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Homotopy Invariants of Repeller–Attractor Pairs. I. The Püppe Sequence of an R–A Pair

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1. PRELIMINARIES

1.1. *Introduction*

In 1947, Wazewski published a paper [26] containing a result, which Conley calls Wazewski's lemma in [3], which in the version Conley proves says roughly that given a subset of phase space which is closed along the flow lines, those points which leave the subset eventually can be deformed by the flow in finite positive time into those points which leave immediately [cf. 3, 5, 15, 26, 27]. This lemma is exploited by use of the contrapositive; if the whole subset cannot be continuously deformed into those points which leave immediately, then some point must stay in the subset for all positive time. By a clever choice of subset(s), one can prove existence of solutions to boundary value problems [cf. 5–7, 16, 17, 31].

It is the basic aim of this paper and its sequel to develop structures which extend in a systematic way the underlying idea of Wazewski's lemma; i.e., what can we discover about the nature of certain distinguished classes of orbits from a partial knowledge of how certain distinguished subsets of phase space are deformed by the flow.

The class of orbits on which we focus attention are the isolated invariants sets, a notion first mentioned by Ura in [32]. An invariant set is isolated if it is the largest invariant set in a closed neighborhood of itself, and such neighborhoods are called isolating neighborhoods.

Subsequently [8], Conley and Easton defined an homology sequence for an isolated invariant set of a smooth flow on a manifold by developing the notion of an isolating block for an isolated invariant set. A compact manifold with corners is an isolating block if the flow is everywhere transverse to its boundary [cf. 28]. Churchill extended the above ideas to the context of continuous flows on a compact metric space [2], and in this context, Montgomery [23] defined a local homotopy index for isolated invariant sets which is the pointed homotopy type of the quotient space of an

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isolating block obtained by collapsing the exit set (the points which leave immediately in positive time) of an isolating block to the distinguished point. Conley in [3] develops the notion of a locally compact Hausdorff local semi-flow and extends the definition of homotopy index to include this context by use of index pairs for an isolated invariant set (a block and its exit set is a special case of an index pair); and it is in this context that the present work is written.

This homotopy index, as well as the sequence developed by Conley and Easton, and Churchill, is a local phenomenon because it can be represented by an isolating block in an arbitrarily small neighborhood of the invariant set (such blocks always exist), and generalizes the Morse index of an isolated critical point of a smooth flow on a manifold. The Morse index of an isolated critical point is the dimension of the unstable manifold to the point, whereas the homotopy index of an isolated critical point is the homotopy type of a pointed n-sphere where n is the dimension of the unstable manifold to the point. Both the Morse and homotopy indices are stable under small perturbations of the flow, but whereas the Morse index of an isolated critical point requires for its definition that certain non-degeneracy hypotheses be satisfied, the homotopy index of an arbitrary isolated invariant set does not.

Using the local index as a foundation, the present work develops structures (called connected simple systems) giving a more global notion of index in that they are stable under continuation along paths in a canonical way, and develops these, not only for a single isolated invariant set, but also for special finite sequences of inter-related isolated invariant sets. Calling the first element in one of these sequences S and the subsequent members $M_1,\ldots,M_n$, the M_i $(i=1,\ldots,n)$ are pairwise disjoint compact invariant subsets of S called by Conley [3, 4] a Morse decomposition for S, a notion developed by him generalizing the decomposition of a compact boundaryless manifold M arising from the flow of a gradient vectorfield of a Morse function on M. For the most part, the present work is concerned only with those Morse decompositions with two elements, called a repeller–attractor pair for S and hereinafter abbreviated to R–A pair and denoted (A^*, A). Roughly, a set A is an attractor in S if all positive semi-orbits in S starting near A stay near and go towards A; a set A^* is a repeller if it is an attractor for the time-reversed flow, and (A^*, A) is an R–A pair if A^* is the largest invariant subset of S disjoint from the attractor A.

For an R–A pair (A^*, A) of S, there exist nested index triples of subsets of phase space, $N_1 \supset N_2 \supset N_3$, so that the quotient spaces N_1/N_3, N_1/N_2, N_2/N_3 represent respectively the homotopy indices of S, A^*, A; also the inclusion induced sequence

$$N_2/N_3 \to N_1/N_3 \to N_1/N_2$$

embeds in a long coexact Püppe sequence

$$N_2/N_3 \to N_1/N_3 \to N_1/N_2 \xrightarrow{c} S(N_2/N_3) \to S(N_1/N_3) \to \cdots \tag{1}$$

denoted $s(N_1, N_2, N_3)$, where c is called the connection map of the sequence and $S(\cdot)$ denotes the reduced suspension functor. The category of all such possible sequences together wth the collection of infinite homotopy commutative ladders of homotopy equivalences between them induced by inclusions and the flow form a connected simple system, called the connection index of the R–A pair. The global nature of this index is expressed by the fact that to every path $(S(t), A^*(t), A(t))$ is associated in a functorial way an infinite homotopy commutative ladder between the long coexact sequences $s(N_1(0), N_2(0), N_3(0))$ and $s(N_1(1), N_2(1), N_3(1))$ associated to the ends of the path. As the space of isolated invariant sets embeds in the space of R–A pairs by mapping an invariant set S to $(S, S, \varnothing)$, the restriction of the structure of the connection index to the image of this embedding defines for each isolated invariant set a connected simple system, which Conley calls the Morse index of the isolated invariant set and for which he gives a direct construction in [3]. Many of the properties of the connection index are direct analogues for properties of the Morse index as stated (but not all proved) by Conley in [3].

Whenever an isolated invariant set is the disjoint union of a repeller and an attractor, the connection map of the sequence (1) above is null-homotopic, and there exists a splitting map

$$\mu: N_1/N_2 \to N_1/N_3$$

of the sequence. The existence of the splitting map is used much the same way and geometrically is interpreted much the same way as Wazewski's lemma. Given a path in the repeller–attractor space for which the invariant sets at the ends are the disjoint union of the repeller and the attractor there exists a diagram

$$\begin{array}{ccc} N_1(0)/N_3(0) & \xleftarrow{\mu(0)} & N_1(0)/N_2(0) \\ \downarrow & & \downarrow \\ N_1(1)/N_3(1) & \xleftarrow{\mu(1)} & N_1(1)/N_2(1) \end{array}$$

relating the splitting maps at the ends where the vertical arrows are given by the continuation of the connection index along the path. This diagram *need not be commutative.* However, the splitting maps are natural relative to the continuation of the connection index along paths in the appropriate sense: if at every point of the path the invariant set is the disjoint union of the repeller and the attractor, then the diagram of (2) is commutative. Hence, if the splitting maps are defined at the ends of a path, but diagram (2) does not commute, then for some point on the path the isolated invariant set is not the disjoint union of the repeller and the attractor; whence there is an orbit connecting the repeller to the attractor.

Of course one way of showing that the diagram does not commute is to examine what the splitting maps do on homology. An appropriate choice of arc, disk, or more generally, singular chain in phase space will represent non-trivial homology, say α, in $N_1(0)/N_2(0)$, the homotopy index of the repeller at the initial point of the path. The diagram (2) affords two ways of mapping α into the homology of $N_1(1)/N_3(1)$, the homotopy index of the ambient isolated invariant set at the end of the path. Call the image of α by the high road $\mu_0'\alpha$, and its image by the low road $\mu_1'\alpha$. Diagram (2) does not commute when $\mu_0'\alpha - \mu_1'\alpha \neq 0$. Moreover this non-zero difference in homology classes is very closely related to what one looks for geometrically when one tries to employ a "shooting" method in differential equations.

At present, all applications of the structures described above are to fast–slow systems of differential equations, i.e., to first-order systems of differential equations where the independent variable is thought of as time and where the dependent variables divide into at least two classes with the variables in each class having velocities of the same order of magnitude, but the velocities of variables in distinct classes having sharply different orders of magnitude with the difference in magnitude governed by a (small) parameter ε. Application of these structures depends heavily on the particular equation under study, and no applications will be presented here, but will be presented in [21, 22]. However, we presently give a brief gestalt of how the structures are used in the two applications of [21, 22].

Both applications assume that there is a path in the repeller–attractor space associated to the fast systems ($\varepsilon = 0$ in a stretched time scale) of the fast–slow systems with the splitting maps defined at either end, but with diagram (2) not commuting by virtue of the existence of an homology class α as above with $\mu_0'\alpha - \mu_1'\alpha \neq 0$. In the technique developed in [22], the path in the repeller–attractor space of the fast systems corresponds to the varying of an external parameter of the fast–slow system, and under appropriate hypotheses, a topological perturbation argument shows that there is a corresponding path in the repeller–attractor space of the full fast–slow system for ε small enough and that the splitting maps are defined at either end of the path, but the corresponding diagram (2) does not commute. In the technique presented in [21], the path in the repeller–attractor space of the fast systems corresponds to varying the slow variable along an orbit of the slow system. Here thinking of α as its representative singular chain, under appropriate hypotheses, a perturbation argument establishes that a sub-arc, sub-disk, or, more generally, sub-chain of α is carried by the flow to a singular chain which represents the non-zero homology $\mu_0'\alpha - \mu_1'\alpha$ which in fact represents homology in $N_2(1)/N_3(1)$, the index of the attractor at the end of the path. Thus, for example, with the above hypotheses, an appropriate arc near the repeller of the fast systems has a sub-arc carried by the full flow to an arc near the attractor of the fast systems. The idea is that the starting arc should consist of points satisfying the left end boundary conditions with a sub-arc carried to an arc which is topologically transverse to the right end boundary conditions; a point of intersection followed backwards in time gives existence of a solution to the boundary value problem on the finite interval.

In this paper it is shown that for any repeller–attractor pair (A^*, A) of an isolated invariant set S there exists a triple of spaces $N_1 \supset N_2 \supset N_3$ so that N_1/N_3, N_1/N_2, and N_2/N_3 are index spaces for S, A^*, and A respectively, and that the sequence of inclusion induced maps $N_2/N_3 \to N_1/N_3 \to N_1/N_2$ is coexact and can be embedded in the long coexact Püppe sequence (1) which is functorial relative to the Morse indices $\mathscr{I}(S)$, $\mathscr{I}(A^*)$, $\mathscr{I}(A)$, this allows for the new definition of a connected simple system $\mathscr{J}(S; A^*, A)$ associated to each repeller–attractor pair in an isolated invariant set S with long coexact sequences as objects and infinite homotopy commutative ladders as morphisms. Along the way generalizations of the above are indicated for Morse decompositions. Finally, when $S = A^* \cup A$, each long coexact sequence of $\mathscr{J}(S; A^*, A)$ has a natural (relative to the morphisms of $\mathscr{J}(S; A^*, A)$) flow induced splitting which can be exploited as indicated above. The sequel to this paper [20] describes the space of Morse decompositions and in particular the space of R–A pairs and describes how $\mathscr{J}(S; A^*, A)$ continues along paths; in particular a functor $\mathscr{J}$ is constructed from the fundamental groupoid of the space of R–A pairs into an appropriate category of which $\mathscr{J}(S; A^*, A)$ is an object.

The existence of the triple $N_1 \supset N_2 \supset N_3$ is stated in [3, Theorem III. 7.2] without proof. Also, the standard proof of the Püppe sequence is outlined there but does not in general apply to the sequence $N_2/N_3 \to N_1/N_3 \to N_1/N_2$ because N_2/N_3 is *not* in general a strong deformation retract of a closed neighborhood of itself in N_1/N_3; the way around this technicality is noted in Section 3. The functoriality of the sequence relative to the Morse indices is new as is the existence of the splitting. Finally [3, Theorem III.7.2.C] is stated without proof; a proof is given here as Proposition 3.3.

The results of this paper and its sequel were part of the author's Ph.D. dissertation done under the direction of C. Conley. The development of the structures in this paper and its sequel was motivated by the desire (which saw fruition in [21, 22]) to show existence of certain "transisition solutions" to singular perturbation problems where existence is motivated by the changing geometrical structure of orbits in phase space as a parameter is varied, as discussed above. These problems, which arose from two reaction–diffusion equations, were first shown to the author and C. Conley by P. Fife who discusses these any many related equations in [14, 33–35].

1.2. *Notation*

The basic mathematical notation is standard with the following possible exception: $\mathbf{R}$, $\mathbf{R}^+$, and $\mathbf{R}^-$ denote respectively the real numbers, the non-negative reals, and the non-positive reals; for $a, b \in \mathbf{R}$, $]a, b[$ denotes the open interval from a to b, $[a, b]$ the corresponding closed interval, and $[a, b[$ and $]a, b]$ denote the half-open intervals, open on the right and left respectively; for A and B sets, their set theoretic difference is $A \setminus B = \{x : x \in A$ and $x \notin B\}$.

We use $\mathrm{cl}_X(A)$, $\mathrm{int}_X(A)$, and $\partial_X(A)$ to denote the closure, interior, and boundary respectively of a subset A of a topological space X. Other topological notation, if not noted otherwise below, follows that of [24]. All homotopies unless explicitly described otherwise are considered in the category of topological spaces with base point and base point preserving maps. Thus for two maps f and g mapping a pointed space X to another pointed space Y, $f \sim g$ means there is a homotopy from f to g which maps base point to base point throughout the deformation. Also we use the notation $\mathscr{T}$ and $\mathscr{T}^*$ to denote the categories of topological spaces and topological spaces with base point, resp., and use $\mathscr{H}\mathscr{T}$ and $\mathscr{H}\mathscr{T}^*$ to denote the corresponding homotopy categories. When spaces X' and X'' are functorially produced from the same object, $X' \simeq X''$ means that X' and X'' are naturally equivalent in $\mathscr{T}$, $\mathscr{T}^*$, $\mathscr{H}\mathscr{T}$, or $\mathscr{H}\mathscr{T}^*$ as appropriate from context.

For a pointed space X, we define the reduced cone on X, C^*X, by $C^*X = X \wedge (I, 1)$ as an object of $\mathscr{T}^*$; i.e., we are using 1 as base point for I so that the usual geometrical picture of a cone corresponds with the definition. If we forget the base point of X, CX denotes the unreduced cone defined as the quotient space $X \times I/X \times \{1\}$ as an object of $\mathscr{T}$. With the above definitions of cones, for a morphism f in $\mathscr{T}$ (resp. $\mathscr{T}^*$), C_f (resp. C_f^*) denotes the usual mapping cone, but when f is an inclusion from A into X, the usual $X \cup CA$ (resp. $X \cup C^*A$) is used instead.

Finally, throughout Γ is a fixed topological space which admits a continuous flow which will be denoted as a right action of the additive group of real numbers on Γ: each $\gamma \in \Gamma$ is moved to $\gamma \cdot t$ for each $t \in \mathbf{R}$. Γ need not be Hausdorff, but we assume Γ contains a non-empty open subset Γ_0 which is Hausdorff in the inherited topology. The motivation for this context is given in [3].

1.3. *Some Remarks on Background Definitions and Results, and Descrepancies in Notation between* [3] *and* [19].

The reader is referred to [19] for definitions of the following: a local flow (or semi-flow) $\Phi \subset \Gamma_0$, an invariant, positively invariant, negatively invariant, or relatively positively invariant subset of Γ, the ω and ω^* limit sets of a subset of Γ, an isolated invariant set, an isolating neighborhood, an index pair, an index space, a connected simple system considered as a category, the Morse index of an isolated invariant set viewed as a category with index spaces as objects and certain homotopy classes of maps as morphisms, and the exit and entrance time maps $\sigma \mid N$ and $\sigma^* \mid N$ defined on N into $[0, \infty]$ for any $N \subset \Gamma$ which is compact or has an upper semi-continuous decomposition by compact sets. Most of these definitions are given or have prototypes given in [3], and a discussion of the discrepancies between [3] and [19] follows.

Note that what in [19] is called a local semi-flow, in [3] is called a local flow, and what in [19] is called a local flow, in [3] is called a two-sided local flow. In this paper, we shall follow the usage of [19] in this regard. Unless stated explicitly to the contrary, we assume that *all local* (*semi-*) *flows* $\Phi \subset \Gamma_0$ *are locally compact.*

It is observed in [19] that if $\varnothing \neq A \subset Y \subset \Gamma_0$ with Y compact and positively invariant, then in general $\omega(A)$ is not a subset of Y since, due to the non-Hausdorffness of Γ, Y is not generally a closed subset of Γ. This fact was overlooked in [3], but does not affect the validity of any of the results given there since in all cases the set of real interest is $\omega(A) \cap \Gamma_0$, as is the case in [19] and this paper. Consequently, to simplify notation, we *use* $\omega(A)$ *as a gloss for* $\omega(A) \cap \Gamma_0$ *whenever* $A \subset Y \subset \Gamma_0$ *as above*, unless stated explicitly to the contrary. In particular, this differs from the notation of [19] where $\omega(A) \cap Z \equiv \omega(A; Z)$ whenever Z or a subset of Z is compact, Hausdorff, positively invariant and contains A; however, we shall revert to the notation of [19] whenever clarity demands. Note that if A above is connected, then $\omega(A) \cap Z$ is connected even if Z is not connected because semi-orbits emanating from A and their closures relative to Z are all connected and must lie in the component of Z which contains A.

Analogously, $\omega^*(A)$ *will be used as a gloss for* $\omega^*(A) \cap \Gamma_0$. *Whenever* $A \subset Y \subset \Gamma_0$ *with* Y *compact and negatively invariant*, again, there is the analogous reversion to the notation of [19] whenever clarity demands, and the analogous remark about connectivity holds for ω^*-limit sets.

We refer the reader to [19] for a discussion of the apparent discrepancies and their resolution between the definitions of [3] and [19] for the objects and morphisms of the Morse index of an isolated invariant set when viewed as a category. We will follow the notation and usage of [19] in this matter;

in particular, note the definition of a general inclusion induced map between index spaces given there.

Exit and entrance time maps are not defined in [3] although they were defined by Churchill [2] and Easton [13] for isolating blocks. In the context mentioned above, in [19], $\sigma \mid N$ and $\sigma^* \mid N$ are shown to be upper semi-continuous and it follows that sets $N \subset \Gamma_0$ which are compact or have upper semi-continuous decompositions by compact sets are closed along orbits in the same sense that Wazewski sets are; cf. [3, Chap. II, definition 2.2, condition (a)]. Recent work of Rybakowski [33–35] suggests that upper semi-continuity of exit and entrance time maps and being closed along orbits are the important properties to focus on rather than versions of compactness, although these may serve as a means to show these properties.

The reader is referred to [3] for the definitions of *attractor*, *repeller*, *Morse set*, and *Morse decomposition* of a compact Hausdorff invariant set in Γ as used in this paper. The reader is also referred to [3] for the precise definition of the *duality between repellers and attractors* relative to a compact Hausdorff invariant set $S \subset \Gamma$. Note that in [3], if A is an attractor in S and A^* its dual repeller, Conley refers to the attractor–repeller pair (A, A^*) where as indicated by the Introduction we shall use the reverse ordering and refer to repeller–attractor pairs. It may amuse the reader to note that Conley had originally intended to use the ordering "repeller–attractor" in [3], and this author had originally intended to use the reverse ordering in [18], but subsequent to a conversation over lunch one day both of us, unbeknownst to each other and to our mutual chagrin, decided in favor of the other's ordering. Conley's argument which convinced this author to use R–A pairs is that orbits travel from the repeller to the attractor so that repeller should come before attractor in the ordering.

In keeping with the above choice of ordering, for an R–A pair (A^*, A) of S, denote the *set of connecting orbits from*: A^* *to* A *in* S by $C(A^*, A)$; i.e., $C(A^*, A) \equiv S \backslash (A^* \cup A)$.

In conformity with the choice of ordering "repeller–attractor" and because it simplifies the indexing in several of the proofs given here and in [20], we have chosen when describing Morse decompositions to refer to a descending sequence of attractors, $S = A_0 \supset A_1 \supset \cdots \supset A_n = \varnothing$ with Morse sets in the decomposition given by $M_i \equiv A_{i-1} \cap A_i^*$, $i = 1, \ldots, n$. In particular, corresponding to a sequence $S = A_0 \supset A_1 \supset A_2 = \varnothing$ is the Morse decomposition given by $M_1 = A^*$, $M_2 = A$ so that as just remarked the order of the subscripts conforms with the order "repeller–attractor." In [3] the sequence of attractors is indexed in the reverse ordering so that the Morse sets also have the reverse ordering.

Finally, note the following trivial observation which shall be used below without further comment. If A_1 and A_2 are attractors relative to S and $A_2 \subset A_1$, then A_2 is an attractor relative to A_1 and the dual repeller of A_2 in

A_1 is $A_2^* \cap A_1$ where A_2^* is the dual in S. The converse is also true [3, II, 5.3.D], but requires the following lemma (1.4) for which we have considerable other uses. The following lemma is a slight variant of one appearing in [3]. The interested reader can easily supply the modifications to the proof, or alternatively, consult [18] for a detailed proof.

1.4. LEMMA [cf. 3, II. 5.1.D]. *Suppose* (i) $Y \subset \Gamma_0$ *is compact and positively invariant,* (ii) $\mathscr{N}$ *is a compact Y-neighborhood such that for* $\gamma \in \partial_Y \mathscr{N}$ *there exists* $t_\gamma > 0$ *so that* $\gamma \cdot -t_\gamma \notin \mathscr{N}$. *Let* A *be the maximal invariant set contained in* $\mathscr{N}$, *and set* $\mathscr{N}^+ = \{\gamma \in \mathscr{N} : \gamma \cdot R^+ \subset \mathscr{N}\}$. *Then* $\mathscr{N}^+$ *is a compact positively invariant Y-neighborhood of* A.

Remark. If Y is invariant, it follows that $\mathscr{N}^+$ is an attractor neighborhood with attractor A.

2. INDEX TRIPLES FOR REPELLER–ATTRACTOR PAIRS

The existence of an index triple for an R–A pair in an isolated invariant set S is proved in Proposition 2.4 below. To outline the proof, first an index pair $\langle N_1, N_3 \rangle$ for S is chosen relative to some isolating neighborhood N; the technical Lemma 2.1 shows that those portions of the forward and backward asymptotic sets of N asymptotic respectively to the repeller and attractor are closed and disjoint, and hence can be separated; in particular, from Proposition 2.2 it follows that there is an isolating neighborhood of the attractor whose backward asymptotic set is precisely that portion of the backward asymptotic set of N asymptotic to the attractor, and an application of the results of [3, Chap. 3, Sect. 4] guarantees a compact, relatively positively invariant N-neighborhood N_2 of this portion of the backward asymptotic set. The triple $\langle N_1, N_2, N_3 \rangle$ is an index triple for the R–A pair as defined in Definition 2.5; roughly, $\langle N_2, N_3 \rangle$, $\langle N_1, N_3 \rangle$, $\langle N_1, N_2 \rangle$ are index pairs for the attractor, S, and the repeller respectively. An index triple is illustrated in Fig. 1 for a saddle connection between two hyperbolic critical points for a flow in the plane.

In what follows, S is an isolated invariant set relative to a local semi-flow $\Phi \subset \Gamma_0$, and N is an isolating Φ-neighborhood of S. Appeals to dualization in the proofs of Lemma 2.1 and Proposition 2.2 below are legitimate because the local semi-flow property of Φ is not called upon.

2.1. LEMMA. *Let* (A^*, A) *be an R–A pair in S. Define*

$$\mathscr{A}_1^+(A^*, A) \equiv \{\gamma \in A^+(N) : \omega(\gamma) \subset A^*\},$$

$$\mathscr{A}_2^-(A^*, A) \equiv \{\gamma \in A^-(N) : \omega^*(\gamma) \subset A\},$$

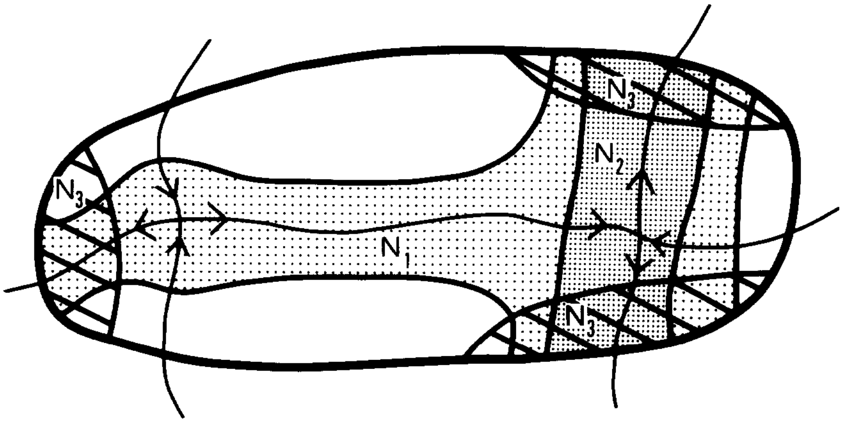

FIG. 1. An index triple for a saddle connection between hyperbolic critical points as constructed in Proposition 2.4. N_1 is the union of the lightly and heavily dotted regions, N_2 is the heavily dotted region, and N_3 is the union of the three lined regions.

$$\mathscr{A}_1^-(A^*, A) \equiv \{\gamma \in A^-(N): \omega^*(\gamma) \subset A^*\},$$
$$\mathscr{A}_2^+(A^*, A) \equiv \{\gamma \in A^+(N): \omega(\gamma) \subset A\}.$$

Then

(a) $\mathscr{A}_1^+$ *and* $\mathscr{A}_2^-$ *are compact and disjoint*;

(b) *if* $S = A^* \cup A$, *then* $\mathscr{A}_1^-$ *and* $\mathscr{A}_2^+$ *are compact and disjoint*;

(c) $(\mathscr{A}_1^+ \cup \mathscr{A}_1^-) \cap (\mathscr{A}_2^+ \cup \mathscr{A}_2^-) = C(A^*, A)$.

Proof. In proving (a) first, it will be shown that $\mathscr{A}_1^+$ is compact; by "dualizing" the argument, i.e., reversing the flow and interchanging the roles played by A^* and A, it then follows that $\mathscr{A}_2^-$ is compact.

Choose M^* and M to be disjoint Φ-isolating neighborhoods of A^* and A respectively, both Φ-interior to N. To simplify notation set $Y = A^+(N)$. Because $\partial_Y(M \cap Y) \subset \partial_\Phi M$, it follows that for each $\gamma \in \partial_Y(M \cap Y)$ there exists $t_\gamma > 0$ so that $\gamma \cdot -t_\gamma \notin M \cap Y$ since either $\gamma \notin A^-(N)$ so a fortiori $\gamma \notin A^-(M \cap Y)$ or $\gamma \in A^-(N)$ whence $\gamma \in S \backslash (A^* \cup A) = C(A^*, A) \equiv C$ so that $\omega^*(\gamma) \subset A^*$ by [3, II, 5.1.A]. Thus by Lemma 1.4, setting $\mathscr{M} = (M \cap Y)^+$, we have that $\mathscr{M}$ is a compact positively invariant Y-neighborhood of A which is Φ-interior to N and disjoint from M^*. As $A^+(N)$ is compact, the compactness of $\mathscr{A}_1^+$ follows immediately from the following sublemma.

SUBLEMMA. $\mathscr{A}_2^+$ *is* Y-*open and* $\mathscr{A}_1^+ = Y \backslash \mathscr{A}_2^+$.

Suppose $\eta \in \mathscr{A}_2^+$. Then $\omega(\eta) \subset A \subset \text{int}_Y \mathscr{M}$ so that for some $t_0 > 0$, $\eta \cdot t_0 \in \text{int}_Y \mathscr{M}$. Because Y is positively invariant, the flow restricted to $Y \times R^+$ is a continuous map into Y; hence for some Y-neighborhood V of η; $V \cdot t_0 \subset \mathscr{M}$. As $\mathscr{M}$ is positively invariant with maximal invariant set A, it follows that each point of V has its ω-limit set contained in A showing $\mathscr{A}_2^+$ is Y-open.

Next, clearly $\mathscr{A}_1^+ \subset Y \backslash \mathscr{A}_2^+$. To show the reverse inclusion, because $\gamma \in Y$ implies $\omega(\gamma) \subset S = A^* \cup C \cup A$, it suffices to show (i) ω-limit sets of points of Y do not intersect C and (ii) ω-limit sets of points of Y which intersect A necessarily are contained in A. Now if $\gamma \in Y$ and some $\eta \in C \cap \omega(\gamma)$, then, as in the proof of openness of $\mathscr{A}_2^+$, each point of some Y-neighborhood V of η has ω-limit set contained in A; this is impossible since $\gamma \cdot t_1 \in V$ for some $t_1 > 0$ forcing $\eta \in \omega(\gamma) \subset A$. Thus (i) holds. Because A^* and A are disjoint sets separated in S and because ω-limit sets of singletons are connected, (ii) follows from (i), completing the proof of the sublemma.

To finish the proof of (a), if $\gamma \in \mathscr{A}_1^+ \cap \mathscr{A}_2^-$, then

$$\gamma \in A^+(N) \cap A^-(N) \backslash C = S \backslash C = A^* \cup A;$$

whence $\gamma \in A^* \cap \mathscr{A}_2^-$ or $\gamma \in A \cap \mathscr{A}_1^+$ which are both impossible since the invariance of A^* and A implies that in the first case $\varnothing \neq \omega^*(\gamma) \subset A^* \cap A$ and in the second $\varnothing \neq \omega(\gamma) \subset A^* \cap A$, but A^* and A are disjoint. Thus $\mathscr{A}_1^+$ and $\mathscr{A}_2^-$ are disjoint.

For (b) assume $S = A^* \cup A$. However then, as A^* and A are disjoint and closed relative to S, they are also both open relative to S, and it follows that (A, A^*) is an R–A pair relative to S—A serves as its own repeller neighborhood and A^* as its own attractor neighborhood relative to S. The conclusion of (b) then follows by applying (a) to this R–A pair since

$$\mathscr{A}_1^+(A, A^*) = \mathscr{A}_2^+(A^*, A) \qquad \text{and} \qquad \mathscr{A}_2^-(A, A^*) = \mathscr{A}_1^-(A^*, A).$$

Finally to prove (c) note that the left-hand side of the equality is the union of the four sets $\mathscr{A}_1^\pm \cap \mathscr{A}_2^\pm$ and $\mathscr{A}_1^\pm \cap \mathscr{A}_2^\mp$ (the four possible choices of plus and minus signs) and that $\mathscr{A}_1^\pm \cap \mathscr{A}_2^\pm = \varnothing$ by the sublemma and its dual statement and that $\mathscr{A}_1^+ \cap \mathscr{A}_2^- = \varnothing$ by (a). On the other hand, $C(A^*, A) \subset \mathscr{A}_1^- \cap \mathscr{A}_2^+$ by [3, II, 5.1.A], and the reverse inclusion holds because $\mathscr{A}_1^- \cap \mathscr{A}_2^+ \subset A^-(N) \cap A^+(N) \backslash (A^* \cup A)$ since A^* and A contain both the ω and ω^* limit sets of each of their points. ∎

2.2 Proposition. *Let N be an isolating neighborhood for S and suppose (A^*, A) is an R–A pair of S. Let M be a closed N-neighborhood of A disjoint from A^*. Then M is an isolating neighborhood of A. Furthermore, if U is N-open and $A \subset U \subset M$, then $N \backslash U$ is an isolating neighborhood of A^*.*

Proof. As N isolates S, N is a Φ-neighborhood of A; hence M is a Φ-neighborhood of A since $A \subset \operatorname{int}_N(M) \cap \operatorname{int}_\Phi(N) = \operatorname{int}_\Phi(M)$.

Suppose B is invariant and $B \subset M$. Then $B \subset N$, and since N isolates S, $B \subset S = A^* \cup C(A^*, A) \cup A$. As M is disjoint from A^*, $B \subset C(A^*, A) \cup A$. However, if $\gamma \in C(A^*, A)$, then $\omega^*(\gamma) \subset A^*$; but if $\gamma \in B$, then $\omega^*(\gamma) \subset \operatorname{cl}_\Phi(B) \subset M$, for B invariant implies $\operatorname{cl}(B)$ invariant and since M is Φ-closed, $\operatorname{cl}_\Phi(B) \subset M$. Thus $B \subset A$; whence A is the maximal invariant set in M. This and the conclusion of the preceding paragraph show that M isolates A.

The second statement of the proposition follows from the first by dualizing; i.e., since A^* and M are disjoint closed sets in N, $A^* \subset N \backslash M \subset N \backslash U$ which yields that $N \backslash U$ is a closed N-neighborhood of A^* disjoint from A. Setting $M^* = N \backslash U$, it follows that M^* satisfies the hypothesis for the first statement of the proposition relative to the time-reversed flow, hence that M^* isolates A^* for the reverse flow, hence for the original flow. ∎

2.3. PROPOSITION. *Let (A^*, A) be a repeller–attractor pair relative to S. Then there are subsets, N_1, N_2, N_3 closed and positively invariant relative to N, satisfying:*

(i) $\langle N_1, N_2 \rangle$ *is an (isolating) index pair for S relative to N;*

(ii) $\langle N_2, N_2 \cap N_3 \rangle$ *is an (isolating) index pair for A relative to N_2;*

(iii) *for each N-open U, if $A \subset U \subset N_2$, then $\langle N_1 \backslash U, (N_2 \cup N_3) \backslash U \rangle$ is an (isolating) index pair for A^* relative to $N \backslash U$. Note that as N_2 is an N-neighborhood of A, there always exists such a U.*

Proof. Choose $\langle N_1, N_3 \rangle$ to be an (isolating) index pair for S relative to N with N_1 an N-neighborhood of $A^-(N)$. To choose N_2, first choose M to be a closed N-neighborhood of $\mathscr{A}_2^-$ which is N-interior to N_1 and disjoint from $\mathscr{A}_1^+$ which can be done because N is a normal space containing the disjoint N-closed sets $\mathscr{A}_1^+$ and $\mathscr{A}_2^-$ with the latter in the N-interior of N_1. Because $A^* \subset \mathscr{A}_1^+$, M is disjoint from A^*. By Proposition 2.2, M is a Φ-isolating neighborhood of A, and it is then clear that $A^-(M) = \mathscr{A}_2^-$. Note that by the choice of M, for some Γ-open W, $\mathscr{A}_2^- \subset W \cap N = \operatorname{int}_N(M)$. By [3, III, 4.1.C] choose $N_2 \subset W$ to be an M-neighborhood of $\mathscr{A}_2^-$ which is closed and positively invariant relative to M.

Then N_2 is an N-neighborhood of $\mathscr{A}_2^-$ which is closed and positively invariant relative to N. For certainly N_2 is N-closed and it is an N-neighborhood of $\mathscr{A}_2^-$ as $\mathscr{A}_2^- \subset \operatorname{int}_M(N_2) \cap \operatorname{int}_N(M) = \operatorname{int}_N(N_2)$. To see the relative positive invariance, let $\gamma \in N_2$ and note that $\sigma \mid N_2(\gamma) \leqslant \sigma \mid M(\gamma) \leqslant \sigma \mid N(\gamma)$ by the containment relation that holds between N_2, M, and N. Now if $\sigma \mid N_2(\gamma) < \sigma \mid N(\gamma)$, then as N_2 is N-interior to M it follows that the orbit segment $\gamma \cdot [0, \sigma \mid N_2(\gamma)]$ can be extended in the forward time direction and still lie in M which is impossible since N_2 is positively invariant relative to M.

Claim $\langle N_2, N_3 \cap M\rangle$ is an index pair for A relative to M; it follows that $\langle N_2, N_3 \cap N_2\rangle$ is an index pair for A relative to N_2. For certainly both elements of the pair are compact, and the relative positive invariance property holds with respect to M since it does with respect to N. That the isolating property is satisfied follows easily from the facts that $\langle N_1, N_3\rangle$ has the property for $S \supset A$ and that A is both N-interior to N_2 and Φ-interior to N so Φ-interior to N_2. Finally, the exit property holds because as seen above $\sigma \mid N_2(\gamma)$, $\sigma \mid M(\gamma)$, and $\sigma \mid N(\gamma)$ are all equal for $\gamma \in N_2$, and $N_2 \subset N_1$ and $\langle N_1, N_3\rangle$ has the exit property relative to N.

To show (iii) of the proposition, suppose U is N-open and $A \subset U \subset N_2$. By Proposition 2.2, $N \backslash U$ is a Φ-isolating neighborhood of A^*. It is also trivial to verify that $N_1 \backslash U$ and $(N_2 \cup N_3)\backslash U$ are compact and positively invariant relative to $N \backslash U$. The proposed index pair has the isolating property because $\langle N_1, N_3\rangle$ does for S and because A^* is disjoint from N_2 and $\mathrm{cl}_\Phi(U)$. It remains to show that the exit property is satisfied. Accordingly, suppose $\gamma \in N_1 \backslash U$ and suppose $\sigma \mid N\backslash U(\gamma)$ is finite. There are two cases to consider: (i) $\sigma \mid N\backslash U(\gamma) = \sigma \mid N(\gamma)$, (ii) $\sigma \mid N\backslash U(\gamma) < \sigma \mid N(\gamma)$. Set $t' = \sigma \mid N\backslash U(\gamma)$. In the first case, because $\langle N_1, N_3\rangle$ has the exit property it follows that $\gamma \cdot t' \in N_3 \backslash U$. In the second case, it follows that for some sequence of positive numbers ε_i decreasing to zero $\gamma \cdot (t' + \varepsilon_i) \in U$; hence $\gamma \cdot t' \in N_2 \backslash U$ as U is N-interior to N_2 and N_2 is N-closed. Thus in either case $\gamma \cdot t' \in (N_2 \cup N_3)\backslash U$ which shows that the exit property is satisfied. ∎

2.4. Proposition. *Given an isolating neighborhood N with maximal invariant set S and closed subsets N_1, N_2, N_3 of N, positively invariant relative to N satisfying*

(i) *$\langle N_1, N_3\rangle$ is an (isolating) index pair for the invariant set S;*

(ii) *N_2 is an isolating neighborhood and $\langle N_2, N_2 \cap N_3\rangle$ is an index pair relative to N_2 for the maximal invariant set A of N_2;*

(iii) *For some N-open U, with $A \subset U \subset N_2$, $N\backslash U$ is an isolating neighborhood and $\langle N_1 \backslash U, (N_2 \cup N_3)\backslash U\rangle$ is an index pair relative to $N\backslash U$ for the maximal invariant set A' of $N\backslash U$.*

Then (A', A) is a repeller–attractor pair of S.

Proof. Since N_2 is positively invariant relative to N and isolates A, it follows immediately that $\omega(S \cap N_2) = A$ which shows that A is an attractor. Let A^* be the dual repeller of A in S,

$$A^* = \{\gamma \in S \colon \omega(\gamma) \cap A = \varnothing\}.$$

It must be shown that $A' = A^*$. Now if $\gamma \in U \cap S$, then it follows that

$\omega(\gamma) \subset A$; hence $A^* \subset N \backslash U$. As A^* is invariant and since $N \backslash U$ isolates A', it follows that $A^* \subset A'$. On the other hand, if $\gamma \in S \backslash A^*$, then $\omega(\gamma) \subset A$. As A' is disjoint from A and as $\omega(A') = A'$, it follows that $A' \subset A^*$. ∎

2.5. Definition. *Let N be an isolating neighborhood with maximal invariant set S and suppose $\langle N_1, N_2, N_3 \rangle$ is an ordered triple of closed subsets of N, positively invariant relative to N, satisfying properties* (i), (ii), *and* (iii) *of the previous proposition. Then $\langle N_1, N_2, N_3 \rangle$ is called an* (*isolating*) *index triple for the repeller–attractor pair* (A^*, A) *of S. If $N_1 \supset N_2 \supset N_3$, it is called a nested* (*isolating*) *index triple. For nested* (*isolating*) *triples, it will be assumed that $N = N_1$, unless explicitly stated otherwise.*

Remark. For the same reasons given in [19, Sect. 4] when Φ is only a local semi-flow $\langle N_1^t, N_2^t, N_3 \rangle$ and $\langle N_1^t, N_2^t, N_3^{-t} \rangle$ need not be index triples because N_1^t and N_2^t may fail to be neighborhoods of S and A. The remedy is also the same and the precise formulation is safely left to the reader.

Proposition 2.5 above and the results of Section 3 to follow could be stated and proved to include non-isolating triples for local semi-flows, but we shall not bother.

2.6. Proposition. *Let S be an isolated invariant set with isolating neighborhood N, and let $\langle N_1, N_2, N_3 \rangle$ be an index triple for the R–A pair (A^*, A) of S. Define*

$$\langle N_1', N_2', N_3' \rangle \equiv \langle N_1, N_1 \cap (N_2 \cup N_3), N_1 \cap N_3 \rangle.$$

Then $\langle N_1', N_2', N_3' \rangle$ is a nested index triple for (A^*, A) relative to N. (*Hence also relative to N_1' if N_1' isolates S*), *and*

(i) $N_1'/N_3' = N_1/N_3$,

(ii) $N_1'/N_2' = N_1/(N_2 \cup N_3)$,

(iii) *there is an inclusion induced homotopy equivalence $N_2'/N_3' \to N_2/N_3$ which is a homeomorphism if $N_1 \supset N_2$.*

Proof. The straightforward check that $\langle N_1', N_2', N_3' \rangle$ is an index triple for (A^*, A) is omitted. Also (i) and (ii) are obvious; (iii) follows since

$$N_1 \cap (N_2 \cup N_3)/N_1 \cap N_3 \simeq N_1 \cap N_2/N_1 \cap N_2 \cap N_3$$

and since $(N_1 \cap N_2, N_1 \cap N_2 \cap N_3) \subset (N_2, N_3 \cap N_2)$ with equality holding if $N_1 \supset N_2$. The inclusion induced map

$$N_1 \cap N_2/N_1 \cap N_2 \cap N_3 \to N_2/N_3$$

is a homotopy equivalence since it is in the simple system for A. ∎

2.7. PROPOSITION. *Let S be an isolated invariant set with isolating Φ-neighborhood N, and let $S = A_0 \supset A_1 \supset \cdots \supset A_n = \varnothing$ be a sequence of attractors in S with Morse decomposition $\{M_1, ..., M_n\}$, $n \geqslant 2$. Then there exist $n+1$ sets, $N_1 \supset N_2 \supset \cdots \supset N_{n+1}$, compact and positively invariant relative to N so that N_i is an isolating Φ-neighborhood of A_{i-1} and $\langle N_i, N_{i+1}, N_{n+1} \rangle$ is an index triple for the R–A pair (M_i, A_i) of A_{i-1}, $i = 1, ..., n$.*

Proof. The proof is by induction and for $n = 2$ follows from Proposition 2.3 and 2.6. Assume the proposition true for a sequence of attractors of length $k \geqslant 2$, and suppose that

$$S = A_0 \supset A_1 \supset \cdots \supset A_{k+1} = \varnothing$$

is a sequence of attractors of length $k+1$. Set $A_i' = A_i$, $0 \leqslant i < k$, and set $A_k' = \varnothing$. By induction there are $k+1$ sets $N_1' \supset N_2' \supset \cdots \supset N_{k+1}'$ compact and positively invariant relative to N so that N_i' Φ-isolates A_{i-1}' and $\langle N_i', N_{i+1}', N_{i+2}' \rangle$ is an index triple for the R–A pair (M_i', A_i') of A_{i-1}', $i = 1, ..., k$. Set $N_i = N_i'$ for $1 \leqslant i \leqslant k$ and set $N_{k+2} = N_{k+1}'$. Then as N_k Φ-isolates $A_{k-1}' = A_{k-1}$ and as $\langle N_k', N_{k+1}' \rangle = \langle N_k, N_{k+2} \rangle$ is an index pair for A_k with

$$A^+(N_k) \subset N_k = \operatorname{int}_{N_k}(N_k),$$

applying the proof of Proposition 2.4 yields N_{k+1} so that $\langle N_k, N_{k+1}, N_{k+2} \rangle$ is an (isolating) index triple for (M_k, A_k), and by Proposition 2.6 (as already $N_k \supset N_{k+2}$) it can be assumed that $N_k \supset N_{k+1} \supset N_{k+2}$. Because it is clear that $(M_i', A_i') = (M_i, A_i)$ for $i = 1, ..., k$, this completes the proof. ∎

3. THE LONG COEXACT SEQUENCE OF AN INDEX TRIPLE

The reader is referred to [25] for the definition of a coexact sequence of maps of topological spaces with base point.

The following lemma provides a sufficient condition for $h(X, A)$ to be isomorphic to $h(X/A)$ where (X, A) is a topological pair and h is any reduced (co)homology theory. The condition is similar yet distinct from a condition given by Young [29] for an inclusion $A \subset X$ to be a cofibration [cf. 24, p. 57], and provides the technical means to construct a Püppe sequence from an index triple.

3.1. LEMMA. *Let (X, A) be a topological pair with A closed in X. Suppose there exists a continuous deformation $D: X \times I \to X$ (so $D(x, 0) = x$ for each $x \in X$) and a closed neighborhood U of A so that $D \mid U \times I$ is a weak deformation retraction of U into A (i.e., $D(U \times I) \subset U$, $D(A \times I) \subset A$,* *and $D(U \times \{1\}) \subset A$); and suppose also that there exists $\varphi: X \to [0, 1]$, a continuous map, with $A \subset \varphi^{-1}(1)$ and $X \backslash U \subset \varphi^{-1}(0)$. Then the composition $X \cup CA \to^{p_0} X \cup CA/CA \simeq X/A$ is a homotopy equivalence in $\mathscr{H}\mathscr{T}$ where p_0 is the quotient map. If in addition D preserves a base point $a_0 \in A$ (i.e., $D(a_0, t) = a_0$ for each $t \in I$) then the composition $X \cup C^*A \to^{p_0} X \cup C^*A/C^*A \simeq X/A$ is an equivalence in $\mathscr{H}\mathscr{T}^*$.*

Proof. Define $\Phi \equiv \{(x, t) \in X \times I: 0 \leqslant t \leqslant \varphi(x)\}$ and $Z \equiv X \times \{0\} \cup U \times I$. Since $\varphi(A) = 1$, $A \times \{1\} \subset A \times I \subset \Phi$, and since $X \backslash U \subset \varphi^{-1}(0)$, $\Phi \subset Z$. It follows that there is a sequence of inclusions $(X \times 0 \cup A \times I, A \times \{1\}) \to^{\iota} (\Phi, A \times \{1\}) \to^{j} (Z, A \times \{1\}) \to^{k} (X \times I, A \times \{1\})$ and hence a sequence of inclusion induced embeddings on the quotients $X \cup CA \to^{\bar{\iota}} \Phi/A \times \{1\} \to^{\bar{j}} Z/A \times \{1\} \to^{\bar{k}} X \times I/A \times \{1\}$ such that $\bar{j} \circ \bar{\iota} = \bar{l}$ and $\bar{k} \circ \bar{j} = \bar{m}$ where $\bar{l}$ and $\bar{m}$ are the embeddings induced by the inclusions $(X \times \{0\} \cup A \times I, \; A \times \{1\}) \to^{l} (Z, A \times I)$ and $(\Phi, A \times \{1\}) \to^{m} (X \times I, A \times \{1\})$, resp. Now the inclusions l and m are weak deformation retracts. For define $H: (Z, A \times \{1\}) \times I \to (Z, A \times \{1\})$ by $H(x, s, t) = (D(x, t), s)$ and $K: (X \times I, A \times \{1\}) \times I \to (X \times I, A \times \{1\})$ by $K(x, s, t) = (x, (1-t)s + ts\varphi(x))$. Now clearly H is continuous as a map into $X \times I$ (it is the restriction of a map defined on $X \times I \times I$ into $X \times I$) with image in $Z \times I$ by the hypothesis on D, and hence is continuous as a map into $Z \times I$. Also, clearly K is continuous, and it is straightforward to check that H and K are the required weak deformation retractions of the pairs. It follows that there are induced maps $\bar{H}: (Z/A \times \{1\}) \times I \to Z/A \times \{1\}$ and $\bar{K}: (X \times I/A \times \{1\}) \times I \to X \times I/A \times \{1\}$ which show that $\bar{l}$ and $\bar{m}$ are weak deformation retracts. Thus $\bar{l}$ and $\bar{m}$ are homotopy equivalences, whence from [19, Proposition 2.14] it follows that $X \cup CA$ is homotopy equivalent to $X \times I/A \times \{1\}$. However, $X/A \simeq X \times \{1\}/A \times \{1\}$ and $X \times \{1\}/A \times \{1\}$ is a strong deformation retract of $X \times I/A \times \{1\}$ where the deformation is given by $\bar{J}([x, s], t) = [x, (1-t)s + t]$. Thus $X \cup CA \sim X/A$. If D preserves a base point $a_0 \in A$, replace $A \times \{1\}$ by $\{a_0\} \times I \cup A \times \{1\}$ in the definitions of $\iota, j, k, \bar{\iota}, \bar{j}, \bar{k}$, and $\bar{J}$. The deformations $\bar{H}$, $\bar{K}$ and $\bar{J}$ are then base point preserving homotopies and this gives $X \cup C^*A \sim X/A$ as pointed spaces. To get that the homotopy equivalence is given (in the unpointed case) by $X \cup CA \to X \cup CA/CA \simeq X \times \{0\}/A \times \{0\}$ consider the commutative diagram

$$\begin{tikzcd}
X \cup CA \arrow[rrr, "\bar{n} = \bar{k} \circ \bar{j} \circ \bar{\iota}"] \arrow[d, "p_0"'] & & & X \times I/A \times \{1\} \arrow[rr, "\bar{r}"] \arrow[dr, "p_1"'] & & X \times \{1\}/A \times \{1\} \\
X \cup CA/CA \arrow[r, "a", "\simeq"'] & X \times \{0\}/A \times \{0\} \arrow[r, "n'"] & X \times I/A \times I \arrow[rr, "b", "\simeq"'] & & (X \times I/A \times \{1\})/(A \times I/A \times \{1\}) \arrow[ur, "\bar{\bar{r}}"'] &
\end{tikzcd}$$

Here $\bar{\bar{r}} = \bar{r} \circ p_1^{-1}$ where $\bar{r}$ is the retraction $[x, t] \to [x, 1]$. Now n' is a homotopy equivalence as $J'(x, s, t) = (x, (1-t)s)$ induces a strong deformation retraction of $X \times I/A \times I$ to $X \times \{0\}/A \times \{0\}$. Also the composition $\bar{r}bn' : X \times \{0\}/A \times \{0\} \to X \times \{1\}/A \times \{1\}$ coincides with the natural homeomorphism, $X \times \{0\}/A \times \{0\} \simeq X \times \{1\}/A \times \{1\}$. It follows that $\bar{\bar{r}}$ is an equivalence; hence p_1 is as $\bar{r}$ is, and this yields that p_0 is an equivalence as $\bar{n}$ and $bn'a$ are. The analogous argument yields the pointed case. ∎

COROLLARY. *Let (X, A) and (Y, B) be topological pairs with A and B closed in X and Y resp. Suppose $D^X : X \times I \to X$ and $D^Y : Y \times I \to Y$ are continuous deformations, U and V are closed neighborhoods of A and B resp., and $\varphi : X \to [0, 1]$ and $\psi : Y \to [0, 1]$ are continuous maps such that the quintuples (X, U, A, D^X, φ), (Y, V, B, D^Y, ψ) satisfy the hypothesis of the lemma. Then $X \times Y \cup C(X \times B \cup A \times Y) \sim X \times Y/X \times B \cup A \times Y$ and if D^X and D^Y preserve base points $a_0 \in A$ and $b_0 \in B$, resp. Then $X \times Y \cup C^*(X \times B \cup A \times Y) \sim X \times Y/X \times B \cup A \times Y$ as pointed spaces, and in both cases the equivalence is given by the quotient map.*

Proof. Define $\chi : X \times Y \to [0, 1]$ by $\chi = (1-\psi)\varphi + \psi \cdot 1$ and define $D^{X \times Y} : X \times Y \times I \to X \times Y$ by $D^{X \times Y} = (D^X, D^Y)$. Then $D^{X \times Y}$ is a deformation of $X \times Y$ and a weak deformation retraction of the open $X \times V \cup U \times Y$ into the closed $X \times B \cup A \times Y$ and χ plays the role of φ in the lemma relative to these two sets. ∎

For an index triple $\langle N_1, N_2, N_3 \rangle$ with $N_1 \supset N_2$ there is an inclusion induced sequence

$$N_2/N_3 \to N_1/N_3 \to N_1/(N_2 \cup N_3). \tag{1}$$

If the triple is nested, this reduces to

$$N_2/N_3 \to N_1/N_3 \to N_1/N_2, \tag{2}$$

and we shall occasionally abuse notation and write (2) when (1) should be written but the meaning is clear from context. This abuse is not serious in view of Proposition 2.6 above. Below we will show that sequence (2) embeds in a functorial long coexact sequence, and via Proposition 2.6 so too does (1). This sequence is useful in determining whether or not

$$C(A^*, A) \equiv S \backslash A^* \cup A \neq \varnothing.$$

It follows easily from [19, Proposition 2.9] that for each $t > 0$, $\langle N_1, N_2, N_3^{-t} \rangle$, $\langle N_1^t, N_2^t, N_3 \rangle$, and $\langle N_1^t, N_2^t, N_3^{-t} \rangle$ are index triples whenever $\langle N_1, N_2, N_3 \rangle$ is, and for the same repeller attractor pair. However, even if $\langle N_1, N_2, N_3 \rangle$ is nested, the above three need not be although $M_1 \supset M_2$, where $\langle M_1, M_2, M_3\rangle$ is any one of the three triples derived from $\langle N_1, N_2, N_3\rangle$. Hence, there is an inclusion induced commutative diagram

$$\begin{array}{ccccc} M_2/M_3 & \to & M_1/M_3 & \to & M_1/M_2 \\ \wr| & & \wr| & & \wr| \\ M_2'/M_3' & \to & M_1'/M_3' & \to & M_1'/M_2' \end{array}$$

where $M_1' \supset M_2' \supset M_3'$ is derived from $\langle M_1, M_2, M_3\rangle$ as in Proposition 2.6, and it is principally in regard to the triples $\langle M_1, M_2, M_3\rangle$ that the substitution of (2) for (1) occurs. The above diagram renders this harmless.

In connection with the long coexact sequence to be developed below, we shall be interested in diagrams of the form

$$\begin{array}{ccccc} N_2/N_3 & \longrightarrow & N_1/N_3 & \longrightarrow & N_1/N_2 \\ {\scriptstyle h_A}\downarrow & & {\scriptstyle h_S}\downarrow & & {\scriptstyle h_{A^*}}\downarrow \\ M_2/M_3 & \longrightarrow & M_1/M_3 & \longrightarrow & M_1/M_2 \end{array}$$

where $\langle N_1, N_2, N_3\rangle$ and $\langle M_1, M_2, M_3\rangle$ are (nested) triples relative to N and M respectively for an R–A pair (A^*, A) of S, where h_A, h_S, and h_{A^*} are morphisms in $\mathscr{I}(A)$, $\mathscr{I}(S)$, and $\mathscr{I}(A^*)$ respectively. Note that it is an abuse of notation to say that h_{A^*} is a morphism of $\mathscr{I}(A^*)$ because strictly speaking N_1/N_2 and M_1/M_2 are not objects of $\mathscr{I}(A^*)$. However, we shall say this anway as a gloss for the statement that there is a (homotopy) commutative diagram

$$\begin{array}{ccc} N_1\backslash U/N_2\backslash U & \simeq & N_1/N_2 \\ {\scriptstyle h'_{A^*}}\downarrow & & \downarrow{\scriptstyle h_{A^*}} \\ M_1\backslash V/M_2\backslash V & \simeq & M_1/M_2 \end{array}$$

where $A \subset V \subset M_2$, $A \subset U \subset N_2$ and V and U are open relative to M and N respectively so that $\langle N_1\backslash U, N_2\backslash U\rangle$ and $\langle M_1\backslash V, M_2\backslash V\rangle$ are index pairs for A^* relative to $N\backslash U$ and $M\backslash V$ respectively, and where h'_{A^*} is a morphism in $\mathscr{I}(A^*)$. In this regard it is useful to note that for each $t > 0$, there is an inclusion induced homeomorphism

$$(N_1\backslash U)^t/(N_2 \cup N_3)\backslash U \simeq N_1^t\backslash U/(N_2 \cup N_3)\backslash U$$

—the inclusion from left to right is obvious; that the induced map is onto follows from the positive invariance of N_2 relative to N_1.

3.2. THEOREM. *Let N be an isolating neighborhood for S and suppose that $\langle N_1, N_2, N_3\rangle$ is a nested index triple for the R–A pair (A^*, A) of S relative to N. Then there is a functorial long coexact sequence*

$$N_2/N_3 \xrightarrow{\iota} N_1/N_3 \xrightarrow{p} N_1/N_2 \xrightarrow{c} S(N_2/N_3) \xrightarrow{S(\iota)} S(N_1/N_3)$$
$$\xrightarrow{S(p)} S(N_1/N_2) \xrightarrow{S(c)} S^2(N_2/N_3) \cdots \tag{3}$$

where $S(\cdot)$ is reduced suspension functor, $S(X) = X \wedge S^1$, and c is the composite

$$\begin{aligned} N_1/N_2 &\simeq (N_1/N_3)/(N_2/N_3) \simeq N_1/N_3 \cup C^*(N_2/N_3)/C^*(N_2/N_3) \\ &\xrightarrow{p_0^{-1}} N_1/N_3 \cup C^*(N_2/N_3) \xrightarrow{k} N_1/N_3 \cup C^*(N_2/N_3)/(N_1/N_3) \\ &\simeq C^*(N_2/N_3)/(N_2/N_3) \simeq S(N_2/N_3) \end{aligned}$$

with p_0 and k being the indicated quotient maps and p_0^{-1} a homotopy inverse for p_0. The composite c is called the connection map of the triple.

In particular, (3) *is natural with respect to $\mathscr{I}(A)$, $\mathscr{I}(S)$ and $\mathscr{I}(A^*)$; i.e., if $\tilde N_1 \supset \tilde N_2 \supset \tilde N_3$ is another index triple for the R–A pair (A^*, A) of S, then there is a homotopy commutative infinite ladder*

$$\begin{tikzcd} N_2/N_3 \arrow[r, "\iota"] \arrow[d] & N_1/N_3 \arrow[r, "p"] \arrow[d] & N_1/N_2 \arrow[r, "c"] \arrow[d] & S(N_2/N_3) \arrow[r, "S(\iota)"] \arrow[d] & S(N_1/N_3) \arrow[r, "S(p)"] \arrow[d] & {} \\ \tilde N_2/\tilde N_3 \arrow[r, "\tilde\iota"] & \tilde N_1/\tilde N_3 \arrow[r, "\tilde p"] & \tilde N_1/\tilde N_2 \arrow[r, "\tilde c"] & S(\tilde N_2/\tilde N_3) \arrow[r, "S(\tilde\iota)"] & S(\tilde N_1/\tilde N_3) \arrow[r, "S(\tilde p)"] & {} \end{tikzcd}$$

where each of the first three vertical arrows on the left is the unique morphism in $\mathscr{I}(A)$, $\mathscr{I}(S)$, or $\mathscr{I}(A^*)$ respectively between the indicated objects and where each of the remaining vertical arrows is the appropriate suspension of one of these.

In particular some connection map is essential if, and only if, every connection map is essential.

3.3. COROLLARY. *If some connection map is essential then $S \neq A^* \cup A$.*

Proof. Suppose $S = A^* \cup A$. Since $A^* \cap A = \varnothing$, it follows that there are index pairs $N_1^* \supset N_2^*$ and $N_1 \supset N_2$ for A^* and A respectively with $N_1^* \cap N_1 = \varnothing$. Hence setting $M_1 = N_1^* \cup N_1$, $M_2 = N_2^* \cup N_1$, $M_3 = N_2^* \cup N_2$, $M_1 \supset M_2 \supset M_3$ is a nested index triple for the R–A pair (A^*, A) of S.

Moreover, $M_1/M_3 = N_1^*/N_2^* \vee N_1/N_2$, $M_1/M_2 = N_1^*/N_2^*$, and $M_2/M_3 = N_1/N_2$ so that

$$M_1/M_3 \cup C^*(M_2/M_3) = N_1^*/N_2^* \vee C^*(N_1/N_2)$$

and

$$M_1/M_3 \cup C^*(M_2/M_3)/(M_1/M_3) = N_1^*/N_2^* \vee C^*(N_1/N_2)/N_1^*/N_2^* \vee N_1/N_2.$$

Thus the map k in the definition of c is the quotient map

$$N_1^*/N_2^* \vee C^*(N_1/N_2) \to N_1^*/N_2^* \vee C^*(N_1/N_2)/N_1^*/N_2^* \vee N_1/N_2.$$

Let H be a base point preserving homotopy from the identity map of $N_1^*/N_2^* \vee C^*(N_1/N_2)$ which fixes N_1^*/N_2^* and contracts $C^*(N_1/N_2)$ to its vertex. Then k is homotopic to $k \circ H_1$ which is clearly a constant map; hence c is inessential. ∎

The following lemma is needed for the Proof of Theorem 3.2, and the notation is that of Theorem 3.2.

3.4. LEMMA. *The quotient map*

$$p_0 : N_1/N_3 \cup C^*(N_2/N_3) \to N_1/N_3 \cup C^*(N_2/N_3)/C^*(N_2/N_3)$$

is a homotopy equivalence of pointed spaces.

Proof. By Lemma 3.1 it suffices to show that there is a closed N_1/N_3-neighborhood, Q, of N_2/N_3, a base point preserving deformation D of N_1/N_3 such that $D \mid Q \times I$ is a weak deformation retraction of Q into N_2/N_3, and a continuous function $\varphi: N_1/N_3 \to [0, 1]$ with $N_2/N_3 \subset \varphi^{-1}(1)$ and $N_1/N_3 \backslash Q \subset \varphi^{-1}(0)$.

The candidate for the neighborhood is N_2^{-t}/N_3 for some $t > 0$, where $N_2^{-t} \equiv \{\gamma \in N_1 : \exists t',\ 0 \leqslant t' \leqslant t \text{ and } \gamma \cdot [0, t'] \subset N_1 \text{ and } \gamma \cdot t' \in N_2\}$. It is immediate that N_2^{-t}/N_3 is a closed neighborhood of N_2/N_3 if it can be shown that N_2^{-t} is a closed neighborhood of N_2 since $N_2 \supset N_3$. Assuming this for the moment, let $f^t: N_1/N_3 \times I \to N_1/N_3$ be defined as in [19, Proposition 3.1] ($\langle N_1, N_3 \rangle$ is an index pair for S). Then f^t is a base point preserving deformation of N_1/N_3 such that the restriction of f^t to N_2^{-t}/N_3 is a weak deformation retraction into N_2/N_3; and since N_1/N_3 is a normal space (being compact and Hausdorff), Urysohn's lemma assures the existence of a continuous $\varphi: N_1/N_3 \to [0, 1]$ with $N_2/N_3 \subset \varphi^{-1}(1)$ and $N_1/N_3 \backslash N_2^{-t}/N_3 \subset \varphi^{-1}(0)$.

To finish showing that p_0 is an equivalence it remains to show that for some $t > 0$, N_2^{-t} is an N_1-neighborhood of N_2. Unfortunately, it does not follow immediately from [19, Proposition 2.9(2)] that N_2^{-t} is a closed N_1-

neighborhood of N_2 since $\langle N_1, N_2\rangle$ is not an index pair for A^* (N_1 does not isolate A^*). However, by the definition of an index triple there exists U, N_1-open, with $A \subset U \subset N_2$ so that $\langle N_1\backslash U, N_2\backslash U\rangle$ is an index pair for A^* relative to $N_1\backslash U$. By [19, Proposition 2.9(2)], a $t>0$ can be chosen so that $(N_2\backslash U)^{-t}$ is a closed $N_1\backslash U$-neighborhood of $N_2\backslash U$. Choosing such a $t>0$, observe that $N_2^{-t}\backslash U = (N_2\backslash U)^{-t}$: if $\gamma \in (N_2\backslash U)^{-t}$, for some t', $0 \leqslant t' \leqslant t$ and $\gamma\cdot[0,t'] \subset N_1\backslash U \subset N_1$ and $\gamma\cdot t' \in N_2\backslash U \subset N_2$ whence $\gamma \in N_2^{-t}\backslash U$; and on the other hand, if $\gamma \in N_2^{-t}\backslash U$, then $\gamma \in N_1\backslash U$ and for some t', $0 \leqslant t' \leqslant t$ and $\gamma\cdot[0,t'] \subset N_1$ and $\gamma\cdot t' \in N_2$ whence if $\gamma\cdot[0,t'] \subset N_1\backslash U$ it follows immediately that $\gamma \in (N_2\backslash U)^{-t}$, and if not, taking $t'' = \sigma \mid N_1\backslash U(\gamma)$, by the exit property of the index pair $\langle N_1\backslash U, N_2\backslash U\rangle$, it follows that $0 \leqslant t'' < t' \leqslant t$ and $\gamma\cdot[0,t''] \subset N_1\backslash U$ and $\gamma\cdot t'' \in N_2\backslash U$ whence $\gamma \in (N_2\backslash U)^{-t}$.

Then for some N_1-open V,

$$N_2\backslash U \subset V \cap N_1\backslash U \subset (N_2\backslash U)^{-t} = N_2^{-t}\backslash U;$$

whence $N_2 \subset V \cup U \subset N_2^{-t}$ (recall $U \subset N_2 \subset N_2^{-t}$) which shows that N_2^{-t} is an N_1-neighborhood of N_2, and it is closed since

$$U \subset \mathrm{cl}_\Phi(U) \subset N_2 \subset N_2^{-t},$$

and since $N_2^{-t}\backslash U$ is closed being equal to $(N_2\backslash U)^{-t}$, this latter being closed by [19, Proposition 2.9] so that

$$\mathrm{cl}(N_2^{-t}) \subset N_2^{-t}\backslash U \cup \mathrm{cl}(U) \subset N_2^{-t},$$

which completes showing that p_0 is an equivalence. ∎

Proof of 3.2. It is a standard result of homotopy theory ([24, VII] or [25, II]) that given a map $f\colon (X, x_0) \to (Y, y_0)$ there is a functorial long coexact sequence in $\mathscr{T}^*$

$$X \xrightarrow{f} Y \xrightarrow{i} C_f^* \xrightarrow{k'} SX \xrightarrow{S(f)} SY \xrightarrow{S(i)} S(C_f^*) \xrightarrow{S(k')} S^2X \xrightarrow{S^2(f)} \cdots, \tag{5}$$

where k' is the composite

$$C_f^* \xrightarrow{k} C_f^*/Y \overset{\psi}{\simeq} C^*X/X \overset{\lambda}{\simeq} SX.$$

It is also well known (and is usually left as an exercise) that if the map f is a cofibration; i.e., is (essentially) an inclusion of a closed subspace which has the homotopy extension property with respect to every space; then sequence (5) leads to a functorial long coexact sequence in $\mathscr{H}\mathscr{T}^*$

$$X \xrightarrow{f} Y \xrightarrow{p} Y/X \xrightarrow{c} SX \xrightarrow{Sf} SY \xrightarrow{Sp} S(Y/X) \xrightarrow{Sc} S^2X \xrightarrow{S^2(f)} \cdots \tag{6}$$

where p is the quotient map and c is the composite

$$Y/X \simeq C_f^*/C^*X \xrightarrow{p_0^{-1}} C_f^* \xrightarrow{k'} SX$$

with p_0^{-1} any homotopy inverse of the quotient $p_0\colon C_f^* \to C_f^*/C^*X$, and the entire purpose of assuming that the inclusion f is a cofibration is to guarantee that p_0 is indeed a homotopy equivalence [cf. 15, Sections 18.9–18.12; 24, VII, Exercises A1–A3].

Hence the existence and coexactness of sequence (3) follows from this exercise when applied to the inclusion of the closed subspace $\iota\colon N_2/N_3 \to N_1/N_3$ using Lemma 3.4 and the natural identification $(N_1/N_3)/(N_2/N_3) \simeq N_1/N_2$.

Because the construction of sequence (6) is functorial and, in particular, because the connecting map c is a natural transformation, to show that there is an infinite homotopy commutative ladder as described by (4) it suffices to show that there is a homotopy commutative diagram

$$\begin{array}{ccccc} N_2/N_3 & \longrightarrow & N_1/N_3 & \longrightarrow & N_1/N_2 \\ {\scriptstyle h_A}\downarrow & & {\scriptstyle h_S}\downarrow & & {\scriptstyle h_{A^*}}\downarrow \\ \tilde{N}_2/\tilde{N}_3 & \longrightarrow & \tilde{N}_1/\tilde{N}_3 & \longrightarrow & \tilde{N}_1/\tilde{N}_2 \end{array} \tag{7}$$

where h_A, h_S, h_{A^*} are morphisms in $\mathscr{I}(A)$, $\mathscr{I}(S)$, and $\mathscr{I}(A^*)$ respectively, and where the horizontal rows are the initial segments of sequences (3) corresponding to the two nested index triples $\langle N_1, N_2, N_3\rangle$ and $\langle \tilde{N}_1, \tilde{N}_2, \tilde{N}_3\rangle$ for the R–A pair (A^*, A) of S. For two such triples, set $M_i = N_i \cap \tilde{N}_i$, $i = 1, 2, 3$. It is then easy to check that $M_1 \supset M_2 \supset M_3$ is a nested index triple for (A^*, A). There is then a commutative diagram of inclusions

$$\begin{array}{ccccc} N_2/N_3 & \longrightarrow & N_1/N_3 & \longrightarrow & N_1/N_2 \\ \uparrow & & \uparrow & & \uparrow \\ M_2/M_3 & \longrightarrow & M_1/M_3 & \longrightarrow & M_1/M_2 \\ \downarrow & & \downarrow & & \downarrow \\ \tilde{N}_2/\tilde{N}_3 & \longrightarrow & \tilde{N}_1/\tilde{N}_2 & \longrightarrow & \tilde{N}_1/\tilde{N}_2 \end{array}$$

with all arrows being functorially inclusion induced. Because each vertical arrow represents a morphism in a connected simple system, there is thus a commutative diagram (7) above in $\mathscr{H}\mathscr{T}^*$ with each vertical arrow being the unique morphism in $\mathscr{I}(A)$, $\mathscr{I}(S)$, or $\mathscr{I}(A^*)$ respectively, between its domain and range. ∎

Remark. There are, in fact, representatives h_A, h_S, h_{A^*} in $\mathscr{I}(A)$, $\mathscr{I}(S)$, $\mathscr{I}(A^*)$ respectively that make diagram (7) above commutative pointwise, not just up to homotopy, but the proof is much more difficult and quite laborious. As we have no need for this fact we shall not prove it.

3.5. Definition of the Connection Index. $\mathscr{J}(S, A^*, A)$. If S is a Φ-isolated invariant set for a local semi-flow $\Phi \subset \Gamma_0$ and (A^*, A) is an R–A pair of S, define $\mathscr{J}(S; A^*, A)$ to be the category whose objects are long coexact sequences as in (3) of Theorem 3.2 for nested index triples $N_1 \supset N_2 \supset N_3$ and whose morphisms are infinite ladders as in (4) of Theorem 3.2. Because $\mathscr{I}(A)$, $\mathscr{I}(A^*)$, and $\mathscr{I}(S)$ are connected simple systems, it is immediate that $\mathscr{J}(S; A^*, A)$ is too.

Remark. For a Morse Decomposition $\{M_1, M_2, \dots, M_n\}$ corresponding to a sequence of attractors $S = A_0 \supset A_1 \supset \cdots \supset A_n = \varnothing$, with the notation of 2.7, there is a sequence of inclusions

$$* \subset N_n/N_{n+1} \subset N_{n-1}/N_{n+1} \subset \cdots \subset N_2/N_{n+1} \subset N_1/N_{n+1},$$

where $\langle N_i, N_{i+1}, N_{n+1} \rangle$ is a nested index triple for the R–A pair (M_i, A_i) of A_{i-1}, $i = 1, \dots, n$. The appropriate connected simple system $\mathscr{H}(S; M_1, \dots, M_n)$ to construct here is probably the spectral (co)homology sequence of the above space filtration of N_1/N_3. The E^1 terms then correspond to the indices of the M_i, N_i/N_{i+1}.

4. The Splitting Class μ for $\mathscr{J}(S; A^*, A)$

Before giving the formal development of the definition of the splitting class we indicate the geometric motivation for the definition. Consider a pair of hyperbolic critical ponts (A^*, A) for a flow in the plane as sketched in Fig. 2, which as the notation suggests form an R–A pair in the isolated invariant set $S = A^* \cup A$.

Assume there exists a connected isolating block N_1 (the flow is transverse to the boundary) for S as schematically indicated in Fig. 2 which is the union of two "squares" B_1 and B_2 which are isolating blocks for A^* and A respectively [cf. 13; 21]. As drawn the two vertical components of the boundary of B_1 comprise its exit set, and the horizontal components of the boundary of B_2 comprise its exit set. Also B_1 and B_2 intersect in the right-hand component of the exit set of B_1 (which is the left-hand component of the entrance set of B_2), and note that the exit set of N_1 consists of the left-hand vertical component of the exit set of B_1 together with the exit set of B_2. Setting N_2 equal to the union of B_2 and the left-hand component of the exit

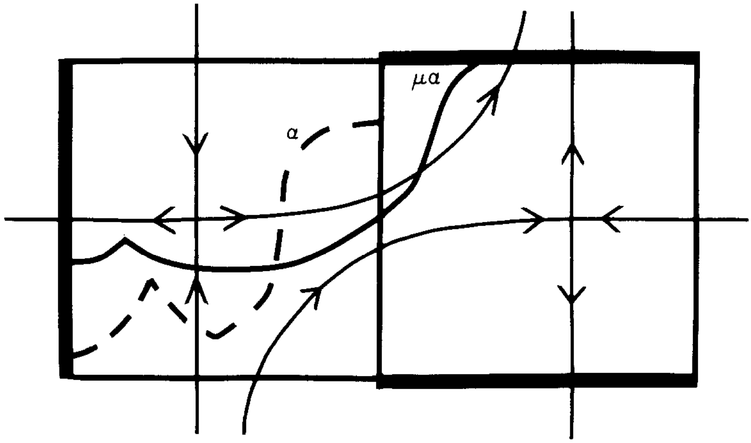

FIG. 2. A schematic of the splitting map on homology.

set of B_1 and setting N_3 equal to the exit set of N_1, note that $N_1 \supset N_2 \supset N_3$ is an index triple for (A^*, A).

Now any arc α in B_1 which has each endpoint in a different component of the exit set of B_1 represents the generating class of the 1–dimensional homology of N_1/N_2, the index of A^*, which has the homotopy type of S^1. Note that one such representing arc is that portion of the local unstable manifold of A^* relative to N_1 (in Lemma 2.1 this was designated $\mathscr{A}_1^-$) which lies in B_1. Note too that the local unstable manifold of A^* relative to N_1 represents non-trivial homology in N_1/N_3, the index of S which has the homotopy type of $S^1 \vee S^1$, a figure eight. The idea of the splitting map is that arcs (more generally singular chains) which represent the same homology in N_1/N_2 as does $\mathscr{A}_1^- \cap B_1$ get carried by the flow in finite time into an arc which represents the same homology in N_1/N_3 as does $\mathscr{A}_1^-$. The next proposition gives the technical necessary and sufficient condition actually needed to define the splitting map. The idea roughly is that given an arc in B_1 representing non-zero homology in N_1/N_2, perhaps this arc intersects the local stable manifold of A relative to N_1 (this was designated $\mathscr{A}_2^+$ in Lemma 2.1), in which case following the arc for finite time under the action of the flow does not give the appropriate homology class in N_1/N_3; however, if first the arc is followed for a long enough time stopping those points which hit the boundary of B_1, and if from this image are excised all those points in the exit set of B_1 which lie outside a neighborhood of $\mathscr{A}_1^-$ which is disjoint frop $\mathscr{A}_2^+$, then the remainder is carried by the flow in finite time into an arc which represents the same homology in N_1/N_3 as does $\mathscr{A}_1^-$.

4.1. PROPOSITION. *Let S be a Φ-isolated invariant set and $\langle N_1, N_2, N_3\rangle$*

a nested index triple for the R–A pair (A^*, A) *of* S. *Then* $S = A^* \cup A$ *if, and only if,* (*) *for each* N_1*-open* U *with* $A \subset U \subset N_2$, *there exists* $T > 0$ *so that for each* $t \geqslant T$,

$$(N_1 \backslash U)^t \cap N_2 \backslash U \subset N_3^{-t}$$

hence for $t \geqslant T$ *there is an inclusion induced map*

$$(N_1 \backslash U)^t / N_2 \backslash U \to N_1 / N_3^{-t}.$$

Proof. First sufficiency of condition (*) is shown. Suppose $\varnothing \neq C(A^*, A) \equiv S \backslash (A^* \cup A)$; choose $\gamma \in C(A^*, A)$; and suppose $A \subset U \subset N_2$, U N_1-open. Because $\omega(\gamma) \subset A$ and N_2 isolates A, for some $t > 0$, $\gamma \cdot t \in N_2$, and set

$$\gamma_0 = \gamma \cdot t \cdot -\sigma^* \mid N_2(\gamma \cdot t) \in \partial_{N_1} N_2 \subset N_2.$$

Because $\gamma_0 \in S$, $\gamma_0 \cdot \mathbf{R}^- \subset N_1$; and as $U \subset \operatorname{int}_{N_1}(N_2)$ and as N_2 is positively invariant relative to N_1, it follows that $\gamma_0 \cdot \mathbf{R}^- \subset N_1 \backslash U$; whence for each $t > 0$,

$$\gamma_0 \in (N_1 \backslash U)^t \cap N_2 \backslash U.$$

However, for each $t > 0$, $N_2 \backslash N_3^{-t}$ is a Φ-neighborhood of A, and since $\omega(\gamma) \subset A$ and N_3^{-t} is positively invariant relative to N_1, it follows that $\gamma_0 \notin N_3^{-t}$; thus $(N_1 \backslash U)^t \cap N_2 \backslash U \not\subset N_3^{-t}$.

Conversely, suppose $S = A^* \cup A$ and U is N_1 – open with $A \subset U \subset N_2$. Using the notation of Lemma 2.1 (N_1 is the isolating neighborhood), $\mathscr{A}_1^-$, $\mathscr{A}_1^+$, $\mathscr{A}_2^-$, and $\mathscr{A}_2^+$ are closed relative to N_1 and $(\mathscr{A}_1^- \cup \mathscr{A}_1^+) \cap (\mathscr{A}_2^- \cup \mathscr{A}_2^+) = \varnothing$, and since N_2 isolates A and is positively invariant relative to N_1, also $A^+(N_2) = N_2 \cap \mathscr{A}_2^+$. Then by normality of N_1, choose disjoint N_1 – open sets $V \supset \mathscr{A}_2^-$ and $V^* \supset \mathscr{A}_1^-$ satisfying

$$(\operatorname{cl}_{N_1}(V^*) \cup \mathscr{A}_1^+) \cap (\operatorname{cl}_{N_1}(V) \cup \mathscr{A}_2^+) = \varnothing,$$

and without loss of generality also assume that $\mathscr{A}_2^+ \cap \operatorname{cl}_{N_1}(V) \backslash U = \varnothing$, for if not replace V by $V \backslash K$ where $K \equiv \operatorname{cl}_{N_1}(V) \cap \mathscr{A}_2^+ \backslash U$. Then by the sublemma of Lemma 2.1, $A^-(N_1) \subset V^* \cup V$, and it follows that $\sigma^* \mid N_1 \backslash (V^* \cup V)$ is bounded; i.e.; for some $T_0 > 0$, $t \geqslant T_0$ implies $\gamma \cdot [-t, 0] \not\subset N_1$ for each $\gamma \in N_1 \backslash (V^* \cup V)$; hence $t \geqslant T_0$ implies $N_1^t \subset V^* \cup V$. Also,

$$\operatorname{cl}_{N_1}(V^* \cup V) \cap N_2 \backslash U \subset N_2 \backslash A^+(N_2)$$

for

$$\operatorname{cl}_{N_1}(V^*) \cap A^+(N_2) \subset \operatorname{cl}_{N_1}(V^*) \cap \mathscr{A}_2^+ \cap N_2 \subset \operatorname{cl}(V^*) \cap \mathscr{A}_2^+ = \varnothing$$

and

$$\mathrm{cl}_{N_1}(V) \cap A^+(N_2) \cap N_2 \backslash U = N_2 \cap \mathscr{A}_2^+ \cap \mathrm{cl}_{N_1}(V) \backslash U \subset \mathscr{A}_2^+ \cap \mathrm{cl}_{N_1}(V) \backslash U = \varnothing;$$

it follows that

$$\sigma \mid \mathrm{cl}_{N_1}(V^* \cup V) \cap N_2 \backslash U$$

is bounded. Therefore, for some $T_1 > 0$, $t \geqslant T_1$ implies $\gamma \cdot [0, t] \not\subset N_2$ for each $\gamma \in \mathrm{cl}_{N_1}(V^* \cup V) \cap N_2 \backslash U$; whence by the exit property for index pairs, $t \geqslant T_1$ implies that

$$\mathrm{cl}(V^* \cup V) \cap N_2 \backslash U \subset N_3^{-t}.$$

Set $T = \max\{T_0, T_1\}$; then $t \geqslant T$ implies $N_1^t \cap N_2 \backslash U \subset N_3^{-t}$, and as $(N_1 \backslash U)^t \subset N_1^t$, $t \geqslant T$, implies $(N_1 \backslash U)^t \cap N_2 \backslash U \subset N_3^{-t}$, showing that (*) holds; that there is an inclusion induced map is then immediate. ∎

4.2. Definition and Proposition. Let S be an isolated invariant set relative to Φ, let $\langle N_1, N_2, N_3 \rangle$ be a nested index triple for the R–A pair (A^*, A) of S, and let U be N_1-open with $A \subset U \subset N_2$.

For each $t \in \mathbf{R}^+$ such that there is an inclusion induced map

$$(N_1 \backslash U)^t / N_2 \backslash U \to N_1 / N_3^{-t},$$

define $\mu_{t,U}: N_1/N_2 \to N_1/N_3$ to be the composition

$$N_1/N_2 \simeq N_1 \backslash U / N_2 \backslash U \xrightarrow{\hat{f}_1^t} (N_1 \backslash U)^t / N_2 \backslash U \to N_1 / N_3^{-t} \xrightarrow{g} N_1^t / N_3 \xrightarrow{\iota} N_1 / N_3.$$

Then,

(1) If for some $t \in \mathbf{R}^+$, $\mu_{t,U}$ is defined, then for each $s \geqslant t$, $\mu_{s,U}$ is defined, and for each N_1-open V with $A \subset V \subset N_2$, for some $t' \in \mathbf{R}^+$, $\mu_{t',V}$ is defined.

(2) If $s, t \in \mathbf{R}^+$ and $\mu_{t,U}$ and $\mu_{s,U}$ are both defined then $\mu_{t,U}$ is homotopic to $\mu_{s,U}$.

(3) If $\mu_{t,U}$ is defined and $A \subset V \subset N_2$, V N_1-open, and $\mu_{s,V}$ is defined then $\mu_{t,U}$ is homotopic to $\mu_{s,V}$;

(4) where $p: N_1/N_3 \to N_1/N_2$ is the inclusion induced map, if $\mu_{t,U}$ is defined, then $p \circ \mu_{t,U}$ is homotopic to $1_{N_1/N_2}$; hence $\mu_{t,U}$ is a splitting of the coexact sequence

$$N_2/N_3 \to N_1/N_3 \xrightarrow{p} N_1/N_2.$$

Proof. Suppose $\mu_{t,U}$ is defined and $s \geqslant t$. Then by the preceding proposition, $(N_1 \backslash U)^t \cap N_2 \backslash U \subset N_3^{-t}$, and because $s \geqslant t$,

$$(N_1 \backslash U)^s \cap N_2 \backslash U \subset (N_1 \backslash U)^t \cap N_2 \backslash U \subset N_3^{-t} \subset N_3^{-s};$$

whence the preceding proposition implies that $\mu_{s,U}$ is defined. Also if $A \subset V \subset N_2$, V N_1-open then it follows immediately from the preceding proposition that for some $t' \in \mathbf{R}^+$, $\mu_{t',V}$ is defined. This shows (1).

Suppose $\mu_{s,U}$ and $\mu_{t,U}$ are both defined, and for definiteness assume $s \geqslant t$. Consider the diagram

$$\begin{tikzcd}
N_1 \backslash U/N_2 \backslash U \arrow[r, "f_1^t"] \arrow[dr, "f_1^s"'] & (N_1 \backslash U)^t/N_2 \backslash U \arrow[r] \arrow[d, phantom, "\mathrm{I}" very near start, shift right=3em] \arrow[dr, phantom, "\mathrm{II}"] & N_1/N_3^{-t} \arrow[r, "g"] \arrow[d] \arrow[dr, phantom, "\mathrm{III}"] & N_1^t/N_3 \arrow[r] & N_1/N_3 \\
& (N_1 \backslash U)^s/N_2 \backslash U \arrow[r] \arrow[u] & N_1/N_3^{-s} \arrow[r] & N_1^s/N_3 \arrow[u] \arrow[ur] \arrow[u, phantom, "\mathrm{IV}", shift left=3em] &
\end{tikzcd}$$

where the vertical arrows are all inclusion induced. Then Triangle I is homotopy commutative because all the arrows are maps of the connected simple system $\mathscr{I}(A^*)$ and because there is a unique homotopy class in $\mathscr{I}(A^*)$ from $N_1 \backslash U/N_2 \backslash U$ to $(N_1 \backslash U)^t/N_2 \backslash U$. Similarly Square III and Triangle IV are homotopy commutative as the arrows in these diagrams are maps of the connected simple system $\mathscr{I}(S)$. Finally, because all the arrows in Square II are induced by functorial inclusions, it follows that Square II is commutative. Then an easy diagram chase shows that the two paths on the perimeter of the diagram from $N_1 \backslash U/N_2 \backslash U$ to N_1/N_3 are the same up to homotopy. Hence $\mu_{t,U}$, the top row, is homotopic to $\mu_{s,U}$, the lower path, showing that (2) holds.

Suppose $\mu_{t,U}$ is defined and also suppose $A \subset V \subset N_2$, V N_1-open, and $\mu_{s,V}$ is defined. Then $A \subset U \subset V \subset N_2$ and $U \cap V$ is N_1-open. By (1) choose r large enough to that $\mu_{r,U}$, $\mu_{r,V}$, and $\mu_{r,U\cap V}$ are defined, and setting $W = U \cap V$, consider the diagram

$$\begin{tikzcd}
N_1 \backslash U/N_2 \backslash U \arrow[r, "f_1^r"] \arrow[d] \arrow[dr, phantom, "\mathrm{I}"] & (N_1 \backslash U)^r/N_2 \backslash U \arrow[r] \arrow[d] & N_1/N_3^{-r} \\
N_1 \backslash W/N_2 \backslash W \arrow[r, "f_1^r"] & (N_1 \backslash W)^r/N_2 \backslash W \arrow[ur] \arrow[ur, phantom, "\mathrm{II}", shift left=1.5em] &
\end{tikzcd}$$

where the unlabeled arrows are inclusion induced. Square I is homotopy commutative because all the arrows are maps for the simple system $\mathscr{I}(A^*)$, and Triangle II is commutative because all the arrows in it are induced by functorial inclusions; hence the diagram is homotopy commutative and it follows immediately that $\mu_{r,U} \sim \mu_{r,U\cap V}$. Similarly $\mu_{r,U\cap V} \sim \mu_{r,V}$ so that

$\mu_{r,U} \sim \mu_{r,V}$ by transitivity; then by (2) it follows that $\mu_{t,U} \sim \mu_{s,V}$ showing that (3) holds.

Suppose $\mu_{t,U}$ is defined, and let $e: N_1\backslash U/N_2\backslash U \simeq N_1/N_2$ be the excision induced homeomorphism. To show that $p \circ \mu_{t,U}$ is homotopic to $1_{N_1/N_2}$, it clearly suffices to show that $e^{-1} \circ p \circ \mu_{t,U} \circ e$ is homotopic to $1_{N_1\backslash U/N_2\backslash U}$. As $e^{-1} \circ p \circ \mu_{t,U} \circ e$ is given by the composition

$$N_1\backslash U/N_2\backslash U \xrightarrow{\hat{f}_1^t} (N_1\backslash U)^t/N_2\backslash U \xrightarrow{k} N_1/N_3^{-t} \xrightarrow{g} N_1^t/N_3 \xrightarrow{\iota} N_1/N_3$$
$$\xrightarrow{p} N_1/N_2 \xrightarrow{e^{-1}} N_1\backslash U/N_2\backslash U,$$

calling this composition φ, it is clear that for each $\gamma \in N_1\backslash U$ either

(i) $\varphi[\gamma] = [\gamma \cdot 2t]$ or

(ii) $\varphi[\gamma] = [N_2\backslash U]$;

however, the end map of the deformation,

$$f_1^{2t}: N_1\backslash U/N_2\backslash U \to N_1\backslash U/N_2\backslash U,$$

is defined by $f_1^{2t}[\gamma] = [\gamma \cdot 2t]$ if $\gamma \in (N_1\backslash U)\backslash(N_2\backslash U)^{-2t}$ and $f_1^{2t}[\gamma] = [N_2\backslash U]$ otherwise, and the deformation f^{2t} provides a homotopy from $1_{N_1\backslash U/N_2\backslash U}$ to f_1^{2t}. Hence it suffices to show that $\varphi = f_1^{2t}$.

Accordingly, if $\gamma \in (N_1\backslash U)\backslash(N_2\backslash U)^{-2t}$, then

$$\gamma \cdot [0, 2t] \subset (N_1\backslash U)\backslash(N_2\backslash U) = N_1\backslash N_2 \subset N_1\backslash N_3$$

and it follows easily from this that $\varphi[\gamma] = [\gamma \cdot 2t] \neq [N_2\backslash U]$; whereas if $\gamma \in (N_1\backslash U) \cap (N_2\backslash U)^{-2t}$, then for some s, $0 \leqslant s < 2t$ and $\gamma \cdot [0, s] \subset N_1\backslash U$ and $\gamma \cdot s \in N_2\backslash U$; and if $0 \leqslant s \leqslant t$, then $\hat{f}_1^t[\gamma] = [N_2\backslash U]$ so that $\varphi[\gamma] = [N_2\backslash U]$ as $\hat{f}_1^t$ is a factor of φ. On the other hand if $t < s \leqslant 2t$, either $\gamma \cdot t \in N_3^{-t}$ or $\gamma \cdot t \notin N_3^{-t}$, and if it is, then $k\hat{f}_1^t[\gamma] = [N_3^{-t}]$ whence it follows that $\varphi[\gamma] = [N_2\backslash U]$, and if it is not, then $\gamma \cdot [0, 2t] \subset N_1\backslash N_3$, but as $\gamma \cdot s \in N_2\backslash U \subset N_2$ and as N_2 is positively invariant relative to N_1, it follows that $\gamma \cdot 2t \in N_2$, hence that

$$p\,\iota\, gk\hat{f}_1^t[\gamma] = [N_2]$$

and in this case too $\varphi[\gamma] = [N_2\backslash U]$, which completes showing that φ coincides with f_1^{2t}. ∎

4.3. Definition of the Splitting Class. With the notation as in the previous subsection, if for some $t \geqslant 0$, and for some N_1-open U, $A \subset U \subset N_2$ and

$$\mu_{t,U}: N_1/N_2 \to N_1/N_3$$

is defined, then define $\mu \in [N_1/N_2, N_1/N_3]$ to be the homotopy class of $\mu_{t,U}$. By the previous proposition, the definition is independent of U and t within the restrictions that U is N_1-open, $A \subset U \subset N_2$, and $t > 0$ is large enough depending upon U. μ will be called the splitting class of the sequence $N_2/N_3 \to N_1/N_3 \to N_1/N_2$.

4.4. PROPOSITION. *Let (A^*, A) be an R–A pair in $S = A^* \cup A$, and suppose $N_1 \supset N_2 \supset N_3$ and $M_1 \supset M_2 \supset M_3$ are index triples for (A^*, A). Then there is a homotopy commutative diagram*

$$\begin{array}{ccc} N_1/N_3 & \xrightarrow{\mu_N} & N_1/N_2 \\ {\scriptstyle h_S}\uparrow & & \uparrow{\scriptstyle h_A} \\ M_1/M_3 & \xrightarrow{\mu_M} & M_1/M_2 \end{array}$$

where μ_N and μ_M are the splitting classes for

$$N_2/N_3 \to N_1/N_3 \to N_1/N_2$$

and

$$M_2/M_3 \to M_1/M_3 \to M_1/M_2$$

respectively, and where h_S and h_A are respectively morphisms of $\mathscr{I}(S)$ and $\mathscr{I}(A)$. Hence μ is natural relative to $\mathscr{J}(S; A^*, A)$.

Proof. By the uniqueness of the morphisms between two objects in the connected simple system and since all are homotopy equivalences, by taking intersections $M_i \cap N_i$ $(i = 1, 2, 3)$, we may assume $M_i \subset N_i$ $(i = 1, 2, 3)$, and hence the diagram upon expansion is given by

$$\begin{array}{ccccccccccc} N_1/N_2 & \simeq & N_1\backslash U/N_2\backslash U & \to & (N_1\backslash U)^t/N_2\backslash U & \to & N_1/N_3^{-t} & \to & N_1^t/N_3 & \to & N_1/N_3 \\ \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\ M_1/M_2 & \simeq & M_1\backslash U/M_2\backslash U & \to & (M_1\backslash U)^t/M_2\backslash U & \to & M_1/M_3^{-t} & \to & M_1^t/M_3 & \to & M_1/M_3 \end{array}$$

where U is Φ-open and $A \subset U \subset M_2 \subset N_2$ and $t > 0$ is large enough so that $\mu_{t,U}^M$ and $\mu_{t,U}^N$ are both defined. As, all the vertical arrows can be assumed to be induced by functorial inclusions, it is trivial to verify that the diagram is commutative. ∎

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