j(x) = \{j'(x)\}$.

Let us introduce the following spaces, needed in the sequel:

$$\begin{aligned}
&H = L^2(\Omega),\\
&V = H^1(\Omega) = \bigl\{v\colon\ v \in L^2(\Omega),\ D^\alpha v \in L^2(\Omega) \text{ for } 0 \leqslant |\alpha| \leqslant 1\bigr\},\\
&V' = V'(\Omega) = \bigl[H^1(\Omega)\bigr]'.
\end{aligned}$$

It is well-known that $V \subset H \subset V'$ form an evolution triple. By c_H^V we will denote "the continuity constant" for the embedding $V \subseteq H$ (so also for the embedding $H \subseteq V'$).

In our evolution case, we will also make use of the following spaces:

$$\begin{aligned}
&\mathcal{H} = L^2(0,T;H) = L^2((0,T) \times \Omega),\\
&\mathcal{V} = L^2(0,T;V),\\
&\mathcal{W} = \{v\colon\ v \in \mathcal{V},\ v' \in \mathcal{V}'\}.
\end{aligned}$$

3. Hyperbolic hemivariational inequality

Let $T > 0$ be any positive real number and let $N \geqslant 1$. By $\Omega \subset \mathbb{R}^N$ we will denote any open and bounded set. We consider the following hyperbolic hemivariational inequality:

Find $u \in C([0,T]; V) \cap C^1([0,T]; H)$ with $u'' \in \mathcal{V}'$ and $\chi \in \mathcal{H}$, such that

$$\text{(HVI)}\quad \begin{cases} u''(t) + Au'(t) + Bu(t) + \chi(t) = f(t) \\ \qquad\qquad\qquad\qquad \text{in } V', \text{ for a.a. } t \in (0,T), \\ u(0) = \psi_0, \quad u'(0) = \psi_1 \quad \text{in } \Omega, \\ \chi(t,x) \in \partial j(u(t,x)) \quad \text{for a.a. } (t,x) \in (0,T) \times \Omega, \end{cases}$$

where $A \in \mathcal{L}(H, V')$, $B \in \mathcal{L}(V, V')$, $j : \mathbb{R} \to \mathbb{R}$, $\psi_0, \psi_1 : \Omega \to \mathbb{R}$ and $f : (0,T) \to V'$ are given.

For our existence result, we will need the following assumptions:

H(j) $j : \mathbb{R} \to \mathbb{R}$ is a locally Lipschitz function, such that
- (i) $j(\xi) = \int_0^\xi \beta(s)\,ds$, where $\beta \in L^\infty_{\mathrm{loc}}(\mathbb{R})$;
- (ii) for every $\xi \in \mathbb{R}$ there exist limits $\lim_{\zeta \to \xi^\pm} \beta(\zeta)$;
- (iii) for every $\xi \in \mathbb{R}$, we have $|\beta(\xi)| \leqslant c_0(1 + |\xi|^r)$, with some $c_0 > 0$ and $0 \leqslant r < 1$.

H(A) $A : H \to V'$ is a linear operator, such that
- (i) A is continuous, i.e., there exists $\alpha_A > 0$, such that for all $v \in H$, we have $\|Av\|_{V'} \leqslant \alpha_A \|v\|_H$;
- (ii) $A|_V$ is nonnegative, i.e., for all $v \in V$, we have $\langle Av, v\rangle_{V \times V'} \geqslant 0$.

H(B) $B : V \to V'$ is a linear operator, such that
- (i) B is continuous, i.e., there exists $\alpha_B > 0$, such that for all $v \in V$, we have $\|Bv\|_{V'} \leqslant \alpha_B \|v\|_V$;
- (ii) B is coercive, i.e., there exists $\beta_B > 0$, such that for all $v \in V$, we have $\langle Bv, v\rangle_{V' \times V} \geqslant \beta_B \|v\|_V^2$;
- (iii) B is symmetric, i.e., for all $v, w \in V$, we have $\langle Bv, w\rangle_{V' \times V} = \langle Bw, v\rangle_{V' \times V}$.

H(f, ψ) $f \in \mathcal{H}$, $\psi_0 \in V$, $\psi_1 \in H$.

Now we can state our main result.

Theorem 3.1. *If hypotheses* H(j), H(A), H(B) *and* H(f, ψ) *hold, then* (HVI) *admits a solution.*

First, for any $\varepsilon > 0$ we consider the following regularized hyperbolic hemivariational inequality:

Find $u_\varepsilon \in C([0,T]; V)$ with $u'_\varepsilon \in \mathcal{W}$ and $\chi_\varepsilon \in \mathcal{H}$, such that

$$(\text{HVI}_\varepsilon)\quad \begin{cases} u''_\varepsilon(t) + Au'_\varepsilon(t) + \varepsilon Bu'_\varepsilon(t) + Bu_\varepsilon(t) + \chi_\varepsilon(t) = f(t), \\ u_\varepsilon(0) = \psi_0, \qquad u'_\varepsilon(0) = \psi_1, \\ \chi_\varepsilon(t,x) \in \partial j(u_\varepsilon(t,x)). \end{cases}$$

Lemma 3.2. *If hypotheses* H(j), H(A), H(B) *and* H(f, ψ) *hold, then for any* $\varepsilon > 0$ *there exists at least one solution* u_ε *of* (HVI$_\varepsilon$).

Proof. This is a consequence of the result of Gasiński (see [11] or [12]). To this end note that operator $\overline{A}:(0,T)\times V\mapsto V'$ defined by $\overline{A}(t,v)=A|_V v+\varepsilon Bv$ is pseudomonotone (with respect to v-variable), bounded (in a sense that for a.a. $t\in(0,T)$ and all $v\in V$, we have $\|\overline{A}(t,v)\|_{V'}\leqslant \bar{a}_1(t)+\bar{c}_1\|v\|_V$, with some $\bar{a}_1\in L^2(0,T)$, and $\bar{c}_1>0$) and coercive (namely for a.a. $t\in(0,T)$ and all $v\in V$, we have $\langle\overline{A}(t,v),v\rangle_{V'\times V}\geqslant \varepsilon\beta_B\|v\|_V^2$). Thus, using Theorem 3.1 and Remark 3.3 of [12], we obtain our lemma. □

In the next lemma we show an estimate on selections of $\partial j(u)$.

Lemma 3.3. *If hypotheses* H(j) *hold and* $u\in C([0,T];V)$ *with* $u'\in\mathcal{W}$ *and* $\eta\in\mathcal{H}$ *are such that* $\eta(t,x)\in\partial j(u(t,x))$ *for almost all* $(t,x)\in(0,T)\times\Omega$, *then*

$$\|\eta\|_{\mathcal{H}}\leqslant \bar{c}(1+\|u\|_{\mathcal{H}}),\tag{2}$$

with some constant $\bar{c}=\bar{c}(\Omega,T,c_0)>0$ *not depending on* u, η *and* r.

Proof. Using hypothesis H(j)(iii), we obtain

$$\begin{aligned}\|\eta\|_{\mathcal{H}}^2&=\int_0^T\|\eta(t)\|_H^2\,dt=\int_0^T\int_\Omega|\eta(t,x)|^2\,dx\,dt\\&\leqslant\int_0^T\int_\Omega 4c_0^2\big(1+|u(t,x)|\big)^2\,dx\,dt\leqslant 8c_0^2\int_0^T\big(|\Omega|+\|u(t)\|_H^2\big)\,dt\\&\leqslant 8c_0^2\big(T|\Omega|+\|u\|_{\mathcal{H}}^2\big),\end{aligned}$$

so estimate (2) holds with $\bar{c}\overset{\text{def}}{=}c_0 2\sqrt{2}\max\{\sqrt{T|\Omega|},1\}$. □

The following lemma gives some estimates on the solutions of (HVI$_\varepsilon$).

Lemma 3.4. *If hypotheses* H(j), H(A), H(B), H(f,ψ) *hold and* u_ε *is a solution of* (HVI$_\varepsilon$), *then for any* $\varepsilon\in(0,1)$ *we have*

$$\begin{aligned}&\max_{t\in[0,T]}\big(\|u_\varepsilon(t)\|_V+\|u_\varepsilon'(t)\|_H\big)+\sqrt{\varepsilon}\|u_\varepsilon'\|_{\mathcal{V}}+\|u_\varepsilon''\|_{\mathcal{V}'}\\&\quad\leqslant\bar{\bar{c}}\big(1+\|\psi_0\|_V+\|\psi_1\|_H+\|f\|_{\mathcal{H}}\big),\end{aligned}\tag{3}$$

where $\bar{\bar{c}}=\bar{\bar{c}}(\Omega,T,c_0,\alpha_A,\alpha_B,\beta_B)>0$ *is a constant not depending on* ε, ψ_0, ψ_1, A, B, f, j *and* r.

Proof. As $u_\varepsilon,u_\varepsilon'\in\mathcal{V}$, so in particular u_ε is an absolutely continuous function and

$$u_\varepsilon(t)=\int_0^t u_\varepsilon'(s)\,ds+\psi_0\quad\text{for all } t\in(0,T)$$

(see Barbu [3, p. 19, Theorem 2.2]). Thus for any $s \in (0, T)$ we have

$$\|u_\varepsilon(s)\|_H^2 \leqslant 2T \int_0^s \|u'_\varepsilon(\tau)\|_H^2 \, d\tau + 2\|\psi_0\|_H^2. \tag{4}$$

From the equality in (HVI_ε), taking the duality brackets on $u'_\varepsilon(s)$ and integrating over interval $(0, t)$, for any $t \in (0, T)$ we obtain

$$\begin{aligned} &\int_0^t \langle u''_\varepsilon(s), u'_\varepsilon(s)\rangle_{V'\times V} \, ds + \int_0^t \langle Au'_\varepsilon(s), u'_\varepsilon(s)\rangle_{V'\times V} \, ds \\ &\quad + \varepsilon \int_0^t \langle Bu'_\varepsilon(s), u'_\varepsilon(s)\rangle_{V'\times V} \, ds + \int_0^t \langle Bu_\varepsilon(s), u'_\varepsilon(s)\rangle_{V'\times V} \, ds \\ &\quad + \int_0^t \langle \chi_\varepsilon(s), u'_\varepsilon(s)\rangle_{V'\times V} \, ds = \int_0^t \langle f(s), u'_\varepsilon(s)\rangle_{V'\times V} \, ds. \end{aligned} \tag{5}$$

We will estimate separately each term in (5). First, we have

$$\begin{aligned} \int_0^t \langle u''_\varepsilon(s), u'_\varepsilon(s)\rangle_{V'\times V} \, ds &= \frac{1}{2}\|u'_\varepsilon(t)\|_H^2 - \frac{1}{2}\|u'_\varepsilon(0)\|_H^2 \\ &= \frac{1}{2}\|u'_\varepsilon(t)\|_H^2 - \frac{1}{2}\|\psi_1\|_H^2 \end{aligned}$$

(compare Zeidler [32, pp. 422–423, Proposition 23.23(iv)]). From hypothesis H(A)(ii), we have

$$\int_0^t \langle Au'_\varepsilon(s), u'_\varepsilon(s)\rangle_{V'\times V} \, ds \geqslant 0.$$

Next, hypothesis H(B)(ii) implies

$$\varepsilon \int_0^t \langle Bu'_\varepsilon(s), u'_\varepsilon(s)\rangle_{V'\times V} \, ds \geqslant \varepsilon\beta_B \int_0^t \|u'_\varepsilon(s)\|_V^2 \, ds.$$

Using the differentiation formula (see Zeidler [32, p. 881, Proof of Theorem 32.E(III)]) and hypotheses H(B)(i) and (ii), we obtain

$$\int_0^t \langle Bu_\varepsilon(s), u'_\varepsilon(s)\rangle_{V'\times V} \, ds = \frac{1}{2} \int_0^t \frac{d}{ds}\langle Bu_\varepsilon(s), u_\varepsilon(s)\rangle_{V'\times V} \, ds$$

$$
\begin{aligned}
&= \frac{1}{2}\big\langle Bu_\varepsilon(t), u_\varepsilon(t)\big\rangle_{V'\times V} - \frac{1}{2}\big\langle Bu_\varepsilon(0), u_\varepsilon(0)\big\rangle_{V'\times V} \\
&\geqslant \frac{\beta_B}{2}\|u_\varepsilon(t)\|_V^2 - \frac{\alpha_B}{2}\|\psi_0\|_V^2.
\end{aligned}
$$

Next, using hypothesis H(j)(iii), the Young inequality, estimate (4) and the continuity of the embedding $V \subset H$, for all $t \in (0,T)$ we have

$$
\begin{aligned}
&\int_0^t \big\langle \chi_\varepsilon(s), u_\varepsilon'(s)\big\rangle_{V'\times V}\,ds = \int_0^t \big(\chi(s), u_\varepsilon'(s)\big)_H\,ds \\
&\quad\geqslant -\int_0^t \|\chi(s)\|_H \|u_\varepsilon'(s)\|_H\,ds \\
&\quad\geqslant -\frac{1}{2}\int_0^t \|u_\varepsilon'(s)\|_H^2\,ds - \frac{1}{2}\int_0^t\int_\Omega c_0^2\big(1+|u_\varepsilon(s,x)|\big)^2\,dx\,ds \\
&\quad\geqslant -\frac{1}{2}\int_0^t \|u_\varepsilon'(s)\|_H^2\,ds - c_0^2\int_0^t \big(|\Omega| + \|u_\varepsilon(s)\|_H^2\big)\,ds \\
&\quad\geqslant -\frac{1}{2}\int_0^t \|u_\varepsilon'(s)\|_H^2\,ds - c_0^2 T|\Omega| \\
&\qquad - c_0^2\int_0^t \left(2T\int_0^s \|u_\varepsilon'(\tau)\|_H^2\,d\tau + 2\|\psi_0\|_H^2\right) ds \\
&\quad\geqslant -\frac{1}{2}\int_0^t \|u_\varepsilon'(s)\|_H^2\,ds - 2Tc_0^2\int_0^t\int_0^s \|u_\varepsilon'(\tau)\|_H^2\,d\tau\,ds \\
&\qquad - Tc_0^2\big(|\Omega| + 2c_H^V\|\psi_0\|_V^2\big).
\end{aligned}
$$

Finally, from the Young inequality, for all $t \in (0,T)$ we have

$$
\begin{aligned}
&\int_0^t \big\langle f(s), u_\varepsilon'(s)\big\rangle_{V'\times V}\,ds \leqslant \int_0^t \big(f(s), u_\varepsilon'(s)\big)_H\,ds \\
&\quad\leqslant \int_0^t \|f(s)\|_H \|u_\varepsilon'(s)\|_H\,ds
\end{aligned}
$$

$$\leqslant \frac{1}{2}\int_0^t \|u_\varepsilon'(s)\|_H^2\,ds + \frac{1}{2}\int_0^t \|f(s)\|_H^2\,ds$$

$$\leqslant \frac{1}{2}\int_0^t \|u_\varepsilon'(s)\|_H^2\,ds + \frac{1}{2}\|f\|_{\mathcal{H}}^2.$$

Putting all the above estimates into (5), for all $t \in (0,T)$ we obtain

$$\frac{1}{2}\|u_\varepsilon'(t)\|_H^2 + \frac{\beta_B}{2}\|u_\varepsilon(t)\|_V^2 + \varepsilon\beta_B\int_0^t \|u_\varepsilon'(s)\|_V^2\,ds$$
$$\leqslant c_1 + c_2\|\psi_0\|_V^2 + \frac{1}{2}\|\psi_1\|_H^2 + \frac{1}{2}\|f\|_{\mathcal{H}}^2$$
$$+ \int_0^t \|u_\varepsilon'(s)\|_H^2\,ds + 2Tc_0^2\int_0^t\int_0^s \|u_\varepsilon'(\tau)\|_H^2\,d\tau\,ds,$$

with $c_1 \stackrel{\text{def}}{=} Tc_0^2|\Omega|$ and $c_2 \stackrel{\text{def}}{=} 2Tc_H^Vc_0^2 + \alpha_B/2$. Thus, for all $t \in (0,T)$ we have

$$\frac{1}{2}\|u_\varepsilon'(t)\|_H^2 + \frac{\beta_B}{2}\|u_\varepsilon(t)\|_V^2 + \varepsilon\beta_B\int_0^t \|u_\varepsilon'(s)\|_V^2$$
$$\leqslant c_3\big(1 + \|\psi_0\|_V^2 + \|\psi_1\|_H^2 + \|f\|_{\mathcal{H}}^2\big)$$
$$+ c_4\int_0^t \|u_\varepsilon'(s)\|_H^2\,ds + c_4\int_0^t\int_0^s \|u_\varepsilon'(\tau)\|_H^2\,d\tau\,ds, \tag{6}$$

where $c_3 \stackrel{\text{def}}{=} \max\{1/2, c_1, c_2\}$ and $c_4 \stackrel{\text{def}}{=} \max\{1, 2Tc_0^2\}$. Now, using the generalization of the Gronwall–Bellman inequality (see Pachpatte [21, p. 758, Theorem 1]), for all $t \in (0,T)$ we obtain

$$\|u_\varepsilon'(t)\|_H^2 \leqslant c_5\big(1 + \|\psi_0\|_V^2 + \|\psi_1\|_H^2 + \|f\|_{\mathcal{H}}^2\big), \tag{7}$$

where $c_5 \stackrel{\text{def}}{=} 2c_3\big(1 + 2Tc_4e^{T(2c_4+1)}\big)$; so

$$\|u_\varepsilon'(t)\|_H \leqslant c_6\big(1 + \|\psi_0\|_V + \|\psi_1\|_H + \|f\|_{\mathcal{H}}\big), \tag{8}$$

where $c_6 \stackrel{\text{def}}{=} \sqrt{c_5}$, and also

$$\|u_\varepsilon'\|_{\mathcal{H}}^2 \leqslant c_7\big(1 + \|\psi_0\|_V^2 + \|\psi_1\|_H^2 + \|f\|_{\mathcal{H}}^2\big), \tag{9}$$

where $c_7 \stackrel{\text{def}}{=} Tc_5$. Applying (7) to (6), we obtain

$$\|u_\varepsilon(t)\|_V^2 \leqslant c_8\big(1 + \|\psi_0\|_V^2 + \|\psi_1\|_H^2 + \|f\|_{\mathcal{H}}^2\big) \tag{10}$$

and

$$\varepsilon\|u_\varepsilon'\|_{\mathcal{V}}^2 \leqslant c_8\big(1+\|\psi_0\|_V^2+\|\psi_1\|_H^2+\|f\|_{\mathcal{H}}^2\big), \tag{11}$$

where $c_8 \overset{\text{def}}{=} (2/\beta_B)(c_3+Tc_4c_5(T/2+1))$. Hence

$$\|u_\varepsilon(t)\|_V \leqslant c_9\big(1+\|\psi_0\|_V+\|\psi_1\|_H+\|f\|_{\mathcal{H}}\big) \tag{12}$$

and

$$\sqrt{\varepsilon}\|u_\varepsilon'\|_{\mathcal{V}} \leqslant c_9\big(1+\|\psi_0\|_V+\|\psi_1\|_H+\|f\|_{\mathcal{H}}\big), \tag{13}$$

where $c_9 \overset{\text{def}}{=} \sqrt{c_8}$. Using Lemma 3.3, continuity of the embedding $\mathcal{V}\subset\mathcal{H}$ and estimate (10), for any $\varepsilon>0$ we have

$$\begin{aligned}\|\chi_\varepsilon\|_{\mathcal{H}}^2 &\leqslant 2\bar{c}^2\big(1+\|u_\varepsilon\|_{\mathcal{H}}^2\big) \leqslant 2\bar{c}^2\left(1+\big(c_H^V\big)^2\int_0^T\|u_\varepsilon(t)\|_V^2\,dt\right)\\ &\leqslant c_{10}\big(1+\|\psi_0\|_V^2+\|\psi_1\|_H^2+\|f\|_{\mathcal{H}}^2\big),\end{aligned}$$

where $c_{10} \overset{\text{def}}{=} 2\bar{c}^2\max\{1, T(c_H^V)^2c_8\}$.

Finally, using the equation in (HVI$_\varepsilon$), hypothesis H(A)(i), continuity of the embedding $H\subset V'$, inequalities (9)–(11) and the last inequality, for all $\varepsilon\in(0,1)$ we can estimate $\|u_\varepsilon''\|_{\mathcal{V}'}$ as

$$\begin{aligned}\|u_\varepsilon''\|_{\mathcal{V}'}^2 &= \int_0^T\|u_\varepsilon''(t)\|_{V'}^2\,dt\\ &\leqslant 5\int_0^T\big\|A(t,u_\varepsilon'(t))\big\|_{V'}^2\,dt\\ &\quad+5\varepsilon^2\int_0^T\big\|B(u_\varepsilon'(t))\big\|_{V'}^2\,dt+5\int_0^T\big\|B(u_\varepsilon(t))\big\|_{V'}^2\,dt\\ &\quad+5\int_0^T\|\chi_\varepsilon(t)\|_{V'}^2\,dt+5\int_0^T\|f(t)\|_{V'}^2\,dt\\ &\leqslant 5\alpha_A\int_0^T\|u_\varepsilon'(t)\|_H^2\,dt\\ &\quad+5\varepsilon^2\alpha_B^2\int_0^T\|u_\varepsilon'(t)\|_V^2\,dt+5\alpha_B^2\int_0^T\|u_\varepsilon(t)\|_V^2\,dt\end{aligned}$$

$$
\begin{aligned}
&+5\big(c_H^V\big)^2\int_0^T \|\chi_\varepsilon(t)\|_H^2\,dt+5\big(c_H^V\big)^2\int_0^T\|f(t)\|_H^2\,dt\\
&\leqslant 5\alpha_A^2\|u_\varepsilon'\|_{\mathcal{H}}^2+5\varepsilon\alpha_B^2\|u_\varepsilon'\|_{\mathcal{V}}^2+5\alpha_B^2\int_0^T\|u_\varepsilon(t)\|_V^2\,dt\\
&\quad+5\big(c_H^V\big)^2\|\chi_\varepsilon\|_{\mathcal{H}}^2+5\big(c_H^V\big)^2\|f\|_{\mathcal{H}}^2\\
&\leqslant c_{11}\big(1+\|\psi_0\|_V^2+\|\psi_1\|_H^2+\|f\|_{\mathcal{H}}^2\big),
\end{aligned}
$$

where $c_{11} \stackrel{\text{def}}{=} 5\big(\alpha_A^2 c_7+\alpha_B^2 c_8(1+T)+(c_H^V)^2(c_{10}+1)\big)$; so

$$\|u_\varepsilon''\|_{\mathcal{V}'}\leqslant c_{12}\big(1+\|\psi_0\|_V+\|\psi_1\|_H+\|f\|_{\mathcal{H}}\big), \tag{14}$$

with $c_{12}=\sqrt{c_{11}}$.

Finally, from (8) and (12)–(14), we obtain (3), with $\bar{\bar{c}} \stackrel{\text{def}}{=} c_6+2c_9+c_{12}$. □

Now we are in the position to prove our main result.

Proof of Theorem 3.1. From Lemma 3.4, it follows that for any $\varepsilon\in(0,1)$, we have

$$\max_{t\in[0,T]}\big(\|u_\varepsilon(t)\|_V+\|u_\varepsilon'(t)\|_H\big)+\|u_\varepsilon''\|_{\mathcal{V}'}\leqslant c_{13},$$

with some constant $c_{13}>0$ not depending on $\varepsilon\in(0,1)$. Thus, we can choose a sequence $\{\varepsilon_n\}_{n\geqslant 1}\subset(0,1)$, such that $\varepsilon_n\searrow 0$ and

$$u_{\varepsilon_n}\to u \quad \text{weakly* in } L^\infty(0,T;V), \tag{15}$$

$$u_{\varepsilon_n}'\to \bar{u} \quad \text{weakly* in } L^\infty(0,T;H), \tag{16}$$

$$u_{\varepsilon_n}''\to \bar{\bar{u}} \quad \text{weakly in } \mathcal{V}'. \tag{17}$$

But in fact $\bar{u}=u'$ and $\bar{\bar{u}}=u''$.

It is easy to see that (HVI_ε) is equivalent to the following problem:

Find $u_\varepsilon\in C([0,T];V)$ with $u_\varepsilon'\in\mathcal{W}$ and $\chi_\varepsilon\in\mathcal{H}$, such that

$$
(\mathrm{HVI}_\varepsilon')\qquad
\begin{cases}
u_\varepsilon''+\widehat{A}u_\varepsilon'+\varepsilon\widehat{B}u_\varepsilon'+\widehat{B}u_\varepsilon+\chi_\varepsilon=f & \text{in } \mathcal{V}',\\
u_\varepsilon(0)=\psi_0,\quad u_\varepsilon'(0)=\psi_1 & \text{in } \Omega,\\
\chi_\varepsilon(t,x)\in\partial j(u_\varepsilon(t,x)) & \text{for a.a. } (t,x)\in(0,T)\times\Omega,
\end{cases}
$$

where $\widehat{A}:\mathcal{H}\to\mathcal{V}'$ and $\widehat{B}:\mathcal{V}\to\mathcal{V}'$ are the Nemytskii operators corresponding to the operators A and B, respectively. Our aim now is to "pass to the limit" in ($\mathrm{HVI}_\varepsilon'$).

As $\widehat{A}$ and $\widehat{B}$ are linear and bounded operators, from (15) and (16) we have

$$\widehat{A}u'_{\varepsilon_n} \to \widehat{A}u' \quad \text{weakly in } \mathcal{V}', \tag{18}$$

$$\widehat{B}u_{\varepsilon_n} \to \widehat{B}u \quad \text{weakly in } \mathcal{V}'. \tag{19}$$

Next, from Lemma 3.4 we see that the sequence $\{\sqrt{\varepsilon_n}u'_{\varepsilon_n}\}_{n\geqslant 1}$ remains bounded in $\mathcal{V}$; hence

$$\varepsilon_n u'_{\varepsilon_n} \to 0 \quad \text{in } \mathcal{V}.$$

But using hypothesis H(B)(i), we have that $\|\varepsilon_n \widehat{B}u'_{\varepsilon_n}\|_{\mathcal{V}'} \leqslant \alpha_B \|\varepsilon_n u'_{\varepsilon_n}\|_{\mathcal{V}}$; thus in fact

$$\varepsilon_n \widehat{B}u'_{\varepsilon_n} \to 0 \quad \text{in } \mathcal{V}'. \tag{20}$$

From (15), (16) and the compactness of the embedding $\mathcal{W} \subset \mathcal{H}$ we obtain

$$u_{\varepsilon_n} \to u \quad \text{in } \mathcal{H},$$

and, in particular, possibly passing to a subsequence,

$$u_{\varepsilon_n}(t,x) \to u(t,x) \quad \text{for a.a. } (t,x) \in (0,T) \times \Omega. \tag{21}$$

Using convergence (15), Lemma 3.3 and extracting a new subsequence if necessary, we obtain

$$\chi_{\varepsilon_n} \to \chi \quad \text{weakly in } \mathcal{H}, \tag{22}$$

with some $\chi \in \mathcal{H}$; hence also

$$\chi_{\varepsilon_n} \to \chi \quad \text{weakly in } L^1((0,T) \times \Omega). \tag{23}$$

Now, because of (17)–(20) and (22), we can "pass to the limit" in the equation in (HVI$'_\varepsilon$) and obtain

$$u'' + \widehat{A}u' + \widehat{B}u + \chi = f \quad \text{in } \mathcal{V}'. \tag{24}$$

Since for all $n \geqslant 1$ we have that $\chi_{\varepsilon_n}(t,x) \in \partial j(u_{\varepsilon_n}(t,x))$ for almost all $(t,x) \in (0,T) \times \Omega$, thus, using convergences (21) and (23) and applying Theorem 7.2.2 on p. 273 of Aubin and Frankowska [2] (recall that ∂j is a lower semicontinuous multifunction with convex and closed values), we get

$$\chi(t,x) \in \partial j(u(t,x)) \quad \text{for a.a. } (t,x) \in (0,T) \times \Omega. \tag{25}$$

Finally, from (15) and (16) we have that $u_{\varepsilon_n} \to u$ weakly in $H^1(0,T;H)$, hence also weakly in $C([0,T];H)$. Analogously, from (16) and (17) we have that $u'_{\varepsilon_n} \to u'$ weakly in $H^1(0,T;V')$, hence also weakly in $C([0,T];V')$. In particular, we have that

$$u_{\varepsilon_n}(0) \to u(0) \quad \text{weakly in } H,$$

$$u'_{\varepsilon_n}(0) \to u'(0) \quad \text{weakly in } V'. \tag{26}$$

To end our proof it remains to show that

$$u \in C([0,T];V) \cap C^1([0,T];H). \tag{27}$$

For this purpose let us recall the definition of the following function space introduced in the book of Lions and Magenes [15]:

$$C_s([0,T];X) \overset{\text{def}}{=} \bigl\{u \in L^\infty(0,T;X): \langle u^*, u(\cdot)\rangle_{X'\times X} \text{ is continuous } \forall u^* \in X'\bigr\}. \tag{28}$$

Of course, one has

$$C([0,T];X) \subset C_s([0,T];X).$$

Moreover, if X and Y are two Banach spaces, X being reflexive, with the dense embedding $X \subset Y$, from [15, p. 297, Lemma 8.1] we know that

$$C_s([0,T];Y) \cap L^\infty(0,T;X) = C_s([0,T];X). \tag{29}$$

In our case, due to (15)–(17) we have that

$$u \in C([0,T];H) \cap L^\infty(0,T;V),$$
$$u' \in C([0,T];V') \cap L^\infty(0,T;H);$$

hence, from (29) we obtain

$$u \in C_s([0,T];V), \tag{30}$$
$$u' \in C_s([0,T];H). \tag{31}$$

Next, using the same argument as in the proof of Lemma 3.4 (see (5) and the sequel), for any $t \in [0,T]$ we can prove the following energy equality

$$\|u'(t)\|_H^2 + \bigl\langle Bu(t), u(t)\bigr\rangle_{V'\times V} = \|\psi_1\|_H^2 + \langle B\psi_0, \psi_0\rangle_{V'\times V} + 2\int_0^t \bigl\langle f(s) - Au'(s) - \chi(s), u'(s)\bigr\rangle_{V'\times V}\, ds.$$

This shows that the function

$$E : [0,T] \ni t \mapsto \|u'(t)\|_H^2 + \bigl\langle Bu(t), u(t)\bigr\rangle_{V'\times V} \in \mathbb{R}$$

is continuous.

Take $t_n, t \in [0,T]$ such that $t_n \to t$ and put

$$\delta_n = \bigl\|u'(t_n) - u'(t)\bigr\|_H^2 + \bigl\langle Bu(t_n) - Bu(t), u(t_n) - u(t)\bigr\rangle_{V'\times V}$$
$$= E(t_n) + E(t) - 2\bigl\langle Bu(t), u(t_n)\bigr\rangle_{V'\times V} - 2\bigl(u'(t_n), u'(t)\bigr)_H.$$

Thanks to (30), (31) and the continuity of E we have that

$$\delta_n \to 2E(t) - 2\bigl\langle Bu(t), u(t)\bigr\rangle_{V'\times V} - 2\|u'(t)\|_H^2 = 0,$$

which, together with the inequality

$$\delta_n \geqslant \bigl\|u'(t_n) - u'(t)\bigr\|_H^2 + \beta_B \bigl\|u(t_n) - u(t)\bigr\|_V^2,$$

gives us (27). Now, from (24)–(27) we obtain that u is a solution of the following problem:

Find $u \in C([0,T];V) \cap C^1([0,T];H)$ with $u'' \in \mathcal{V}'$ and $\chi \in \mathcal{H}$, such that

$$(\mathrm{HVI}') \qquad \begin{cases} u'' + \widehat{A}u' + \widehat{B}u + \chi = f & \text{in } \mathcal{V}', \\ u(0) = \psi_0, \quad u'(0) = \psi_1 & \text{in } \Omega, \\ \chi(t,x) \in \partial j(u(t,x)) & \text{for a.a. } (t,x) \in (0,T) \times \Omega, \end{cases}$$

and in particular u is a solution of (HVI). □

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4. Applications and examples

As mentioned in the introduction the classical model for our considerations was the sine-Gordon equation

$$(\mathrm{SGE}) \qquad \begin{cases} u'' + \alpha u' - \Delta u + \gamma \sin u = f & \text{in } \Omega \times \mathbb{R}, \\ u(0) = \psi_0, \quad u'(0) = \psi_1 & \text{in } \Omega. \end{cases}$$

Of course, as $|\sin u| \leqslant 1$ the assumptions of Theorem 3.1 are satisfied if $\alpha \geqslant 0$. This is a "single-valued" case where $j(u) = -\cos u$ and $\partial j(u) = \{\sin u\}$.

In the sequel we present some examples of "dampings", which are admissible in our framework. First let us consider a slight generalization of the one used in the sine-Gordon equation; namely

$$Av = av, \quad \text{with } a \in L^\infty(\Omega),\ a \geqslant 0 \text{ a.e.}$$

In this case (HVI) has the form

$$\frac{\partial^2 u}{\partial t^2} + a(x)\frac{\partial u}{\partial t} + Bu + \partial j(u) \ni f.$$

The above "damping" operators map H into itself. The next two have values in V'. This time take $a \in W^{1,\infty}(\Omega;\mathbb{R}^N)$ and consider

$$A_1 v = \sum_{i=1}^N \frac{\partial}{\partial x_i}\bigl(a_i(x)v\bigr), \qquad A_2 v = \sum_{i=1}^N a_i(x)\frac{\partial v}{\partial x_i}.$$

It means this time (HVI) has the form

$$\frac{\partial^2 u}{\partial t^2} + \sum_{i=1}^N \frac{\partial}{\partial x_i}\left(a_i(x)\frac{\partial u}{\partial t}\right) + Bu + \partial j(u) \ni f$$

or, respectively,

$$\frac{\partial^2 u}{\partial t^2} + \sum_{i=1}^N a_i(x)\frac{\partial^2 u}{\partial t\,\partial x_i} + Bu + \partial j(u) \ni f.$$

The nonnegativity condition for A_1 yields

$$\begin{cases} \operatorname{div} a \geqslant 0 & \text{in } \Omega, \\ (a, n) \geqslant 0 & \text{on } \partial\Omega, \end{cases}$$

where n is the outer normal to the boundary of Ω. In the case $V = H_0^1(\Omega)$ we could drop the second inequality. Similarly, A_2 is nonnegative provided that

$$\begin{cases} \operatorname{div} a = 0 & \text{in } \Omega, \\ (a, n) = 0 & \text{on } \partial\Omega. \end{cases}$$

This time, if we took $V = H_0^1(\Omega)$ we would need only the inequality

$$\operatorname{div} a \leqslant 0 \quad \text{a.e. in } \Omega.$$

Acknowledgments

This work has been partially supported by the State Committee for Scientific Research of the Republic of Poland (KBN) under research grants No. 7 T07A 047 18 and No. 2 P03A 004 19.

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