General Topology and its Applications 2 (1972) 67–73. North-Holland Publishing Company

A TOPOLOGICAL CHARACTERIZATION OF PRODUCTS OF COMPACT TOTALLY ORDERED SPACES

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Received 26 May 1971

AMS Subj. Class.: Primary 5452, 5425, 5456; Secondary 0610

compact totally ordered spaces
topological products
2-compactness
supercompactness

1. Introduction

In [1], a topological characterization of the n-cell, I^n, and Hilbert cube, I^∞, is presented and commented upon. The authors realized that the idea of the proof can be used to obtain a more general result as expressed in the title of this paper. Moreover, the proof has been shortened and the presentation improved.

The main theorem (Theorem 2.10) states that the existence of an open subbase of a special kind is required (a so-called 2-compact subbase, see the definitions below). About the strength of this condition, it may be remarked that *every* compact metrizable space has a subbase which satisfies condition (S.1) below by O'Connor's Theorem [2], and a subbase satisfying (S.2) vacuously (by using open spheres of small radius). It is precisely the existence of a subbase satisfying both conditions simultaneously which forces a T_1 space to be the product of compact totally ordered spaces.

2. Main result

Definition 2.1. A family S of subsets of a set X is *2-ccompact* (read: 2-c-compact) iff

(S.1) every cover of X by elements of S has a two-element subcover; and

(S.2) if $S_0 \cup S_1 = X = S_0 \cup S_2$ with $S_i \in S$ for $i = 0, 1, 2$, then either $S_1 \subset S_2$ or $S_2 \subset S_1$ (i.e., S_1 and S_2 are *comparable* by inclusion).

Definition 2.2. A topological space X is *2-ccompact* iff there exists an open subbase S which generates the topology on X and is a 2-ccompact family. Like Bourbaki, but unlike Kelley, we allow $S = \emptyset$ as a subbase for the trivial topology on X. Thus $S = \emptyset$ is a 2-ccompact subbase for a one-point space. When we wish to make the 2-ccompact subbase explicit, we say that "(X, S) is a 2-ccompact space".

Remark. Every 2-ccompact space is compact by (S.1) and Alexander's subbase theorem.

Convention. Note that S and $S\backslash\{\emptyset, X\}$ generate the same topology, and that S is 2-ccompact iff $S\backslash\{\emptyset,X\}$ is 2-ccompact. We shall assume without loss of generality that $\emptyset \notin S$ and $X \notin S$. This convention eliminates some trivial cases in the exposition.

Proposition 2.3. The product of 2-ccompact spaces is 2-ccompact.
Proof. Given a family $\{(X_\alpha, S_\alpha) | \alpha \in A\}$ of 2-ccompact spaces. Let $X = \Pi_{\alpha\in A} X_\alpha$ and let $S = \{\pi_\alpha^{-1}[S] | S \in S_\alpha \text{ and } \alpha \in A\}$. It is easily verified that S is a subbase for the usual product topology on X. If $C \subset S$ covers X, then there exists some $\alpha_0 \in A$ such that C_{α_0} covers X_{α_0}, where $C_\alpha = \{S \in S_\alpha | \pi_\alpha^{-1}[S] \in C\}$. For otherwise we may choose $x \in X \setminus \bigcup C$ so that $\pi_\alpha(x) \in X_\alpha \setminus \bigcup C_\alpha$ for every α. Thus there exist elements S and S' in C_{α_0} such that $S \cup S' = X_{\alpha_0}$. Hence $\{\pi_{\alpha_0}^{-1}[S], \pi_{\alpha_0}^{-1}[S']\}$ is a subcover of C, and (S.1) holds. If $\pi_\alpha^{-1}[S_0] \cup \pi_\beta^{-1}[S_1] = X = \pi_\alpha^{-1}[S_0] \cup \pi_\gamma^{-1}[S_2]$, then, as above, we can see that $\alpha = \beta = \gamma$ and $S_0 \cup S_1 = X_\alpha = S_0 \cup S_2$. Hence, since S_1 and S_2 are comparable, so are $\pi_\beta^{-1}[S_1]$ and $\pi_\gamma^{-1}[S_2]$, and (S.2) holds.

Proposition 2.4. Let (X, S) be a non-degenerate 2-ccompact T_1 space. Then, if $x \in S \in S$, there exists $S' \in S$ such that $x \in X \setminus S' \subset S$.

Proof. Since X is T_1, for each $y \in X \setminus S$ there exists a subbase element S_y with $y \in S_y$ and $x \notin S_y$. Thus $\mathcal{C} = \{S\} \cup \{S_y \mid y \in X \setminus S\}$ covers X and by (S.1) has a subcover by two elements. Clearly one of them must be S. Hence, for some $y_0 \in X \setminus S$, we have $X = S \cup S_{y_0}$ and $x \in X \setminus S_{y_0} \subset S$.

Notation. Consider the comparability relation on $\mathcal{S}$. We denote this relation by "$\sim$". Thus for S_1 and S_2 elements of $\mathcal{S}$ we have: $S_1 \sim S_2$ iff $S_1 \subset S_2$ or $S_2 \subset S_1$.

Obviously, $\sim$ is reflexive and symmetric for any family. Crucial is the following proposition:

Proposition 2.5. Let $(X, \mathcal{S})$ be a 2-ccompact T_1 space. Then $\sim$ is an equivalence relation on $\mathcal{S}$.
Proof. It suffices to prove that $\sim$ is transitive. Suppose $S_1 \sim S_2$ and $S_2 \sim S_3$. Without loss of generality we consider only two cases, the other two cases being obvious.

Case 1. $S_1 \subset S_2$ and $S_3 \subset S_2$. Suppose $x_1 \in S_1 \setminus S_3$ and $x_3 \in S_3 \setminus S_1$. By Proposition 2.4, there exist sets S_1' and S_3' such that $x_1 \in X \setminus S_1' \subset S_1$ and $x_3 \in X \setminus S_3' \subset S_3$. Thus $S_2 \cup S_1' = X = S_2 \cup S_3'$. By (S.2), $S_1' \sim S_3'$. However, this is impossible since $x_3 \in S_1' \setminus S_3'$ and $x_1 \in S_3' \setminus S_1'$. Hence $S_1 \sim S_3$.

Case 2. $S_2 \subset S_1$ and $S_2 \subset S_3$. Since $S_2 \neq \emptyset$, we can use Proposition 2.4 to obtain $S_0 \in \mathcal{S}$ such that $S_0 \cup S_2 = X$. But then $S_0 \cup S_1 = X = S_0 \cup S_3$, and $S_1 \sim S_3$ by (S.2).

Notation. Let $(X, \mathcal{S})$ be a 2-ccompact T_1 space and let $\mathcal{P} = \{\mathcal{S}_\alpha \mid \alpha \in A\}$ be the partition of $\mathcal{S}$ into $\sim$-equivalence classes (or $\sim$-classes, for short).

Lemma 2.6. For each $\sim$-class $\mathcal{S}_\alpha \in \mathcal{P}$ and $S \in \mathcal{S}_\alpha$, there exists a complementary class $\mathcal{S}_\alpha' \in \mathcal{P}$ and $S' \in \mathcal{S}_\alpha'$ such that $S \cup S' = X$. $\mathcal{S}_\alpha'$ is necessarily unique and independent of the choice of $S \in \mathcal{S}_\alpha$; $(\mathcal{S}_\alpha')' = \mathcal{S}_\alpha$.
Proof. Given $S \in \mathcal{S}_\alpha$, the existence of an $S' \in \mathcal{S}$ such that $S \cup S' = X$ follows from Proposition 2.4. Suppose that S_1 and S_2 are elements of $\mathcal{S}_\alpha$, and $S_1 \cup S_1' = X = S_2 \cup S_2'$. Since S_1 and S_2 belong to the same $\sim$-class, they are comparable, say $S_1 \subset S_2$. But then $S_2 \cup S_1' = X = S_2 \cup S_2'$, so $S_1' \sim S_2'$. Clearly, $(\mathcal{S}_\alpha')' = \mathcal{S}_\alpha$.

Lemma 2.7. The following statements are equivalent:

(1). $\mathcal{S}_\beta = \mathcal{S}'_\alpha$.

(2). For all $S \in \mathcal{S}_\alpha$ and all $S' \in \mathcal{S}_\beta$, either $S \cup S' = X$ or $S \cap S' = \emptyset$.

(3). There exist $S \in \mathcal{S}_\alpha$ and $S' \in \mathcal{S}_\beta$ such that either $S \cup S' = X$ or $S \cap S' = \emptyset$.

Proof. (1) ⇒ (2). Suppose that $\mathcal{S}_\beta = \mathcal{S}'_\alpha$, $S \in \mathcal{S}_\alpha$ and $S' \in \mathcal{S}_\beta$. Then, if $x \in S \cap S'$, by Proposition 2.4 there exists $S'' \in \mathcal{S}$ such that $x \notin S''$ and $S \cup S'' = X$. Thus $S'' \in \mathcal{S}'_\alpha = \mathcal{S}_\beta$ and $S' \sim S''$. Since $x \in S' \setminus S''$, we must have $S'' \subset S'$, and so $S \cup S' = X$.

(2) ⇒ (3). Clear.

(3) ⇒ (1). Suppose that $S \in \mathcal{S}_\alpha$ and $S' \in \mathcal{S}_\beta$. By Lemma 2.6, there exists $S'' \in \mathcal{S}_\alpha$ such that $S \cup S'' = X$. If $X = S \cup S'$, then by (S.2), $S' \sim S''$. If $S \cap S' = \emptyset$, then $S' \subset S''$, so $S' \sim S''$. Thus in either case $S' \in \mathcal{S}_\beta \cap \mathcal{S}'_\alpha$ hence $\mathcal{S}_\beta = \mathcal{S}'_\alpha$.

Theorem 2.8. Let X be a non-degenerate T_1 space. Then the topology of X is compact and induced by a total ordering iff there is a subbase $\mathcal{S}$ for the topology of X such that $(X, \mathcal{S})$ is a 2-ccompact space and $\sim$ partitions $\mathcal{S}$ into exactly two $\sim$-classes.

Proof. (⇒). Suppose that $\leqslant$ is a total ordering inducing the compact topology of X. Take as a subbase, $\mathcal{S}$, of the order topology the family of all "left-rays" $L_y = \{x \in X \mid x < y\}$ and all "right-rays" $R_x = \{y \in X \mid x < y\}$ Clearly, $(X, \mathcal{S})$ is a 2-ccompact space with exactly two $\sim$-classes.

(⇐). Conversely, suppose that X is 2-ccompact with exactly two $\sim$-classes $\mathcal{L}$ and $\mathcal{L}' = \mathcal{R}$. Define an ordering $\leqslant$ on X by: $x \leqslant y$ iff $(y \in L \in \mathcal{L} \Rightarrow x \in L)$. Clearly $\leqslant$ is reflexive and transitive. Suppose that $x \neq y$. Then there exists an open subbase element S such that $x \in S$ and $y \notin S$. If $S \in \mathcal{L}$, then $x \not\geqslant y$. If $S \in \mathcal{R}$, then there exists $S' \in \mathcal{L} = \mathcal{R}'$ such that $y \in S'$ and $x \notin S'$. Thus $x \not\leqslant y$. Hence $\leqslant$, being also anti-symmetric, is a partial order. Suppose that $x \not\leqslant y$. Then there exists $L \in \mathcal{L}$ such that $y \in L$ and $x \notin L$. Now, if $x \in \tilde{L} \in \mathcal{L}$, then $L \sim \tilde{L}$, and we can only have $y \in L \subset \tilde{L}$. Thus $y \leqslant x$, and $\leqslant$ is a total ordering. Note that $x \leqslant y$ iff $(x \in R \in \mathcal{R} \Rightarrow y \in R)$; we omit the proof.

We show that each $L_y = \{x \in X \mid x < y\}$ is open in X. (The proof that each right-ray is open being similar.) Let $x \in L_y$. Then by the anti-symmetry, $y \not\leqslant x$, so there exists $L^x \in \mathcal{L}$ such that $x \in L^x$ and $y \notin L^x$. Now, if $z \in L^x$ and $y \leqslant z$, then $y \in L^x$, a contradiction; thus $z < y$, that is, $L^x \subset L_y$. Since L_y is clearly the union of the $L^x \in \mathcal{L}$ with $x < y$, L_y is open in X. Since the order topology is necessarily T_2 and, as we have just shown, weaker than the given compact topology on X, the two topologies coincide.

Corollary 2.9. The product of totally ordered compact spaces is 2-ccompact.

Proof. Immediate from Proposition 2.3 and Theorem 2.8.

Now we prove the main theorem of this paper.

Theorem 2.10. X is homeomorphic to the topological product of totally ordered compact spaces iff X is a 2-ccompact T_1 space.

Proof. The necessity was established in Corollary 2.9.

To prove the sufficiency, we suppose without loss of generality that $(X, \mathcal{S})$ is a non-degenerate 2-ccompact T_1 space. For suitable index set A, we may express $\mathcal{P}$, the family of $\sim$-classes, as the disjoint union of two subfamilies as follows: $\mathcal{P} = \{\mathcal{L}_\alpha \mid \alpha \in A\} \cup \{\mathcal{R}_\alpha \mid \alpha \in A\}$, where the $\mathcal{L}_\alpha$'s and $\mathcal{R}_\alpha$'s are complementary $\sim$-classes, i.e., $\mathcal{R}_\alpha = \mathcal{L}'_\alpha$, and for no distinct $\alpha_1, \alpha_2 \in A$ is it true that $\mathcal{L}_{\alpha_1} = \mathcal{L}_{\alpha_2}$ or $\mathcal{L}_{\alpha_1} = \mathcal{R}_{\alpha_2}$.

For each $\alpha \in A$ define an equivalence relation $\equiv_\alpha$ on X by $x \equiv_\alpha y$ iff (for all $L \in \mathcal{L}_\alpha$, $x \in L \Leftrightarrow y \in L$). (Note that we could just as well have defined $\equiv_\alpha$ by $x \equiv_\alpha y$ iff (for all $R \in \mathcal{R}_\alpha$, $x \in R \Leftrightarrow y \in R$). This follows from the fact that if $x \in S \in \mathcal{S}_\gamma$ and $y \notin S$, then there exists $S' \in \mathcal{S}'_\gamma$ such that $x \notin S'$ and $y \in S'$.) Let X_α denote the quotient set $X/{\equiv_\alpha} = \{[x]_\alpha \mid [x]_\alpha \text{ is the } \equiv_\alpha\text{-equivalence class of } x\}$, and let $\eta_\alpha : X \to X_\alpha$ be the canonical projection function. Endow each X_α with the topology generated by the subbase σ_α consisting of sets of the form $\eta_\alpha[L]$ or $\eta_\alpha[R]$, where $L \in \mathcal{L}_\alpha$ and $R \in \mathcal{R}_\alpha$, respectively.

Suppose that $[x]_\alpha \neq [y]_\alpha$ are distinct points of X_α, say for some $L \in \mathcal{L}_\alpha$ we have $x \in L$ and $y \notin L$. Then there exists $R \in \mathcal{R}_\alpha$ such that $y \in R$ and $x \notin R$. Hence $[x]_\alpha \in \eta_\alpha[L] \subset X_\alpha \setminus [y]_\alpha$ and $[y]_\alpha \in \eta_\alpha[R] \subset X_\alpha \setminus [x]_\alpha$, so X_α is T_1. Clearly, $\eta_\alpha^{-1}[\eta_\alpha[L]] = L$ and $\eta_\alpha^{-1}[\eta_\alpha[R]] = R$ for each $L \in \mathcal{L}_\alpha$ and each $R \in \mathcal{R}_\alpha$, and from this we can draw several conclusions. Since η_α^{-1} carries the subbase σ_α on X_α into the subbase $\mathcal{S}$ on X, η_α is continuous. If $\mathcal{C}$ is a cover of X_α by elements of σ_α, then $\eta_\alpha^{-1}[\mathcal{C}]$ is a cover of X by elements from $\mathcal{L}_\alpha \cup \mathcal{R}_\alpha \subset \mathcal{S}$. Hence there exist two elements $L \in \mathcal{L}_\alpha$ and $R \in \mathcal{R}_\alpha$ such that $X = L \cup R$ and $\eta_\alpha[L], \eta_\alpha[R] \in \mathcal{C}$. Thus, $\{\eta_\alpha[L], \eta_\alpha[R]\}$ is a two-element subcover of $\mathcal{C}$. Similarly, we can prove that σ_α satisfies (S.2). Since X_α is a 2-ccompact T_1 space with exactly two $\sim$-classes $\eta_\alpha[\mathcal{L}_\alpha]$ and $\eta_\alpha[\mathcal{R}_\alpha]$, it follows from Theorem 2.8 that

X_α is a compact totally ordered T_2 space. (The order on X_α is given by $[x]_\alpha \leqslant [y]_\alpha$ iff $([y]_\alpha \in \eta_\alpha[L] \Rightarrow [x]_\alpha \in \eta_\alpha[L]$ for all $L \in \mathsf{L}_\alpha$).)

Let $T = \Pi_{\alpha \in A} X_\alpha$ be the topological product of the X_α, and let π_α: $T \to X_\alpha$ be the projection maps. There exists a continuous map $f: X \to T$ such that $\pi_\alpha \circ f = \eta_\alpha$ for each $\alpha \in A$. Since X is compact and T is T_2, the map f is closed. To see that f is injective, suppose that x_1 and x_2 are distinct points of X. Since X is T_1, there exists an $\alpha \in A$ and $L \in \mathsf{L}_\alpha$ such that $x_1 \in L$ and $x_2 \notin L$ or vice versa. Thus $\pi_\alpha \circ f(x_1) = \eta_\alpha(x_1) = [x_1]_\alpha \neq [x_2]_\alpha = \eta_\alpha(x_2) = \pi_\alpha \circ f(x_2)$, and consequently $f(x_1) \neq f(x_2)$.

Having embedded X as a closed subset of T, we proceed to show that $f[X]$ is dense in T. Let $t \in U \subset T$, where U is open in T. Then, there exist finitely many distinct indices, say $\alpha_1, \alpha_2, ..., \alpha_n \in A$ such that $t \in \bigcap_{i=1}^n \pi_{\alpha_i}^{-1} [U_{\alpha_i}] \subset U$ with U_{α_i} an open basis element in X_{α_i}, $i = 1, 2, ..., n$. Thus, either

(1) $U_{\alpha_i} = \eta_{\alpha_i}[L_i]$ for $L_i \in \mathsf{L}_{\alpha_i}$; or

(2) $U_{\alpha_i} = \eta_{\alpha_i}[R_i]$ for $R_i \in \mathsf{R}_{\alpha_i}$; or

(3) $U_{\alpha_i} = \eta_{\alpha_i}[L_i] \cap \eta_{\alpha_i}[R_i]$ for $L_i \in \mathsf{L}_{\alpha_i}$ and $R_i \in \mathsf{R}_{\alpha_i}$, $i = 1, 2, ..., n$.

We define a family F of sets whose members will be denoted by F_i or F_i^1 or F_i^2. We want to show that there exists an x common to the L_i's and R_i's; the point we obtain will actually be a member of $\bigcap \mathsf{F}$. We will then have $f(x) \in U$, proving that $f[X]$ is dense in T.

Consider first the case that U_{α_i} is of type (3). Since $\pi_{\alpha_i}(t) \in U_{\alpha_i}$ and

$$\emptyset \neq \eta_{\alpha_i}^{-1} [U_{\alpha_i}] = \eta_{\alpha_i}^{-1} [\eta_{\alpha_i}[L_i] \cap \eta_{\alpha_i}[R_i]]$$

$$= \eta_{\alpha_i}^{-1} \circ \eta_{\alpha_i}[L_i] \cap \eta_{\alpha_i}^{-1} \circ \eta_{\alpha_i}[R_i] = L_i \cap R_i,$$

we can find a point, say x_i, with $x_i \in L_i \cap R_i$. By Proposition 2.4, there exist $L_i' \in \mathsf{R}_{\alpha_i}$ and $R_i' \in \mathsf{L}_{\alpha_i}$ such that $x_i \in X \setminus L_i' \subset L_i$ and $x_i \in X \setminus R_i' \subset R_i$. Let $F_i^1 = X \setminus L_i'$ and $F_i^2 = X \setminus R_i'$ in this case.

In the case that U_{α_i} is of type (1) or type (2), let $F_i = X \setminus L_i' \subset L_i$, $L_i' \in \mathsf{R}_{\alpha_i}$, or let $F_i = X \setminus R_i' \subset R_i$, $R_i' \in \mathsf{L}_{\alpha_i}$. Suppose that $\bigcap \mathsf{F} = \emptyset$. Then X is covered by the L_i''s and R_i''s. But no two such sets suffice to cover x. For two such sets would necessarily belong to complementary $\sim$-classes. Thus one set would be of the form L_i' and the other of the form R_j'. Thus $\mathsf{R}_{\alpha_i} = \mathsf{L}_{\alpha_j}$, so that $i = j$. But then $x_i \notin L_i' \cup R_j'$, a contradiction. Hence there exists $x \in \bigcap \mathsf{F}$. Thus for $1 \leqslant i \leqslant n$, $\pi_{\alpha_i}[f(x)] = \eta_{\alpha_i}(x) \in U_{\alpha_i}$. (For example, in the case that U_{α_i} is of type (3), $\eta_{\alpha_i}(x) \in \eta_{\alpha_i} [F_i^1 \cap F_i^2] \subset \eta_{\alpha_i} [L_i] \cap \eta_{\alpha_i}[R_i] = U_{\alpha_i}$.) Since $f[X]$ is dense and closed in T, f is a homeomorphism of X onto T.

Corollary 2.11. A space X is the product of $\mathfrak{m}$ non-degenerate compact totally ordered spaces iff there is a subbase S for the topology on X such that (X, S) is a 2-ccompact T_1 space and S is partitioned into 2 m [illegible]-classes.

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