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Dilations and rigid factorisations on noncommutative L^p-spaces

Marius Junge [a], Christian Le Merdy [b,*]

[a] *Mathematics Department, University of Illinois, Urbana, IL 61801, USA*

[b] *Département de Mathématiques, Université de Franche-Comté, 25030 Besançon Cedex, France*

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Abstract

We study some factorisation and dilation properties of completely positive maps on noncommutative L^p-spaces. We show that Akcoglu's dilation theorem for positive contractions on classical (= commutative) L^p-spaces has no reasonable analog in the noncommutative setting. Our study relies on nonsymmetric analogs of Pisier's operator space valued noncommutative L^p-spaces that we investigate in the first part of the paper.

Keywords: Noncommutative L^p-spaces; Completely positive maps; Operator spaces; Dilations; Completely positive maps

1. Introduction

Akcoglu's dilation theorem [1,2] for positive contractions on classical L^p-spaces plays a tremendous role in various areas of analysis. The main result of this paper says that there is no 'reasonable' analog of that result for (completely) positive contractions acting on noncommutative L^p-spaces. Recall that Akcoglu's theorem essentially says that for any measure space (Ω, μ), for any $1 < p < \infty$ and for any positive contraction $u : L^p(\Omega) \to L^p(\Omega)$, there is another measure space (Ω', μ'), two contractions $J : L^p(\Omega) \to L^p(\Omega')$ and $Q : L^p(\Omega') \to L^p(\Omega)$, and an invertible isometry $U : L^p(\Omega') \to L^p(\Omega')$ such that $u^n = QU^nJ$ for any in-

* Corresponding author.
E-mail addresses: junge@math.uiuc.edu (M. Junge), clemerdy@univ-fcomte.fr (C. Le Merdy).

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teger $n \geqslant 0$. Let S^p be the pth Schatten space of operators $a : \ell^2 \to \ell^2$ equipped with the norm $\|a\|_p = (tr(|a|^p))^{\frac{1}{p}}$. We show that if $p \neq 2$, there exists a completely positive contraction $u : S^p \to S^p$ which is not dilatable in the noncommutative sense. Namely whenever $L^p(M)$ is a noncommutative L^p-space associated with a von Neumann algebra M, there is no triple (J, Q, U) consisting of contractions $J : S^p \to L^p(M)$ and $Q : L^p(M) \to S^p$, and of an invertible isometry $U : L^p(M) \to L^p(M)$, such that $u^n = QU^nJ$ for any integer $n \geqslant 0$. Let $p' = p/(p-1)$ be the conjugate number of p. We actually show the stronger result that there is no pair (T, S) of isometries $T : S^p \to L^p(M)$ and $S : S^{p'} \to L^{p'}(M)$ such that $u = S^*T$.

The 'need' of a noncommutative version of Akcoglu's theorem (and its semigroup version [7]) came out from some recent work of Q. Xu and the authors devoted to diffusion semigroups on noncommutative L^p-spaces [16]. The lack of a noncommutative Akcoglu's theorem turns out to be a key feature of this topic.

We give two proofs of our main result. In Section 4, we give a nonconstructive one, that is, we show the existence of a completely positive contraction $u : S^p \to S^p$ which is not dilatable without giving an explicit example. In Section 5, we provide a second proof, which is longer but shows an explicit example. Our proofs rely on various properties of a class of operator space valued noncommutative L^p-spaces which we investigate in Sections 2 and 3, and on L^p-matricially normed spaces [15].

We will need a few techniques from operator space theory and we refer the reader to either [5] or [23] for the necessary background on this topic. If E, F are any two operator spaces, we let $CB(E, F)$ denote the space of all completely bounded maps $u : E \to F$. We let $\|u\|_{\mathrm{cb}}$ denote the completely bounded norm of such a map and we say that u is a complete contraction if $\|u\|_{\mathrm{cb}} \leqslant 1$. We let $E \otimes_{\mathrm{h}} F$ and $E \otimes_{\min} F$ denote the Haagerup tensor product and the minimal tensor product of E and F, respectively. Then we let $\| \ \|_{\min}$ denote the norm on $E \otimes_{\min} F$.

For any integer $k \geqslant 1$ we let M_k be the space of all $k \times k$ matrices equipped with the operator norm and for any $1 \leqslant p < \infty$, we let S_k^p be that space equipped with the pth Schatten norm. Also we use the notation S^∞ for the C^*-algebra of compact operators on ℓ^2. Unless stated otherwise, we let $(e_k)_{k \geqslant 1}$ denote the canonical basis of ℓ^2 and for any $i, j \geqslant 1$, we let $E_{ij} : \ell^2 \to \ell^2$ be the matrix unit taking e_j to e_i and taking e_k to 0 for any $k \neq j$. If X is any vector space, we regard as usual $S_k^p \otimes X$ as the space of all $k \times k$ matrices with entries in X, writing $[x_{ij}]$ for $\sum_{i,j} E_{ij} \otimes x_{ij}$ whenever $x_{ij} \in X$.

2. Some noncommutative operator space valued L^p-spaces

In this section we introduce a variant of the noncommutative vector valued L^p-spaces considered by Pisier in [22, Chapter 3] and we establish a few preliminary results. We refer the reader to [11,12] for related constructions. We start with some background and preliminary results on noncommutative L^p-spaces associated with a trace. We shall only give a brief account on theses spaces and we refer to [6,24,25] and the references therein for more details and further information.

We let (M, φ) be a semifinite von Neumann algebra equipped with a normal semifinite faithful trace φ. Then we let

$$V(M) = \bigcup eMe, \tag{2.1}$$

where the union runs over all projections $e \in M$ such that $\varphi(e) < \infty$. This is a $*$-algebra and the semifiniteness of φ ensures that $V(M)$ is w^*-dense in M. Let us write $V = V(M)$ for simplicity and let $V_+ = M_+ \cap V$ denote the positive part of V. Then any $a \in V_+$ has a finite trace.

Let $1 \leqslant p < \infty$. For any $a \in V$, the operator $|a|^p$ belongs to V and we set

$$\|a\|_p = \left(\varphi\left(|a|^p\right)\right)^{\frac{1}{p}}, \quad a \in V.$$

Here $|a| = (a^*a)^{\frac{1}{2}}$ denotes the modulus of a. It turns out that $\| \ \|_p$ is a norm on V. By definition, the noncommutative L^p-space associated with (M, φ) is the completion of $(V, \| \ \|_p)$. It is denoted by $L^p(M)$. For convenience, we also set $L^\infty(M) = M$ equipped with the operator norm $\| \ \|_\infty$.

Assume that $M \subset B(H)$ acts on some Hilbert space H, and let $M' \subset B(H)$ denote the commutant of M. It will be fruitful to have a description of the elements of $L^p(M)$ as (possibly unbounded) operators on H. We say that a closed and densely defined operator a on H is affiliated with M if a commutes with any unitary of M'. Then we say that an affiliated operator a is measurable (with respect to the trace φ) provided that there is a positive real number $\lambda > 0$ such that $\varphi(\epsilon_\lambda) < \infty$, where $\epsilon_\lambda = \chi_{[\lambda,\infty)}(|a|)$ is the projection associated to the indicator function of $[\lambda, \infty)$ in the Borel functional calculus of $|a|$. The set $L^0(M)$ of all measurable operators is a $*$-algebra (see e.g. [25, Chapter I] for a proof and a precise definition of the sum and product on $L^0(M)$).

We recall further properties of $L^0(M)$ that will be used later on. First for any a in $L^0(M)$ and any $0 < p < \infty$, the operator $|a|^p = (a^*a)^{\frac{p}{2}}$ belongs to $L^0(M)$. Second, let $L^0(M)_+$ be the positive part of $L^0(M)$, that is, the set of all selfadjoint positive operators in $L^0(M)$. Then the trace φ extends to a positive tracial functional on $L^0(M)_+$, still denoted by φ, in such a way that for any $1 \leqslant p < \infty$, we have

$$L^p(M) = \left\{a \in L^0(M)\colon \varphi\left(|a|^p\right) < \infty\right\},$$

equipped with $\|a\|_p = (\varphi(|a|^p))^{\frac{1}{p}}$. Furthermore, φ uniquely extends to a bounded linear functional on $L^1(M)$, still denoted by φ. For any $a, c \in L^0(M)$, we have $ac \in L^1(M)$ if and only if $ca \in L^1(M)$ and in this case, $\varphi(ac) = \varphi(ca)$. Furthermore we have

$$\left|\varphi(a)\right| \leqslant \varphi\left(|a|\right) = \|a\|_1$$

for any $a \in L^1(M)$.

Let $1 \leqslant p, q, s \leqslant \infty$ such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{s}$. The so-called noncommutative Hölder inequality asserts that $L^p(M) \cdot L^q(M) \subset L^s(M)$ and that we have

$$\|ac\|_s \leqslant \|a\|_p \|c\|_q, \quad a \in L^p(M),\ c \in L^q(M). \tag{2.2}$$

For any $1 \leqslant p < \infty$, let $p' = p/(p-1)$ be the conjugate number of p. Applying (2.2) with $q = p'$ and $s = 1$, we may define a duality pairing between $L^p(M)$ and $L^{p'}(M)$ by

$$\langle a, c\rangle = \varphi(ac), \quad a \in L^p(M),\ c \in L^{p'}(M). \tag{2.3}$$

This induces an isometric isomorphism

$$L^p(M)^* = L^{p'}(M), \quad 1 \leqslant p < \infty, \ \frac{1}{p} + \frac{1}{p'} = 1.$$

In particular, we may identify $L^1(M)$ with the (unique) predual M_* of M.

We will assume that the reader is familiar with complex interpolation of Banach spaces, for which we refer to [4]. We recall that by means of the embeddings of $L^\infty(M)$ and $L^1(M)$ into $L^0(M)$, one may regard $(L^\infty(M), L^1(M))$ as a compatible couple of Banach spaces and that we have

$$\big[L^\infty(M), L^1(M)\big]_{1/p} = L^p(M), \quad 1 \leqslant p \leqslant \infty, \tag{2.4}$$

where $[\cdot,\cdot]_\theta$ denotes the complex interpolation method.

For any $1 \leqslant p < \infty$, we let $L^p(M)_+ = L^0(M)_+ \cap L^p(M)$ denote the positive part of $L^p(M)$. We recall that the support projection Q of any element $b \in L^p(M)_+$ is the orthogonal projection onto the closure of the range of b, and that $\ker(Q) = \ker(b)$. This projection belongs to M.

Lemma 2.1. *Let $1 \leqslant p, q, s \leqslant \infty$ such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{s}$ and $s < \infty$. Let $b \in L^p(M)_+$ and let Q be its support projection. Then $\overline{bL^q(M)}^{\| \ \|_s} = QL^s(M)$.*

Proof. Let s' be the conjugate number of s. Since $(QL^s(M))^\perp = L^{s'}(M)(1-Q)$, it suffices to show that $(bL^q(M))^\perp = L^{s'}(M)(1-Q)$. If $c \in (bL^q(M))^\perp$, then $\varphi(cba) = 0$ for any $a \in L^q(M)$, hence $cb = 0$. This implies $cQ = 0$, hence $c \in L^{s'}(M)(1-Q)$. This proves one inclusion and the other one is obvious. □

Lemma 2.2. *Assume that $p \geqslant 2$ and let $q \geqslant 2$ be defined by $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$. Let $y \in L^{p'}(M)$, $a \in L^q(M)_+$ and $b \in L^2(M)_+$ such that*

$$\big|\varphi(yzd)\big| \leqslant \|da\|_2 \|bz\|_2$$

for any $z \in M$ and $d \in L^p(M)$. Let Q_a and Q_b be the support projections of a and b, respectively. Then there exists $w \in M$ such that $\|w\| \leqslant 1$, $y = awb$ and $w = Q_a w Q_b$.

Proof. By Lemma 2.1, $aL^p(M)$ and bM are dense subspaces of $Q_a L^2(M)$ and $Q_b L^2(M)$, respectively. Hence according to our assumption, there exists a (necessarily unique) continuous sesquilinear form $\sigma : Q_b L^2(M) \times Q_a L^2(M) \to \mathbb{C}$ such that $\sigma(bz, ad) = \varphi(yzd^*)$ for any $z \in M$ and any $d \in L^p(M)$. Let $\overline{\sigma}$ be the contractive sesquilinear form on $L^2(M)$ defined by $\overline{\sigma}(g,h) = \sigma(Q_b g, Q_a h)$ and let

$$T : L^2(M) \longrightarrow L^2(M)$$

be the associated linear contraction. By construction we have

$$\big\langle T(bz), ad\big\rangle_2 = \varphi(yzd^*), \quad z \in M, \ d \in L^p(M) \tag{2.5}$$

and

$$\big\langle T(g), h\big\rangle_2 = \big\langle T(Q_b g), Q_a h\big\rangle_2, \quad g, h \in L^2(M), \tag{2.6}$$

where $\langle\,,\rangle_2$ denotes the inner product on $L^2(M)$.

We claim that for any $c \in M$ and any $g \in L^2(M)$, we have $T(gc) = T(g)c$. Indeed, for any $z \in M$ and $d \in L^p(M)$ we have

$$\big\langle T(bzc), ad\big\rangle_2 = \varphi(yzcd^*) = \varphi\big(yz(dc^*)^*\big) = \big\langle T(bz), adc^*\big\rangle_2$$

by (2.5). Consequently we have

$$\big\langle T(Q_b gc), Q_a h\big\rangle_2 = \big\langle T(Q_b g), Q_a hc^*\big\rangle_2$$

for any $g, h \in L^2(M)$, and hence

$$\big\langle T(gc), h\big\rangle_2 = \big\langle T(g), hc^*\big\rangle_2 = \big\langle T(g)c, h\big\rangle_2$$

by (2.6). This proves the claim.

Consequently there exists $w \in M$, with $\|w\|_\infty = \|T\| \leqslant 1$, such that $T(g) = wg$ for any $g \in L^2(M)$. Using (2.5) again, we find that

$$\varphi(awbzd^*) = \varphi\big(w(bz)(ad)^*\big) = \varphi(yzd^*)$$

for any $z \in M$ and any $d \in L^p(M)$. This shows that $y = awb$.

The identity (2.6) ensures that $\langle Q_a w Q_b g, h\rangle = \langle wg, h\rangle$ for any $g, h \in L^2(M)$. Hence we have $w = Q_a w Q_b$. $\square$

We introduce a notation which will be used throughout. Suppose that $p, q, r, s \geqslant 1$ satisfy $\frac{1}{q} + \frac{1}{r} + \frac{1}{s} = \frac{1}{p}$. Let X be any vector space, let $y \in L^r(M) \otimes X$ and let $(a_k)_k$ and $(x_k)_k$ be finite families in $L^r(M)$ and X, respectively, such that $y = \sum_k a_k \otimes x_k$. Then for any $c \in L^q(M)$ and $d \in L^s(M)$, we will write cyd for the element of $L^p(M) \otimes X$ defined by

$$cyd = \sum_k ca_k d \otimes x_k.$$

Let F be an operator space, let $1 \leqslant p < \infty$ and let $y \in V \otimes F$. If $p \geqslant 2$, we let

$$\|y\|_{\alpha_p^\ell} = \inf\big\{\|c\|_\infty \|z\|_{\min} \|d\|_p\big\},$$

where the infimum runs over all $c, d \in V$ and all $z \in M \otimes F$ such that $y = czd$. Here $\|z\|_{\min}$ denotes the norm of z in $M \otimes_{\min} F$. Arguing as in the proof of [22, Lemma 3.5], it is not hard to check that $\|\ \|_{\alpha_p^\ell}$ is a norm on $V \otimes F$. The proof of the triangle inequality relies on the convexity condition

$$\big\|\big(d_1^* d_1 + d_2^* d_2\big)^{\frac{1}{2}}\big\|_p \leqslant \big(\|d_1\|_p^2 + \|d_2\|_p^2\big)^{\frac{1}{2}}, \quad d_1, d_2 \in L^p(M),$$

and the latter holds because $p \geqslant 2$.

If $p \leqslant 2$, we let $q \geqslant 2$ be such that $\frac{1}{2} + \frac{1}{q} = \frac{1}{p}$, and we let

$$\|y\|_{\alpha_p^\ell} = \inf\bigl\{\|a\|_q \|z\|_{\min} \|b\|_2\bigr\},$$

where the infimum runs over all $a, b \in V$, and all $z \in M \otimes F$ such that $y = azb$. Arguing again as in [22, Lemma 3.5], we find that $\| \ \|_{\alpha_p^\ell}$ is a norm on $V \otimes F$. Then for any $p \geqslant 1$, we define the space

$$L^p\{M; F\}_\ell$$

as the completion of $V \otimes F$ for the norm $\| \ \|_{\alpha_p^\ell}$.

Likewise, if $p \geqslant 2$, we let

$$\|y\|_{\alpha_p^r} = \inf\bigl\{\|c\|_p \|z\|_{\min} \|d\|_\infty\bigr\},$$

where the infimum runs over all $c, d \in V$ and all $z \in M \otimes F$ such that $y = czd$. Then if $p \leqslant 2$ we let

$$\|y\|_{\alpha_p^r} = \inf\bigl\{\|a\|_2 \|z\|_{\min} \|b\|_q\bigr\},$$

where the infimum runs over all $a, b \in V$, and all $z \in M \otimes F$ such that $y = azb$. We obtain that $\| \ \|_{\alpha_p^r}$ is a norm on $V \otimes F$ as before, and we let

$$L^p\{M; F\}_r$$

be the completion of $V \otimes F$ for that norm.

In the case when $M = M_k$, these definitions reduce to the ones given in [15, Section 2] and we have

$$S_k^p\{F\}_\ell = L^p\{M_k; F\}_\ell \quad \text{and} \quad S_k^p\{F\}_r = L^p\{M_k; F\}_r,$$

where $S_k^p\{F\}_\ell$ and $S_k^p\{F\}_r$ are the spaces introduced in the latter paper.

For any $\eta \in F^*$, the linear mapping $I_V \otimes \eta : V \otimes F \to V$ (uniquely) extends to a bounded map $\bar{\eta} : L^p\{M; F\}_\ell \to L^p(M)$, and we have $\|\bar{\eta}\| = \|\eta\|$. Indeed assume for example that $p \geqslant 2$, and let $y = czd \in V \otimes F$, with $c, d \in V$ and $z \in M \otimes F$. Let $(a_k)_k$ and $(x_k)_k$ be finite families in M and F, respectively, such that $z = \sum_k a_k \otimes x_k$. Then $(I_V \otimes \eta)y = \sum_k \langle \eta, x_k\rangle c a_k d$, hence

$$\bigl\|(I_V \otimes \eta)y\bigr\|_p \leqslant \|c\|_\infty \Bigl\|\sum_k \langle \eta, x_k\rangle a_k\Bigr\|_\infty \|d\|_p \leqslant \|c\|_\infty \|\eta\| \|z\|_{\min} \|d\|_p.$$

Passing to the infimum over all c, d, z factorising y, we obtain that $\|(I_V \otimes \eta)y\|_p \leqslant \|\eta\| \|y\|_{\alpha_p^\ell}$.

Thanks to the above fact, we have a canonical (dense) inclusion

$$L^p(M) \otimes F \subset L^p\{M; F\}_\ell. \tag{2.7}$$

More precisely, the bilinear mapping $V \times F \to V \otimes F \subset L^p\{M; F\}_\ell$ obviously extends to a contractive bilinear mapping $L^p(M) \times F \to L^p\{M; F\}_\ell$, which yields a linear mapping

$\kappa : L^p(M) \otimes F \to L^p\{M; F\}_\ell$. Then we obtain (2.7) by showing that κ is one-to-one. For that purpose, let y in $L^p(M) \otimes F$ and assume that $\kappa(y) = 0$. For any $\eta \in F^*$, we have $(\bar{\eta} \circ \kappa)y = (I_{L^p} \otimes \eta)y$, hence $(I_{L^p} \otimes \eta)y = 0$. This shows that $y = 0$.

The next lemma follows from the above discussion. We omit its easy proof.

Lemma 2.3.

(1) *Assume that $p \geqslant 2$. Then for any $z \in M \otimes F$ and any $d \in L^p(M)$, we have*

$$\|zd\|_{L^p\{M;F\}_\ell} \leqslant \|z\|_{\min} \|d\|_p.$$

(2) *Assume that $p \leqslant 2$, and that $\frac{1}{2} + \frac{1}{q} = \frac{1}{p}$. Then for any $z \in M \otimes F$ and any $a \in L^r(M)$, $b \in L^2(M)$, we have*

$$\|azb\|_{L^p\{M;F\}_\ell} \leqslant \|a\|_q \|z\|_{\min} \|b\|_2.$$

(3) *The embedding* (2.7) *extends to a contractive linear map $L^p(M)\hat{\otimes}F \to L^p\{M; F\}_\ell$, where $\hat{\otimes}$ denotes the Banach space projective tensor product.*

We end this section with an observation regarding opposite structures. We recall that the opposite operator space of F, denoted by F^{op}, is defined as being the vector space F equipped with the following matrix norms. For any $[x_{ij}] \in M_k \otimes F$,

$$\big\|[x_{ij}]\big\|_{M_k(F^{\mathrm{op}})} = \big\|[x_{ji}]\big\|_{M_k(F)}.$$

(See [23, Section 2.10].) Then M^{op} coincides with the von Neumann algebra obtained by endowing M with the reverse product $*$ defined by $a * c = ca$ (for $a, c \in M$). It is clear from the definition that $M \otimes_{\min} F = M^{\mathrm{op}} \otimes_{\min} F^{\mathrm{op}}$ isometrically. We deduce that we have an isometric identification

$$L^p\{M; F\}_r \simeq L^p\big\{M^{\mathrm{op}}; F^{\mathrm{op}}\big\}_\ell. \tag{2.8}$$

Indeed assume for example that $p \geqslant 2$ and let $y \in V \otimes F$. Suppose that the norm of y in $L^p\{M; F\}_r$ is < 1. Then there exist $c, d \in V$ and $z \in M \otimes F$ such that $y = czd$, $\|c\|_p < 1$, $\|d\|_\infty < 1$ and $\|z\|_{M \otimes_{\min} F} < 1$. Let us write $z = \sum_k a_k \otimes x_k$, with $a_k \in M$ and $x_k \in F$, so that $y = \sum_k ca_kd \otimes x_k$. Then $ca_kd = d * a_k * c$ for any k, hence $y = d * (\sum_k a_k \otimes x_k) * c = d * z * c$. Since $\|z\|_{M \otimes_{\min} F} = \|z\|_{M^{\mathrm{op}} \otimes_{\min} F^{\mathrm{op}}}$, this implies that the norm of y in $L^p\{M^{\mathrm{op}}; F^{\mathrm{op}}\}_\ell$ is < 1. Reversing the argument we find that the norms of y in $L^p\{M; F\}_r$ and in $L^p\{M^{\mathrm{op}}; F^{\mathrm{op}}\}_\ell$ actually coincide.

3. Duality for $L^p\{M; F\}_\ell$

We let R and C be the standard row and column Hilbert spaces, and we denote by R_k and C_k their k-dimensional versions, respectively. This section is devoted to various properties of the dual space of $L^p\{M; F\}_\ell$, especially when $F = R$. We will start with a description of the dual space of $S_k^p\{F\}_\ell$ for any F.

We recall that if E_0 and E_1 are any two operator spaces, and if (E_0, E_1) is a compatible couple in the sense of Banach space interpolation theory, then $[E_0, E_1]_\theta$ has a canonical operator space structure. Indeed its matrix norms are given by the isometric identities $M_k([E_0, E_1]_\theta) = [M_k(E_0), M_k(E_1)]_\theta$. See [23, Section 2.7] and [21] for details and complements. For any $\theta \in [0, 1]$, we let

$$R(\theta) = [R, C]_\theta$$

be the Hilbertian operator space obtained by applying this construction to the couple (R, C). Then we both have

$$R(\theta)^* = R(1-\theta) \quad \text{and} \quad R(\theta)^{\mathrm{op}} = R(1-\theta)$$

completely isometrically for any $\theta \in [0, 1]$.

Let F be an operator space. We may identify $S_k^p \otimes F$ with $\ell_k^2 \otimes F \otimes \ell_k^2$ be identifying $e_i \otimes x \otimes e_j$ with $E_{ij} \otimes x$ for any $x \in F$ and any $1 \leqslant i, j \leqslant k$. According to [15], this induces isometric identifications

$$S_k^p\{F\}_\ell \simeq C_k \otimes_{\mathrm{h}} F \otimes_{\mathrm{h}} R_k\left(\frac{2}{p}\right) \quad \text{and} \quad S_k^p\{F\}_r \simeq R_k\left(1 - \frac{2}{p}\right) \otimes_{\mathrm{h}} F \otimes_{\mathrm{h}} R_k \tag{3.1}$$

if $p \geqslant 2$, whereas

$$S_k^p\{F\}_\ell \simeq R_k\left(2\left(1 - \frac{1}{p}\right)\right) \otimes_{\mathrm{h}} F \otimes_{\mathrm{h}} C_k \quad \text{and} \quad S_k^p\{F\}_r \simeq R_k \otimes_{\mathrm{h}} F \otimes_{\mathrm{h}} R_k\left(\frac{2}{p} - 1\right) \tag{3.2}$$

if $p \leqslant 2$.

Proposition 3.1. *Let $1 < p, p' < \infty$ be conjugate numbers and let F be an operator space. Then we have isometric identifications*

$$\left(S_k^p\{F\}_\ell\right)^* \simeq S_k^{p'}\left\{F^{*\mathrm{op}}\right\}_\ell \quad \textit{and} \quad \left(S_k^p\{F\}_r\right)^* \simeq S_k^{p'}\left\{F^{*\mathrm{op}}\right\}_r \tag{3.3}$$

through the duality pairing $(S_k^p \otimes F) \times (S_k^{p'} \otimes F^*) \to \mathbb{C}$ mapping the pair $(a \otimes x, c \otimes \eta)$ to the complex number $tr(ac)\langle \eta, x\rangle$ for any $a \in S_k^p$, $c \in S_k^{p'}$, $x \in F$ and $\eta \in F^*$.

Proof. We will use the fact that if $E_1, \dots, E_n$ are any operator spaces, then $E_1 \otimes_{\mathrm{h}} \cdots \otimes_{\mathrm{h}} E_n$ is isometrically isomorphic to $E_n^{\mathrm{op}} \otimes_{\mathrm{h}} \cdots \otimes_{\mathrm{h}} E_1^{\mathrm{op}}$ via the linear mapping taking $x_1 \otimes \cdots \otimes x_n$ to $x_n \otimes \cdots \otimes x_1$ for any $x_1 \in E_1, \dots, x_n \in E_n$ (see e.g. [23, p. 97]).

We only prove the first identity in (3.3), the second one being similar. We use the self-duality of the Haagerup tensor product (see e.g. [5, Theorem 9.4.7]). Assume that $p \geqslant 2$. By the above observations, we have

$$\begin{aligned}\left(S_k^p\{F\}_\ell\right)^* &\simeq C_k^* \otimes_{\mathrm{h}} F^* \otimes_{\mathrm{h}} R_k\left(\frac{2}{p}\right)^* \\ &\simeq R_k \otimes_{\mathrm{h}} F^* \otimes_{\mathrm{h}} R_k\left(1 - \frac{2}{p}\right)\end{aligned}$$

$$\begin{aligned} &\simeq R_k\left(1-\frac{2}{p}\right)^{\mathrm{op}} \otimes_{\mathrm{h}} F^{*\mathrm{op}} \otimes_{\mathrm{h}} R_k^{\mathrm{op}} \\ &\simeq R_k\left(\frac{2}{p}\right) \otimes_{\mathrm{h}} F^{*\mathrm{op}} \otimes_{\mathrm{h}} C_k \\ &\simeq S_k^{p'}\left\{F^{*\mathrm{op}}\right\}_{\ell}. \end{aligned}$$

Moreover it is not hard to check (left to the reader) that the duality pairing leading to these isometric isomorphisms is the one given in the statement.

The proof for $p \leqslant 2$ is similar. $\square$

Remark 3.2. Let $S_k^p[F]$ denote Pisier's operator space valued Schatten space [22, Chapter 1]. We recall that for any $y \in S_k^p \otimes F$, the norm $\|y\|_{S_k^p[F]}$ is equal to $\inf\{\|c\|_{2p}\|z\|_{\min}\|d\|_{2p}\}$, where the infimum runs over all $c, d \in S_k^{2p}$ and all $z \in M_k(F) = M_k \otimes_{\min} F$ such that $y = czd$. Moreover we have

$$S_k^p[F] \simeq R_k\left(1-\frac{1}{p}\right) \otimes_{\mathrm{h}} F \otimes_{\mathrm{h}} R_k\left(\frac{1}{p}\right) \tag{3.4}$$

isometrically. Then the proof of Proposition 3.1 yields an isometric identification

$$S_k^p[F]^* \simeq S_k^{p'}\left[F^{*\mathrm{op}}\right]. \tag{3.5}$$

Using transposition, the latter result is the same as [22, Corollary 1.8].

We finally observe that in general the identifications in (3.3) are not completely isometric (already with $k = 1$).

Proposition 3.1 leads to a natural duality problem, which turns out to be crucial for our investigations in the next two sections. Let $1 < p, p' < \infty$ be two conjugate numbers, and consider an arbitrary semifinite von Neumann algebra (M, φ). For any operator space F, consider the duality pairing

$$\left(L^p(M) \otimes F\right) \times \left(L^{p'}(M) \otimes F^*\right) \longrightarrow \mathbb{C}$$

defined by

$$(a \otimes x, c \otimes \eta) \longmapsto \varphi(ac)\langle \eta, x\rangle \tag{3.6}$$

for any $a \in L^p(M)$, $c \in L^{p'}(M)$, $x \in F$ and $\eta \in F^*$. In view of Proposition 3.1, it is natural to wonder whether this pairing induces an isometric embedding of $L^{p'}\{M; F^{*\mathrm{op}}\}_\ell$ into $L^p\{M; F\}_\ell^*$. Arguing as in the proof of [22, Theorem 4.1], and using Proposition 3.1, we may obtain that this holds true when M is hyperfinite. However it is false in general, see Remark 3.5(2). In the rest of this section we will focus on the special case when $F = R$ and we will show a positive result in that case.

We recall that $R^* = C$ and that $C^{\mathrm{op}} = R$, so that $R^{*\mathrm{op}} = R$. In Sections 4 and 5, we will use the fact that for any $1 < p < \infty$, the above pairing induces a contraction $L^{p'}\{M; R\}_\ell \to L^p\{M; R\}_\ell^*$. The next theorem is a more precise result that we prove for the sake of completeness.

Theorem 3.3.

(1) *For any* $1 < p \leqslant 2$*, we have*

$$L^{p'}\{M; R\}_\ell \hookrightarrow L^p\{M; R\}_\ell^* \text{ isometrically.}$$

(2) *For any* $2 < p < \infty$*, we have an isometric isomorphism*

$$L^p\{M; R\}_\ell^* \simeq L^{p'}\{M; R\}_\ell.$$

In the sequel we let $(e_n)_{n\geqslant 1}$ denote the canonical basis of R and we recall that for any finite sequence $(z_n)_n$ in M, we have

$$\left\| \sum_n z_n \otimes e_n \right\|_{M \otimes_{\min} R} = \left\| \sum_n z_n z_n^* \right\|_\infty^{\frac{1}{2}}.$$

Lemma 3.4. *Let* $2 \leqslant p < \infty$*. For any finite families* $(d_j)_j$ *in* $L^p(M)$ *and* $(z_{nj})_{n,j}$ *in* M*, we have*

$$\left\| \sum_{n,j} z_{nj} d_j \otimes e_n \right\|_{L^p\{M;R\}_\ell} \leqslant \left\| \left(\sum_j d_j^* d_j \right)^{\frac{1}{2}} \right\|_p \left\| \sum_{n,j} z_{nj} z_{nj}^* \right\|_\infty^{\frac{1}{2}}.$$

Proof. We suppose that $M \subset B(H)$ as before. Let $d = (\sum_j d_j^* d_j)^{1/2}$ and let Q be its support projection. For any j, we have $0 \leqslant d_j^* d_j \leqslant d^2$ hence there exists a (necessarily unique) $w_j \in M$ such that

$$w_j d = d_j \quad \text{and} \quad w_j Q = w_j.$$

Then we have

$$d^2 = \sum_j d_j^* d_j = d\left(\sum_j w_j^* w_j \right) d \quad \text{and} \quad Q\left(\sum_j w_j^* w_j \right) Q = \sum_j w_j^* w_j.$$

This readily implies that $\sum_j w_j^* w_j = Q$. Indeed, these two bounded operators coincide on the range of d and on the kernel of Q. In particular, we have

$$\left\| \sum_j w_j^* w_j \right\|_\infty \leqslant 1.$$

Let $g_1, \dots, g_n, \dots$ and h be elements of H. Then

$$\sum_n \left\langle \left(\sum_j z_{nj} w_j \right) g_n, h \right\rangle = \sum_{n,j} \left\langle w_j(g_n), z_{nj}^*(h) \right\rangle.$$

Hence by Cauchy–Schwarz, we have

$$\begin{aligned}\left|\sum_n\left\langle\left(\sum_j z_{nj}w_j\right)g_n,h\right\rangle\right| &\leqslant \left(\sum_{n,j}\|w_j(g_n)\|^2\right)^{\frac12}\left(\sum_{n,j}\|z_{nj}^*(h)\|^2\right)^{\frac12}\\ &\leqslant \left\|\sum_j w_j^*w_j\right\|_\infty^{\frac12}\left(\sum_n\|g_n\|^2\right)^{\frac12}\left\|\sum_{n,j}z_{nj}z_{nj}^*\right\|_\infty^{\frac12}\|h\|\\ &\leqslant \left\|\sum_{n,j}z_{nj}z_{nj}^*\right\|_\infty^{\frac12}\left(\sum_n\|g_n\|^2\right)^{\frac12}\|h\|.\end{aligned}$$

For any $n \geqslant 1$, let

$$z'_n = \sum_j z_{nj}w_j.$$

The above calculation shows that

$$\left\|\sum_n z'_n\otimes e_n\right\|_{M\otimes_{\min}R} \leqslant \left\|\sum_{n,j}z_{nj}z_{nj}^*\right\|_\infty^{\frac12}.$$

Moreover we have

$$\left\|\sum_{n,j}z_{nj}d_j\otimes e_n\right\|_{L^p\{M;R\}_\ell} = \left\|\left(\sum_n z'_n\otimes e_n\right)d\right\|_{L^p\{M;R\}_\ell} \leqslant \|d\|_p\left\|\sum_n z'_n\otimes e_n\right\|_{M\otimes_{\min}R}$$

by Lemma 2.3(1). The result follows at once. □

Proof of Theorem 3.3. The first step of the proof will consist in showing that for any $2 \leqslant p < \infty$, we have

$$L^{p'}\{M;R\}_\ell \subset L^p\{M;R\}_\ell^* \text{ isometrically.} \tag{3.7}$$

We let $V = V(M)$ be given by (2.1) and we let $\mathcal{H} \subset R$ be the linear span of the e_ns. By Lemma 2.3(3), $V \otimes \mathcal{H}$ is both dense in $L^p\{M;R\}_\ell$ and $L^{p'}\{M;R\}_\ell$. In the sequel we regard $V \otimes \mathcal{H}$ as the space of finite sequences in V. Indeed we identify such a sequence $(y_n)_n$ with $\sum_{n\geqslant 1} y_n \otimes e_n$.

We let $q \geqslant 2$ such that $\frac12 + \frac1q = \frac1{p'}$. Equivalently,

$$\frac1q + \frac1p = \frac12.$$

Let $y = (y_n)_n$ and $y' = (y'_n)_n$ in $V \otimes \mathcal{H}$. Let $c, d \in V$ and let $(z_n)_n$ be a sequence of M such that $y_n = cz_nd$ for any $n \geqslant 1$. Likewise, let $a, b \in V$ and let $(z'_n)_n$ be a sequence of M such that $y'_n = az'_nb$ for any $n \geqslant 1$. The duality pairing $\langle y, y'\rangle$ from (3.6) is given by

$$\langle y, y'\rangle = \sum_n \varphi\big(y_ny'_n\big) = \sum_n \varphi\big(cz_ndaz'_nb\big) = \sum_n \varphi\big(bcz_ndaz'_n\big).$$

By Cauchy–Schwarz, we deduce that

$$\left|\langle y, y'\rangle\right| \leqslant \sum_n \left|\varphi\big(bcz_n daz'_n\big)\right| \leqslant \sum_n \|bcz_n\|_2 \left\|daz'_n\right\|_2$$
$$\leqslant \left(\sum_n \|bcz_n\|_2^2\right)^{\frac{1}{2}} \left(\sum_n \left\|daz'_n\right\|_2^2\right)^{\frac{1}{2}}.$$

Moreover we have

$$\sum_n \|bcz_n\|_2^2 = \sum_n \varphi\big(bcz_n z_n^* c^* b^*\big) = \varphi\left(bc\left(\sum_n z_n z_n^*\right)c^* b^*\right) \leqslant \|c\|_\infty^2 \|b\|_2^2 \left\|\sum_n z_n z_n^*\right\|_\infty.$$

Likewise,

$$\sum_n \left\|daz'_n\right\|_2^2 \leqslant \|da\|_2^2 \left\|\sum_n z'_n {z'_n}^*\right\|_\infty,$$

and hence

$$\sum_{n\geqslant 1} \left\|daz'_n\right\|_2^2 \leqslant \|d\|_p^2 \|a\|_q^2 \left\|\sum_n z'_n {z'_n}^*\right\|_\infty.$$

Altogether we deduce that

$$\left|\langle y, y'\rangle\right| \leqslant \|d\|_p \|a\|_q \|c\|_\infty \|b\|_2 \left\|\sum_n z_n \otimes e_n\right\|_{M\otimes_{\min} R} \left\|\sum_n z'_n \otimes e_n\right\|_{M\otimes_{\min} R}.$$

Passing to the infimum over all possible $a, b, c, d \in V$ and z_n, z'_n in M as above, we deduce that

$$\left|\langle y, y'\rangle\right| \leqslant \|y\|_{L^p\{M;R\}_\ell} \|y'\|_{L^{p'}\{M;R\}_\ell}.$$

This shows that the duality pairing (3.6) for $F = R$ induces a contraction

$$L^{p'}\{M; R\}_\ell \longrightarrow L^p\{M; R\}_\ell^*.$$

To show that this contraction is actually an isometry, we let $y' = (y'_n)_n$ in $V \otimes \mathcal{H}$, we let $\zeta : L^p\{M; R\}_\ell \to \mathbb{C}$ be the corresponding functional and we assume that $\|\zeta\| \leqslant 1$. According to Lemma 3.4 we have

$$\left|\sum_{n,j} \varphi\big(y'_n z_{nj} d_j\big)\right| = \left|\left\langle \zeta, \sum_{n,j} z_{nj} d_j \otimes e_n\right\rangle\right| \leqslant \left\|\left(\sum_j d_j^* d_j\right)^{\frac{1}{2}}\right\|_p \left\|\sum_{n,j} z_{nj} z_{nj}^*\right\|_\infty^{\frac{1}{2}}$$

for any finite families $(d_j)_j$ in $L^p(M)$ and $(z_{nj})_{n,j}$ in M. Multiplying each z_{nj} by an appropriate complex number of modulus one, we deduce that

$$\sum_{n,j}\bigl|\varphi\bigl(y'_n z_{nj} d_j\bigr)\bigr| \leqslant \biggl\|\Bigl(\sum_j d_j^* d_j\Bigr)^{\frac{1}{2}}\biggr\|_p \biggl\|\sum_{n,j} z_{nj} z_{nj}^*\biggr\|_\infty^{\frac{1}{2}}.$$

Note that $\frac{q}{2}$ is the conjugate number of $\frac{p}{2}$ and let K_1 be the positive part of the unit ball of $L^{q/2}(M)$, equipped with the $\sigma(L^{q/2}(M), L^{p/2}(M))$-topology. Likewise, let K_2 be the positive part of the unit ball of M^*, equipped with the w^*-topology. Since $\|d\|_p^2 = \|d^* d\|_{p/2}$ for any $d \in L^p(M)$, it follows from above that for any $(d_j)_j$ in $L^p(M)$ and any $(z_{nj})_{n,j}$ in M, we have

$$2\sum_{n,j}\bigl|\varphi\bigl(y'_n z_{nj} d_j\bigr)\bigr| \leqslant \sup_{A\in K_1} \varphi\biggl(\Bigl(\sum_j d_j^* d_j\Bigr)A\biggr) + \sup_{B\in K_2}\biggl\langle B, \sum_{n,j} z_{nj} z_{nj}^*\biggr\rangle.$$

Since K_1 and K_2 are compact, we deduce from [5, Lemma 2.3.1] (minimax principle) that there exist $A \in K_1$ and $B \in K_2$ such that

$$2\sum_{n,j}\bigl|\varphi\bigl(y'_n z_{nj} d_j\bigr)\bigr| \leqslant \varphi\biggl(\Bigl(\sum_j d_j^* d_j\Bigr)A\biggr) + \biggl\langle B, \sum_{n,j} z_{nj} z_{nj}^*\biggr\rangle$$

for any d_j and z_{nj} as above. Using the classical identity $2st = \inf_{\delta>0} \delta t^2 + \delta^{-1} s^2$ for nonnegative real numbers $s, t \geqslant 0$, we finally deduce that

$$\sum_n\bigl|\varphi\bigl(y'_n z_n d\bigr)\bigr| \leqslant \varphi(d^* d A)^{\frac{1}{2}}\biggl\langle B, \sum_n z_n z_n^*\biggr\rangle^{\frac{1}{2}}, \quad d \in L^p(M),\ z_n \in M. \tag{3.8}$$

We now argue as in the proof of [9, Proposition 2.3] to show that B may be replaced by its normal part in the above estimate. Let B_{sing} be the singular part of B. It is shown in [9] that there is an increasing net $(e_t)_t$ of projections in M converging to 1 in the w^*-topology, such that $B_{\mathrm{sing}}(e_t) = 0$ for any t. This implies that

$$\biggl\langle B_{\mathrm{sing}}, \sum_n (e_t z_n)(e_t z_n)^*\biggr\rangle = \biggl\langle B_{\mathrm{sing}}, e_t\Bigl(\sum_n z_n z_n^*\Bigr)e_t\biggr\rangle = 0.$$

Since $\varphi(y'_n z_n d) = \lim_t \varphi(y'_n e_t z_n d)$, this implies that (3.8) holds true with $B - B_{\mathrm{sing}}$ instead of B.

Thus we may assume that B is normal, and we regard it as an element of $L^1(M)_+$. Let $b = B^{1/2} \in L^2(M)_+$ be its square root. For any $z_1, \ldots, z_n, \ldots$ in M, we have

$$\biggl\langle B, \sum_n z_n z_n^*\biggr\rangle = \sum_n \varphi\bigl(b^2 z_n z_n^*\bigr) = \sum_n \|b z_n\|_2^2.$$

Likewise if we let $a = A^{1/2} \in L^q(M)_+$, then we have $\varphi(d^*dA) = \|da\|_2^2$ for any $d \in L^p(M)$. Consequently, we have

$$\sum_n \bigl|\varphi\bigl(y'_n z_n d\bigr)\bigr| \leqslant \|da\|_2 \biggl(\sum_n \|bz_n\|_2^2\biggr)^{\frac{1}{2}}, \quad d \in L^p(M),\ z_n \in M.$$

Applying Lemma 2.2 to each y'_n, we deduce that there is a finite sequence $(w_n)_n$ in M such that $y'_n = aw_nb$ and $w_n = Q_a w_n Q_b$ for any $n \geqslant 1$, where Q_a and Q_b denote the support projections of a and b, respectively. Since $L^p(M)a$ is dense in $L^2(M)Q_a$, and bM is dense in $Q_bL^2(M)$ (see Lemma 2.1), the above estimate yields

$$\biggl|\varphi\biggl(\sum_n w_n g_n h\biggr)\biggr| \leqslant \|h\|_2 \biggl(\sum_n \|g_n\|_2^2\biggr)^{\frac{1}{2}}, \quad h \in L^2(M)Q_a,\ g_n \in Q_bL^2(M).$$

Since $w_n = Q_a w_n Q_b$ this implies that

$$\biggl|\varphi\biggl(\sum_n w_n g_n h\biggr)\biggr| \leqslant \|h\|_2 \biggl(\sum_n \|g_n\|_2^2\biggr)^{\frac{1}{2}}, \quad h \in L^2(M),\ g_n \in L^2(M).$$

Regarding $M \subset B(L^2(M))$ in the usual way, we deduce that $\bigl\|\sum_n w_n w_n^*\bigr\|_\infty \leqslant 1$. Appealing to Lemma 2.3(2), this proves that $\|y'\|_{L^{p'}\{M;R\}_\ell} \leqslant 1$, and concludes the proof of (3.7).

The latter intermediate result implies that for any $2 \leqslant p < \infty$, we have

$$L^p\{M; R_N\}_\ell^* \simeq L^{p'}\{M; R_N\}_\ell \tag{3.9}$$

for any integer $N \geqslant 1$. Since the above spaces are reflexive, this implies that (3.9) actually holds true for any $1 < p < \infty$. In turn this implies that (3.7) holds true for any $1 < p < \infty$, because $V \otimes \mathcal{H}$ is dense in $L^{p'}\{M; R\}_\ell$. In particular we obtain part (1) of the theorem.

We now turn to the proof of (2), which will consist in showing that for $2 < p < \infty$, the isometry given by (3.7) is onto. Note that according to (2.4), we have

$$L^p(M) = \bigl[M, L^2(M)\bigr]_\theta, \tag{3.10}$$

where $\theta = \frac{2}{p}$. We will now check that for any integer $N \geqslant 1$, we have

$$L^p\{M; R_N\}_\ell \simeq \bigl[M \otimes_{\min} R_N, L^2\{M; R_N\}_\ell\bigr]_\theta \text{ isometrically.} \tag{3.11}$$

For that purpose, let $y \in V \otimes R_N$ and let $\|y\|_\theta$ denote its norm in the above interpolation space.

Assume that $\|y\|_{\alpha_p^\ell} < 1$. There exist $c, d \in V$ and $z \in M \otimes R_N$ such that $y = czd$, $\|z\|_{\min} < 1$, $\|c\|_\infty < 1$ and $\|d\|_p < 1$. Consider the strip

$$\Sigma = \bigl\{\lambda \in \mathbb{C}\colon\ 0 < \mathrm{Re}(\lambda) < 1\bigr\}.$$

According to (3.10), there exists a continuous function $D : \overline{\Sigma} \to M + L^2(M)$ whose restriction to Σ is analytic, such that $D(\theta) = d$, the functions $t \mapsto D(it)$ and $t \mapsto D(1 + it)$ belong to

$C_0(\mathbb{R}; M)$ and $C_0(\mathbb{R}; L^2(M))$, respectively, and such that $\|D(it)\|_\infty < 1$ and $\|D(1+it)\|_2 < 1$ for any $t \in \mathbb{R}$. We define

$$f : \overline{\Sigma} \longrightarrow M \otimes_{\min} R_N + L^2\{M; R_N\}_\ell$$

by letting

$$f(\lambda) = czD(\lambda), \quad \lambda \in \overline{\Sigma}.$$

Then f is continuous, its restriction to Σ is analytic and we have $f(\theta) = y$. Moreover the functions $t \mapsto f(it)$ and $t \mapsto f(1+it)$ belong to $C_0(\mathbb{R}; M \otimes_{\min} R_N)$ and $C_0(\mathbb{R}; L^2\{M; R_N\}_\ell)$, respectively. Further for any $t \in \mathbb{R}$ we have

$$\big\|f(1+it)\big\|_{\alpha_2^\ell} \leqslant \|c\|_\infty \|z\|_{\min} \big\|D(1+it)\big\|_2 < 1$$

by Lemma 2.3(1). Also we have $\|f(it)\|_{\min} < 1$ for any $t \in \mathbb{R}$, hence $\|y\|_\theta < 1$.

Assume conversely that $\|y\|_\theta < 1$ and write $y = (y_1, \ldots, y_N)$. Thus there is an N-tuple $(f_1, \ldots, f_N)$ of continuous functions from $\overline{\Sigma}$ into $M + L^2(M)$ such that $f_n(\theta) = y_n$ and $f_{n|\Sigma}$ is analytic for any $n = 1, \ldots, N$, and such that

$$\Bigg\|\sum_{n=1}^N f_n(it) \otimes e_n\Bigg\|_{M \otimes_{\min} R_N} < 1 \quad \text{and} \quad \Bigg\|\sum_{n=1}^N f_n(1+it) \otimes e_n\Bigg\|_{L^2\{M; R_N\}_\ell} < 1$$

for any $t \in \mathbb{R}$. Let $a, b \in V$ and $z'_1, \ldots, z'_N$ in M such that

$$\Bigg\|\sum_{n=1}^N z'_n \otimes e_n\Bigg\|_{M \otimes_{\min} R_N} < 1, \qquad \|a\|_q < 1, \quad \text{and} \quad \|b\|_2 < 1.$$

Since $[L^2(M), M]_\theta = L^q(M)$, there is a continuous function $A : \overline{\Sigma} \to M + L^2(M)$ whose restriction to Σ is analytic, such that $A(\theta) = a$ and for any $t \in \mathbb{R}$, $\|A(it)\|_2 < 1$ and $\|A(1+it)\|_\infty < 1$. Consider $F : \overline{\Sigma} \to \mathbb{C}$ defined by

$$F(\lambda) = \sum_{n=1}^N \varphi\big(A(\lambda) z'_n b f_n(\lambda)\big), \quad \lambda \in \overline{\Sigma}.$$

Then F is a well-defined continuous function, whose restriction to Σ is analytic. For any $t \in \mathbb{R}$, we have

$$\big|F(1+it)\big| \leqslant \Bigg\|\sum_{n=1}^N f_n(1+it) \otimes e_n\Bigg\|_{L^2\{M; R_N\}_\ell} \Bigg\|\sum_{n=1}^N A(1+it) z'_n b \otimes e_n\Bigg\|_{L^2\{M; R_N\}_\ell},$$

by the first part of the proof of this theorem. Thus $|F(1+it)| < 1$. Likewise, we have

$$\big|F(it)\big| \leqslant \Bigg\|\sum_{n=1}^N f_n(it) \otimes e_n\Bigg\|_{M \otimes_{\min} R_N} \Bigg\|\sum_{n=1}^N A(it) z'_n b \otimes e_n\Bigg\|_{L^1\{M; R_N\}_\ell} < 1$$

for any $t \in \mathbb{R}$. It therefore follows from the three lines lemma that $|F(\theta)| < 1$. Since

$$F(\theta) = \sum_{n=1}^{N} \varphi\big(az'_n b y_n\big)$$

is the action of y on $\sum_{n=1}^{N} az'_n b \otimes e_n$, this shows that the norm of y as an element of $L^{p'}\{M; R_N\}_\ell^*$ is $\leqslant 1$. By (3.9), this means that $\|y\|_{\alpha_p^\ell} \leqslant 1$.

We will conclude our proof of (2) by adapting some ideas from [22, Chapter 1]. We momentarily fix two integers $1 < k < m$ and we let $P : R_m \to R_m$ be the orthogonal projection onto $R_k = \mathrm{Span}\{e_1, \dots, e_k\}$. We let $\overline{P} = I_V \otimes P$ on $V \otimes R_m$. For any $y \in V \otimes R_m$, we have

$$\|y\|_{\min} \leqslant \big(\big\|\overline{P}(y)\big\|_{\min}^2 + \big\|(I - \overline{P})(y)\big\|_{\min}^2\big)^{\frac{1}{2}}.$$

Indeed this assertion simply means that for any $y_1, \dots, y_m$ in M, we have

$$\left\|\sum_{n=1}^{m} y_n y_n^*\right\|^{\frac{1}{2}} \leqslant \left(\left\|\sum_{n=1}^{k} y_n y_n^*\right\| + \left\|\sum_{n=k+1}^{m} y_n y_n^*\right\|\right)^{\frac{1}{2}}.$$

Moreover it is plain that

$$\|y\|_{\alpha_2^\ell} \leqslant \big\|\overline{P}(y)\big\|_{\alpha_2^\ell} + \big\|(I - \overline{P})y\big\|_{\alpha_2^\ell}.$$

Recall that $2 < p < \infty$ and let $s > 1$ be defined by $\frac{1}{s} = \frac{1}{2} + \frac{1}{p}$. By interpolation, using (3.11), we deduce from above that the (well-defined) linear mapping

$$(V \otimes R_k) \oplus \big(V \otimes [R_m \ominus R_k]\big) \longrightarrow V \otimes R_m$$

taking any $(\overline{P}(y), y - \overline{P}(y))$ to y extends to a contraction

$$L^p\{M; R_k\}_\ell \overset{s}{\oplus} L^p\{M; R_m \ominus R_k\}_\ell \longrightarrow L^p\{M; R_m\}_\ell.$$

By (3.9) its adjoint is a contraction

$$L^{p'}\{M; R_m\}_\ell \longrightarrow L^{p'}\{M; R_k\}_\ell \overset{s'}{\oplus} L^{p'}\{M; R_m \ominus R_k\}_\ell$$

and this adjoint maps any $y' \in V \otimes R_m$ to the pair $(\overline{P}(y'), y' - \overline{P}(y'))$.

We deduce that for any finite family $(y'_1, \dots, y'_m)$ in $L^{p'}(M)$ and any $1 < k < m$, we have

$$\left\|\sum_{n=1}^{k} y'_n \otimes e_n\right\|_{\alpha_{p'}^\ell}^{s'} + \left\|\sum_{n=k+1}^{m} y'_n \otimes e_n\right\|_{\alpha_{p'}^\ell}^{s'} \leqslant \left\|\sum_{n=1}^{m} y'_n \otimes e_n\right\|_{\alpha_{p'}^\ell}^{s'}. \tag{3.12}$$

(It should be observed that s' is finite.) Let $\zeta \in L^p\{M;R\}_\ell^*$. For any integer $n \geqslant 1$, let $\zeta_n : L^p(M) \to \mathbb{C}$ be defined by $\zeta_n(y) = \zeta(y \otimes e_n)$. Then ζ_n is represented by some $y'_n \in L^{p'}(M)$, and it is easy to show, using the density of $V \otimes \bigcup_m R_m$ in $L^p\{M;R\}_\ell$, that

$$\|\zeta\|_{L^p\{M;R\}_\ell^*} = \lim_{m\to\infty} \left\| \sum_{n=1}^m y'_n \otimes e_n \right\|_{\alpha_{p'}^\ell}. \tag{3.13}$$

Letting $m \to \infty$ in (3.12), we deduce that

$$\left\| \sum_{n=1}^k y'_n \otimes e_n \right\|_{\alpha_{p'}^\ell}^{s'} + \left\| \zeta - \sum_{n=1}^k y'_n \otimes e_n \right\|_{L^p\{M,R\}_\ell^*}^{s'} \leqslant \|\zeta\|_{L^p\{M,R\}_\ell^*}^{s'}$$

for any $k \geqslant 1$. Using (3.13) again, this implies that

$$\left\| \zeta - \sum_{n=1}^k y'_n \otimes e_n \right\|_{L^p\{M,R\}_\ell^*} \longrightarrow 0$$

when $k \to \infty$. Thus ζ belongs to the closure of $L^{p'}(M) \otimes R$, hence $\zeta \in L^{p'}\{M;R\}_\ell$. $\square$

Remark 3.5. (1) The isometric embedding in Theorem 3.3(1) is not surjective in general. Indeed let $B = B(\ell^2)$ and set $S^2\{R\}_\ell = L^2\{B;R\}_\ell$. As in (3.1), we have

$$S^2\{R\}_\ell \simeq C \otimes_{\mathrm{h}} R \otimes_{\mathrm{h}} C \tag{3.14}$$

and passing to the opposite structures, this yields

$$S^2\{R\}_\ell \simeq R \otimes_{\mathrm{h}} C \otimes_{\mathrm{h}} R.$$

Regard $S^1 = B_*$ as the predual operator space of B. By well-known computations, we deduce that $S^2\{R\}_\ell \simeq S^1 \otimes_{\mathrm{h}} R$ and that $S^2\{R\}_\ell^* \simeq B \otimes_{\mathrm{h}} C$. On the other hand, $S^2\{R\}_\ell \simeq S^\infty \otimes_{\mathrm{h}} C$ by (3.14). Hence the embedding of $S^2\{R\}_\ell$ into its dual corresponds to $\iota \otimes I_C$, where $\iota : S^\infty \hookrightarrow B$ is the canonical embedding of the compact operators into the bounded operators.

Likewise for any $1 < p \leqslant 2$, the embedding of $S^{p'}\{R\}_\ell$ into $S^p\{R\}_\ell^*$ corresponds to

$$\iota \otimes I_{R(2/p')} : S^\infty \otimes_{\mathrm{h}} R\left(\frac{2}{p'}\right) \hookrightarrow B \otimes_{\mathrm{h}} R\left(\frac{2}{p'}\right).$$

(2) Let F be an operator space, let $1 < p < \infty$ and suppose that

$$L^{p'}\{M;F^{*\mathrm{op}}\}_\ell \longrightarrow L^p\{M;F\}_\ell^* \quad \text{contractively}. \tag{3.15}$$

Then we also have

$$M \otimes_{\min} F^{*\mathrm{op}} \longrightarrow L^1\{M;F\}_\ell^* \quad \text{contractively}. \tag{3.16}$$

Indeed assume that $p \geqslant 2$, and let $2 < q \leqslant \infty$ such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$. Let $w \in M \otimes F^{*\mathrm{op}}$ and let $y \in V \otimes F$ with $\|y\|_{\alpha_1^\ell} < 1$. Then we can write $y = azb$ for some $a, b \in V$ and some $z \in M \otimes F$ such that $\|a\|_2 < 1$, $\|b\|_2 < 1$ and $\|z\|_{\min} < 1$. Let us factorise a and b in the form $a = a_1 a_2$ and $b = b_1 b_2$, with $a_1, a_2, b_1, b_2 \in V$ verifying $\|a_1\|_2 < 1$, $\|a_2\|_\infty < 1$, $\|b_1\|_p < 1$, $\|b_2\|_q < 1$. It is plain that

$$\langle w, y\rangle = \langle w, azb\rangle = \langle b_2 w a_1, a_2 z b_1\rangle.$$

Hence by our assumption, we have

$$\begin{aligned} \big|\langle w, y\rangle\big| &\leqslant \|b_2 w a_1\|_{L^{p'}\{M; F^{*\mathrm{op}}\}_\ell} \|a_2 z b_1\|_{L^p\{M;F\}_\ell} \\ &\leqslant \|a_1\|_2 \|a_2\|_\infty \|b_1\|_p \|b_2\|_q \|w\|_{\min} \|z\|_{\min} \leqslant \|w\|_{\min}. \end{aligned}$$

This shows (3.16). It is a well-known consequence of Haagerup's characterization of injectivity [10] that if the von Neumann algebra M is not injective, then (3.16) does not hold true for $F = \ell^\infty$. The above argument shows that for any $1 < p < \infty$, (3.15) cannot hold true either in this case.

(3) Using a standard approximation argument, we deduce from (3.11) that for any $p \geqslant 2$,

$$L^p\{M; R\}_\ell \simeq \big[M \otimes_{\min} R, L^2\{M; R\}_\ell\big]_{2/p} \quad \text{isometrically.}$$

Also, slightly modifying our arguments in the proof of Theorem 3.3, one can show that

$$M \otimes_{\min} R_N \simeq L^1\{M; R_N\}_\ell^*$$

for any $N \geqslant 1$. Details are left to the reader.

(4) Lemma 2.3, Theorem 3.3 and all formulas above have versions for the 'r-case,' i.e. with the spaces $L^p\{M; F\}_r$ in place of $L^p\{M; F\}_\ell$. These versions can be obtained by mimicking the proofs of the 'ℓ-case,' or by applying that 'ℓ-case' together with (2.8). Thus the 'r-version' of Theorem 3.3 says that for any $1 < p < \infty$, we have

$$L^{p'}\{M; C\}_r \hookrightarrow L^p\{M; C\}_r^* \quad \text{isometrically,}$$

and that this embedding is onto if $p > 2$.

4. Rigid factorizations and dilations of L^p operators

In this section we study various properties for bounded linear maps on noncommutative L^p-spaces. We need to introduce the matricial structure of $L^p(M)$. If (M, φ) is any semifinite von Neumann algebra, we equip $M_k(M) = M_k \otimes M$ with the trace $tr \otimes \varphi$ for any $k \geqslant 1$, where tr is the usual trace on M_k. This gives rise to the noncommutative L^p-spaces $L^p(M_k(M))$. According to [23, p. 141], there exists a (necessarily unique) operator space structure on $L^p(M)$ such that

$$S_k^p\big[L^p(M)\big] \simeq L^p\big(M_k(M)\big) \quad \text{isometrically}$$

for any $k \geqslant 1$. (This structure is obtained by interpolation between the predual operator space of M^{op} and M.)

We say that a linear map $u : L^p(M) \to L^p(M)$ is positive if it maps $L^p(M)_+$ into itself. (Note that $L^p(M)$ is spanned by $L^p(M)_+$.) Next we say that u is completely positive if

$$I_{S_k^p} \otimes u : L^p\big(M_k(M)\big) \longrightarrow L^p\big(M_k(M)\big)$$

is positive for any $k \geqslant 1$.

We will consider isometries on noncommutative L^p-spaces, and we will use their description given by Yeadon's theorem (see also Remark 4.2).

Theorem 4.1. *(Yeadon [26].) Let (M, φ) and (N, ψ) be two semifinite von Neumann algebras, let $1 < p \neq 2 < \infty$, and let $T : L^p(N) \to L^p(M)$ be a linear isometry. There exist a one-to-one normal Jordan homomorphism $J : N \to M$, a positive unbounded operator B affiliated with $J(N)' \cap M$ and a partial isometry $W \in M$ such that W^*W is the support projection of B, $\psi(a) = \varphi(B^p J(a))$ for all $a \in N_+$, and*

$$T(a) = WBJ(a), \quad a \in N \cap L^p(N).$$

Remark 4.2. We will need a little information on Jordan homomorphisms, for which we refer e.g. to [17, pp. 773–777]. Let M, N be von Neumann algebras. We recall that a Jordan homomorphism $J : N \to M$ is a linear map satisfying $J(a^2) = J(a)^2$ and $J(a^*) = J(a)^*$ for any $a \in N$. Assume that $J : N \to M$ is a normal Jordan homomorphism, and let $D \subset M$ be the von Neumann algebra generated by the range of J. Then there exist two central projections e_1, e_2 of D such that the map $\pi_1 : N \to M$ defined by $\pi_1(a) = J(a)e_1$ is a $*$-representation, the map $\pi_2 : N \to M$ defined by $\pi_2(a) = J(a)e_2$ is a $*$-anti-representation, and $e_1 + e_2$ is equal to the unit of D. Thus we have $J = \pi_1 + \pi_2$.

Throughout the rest of this section, we fix a number $1 < p \neq 2 < \infty$, and we let p' denote its conjugate number. Let (N, ψ) be a semifinite von Neumann algebra and let $u : L^p(N) \to L^p(N)$ be a linear mapping. We say that u admits a rigid factorisation if there exist another semifinite von Neumann algebra (M, φ) and two linear isometries $T : L^p(N) \to L^p(M)$ and $S : L^{p'}(N) \to L^{p'}(M)$ such that $u = S^*T$:

$$\begin{tikzcd} & L^p(M) \arrow[dr, "S^*"] & \\ L^p(N) \arrow[ur, "T"] \arrow[rr, "u"] & & L^p(N) \end{tikzcd}$$

We note that any completely positive contraction $u : S_k^p \to S_k^p$ is completely contractive. This follows from [20, Proposition 2.2 and Lemma 2.3]. The main result of this section is the following.

Theorem 4.3. *Assume that $1 < p \neq 2 < \infty$. There exist an integer $k \geqslant 1$ and a completely positive contraction $u : S_k^p \to S_k^p$ which does not have a rigid factorisation.*

The origin of this result is the search for a noncommutative analog of Akcoglu's dilation theorem [1,2]. Let (Ω, μ) be a measure space, and let $u : L^p(\Omega) \to L^p(\Omega)$ be a positive contraction.

Akcoglu's theorem asserts that there exist another measure space (Ω', μ'), two contractions

$$J : L^p(\Omega) \longrightarrow L^p(\Omega') \quad \text{and} \quad Q : L^p(\Omega') \longrightarrow L^p(\Omega),$$

and an invertible isometry $U : L^p(\Omega') \to L^p(\Omega')$ such that $u^n = QU^nJ$ for any integer $n \geqslant 0$.

$$\begin{tikzcd} L^p(\Omega') \arrow[r, "U^n"] & L^p(\Omega') \arrow[d, "Q"] \\ L^p(\Omega) \arrow[u, "J"] \arrow[r, "u^n"] & L^p(\Omega) \end{tikzcd}$$

Owing to that statement, we consider a noncommutative L^p-space $L^p(N)$, a linear mapping $u : L^p(N) \to L^p(N)$, and we say that u is *dilatable* if there exist another noncommutative L^p-space $L^p(M)$, two linear contractions $J : L^p(N) \to L^p(M)$ and $Q : L^p(M) \to L^p(N)$, and an invertible isometry $U : L^p(M) \to L^p(M)$ such that $u^n = QU^nJ$ for any integer $n \geqslant 0$. Any dilatable operator is clearly a contraction and Akcoglu's theorem implies that any positive contraction on a commutative L^p-space is dilatable.

If $u : L^p(N) \to L^p(N)$ is a dilatable operator on a noncommutative L^p-space, then QJ is equal to the identity of $L^p(N)$. Since $\|J\| \leqslant 1$ and $\|Q\| \leqslant 1$, this implies that J and Q^* are isometries. Furthermore we have $u = QUJ$, hence $u = S^*T$, with $T = UJ$ and $S = Q^*$. This shows that u admits a rigid factorisation. As a consequence of Theorem 4.3, we therefore obtain the following corollary, saying that there is no direct analog of Akcoglu's theorem on noncommutative L^p-spaces.

Corollary 4.4. *For any $1 < p \neq 2 < \infty$, there is an integer $k \geqslant 1$ and a completely positive contraction $u : S_k^p \to S_k^p$ which is not dilatable.*

We refer the reader to [3] for a related but different notion of factorisation of linear maps as the product of an isometry and of the adjoint of an isometry.

We will give two proofs of Theorem 4.3, one at the end of this section and another one in Section 5. Both will rely on the following decomposition result of independent interest.

Proposition 4.5. *Let $1 < p \neq 2 < \infty$ and let (M, φ) and (N, ψ) be two semifinite von Neumann algebras. Let $T : L^p(N) \to L^p(M)$ be a linear isometry. Then there exist two contractions $T_1, T_2 : L^p(N) \to L^p(M)$ such that*

$$T = T_1 + T_2,$$

and for any operator space F,

$$\left\| T_1 \otimes I_F : L^p\{N; F\}_\ell \longrightarrow L^p\{M; F\}_\ell \right\| \leqslant 1 \tag{4.1}$$

and

$$\left\| T_2 \otimes I_F : L^p\{N; F\}_\ell \longrightarrow L^p\big\{M; F^{\mathrm{op}}\big\}_r \right\| \leqslant 1. \tag{4.2}$$

Proof. Let $T : L^p(N) \to L^p(M)$ be a linear isometry, and let W, B, J be provided by Yeadon's Theorem 4.1, so that $T = WBJ$. We apply Remark 4.2 to the normal Jordan homomorphism $J : N \to M$, and let e_1, e_2, π_1, π_2 be given by this statement. Since B commutes with the range of J, it commutes with e_1, and hence B commutes with the range of π_1.

We define $T_1, T_2 : L^p(N) \to L^p(M)$ by letting

$$T_1(a) = T(a)e_1 \quad \text{and} \quad T_2(a) = T(a)e_2$$

for any $a \in L^p(N)$. By construction, $T = T_1 + T_2$.

Assume that $p < 2$ and let $q > 2$ be such that $\frac{1}{2} + \frac{1}{q} = \frac{1}{p}$. Let $V = V(N)$ and let $y \in V \otimes F$ such that $\|y\|_{\alpha_p^\ell} < 1$. Thus we can write $y = azb$ for some $a, b \in V$ and $z \in N \otimes F$ such that

$$\|a\|_q \leqslant 1, \qquad \|b\|_2 \leqslant 1, \quad \text{and} \quad \|z\|_{\min} \leqslant 1.$$

Let $(c_k)_k$ and $(x_k)_k$ be finite families in N and F, respectively, such that $z = \sum_k c_k \otimes x_k$. Then

$$(T_1 \otimes I_F)y = \sum_k T_1(ac_kb) \otimes x_k.$$

Let $\theta = \frac{p}{2}$, so that $1 - \theta = \frac{p}{q}$. Since $\pi_1 = J(\cdot)e_1$ is a $*$-representation whose range commutes with B, we have

$$T_1(ac_kb) = WB\pi_1(ac_kb) = WB\pi_1(a)\pi_1(c_k)\pi_1(b) = WB^{1-\theta}\pi_1(a)\pi_1(c_k)B^\theta\pi_1(b)$$

for any k. Hence

$$\begin{aligned}(T_1 \otimes I_F)y &= WB^{1-\theta}\pi_1(a)\left(\sum_k \pi_1(c_k) \otimes x_k\right)B^\theta\pi_1(b) \\ &= WB^{1-\theta}\pi_1(a)\big(\pi_1 \otimes I_F\big)(z)B^\theta\pi_1(b).\end{aligned}$$

By Lemma 2.3, we deduce that

$$\big\|(T_1 \otimes I_F)y\big\|_{L^p\{M;F\}_\ell} \leqslant \big\|WB^{1-\theta}\pi_1(a)\big\|_q \big\|(\pi_1 \otimes I_F)(z)\big\|_{\min}\big\|B^\theta\pi_1(b)\big\|_2.$$

Since W is the support projection of B, we have $|WB^{1-\theta}\pi_1(a)| = |B^{1-\theta}\pi_1(a)|$. Since B commutes with the range of π_1, and π_1 is a $*$-representation, we deduce that

$$\big|WB^{1-\theta}\pi_1(a)\big|^q = B^{q(1-\theta)}\big|\pi_1(a)\big|^q = B^p\pi_1\big(|a|^q\big).$$

Thus

$$\big\|WB^{1-\theta}\pi_1(a)\big\|_q^q = \varphi\big(B^p\pi_1\big(|a|^q\big)\big) \leqslant \varphi\big(B^pJ\big(|a|^q\big)\big) = \psi\big(|a|^q\big) = \|a\|_q^q \leqslant 1.$$

Likewise, we have

$$\big\|B^\theta\pi_1(b)\big\|_2 \leqslant \|b\|_2 \leqslant 1.$$

The $*$-representation π_1 is a complete contraction, hence

$$\big\|(\pi_1 \otimes I_F)(z)\big\|_{\min} \leqslant \|z\|_{\min} \leqslant 1.$$

Thus we obtain that $\big\|(T_1 \otimes I_F)y\big\|_{L^p\{M;F\}_\ell} \leqslant 1$. This shows (4.1), that is, $T_1 \otimes I_F$ extends to a contraction from $L^p\{N; F\}_\ell$ into $L^p\{M; F\}_\ell$. The proof for $p \geqslant 2$ is similar.

The inequality (4.2) can be proved by similar arguments. It also follows from the above proof and the identification (2.8). Indeed, saying that $\pi_2 : N \to M$ is an $*$-anti-representation means that π_2 is a $*$-representation from N into M^{op}. $\square$

Remark 4.6. Let $T, T_1, T_2 : L^p(N) \to L^p(M)$ as above. Then we also have

$$\big\|T_1 \otimes I_F : L^p\{N; F\}_r \longrightarrow L^p\{M; F\}_r\big\| \leqslant 1$$

and

$$\big\|T_2 \otimes I_F : L^p\{N; F\}_r \longrightarrow L^p\big\{M; F^{\mathrm{op}}\big\}_\ell\big\| \leqslant 1$$

for any operator space F. These estimates have the same proofs as (4.1) and (4.2). Appealing to (2.8), they can be also viewed as a formal consequence of the latter estimates.

Our first proof of Theorem 4.3 will appeal to L^p-matricially normed spaces and some results from [15]. Let X be a Banach space. For any integers $k, m \geqslant 1$ and any $y \in S_k^p \otimes X$ and $y' \in S_m^p \otimes X$, let

$$y \oplus y' = \begin{bmatrix} y & 0 \\ 0 & y' \end{bmatrix}$$

denote the corresponding block diagonal element of $S_{k+m}^p \otimes X$. Suppose that for any integer $k \geqslant 1$, the matrix space $S_k^p \otimes X$ is equipped with a norm $\|\ \|_\alpha$ and that the natural embedding $y \mapsto y \oplus 0$ from $S_k^p \otimes_\alpha X$ into $S_{k+1}^p \otimes_\alpha X$ is an isometry. Here $S_k^p \otimes_\alpha X$ denotes the vector space $S_k^p \otimes X$ equipped with the norm $\|\ \|_\alpha$ and by the above assumption, there is no ambiguity in the use of a single notation $\|\ \|_\alpha$ (not depending on k) for all these matrix norms. We say that X equipped with $\|\ \|_\alpha$ is an L^p-matricially normed space if $S_1^p \otimes_\alpha X = X$ isometrically and if the following two properties hold.

(P1) For any integer $k \geqslant 1$, for any $c, d \in M_k$ and for any $y \in S_k^p \otimes X$, we have

$$\|cyd\|_\alpha \leqslant \|c\|_\infty \|y\|_\alpha \|d\|_\infty,$$

where $\|\ \|_\infty$ denotes the operator norm.

(P2) For any integers $k, m \geqslant 1$, and for any $y \in S_k^p \otimes X$ and $y' \in S_m^p \otimes X$, we have

$$\|y \oplus y'\|_\alpha = \big(\|y\|_\alpha^p + \|y'\|_\alpha^p\big)^{\frac{1}{p}}.$$

Let $u : S_k^p \to S_k^p$ be a linear map. Following [20], the regular norm of u, denoted by $\|u\|_{\mathrm{reg}}$, is defined as the smallest constant $K \geqslant 0$ such that

$$\left\|u \otimes I_F : S_k^p[F] \longrightarrow S_k^p[F]\right\| \leqslant K$$

for any operator space F.

Theorem 4.7. *(See [15].) Let X, Y be two L^p-matricially normed spaces, with associated norms on the matrix spaces $S_k^p \otimes X$ and $S_k^p \otimes Y$ denoted by $\| \ \|_\alpha$ and $\| \ \|_\beta$, respectively. Let $\sigma : X \to Y$ be a bounded operator, and assume that there is a constant $C \geqslant 0$ such that*

$$\left\|u \otimes \sigma : S_k^p \otimes_\alpha X \longrightarrow S_k^p \otimes_\beta Y\right\| \leqslant C\|u\|_{\mathrm{reg}} \tag{4.3}$$

for any $u : S_k^p \to S_k^p$ and any $k \geqslant 1$. Then there exist an operator space F and two bounded operators

$$\tau : X \longrightarrow F \quad \text{and} \quad \rho : F \longrightarrow Y$$

such that $\sigma = \rho \circ \tau$, τ has dense range and for any $k \geqslant 1$,

$$\left\|I_{S_k^p} \otimes \tau : S_k^p \otimes_\alpha X \longrightarrow S_k^p[F]\right\| \leqslant C \quad \text{and} \quad \left\|I_{S_k^p} \otimes \rho : S_k^p[F] \longrightarrow S_k^p \otimes_\beta Y\right\| \leqslant 1. \tag{4.4}$$

Remark 4.8. (1) Let $\| \ \|_{\alpha_0}$ and $\| \ \|_{\alpha_1}$ be norms on the matrix spaces $S_k^p \otimes X$ such that X equipped with $\| \ \|_{\alpha_0}$ (respectively $\| \ \|_{\alpha_1}$) is an L^p-matricially normed space. We define a norm $\| \ \|_\beta$ on each $S_k^p \otimes X$ by the following formula. For any $y \in S_k^p \otimes X$,

$$\|y\|_\beta = \inf\left\{\left(\|y_0\|_{\alpha_0}^p + \|y_1\|_{\alpha_1}^p\right)^{\frac{1}{p}} :\ y_0, y_1 \in S_k^p \otimes X,\ y = y_0 + y_1\right\}.$$

It turns out that X equipped with $\| \ \|_\beta$ is an L^p-matricially normed space. This structure is obtained as the 'sum' of the ones given by $S_k^p \otimes_{\alpha_0} X$ and $S_k^p \otimes_{\alpha_0} X$, and we simply write

$$S_k^p \otimes_\beta X = S_k^p \otimes_{\alpha_0} X +_p S_k^p \otimes_{\alpha_1} X$$

in this case.

It is obvious that $\| \ \|_\beta$ satisfies (P1) and the inequality "$\leqslant$" in (P2). To prove the reverse inequality "$\geqslant$" in (P2), take $y \in S_k^p \otimes X$ and $y' \in S_m^p \otimes X$ and assume that

$$\left\|\begin{bmatrix} y & 0 \\ 0 & y' \end{bmatrix}\right\|_\beta < 1.$$

Then there exists a decomposition

$$\begin{bmatrix} y & 0 \\ 0 & y' \end{bmatrix} = \begin{bmatrix} y_{11}^0 & y_{12}^0 \\ y_{21}^0 & y_{22}^0 \end{bmatrix} + \begin{bmatrix} y_{11}^1 & y_{12}^1 \\ y_{21}^1 & y_{22}^1 \end{bmatrix} \quad \text{with} \quad \left\|\begin{bmatrix} y_{11}^0 & y_{12}^0 \\ y_{21}^0 & y_{22}^0 \end{bmatrix}\right\|_{\alpha_0}^p + \left\|\begin{bmatrix} y_{11}^1 & y_{12}^1 \\ y_{21}^1 & y_{22}^1 \end{bmatrix}\right\|_{\alpha_1}^p < 1.$$

Since

$$\begin{bmatrix} y_{11}^0 & 0 \\ 0 & y_{22}^0 \end{bmatrix} = \frac{1}{2}\left(\begin{bmatrix} y_{11}^0 & y_{12}^0 \\ y_{21}^0 & y_{22}^0 \end{bmatrix} + \begin{bmatrix} I_k & 0 \\ 0 & -I_m \end{bmatrix}\begin{bmatrix} y_{11}^0 & y_{12}^0 \\ y_{21}^0 & y_{22}^0 \end{bmatrix}\begin{bmatrix} I_k & 0 \\ 0 & -I_m \end{bmatrix}\right),$$

we obtain by applying (P1) and (P2) to $\| \ \|_{\alpha_0}$ that

$$\|y_{11}^0\|_{\alpha_0}^p + \|y_{22}^0\|_{\alpha_0}^p \leqslant \left\| \begin{bmatrix} y_{11}^0 & y_{12}^0 \\ y_{21}^0 & y_{22}^0 \end{bmatrix} \right\|_{\alpha_0}^p .$$

Similarly,

$$\|y_{11}^1\|_{\alpha_1}^p + \|y_{22}^1\|_{\alpha_1}^p \leqslant \left\| \begin{bmatrix} y_{11}^1 & y_{12}^1 \\ y_{21}^1 & y_{22}^1 \end{bmatrix} \right\|_{\alpha_1}^p .$$

Since $y = y_{11}^0 + y_{11}^1$ and $y' = y_{22}^0 + y_{22}^1$, we deduce that

$$\|y\|_\beta^p + \|y'\|_\beta^p \leqslant \|y_{11}^0\|_{\alpha_0}^p + \|y_{22}^0\|_{\alpha_0}^p + \|y_{11}^1\|_{\alpha_1}^p + \|y_{22}^1\|_{\alpha_1}^p < 1,$$

which proves the desired inequality.

(2) Let F be an operator space and recall that we have

$$S_k^p\{F\}_\ell = S_k^p \otimes_{\alpha_p^\ell} F \quad \text{and} \quad S_k^p\{F\}_r = S_k^p \otimes_{\alpha_p^r} F.$$

According to [15, Section 2], F equipped with $\| \ \|_{\alpha_p^\ell}$ (respectively $\| \ \|_{\alpha_p^r}$) is an L^p-matricially normed space. In the sequel we will use the L^p-matricially normed space structure on ℓ^2 defined as the sum of $S_k^p\{R\}_\ell$ and $S_k^p\{C\}_r$.

The following is independent of Theorem 4.7 and will be used in both proofs of Theorem 4.3.

Corollary 4.9. *Let $1 < p \neq 2 < \infty$ and suppose that $u : S_k^p \to S_k^p$ admits a rigid factorisation. Then*

$$\left\| u \otimes I_{\ell^2} : S_k^p\{R\}_\ell \longrightarrow S_k^p\{R\}_\ell +_p S_k^p\{C\}_r \right\| \leqslant 4.$$

Proof. Suppose that $u : S_k^p \to S_k^p$ admits a rigid factorisation. By definition there exist a semifinite von Neumann algebra M and two linear isometries

$$T : S_k^p \longrightarrow L^p(M) \quad \text{and} \quad S : S_k^{p'} \longrightarrow L^{p'}(M)$$

such that $u = S^*T$. According to Proposition 4.5, we have a decomposition $T = T_1 + T_2$ for some $T_1, T_2 : S_k^p \to L^p(M)$ satisfying

$$\left\| T_1 \otimes I_F : S_k^p\{F\}_\ell \to L^p\{M; F\}_\ell \right\| \leqslant 1 \quad \text{and} \quad \left\| T_2 \otimes I_F : S_k^p\{F\}_\ell \to L^p\left\{M; F^{\mathrm{op}}\right\}_r \right\| \leqslant 1$$

for any operator space F. Likewise we have a decomposition $S = S_1 + S_2$ for some $S_1, S_2 : S_k^{p'} \to L^{p'}(M)$ satisfying

$$\left\| S_1 \otimes I_G : S_k^{p'}\{G\}_\ell \to L^{p'}\{M; G\}_\ell \right\| \leqslant 1 \quad \text{and} \quad \left\| S_2 \otimes I_G : S_k^{p'}\{G\}_\ell \to L^{p}\left\{M; G^{\mathrm{op}}\right\}_r \right\| \leqslant 1$$

for any operator space G. By Remark 4.6, we also have

$$\left\| S_1 \otimes I_G : S_k^{p'}\{G\}_r \to L^{p'}\{M; G\}_r \right\| \leqslant 1 \quad \text{and} \quad \left\| S_2 \otimes I_G : S_k^{p'}\{G\}_r \to L^{p}\left\{M; G^{\mathrm{op}}\right\}_\ell \right\| \leqslant 1.$$

Mixing the two decompositions, we have

$$u = S_1^* T_1 + S_2^* T_1 + S_1^* T_2 + S_2^* T_2.$$

Since $S_1 \otimes I_R$ is a contraction from $S_k^{p'}\{R\}_\ell$ into $L^{p'}\{M; R\}_\ell$, it follows from Theorem 3.3 that $S_1^* \otimes I_R$ extends to a contraction from $L^p\{M; R\}_\ell$ into $S_k^p\{R\}_\ell$. Consequently,

$$\left\| S_1^* T_1 \otimes I_R : S_k^p\{R\}_\ell \longrightarrow S_k^p\{R\}_\ell \right\| \leqslant 1.$$

Likewise, since $S_2 \otimes I_R$ is a contraction from $S_k^{p'}\{C\}_r$ into $L^{p'}\{M; R\}_\ell$, it follows from Theorem 3.3 and Proposition 3.1 that $S_2^* \otimes I_R$ extends to a contraction from $L^p\{M; R\}_\ell$ into $S_k^p\{C\}_r$. Consequently,

$$\left\| S_2^* T_1 \otimes I_R : S_k^p\{R\}_\ell \longrightarrow S_k^p\{C\}_r \right\| \leqslant 1.$$

Similarly we obtain that

$$\left\| S_1^* T_2 \otimes I_R : S_k^p\{R\}_\ell \to S_k^p\{C\}_r \right\| \leqslant 1 \quad \text{and} \quad \left\| S_2^* T_2 \otimes I_R : S_k^p\{R\}_\ell \to S_k^p\{R\}_\ell \right\| \leqslant 1.$$

The result follows at once. □

Proof of Theorem 4.3. By duality we may suppose that $p > 2$. Following Remark 4.8, let $\| \ \|_\beta$ denote the matrix norms on ℓ^2 given by

$$S_k^p \otimes_\beta \ell^2 = S_k^p\{R\}_\ell +_p S_k^p\{C\}_r.$$

Assume that for any integer $k \geqslant 1$, every completely positive contraction $S_k^p \to S_k^p$ admits a rigid factorisation. Let $u : S_k^p \to S_k^p$ be an arbitrary linear map. By [20] and [21, Corollary 8.7], one can find four completely positive maps $u_1, u_2, u_3, u_4 : S_k^p \to S_k^p$ such that $u = (u_1 - u_2) + i(u_3 - u_4)$ and for any $j = 1, \ldots, 4$, $\|u_j\| \leqslant \|u\|_{\mathrm{reg}}$. By Corollary 4.9 we deduce that

$$\left\| u \otimes I_{\ell^2} : S_k^p\{R\}_\ell \longrightarrow S_k^p \otimes_\beta \ell^2 \right\| \leqslant 16 \|u\|_{\mathrm{reg}}.$$

Let us apply Theorem 4.7 with $X = Y = \ell^2$, and $\sigma = I_{\ell^2}$. Thus there exist an operator space F and two bounded operators $\tau : \ell^2 \to F$ and $\rho : F \to \ell^2$ such that $\rho \circ \tau = I_{\ell^2}$ and for any $k \geqslant 1$,

$$\left\| I_{S_k^p} \otimes \tau : S_k^p\{R\}_\ell \longrightarrow S_k^p[F] \right\| \leqslant 16 \quad \text{and} \quad \left\| I_{S_k^p} \otimes \rho : S_k^p[F] \longrightarrow S_k^p \otimes_\beta \ell^2 \right\| \leqslant 1.$$

Moreover we can assume that F is equal to the range of τ and hence, $\rho = \tau^{-1}$. We can now conclude and get to a contradiction as in the proof of [15, Theorem 2.6]. We only give a sketch of the argument and refer the reader to the latter paper for details.

By means of (3.1) and (3.4), the above estimates imply that

$$\left\|\tau^{-1}\right\| \leqslant 1 \quad \text{and} \quad \left\| I_{\ell_k^2} \otimes \tau : C_k \otimes_{\mathrm{h}} R \longrightarrow R_k\left(1 - \frac{1}{p}\right) \otimes_{\mathrm{h}} F \right\| \leqslant 16$$

for any $k \geqslant 1$. Using the well-known isometric identifications

$$C_k \otimes_{\mathrm{h}} R_k \simeq M_k \quad \text{and} \quad CB\left(C_k, R_k\left(1 - \frac{1}{p}\right)\right) \simeq S_k^{2p},$$

we can deduce that $\|v\|_{2p} \leqslant 16\|v\|_\infty$ for any linear mapping $v : \ell_k^2 \to \ell_k^2$. This is false if $k > 16^{2p}$. $\square$

Remark 4.10. So far we have only considered noncommutative L^p-spaces associated with a semifinite trace. In fact semifiniteness was necessary to define the spaces $L^p\{M; F\}_\ell$ (or $L^p\{M; F\}_r$), and hence the duality results stated in Section 3 make sense only in the tracial setting. We wish to indicate however that Corollary 4.9 and Theorem 4.3 extend to the nontracial case.

More precisely, let M be an arbitrary von Neumann algebra and for any $1 \leqslant p \leqslant \infty$, let $L^p(M)$ denote the noncommutative L^p-space constructed by Haagerup [8]. We refer the reader to [25] for a complete description of these spaces, and to [24] or [13] for a brief presentation. We recall that if M is semifinite and φ is a n.s.f. trace on M, then Haagerup's space $L^p(M)$ is isometrically isomorphic to the usual tracial L^p-space (see Section 2). Our extension of Corollary 4.9 is as follows: for any $1 < p \neq 2 < \infty$, for any integer $k \geqslant 1$ and for any pair of isometries

$$T : S_k^p \longrightarrow L^p(M) \quad \text{and} \quad S : S_k^{p'} \longrightarrow L^{p'}(M), \tag{4.5}$$

we have

$$\left\| S^*T \otimes I_{\ell^2} : S_k^p\{R\}_\ell \longrightarrow S_k^p\{R\}_\ell +_p S_k^p\{C\}_r \right\| \leqslant 4.$$

Likewise, Theorem 4.3 extends as follows: for $k \geqslant 1$ large enough, there exists a completely positive contraction $u : S_k^p \to S_k^p$ such that whenever M is a (not necessarily semifinite) von Neumann algebra there is no pair (T, S) of isometries as in (4.5) such that $u = S^*T$.

The proofs of these extensions are similar to the ones given above in the tracial case, up to technical details. They require the extension of Yeadon's theorem obtained in [14, Theorem 3.1] as well as the duality techniques from [13, Section 1]. We skip the details.

Remark 4.11. Let (Ω, μ) be a measure space and let $u : L^p(\Omega) \to L^p(\Omega)$ be a contraction (with $1 < p \neq 2 < \infty$). The following assertions are equivalent:

(i) u admits a rigid factorisation.

(ii) There exist a measure space (Ω', μ') and two linear isometries $T : L^p(\Omega) \to L^p(\Omega')$ and $S : L^{p'}(\Omega) \to L^{p'}(\Omega')$ such that $u = S^*T$ (*commutative* rigid factorisation).

(iii) For any integer $k \geqslant 1$,

$$\left\| u \otimes I_{\ell_k^\infty} : L^p(\Omega; \ell_k^\infty) \longrightarrow L^p(\Omega; \ell_k^\infty) \right\| \leqslant 1.$$

(Equivalently, u is regular and $\|u\|_{\mathrm{reg}} \leqslant 1$, see [20].)

(iv) There exists a positive contraction v on $L^p(\Omega)$ such that $|u(f)| \leqslant v(|f|)$ for any $f \in L^p(\Omega)$.

The equivalence of (ii) and (iv) follows from [19, Section 3], and the equivalence of (iii) and (iv) is well known (see e.g. [18]). So we only need to show that (i) implies (iii). For this purpose, assume that $u = S^*T$, where $T : L^p(\Omega) \to L^p(M)$ and $S : L^{p'}(\Omega) \to L^{p'}(M)$ are isometries. For any integer $k \geqslant 1$, let $L^p(M; \ell_k^\infty)$ and $L^{p'}(M; \ell_k^1)$ be the operator space valued spaces introduced in [11]. Arguing as in the proof of Proposition 4.5 it is not hard to show that

$$T \otimes I_{\ell_k^\infty} : L^p(\Omega; \ell_k^\infty) \longrightarrow L^p(M; \ell_k^\infty) \quad \text{and} \quad S \otimes I_{\ell_k^1} : L^{p'}(\Omega; \ell_k^1) \longrightarrow L^{p'}(M; \ell_k^1)$$

are contractions. Using [11, Proposition 3.6], we deduce that $u \otimes I_{\ell_k^\infty}$ is a contraction on $L^p(\Omega; \ell_k^\infty)$.

5. A concrete example

The proof of Theorem 4.3 given above has a serious drawback. Indeed, it does not show any concrete example of a completely positive contraction $u : S_k^p \to S_k^p$ without a rigid factorisation. The aim of this section is to present such an example, thus giving another proof of that theorem. This second proof does not use Theorem 4.7.

Throughout we let $1 < p < \infty$, we consider an integer $k \geqslant 1$. Let $u_1 : S_k^p \to S_k^p$ be defined by letting $u_1(E_{i1}) = k^{-\frac{1}{2p}} E_{ii}$ for any $i \geqslant 1$ and $u_1(E_{ij}) = 0$ for any $j \geqslant 2$ and any $i \geqslant 1$. This can be written as

$$u_1(x) = \sum_{i=1}^k a_i^* x b_i, \quad x \in S_k^p,$$

where

$$a_i = E_{ii} \quad \text{and} \quad b_i = k^{-\frac{1}{2p}} E_{1i}, \quad 1 \leqslant i \leqslant k.$$

Consider the three linear maps $u_2, u_3, u_4 : S_k^p \to S_k^p$ defined by letting

$$u_2(x) = \sum_{i=1}^k b_i^* x a_i, \qquad u_3(x) = \sum_{i=1}^k a_i^* x a_i, \quad \text{and} \quad u_4(x) = \sum_{i=1}^k b_i^* x b_i$$

for any $x \in S_k^p$. Then u_3 is the canonical diagonal projection taking any $x = [x_{ij}] \in S_k^p$ to the diagonal matrix $\sum_i x_{ii} E_{ii}$. Thus $\|u_3\| = 1$. Next, u_4 is the rank one operator taking any $x = [x_{ij}] \in S_k^p$ to $k^{-1/p} x_{11} I_k$, where I_k denotes the identity matrix. Since $\|I_k\|_p = k^{1/p}$, we have $\|u_4\| = 1$. According to [21, Theorem 8.5] and [20], this implies that $\|u_1\|_{\mathrm{reg}} \leqslant 1$. In particular, u_1 is a contraction. Likewise, u_2 is a contraction.

We now consider the average

$$u = \frac{1}{4}\big(u_1 + u_2 + u_3 + u_4\big) \tag{5.1}$$

of these four maps. Then $u : S_k^p \to S_k^p$ is a contraction. Moreover we have

$$u(x) = \frac{1}{4}\sum_{i=1}^{k}(a_i + b_i)^* x(a_i + b_i), \quad x \in S_k^p. \tag{5.2}$$

Hence u is completely positive.

Theorem 5.1. *Assume that $1 < p < \infty$, and let $u : S_k^p \to S_k^p$ be the completely positive contraction defined by* (5.1) *and/or* (5.2).

(1) *We have*

$$\lim_{k\to\infty} \big\| u \otimes I_{\ell_k^2} : S_k^p\{R_k\}_\ell \longrightarrow S_k^p\{R_k\}_\ell +_p S_k^p\{C_k\}_r \big\| = \infty.$$

(2) *Assume that $p \neq 2$. Then for k large enough, the operator u does not admit a rigid factorisation.*

The proof will be given at the end of this section. We need the following elementary lemma.

Lemma 5.2. *Let E_1 and E_2 be two operator spaces with a common finite dimension k. Let $(e_1^1, \ldots, e_k^1)$ and $(e_1^2, \ldots, e_k^2)$ be some bases of E_1 and E_2, respectively. Assume that these bases are* completely 1-unconditional*, in the sense that for any k-tuple $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_k)$ with $\varepsilon_i = \pm 1$, the operators*

$$V_\varepsilon^1 : E_1 \longrightarrow E_1 \quad \textit{and} \quad V_\varepsilon^2 : E_2 \longrightarrow E_2$$

defined by letting $V_\varepsilon^1(e_i^1) = \varepsilon_i e_i^1$ and $V_\varepsilon^2(e_i^2) = \varepsilon_i e_i^2$ for any $1 \leqslant i \leqslant k$ are completely contractive. Let

$$\Delta : E_1 \otimes_{\mathrm{h}} E_2 \longrightarrow E_1 \otimes_{\mathrm{h}} E_2$$

be the 'diagonal' projection defined by letting $\Delta(e_i^1 \otimes e_j^2) = 0$ if $i \neq j$, and $\Delta(e_i^1 \otimes e_i^2) = e_i^1 \otimes e_i^2$ for any $i \geqslant 1$. Then Δ is a complete contraction.

Proof. Let μ be the uniform probability measure on $\Omega = \{-1, 1\}^k$. It is easy to check that

$$\Delta = \int_\Omega V_\varepsilon^1 \otimes V_\varepsilon^2 \, d\mu(\varepsilon).$$

For any $\varepsilon \in \Omega$, we have

$$\big\| V_\varepsilon^1 \otimes V_\varepsilon^2 : E_1 \otimes_{\mathrm{h}} E_2 \longrightarrow E_1 \otimes_{\mathrm{h}} E_2 \big\|_{\mathrm{cb}} \leqslant \big\| V_\varepsilon^1 \big\|_{\mathrm{cb}} \big\| V_\varepsilon^2 \big\|_{\mathrm{cb}} \leqslant 1.$$

Hence

$$\|\Delta\|_{\mathrm{cb}} \leqslant \int_{\Omega} \big\| V_\varepsilon^1 \otimes V_\varepsilon^2 \big\|_{\mathrm{cb}}\, d\mu(\varepsilon) \leqslant 1. \qquad \square$$

We let

$$D_k \subset \ell_k^2 \otimes \ell_k^2 \otimes \ell_k^2$$

be the k-dimensional subspace of $\ell_k^2 \otimes \ell_k^2 \otimes \ell_k^2$ spanned by $\{e_i \otimes e_i \otimes e_i\colon 1 \leqslant i \leqslant k\}$. Then we let

$$P : \ell_k^2 \otimes \ell_k^2 \otimes \ell_k^2 \longrightarrow \ell_k^2 \otimes \ell_k^2 \otimes \ell_k^2$$

be the projection onto D_k defined by letting $P(e_i \otimes e_j \otimes e_m) = 0$ if $\mathrm{card}\{i, j, m\} \geqslant 2$, and $P(e_i \otimes e_i \otimes e_i) = e_i \otimes e_i \otimes e_i$ for any $i \geqslant 1$. If $p \geqslant 2$, then according to the identification

$$S_k^p\{R_k\}_\ell = C_k \otimes_{\mathrm{h}} R_k \otimes_{\mathrm{h}} R_k(2/p) \tag{5.3}$$

given by (3.1), we may regard P as defined on $S_k^p\{R_k\}_\ell$. Using (3.2), we can do the same when $p < 2$.

Lemma 5.3. *We have*

$$\big\| P : S_k^p\{R_k\}_\ell \longrightarrow S_k^p\{R_k\}_\ell \big\| = 1.$$

Moreover, for any complex numbers $\lambda_1, \ldots, \lambda_k$, we have

$$\Bigg\| \sum_{i=1}^k \lambda_i e_i \otimes e_i \otimes e_i \Bigg\|_{S_k^p\{R_k\}_\ell} = \Bigg(\sum_{i=1}^k |\lambda_i|^p \Bigg)^{\frac{1}{p}}.$$

Proof. We assume that $p \geqslant 2$, the proof for $p < 2$ being similar. Let

$$\Delta : \ell_k^2 \otimes \ell_k^2 \longrightarrow \ell_k^2 \otimes \ell_k^2$$

be the diagonal projection (in the sense of Lemma 5.2). Then we can write

$$P = (\Delta \otimes I_{\ell_k^2}) \circ (I_{\ell_k^2} \otimes \Delta), \tag{5.4}$$

which is going to lead us to a two-step proof.

We need several elementary operator space results, for which we refer e.g. to [23, Chapter 5] or [5, Section 9.3]. First, $C_k \otimes_{\mathrm{h}} R_k \simeq M_k$, and the diagonal of $C_k \otimes_{\mathrm{h}} R_k$ coincides with the commutative C^*-algebra ℓ_k^∞. Second, $R_k \otimes_{\mathrm{h}} C_k \simeq M_k^* = S_k^1$, and the diagonal of $R_k \otimes_{\mathrm{h}} C_k$ coincides with the operator space dual of ℓ_k^∞, that is $\mathrm{Max}(\ell_k^1)$ (see e.g. [23, Chapter 3]). Third, $R_k \otimes_{\mathrm{h}} R_k \simeq R_{k^2}$. We deduce from above that

$$\|\Delta : R_k \otimes_{\mathrm{h}} R_k \to R_k \otimes_{\mathrm{h}} R_k\|_{\mathrm{cb}} = 1 \quad \text{and} \quad \|\Delta : R_k \otimes_{\mathrm{h}} C_k \to R_k \otimes_{\mathrm{h}} C_k\|_{\mathrm{cb}} = 1$$

and moreover,

$$\Delta(R_k \otimes_{\mathrm{h}} R_k) \simeq R_k \quad \text{and} \quad \Delta(R_k \otimes_{\mathrm{h}} C_k) \simeq \mathrm{Max}(\ell_k^1)$$

completely isometrically.

Next according to [23, Theorem 5.22], we have

$$R_k \otimes_{\mathrm{h}} R_k(2/p) \simeq [R_k \otimes_{\mathrm{h}} R_k, R_k \otimes_{\mathrm{h}} C_k]_{2/p}$$

completely isometrically. Hence by interpolation,

$$\big\| \Delta : R_k \otimes_{\mathrm{h}} R_k(2/p) \longrightarrow R_k \otimes_{\mathrm{h}} R_k(2/p) \big\|_{\mathrm{cb}} = 1 \tag{5.5}$$

and we have

$$\Delta\big(R_k \otimes_{\mathrm{h}} R_k(2/p)\big) \simeq \big[R_k, \mathrm{Max}\big(\ell_k^1\big)\big]_{2/p} \tag{5.6}$$

completely isometrically.

Now applying Lemma 5.2 with $E_1 = C_k$ and $E_2 = \mathrm{Max}(\ell_k^1)$, we find that

$$\big\| \Delta : C_k \otimes_{\mathrm{h}} \mathrm{Max}\big(\ell_k^1\big) \longrightarrow C_k \otimes_{\mathrm{h}} \mathrm{Max}\big(\ell_k^1\big) \big\|_{\mathrm{cb}} = 1.$$

We claim that

$$\Delta\big(C_k \otimes_{\mathrm{h}} \mathrm{Max}\big(\ell_k^1\big)\big) \simeq \ell_k^2$$

isometrically. Indeed, we have $C_k \otimes_{\mathrm{h}} \mathrm{Max}(\ell_k^1) = C_k \otimes_{\min} \mathrm{Max}(\ell_k^1) \simeq CB(\ell_k^\infty, C_k)$. Hence writing $B = B(\ell^2)$ for simplicity, we have for any $\lambda_1, \dots, \lambda_k$ in $\mathbb{C}$ that

$$\begin{aligned}
\left\| \sum_{i=1}^k \lambda_i e_i \otimes e_i \right\|_{C_k \otimes_{\mathrm{h}} \mathrm{Max}(\ell_k^1)} &= \sup\left\{ \left\| \sum_{i=1}^k \lambda_i e_i \otimes y_i \right\|_{C_k \otimes_{\min} B} :\ y_i \in B,\ \sup_i \|y_i\| \leqslant 1 \right\} \\
&= \sup\left\{ \left\| \sum_{i=1}^k |\lambda_i|^2 y_i^* y_i \right\|_B^{1/2} :\ y_i \in B,\ \sup_i \|y_i\| \leqslant 1 \right\} \\
&= \left(\sum_{i=1}^k |\lambda_i|^2 \right)^{1/2}.
\end{aligned}$$

On the other hand,

$$\|\Delta : C_k \otimes_{\mathrm{h}} R_k \longrightarrow C_k \otimes_{\mathrm{h}} R_k\|_{\mathrm{cb}} = 1 \quad \text{and} \quad \Delta(C_k \otimes_{\mathrm{h}} R_k) \simeq \ell_k^\infty.$$

Since

$$C_k \otimes_{\mathrm{h}} \big[R_k, \mathrm{Max}\big(\ell_k^1\big)\big]_{2/p} = \big[C_k \otimes_{\mathrm{h}} R_k, C_k \otimes_{\mathrm{h}} \mathrm{Max}\big(\ell_k^1\big)\big]_{2/p},$$

we deduce by interpolation that

$$\big\| \Delta : C_k \otimes_{\mathrm{h}} \big[R_k, \mathrm{Max}\big(\ell_k^1\big)\big]_{2/p} \longrightarrow C_k \otimes_{\mathrm{h}} \big[R_k, \mathrm{Max}\big(\ell_k^1\big)\big]_{2/p} \big\|_{\mathrm{cb}} = 1. \tag{5.7}$$

Since $[\ell_k^\infty, \ell_k^2]_{p/2} = \ell_k^p$, we obtain in addition that

$$\Delta\big(C_k \otimes_{\mathrm{h}} \big[R_k, \mathrm{Max}\big(\ell_k^1\big)\big]_{2/p}\big) \simeq \ell_k^p \tag{5.8}$$

isometrically.

Using (5.3) and the composition formula (5.4), we deduce from (5.5)–(5.8) that P is a contraction on $S_k^p\{R_k\}_\ell$, and that its range is equal to ℓ_k^p. □

Proof of Theorem 5.1. The assertion (2) follows from (1) by Corollary 4.9, so we only need to prove (1). As in Section 4, we let

$$S_k^p \otimes_\beta \ell_k^2 = S_k^p\{R_k\}_\ell +_p S_k^p\{C_k\}_r.$$

We observe that Lemma 5.3 holds as well with $S_k^p\{C_k\}_r$ replacing $S_k^p\{R_k\}_\ell$. Namely, P is contractive on $S_k^p\{C_k\}_r$, and $P(S_k^p\{C_k\}_r)$ is equal to ℓ_k^p. We deduce that

$$\big\| P : S_k^p \otimes_\beta \ell_k^2 \longrightarrow S_k^p \otimes_\beta \ell_k^2 \big\| = 1 \tag{5.9}$$

and that for any complex numbers $\lambda_1, \ldots, \lambda_k$, we have

$$\left(\sum_{i=1}^k |\lambda_i|^p\right)^{1/p} \leqslant 2 \left\| \sum_{i=1}^k \lambda_i e_i \otimes e_i \otimes e_i \right\|_{S_k^p \otimes_\beta \ell_k^2}. \tag{5.10}$$

Now consider

$$w = \sum_{i=1}^k e_i \otimes e_i \otimes e_1.$$

By (5.3) we have

$$\|w\|_{S_k^p\{R_k\}_\ell} = \left\| \sum_{i=1}^k e_i \otimes e_i \right\|_{C_k \otimes_{\mathrm{h}} R_k} = \|I_k\|_\infty = 1.$$

Recall that if we regard $S_k^p\{R_k\}_\ell$ as the tensor product $S_k^p \otimes \ell_k^2$, then $e_i \otimes e_j \otimes e_m$ corresponds to $E_{im} \otimes e_j$. Hence we have

$$(u_1 \otimes I_{\ell_k^2})(w) = k^{-\frac{1}{2p}} \sum_{i=1}^k e_i \otimes e_i \otimes e_i;$$

$$(u_2 \otimes I_{\ell_k^2})(w) = k^{-\frac{1}{2p}} e_1 \otimes e_1 \otimes e_1;$$

$$(u_3 \otimes I_{\ell_k^2})(w) = e_1 \otimes e_1 \otimes e_1;$$

$$(u_4 \otimes I_{\ell_k^2})(w) = k^{-\frac{1}{p}} \sum_{i=1}^{k} e_i \otimes e_1 \otimes e_i.$$

Consequently,

$$P(u \otimes I_{\ell_k^2})(w) = \frac{1}{4}\left(k^{-\frac{1}{2p}} \sum_{i=1}^{k} e_i \otimes e_i \otimes e_i + \left(k^{-\frac{1}{2p}} + 1 + k^{-\frac{1}{p}}\right) e_1 \otimes e_1 \otimes e_1\right).$$

Applying (5.10) and (5.9), we deduce that

$$\begin{aligned}\left(\left(2k^{-\frac{1}{2p}} + 1 + k^{-\frac{1}{p}}\right)^p + (k-1)k^{-\frac{1}{2}}\right)^{\frac{1}{p}} &\leqslant 8\left\| P\left(u \otimes I_{\ell_k^2}\right)(w)\right\|_\beta \\ &\leqslant 8\left\| u \otimes I_{\ell_k^2} : S_k^p\{R_k\}_\ell \longrightarrow S_k^p \otimes_\beta \ell_k^2\right\|.\end{aligned}$$

This proves (1). □

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