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On the θ-method for delay differential equations with infinite lag

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Abstract

This paper discusses the θ-method for the numerical solution of delay differential equations with infinite lag. Our analysis is based on the test equation $y'(t)=ay(\lambda t)+by(t)$, where $a,b\in\mathbb{C}$, and $\lambda\in(0,1)$. In order to solve the storage problem involved in studying the long time behaviour of the solution we use a grid with increased stepsizes. Monotonicity, uniform boundedness, asymptotic stability, algebraic decay and global discretisation error of the numerical solution are investigated.

Keywords: Delay differential equations; θ-method; Asymptotic stability; Global error estimate

AMS classification: 65L20

1. Introduction

This paper discusses the numerical solution of the initial value problem of the delay differential equation

$$y'(t)=f(t,y(t),y(\phi(t))),\ \ t>0, \qquad y(t)=y_0(t),\ \ t\in[\inf_{\tau\geqslant 0}\{\phi(\tau)\},0], \tag{1.1}$$

where f, ϕ and y_0 are given functions with $\phi(t)\leqslant t$ for all $t\geqslant 0$. As far as time lag is concerned, the equation can be classified into two categories, namely, those with finite time lag, i.e., $\limsup_{t\to\infty}(t-\phi(t))<\infty$, and those with infinite time lag, i.e., $\limsup_{t\to\infty}(t-\phi(t))=\infty$. Two typical examples are the initial value problems

$$y'(t)=ay(t-\lambda)+by(t),\ \ t>0, \qquad y(t)=y_0(t),\ \ t\in[-\lambda,0] \tag{1.2}$$

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and

$$y'(t) = ay(\lambda t) + by(t), \quad t > 0, \qquad y(0) = y_0, \tag{1.3}$$

where a, b are complex constants, $\lambda \in (0, \infty)$ and $(0, 1)$ in the case of (1.2) and (1.3), respectively.

The numerical method to be considered is the so-called θ-method which is specified by the parameter $\theta \in [0, 1]$. Before applying it to Eq. (1.1), we introduce some notation. Let t_n, $n = 0, 1, \ldots$, be the grid points satisfying

$$0 = t_0 < t_1 < t_2 < \cdots < \infty, \qquad \lim_{n \to \infty} t_n = \infty,$$

and $h_n = t_{n+1} - t_n$, $n = 0, 1, \ldots$, the stepsizes. In order to provide an output, we define the numerical solution $y^h(t)$ as the piecewise linear interpolation

$$y^h(t) = \frac{t_{n+1} - t}{h_n} y_n + \frac{t - t_n}{h_n} y_{n+1}, \qquad t \in [t_n, t_{n+1}), \quad n \geqslant 0,$$

where y_n denotes our approximation to the exact solution $y(t)$ of (1.1) at the grid point t_n. There are two kinds of θ-methods, namely, the linear θ-method and the one-leg θ-method. Upon application to (1.1), they give the recurrence relations

$$y_{n+1} = y_n + h_n\{(1 - \theta) f(t_n, y_n, y^h(\phi(t_n))) + \theta f(t_{n+1}, y_{n+1}, y^h(\phi(t_{n+1})))\},$$
$$n = 0, 1, \ldots$$

and

$$y_{n+1} = y_n + h_n f((1 - \theta)t_n + \theta t_{n+1}, (1 - \theta)y_n + \theta y_{n+1}, (1 - \theta)y^h(\phi(t_n)) + \theta y^h(\phi(t_{n+1}))),$$
$$n = 0, 1, \ldots,$$

respectively. In the case of (1.3), both θ-methods give the same recurrence relation

$$y_{n+1} = y_n + ah_n((1 - \theta)y^h(\lambda t_n) + \theta y^h(\lambda t_{n+1})) + bh_n((1 - \theta)y_n + \theta y_{n+1}), \quad n = 0, 1, \ldots. \tag{1.4}$$

There are remarkable differences, both analytically and numerically, between delay differential equations with infinite lags and those with finite lags. Let us compare (1.3) with (1.2). The solution of (1.3) is an analytic function on $[0, \infty)$, whereas the solution of (1.2) is initially nonsmooth but becomes smoother with increasing t. In the case where $b + |a| < 0$, the solution of (1.3) decays algebraically, whereas the solution of (1.2) decays exponentially. The most significant difference is in storage. In order to calculate all the future values of $y(t)$ beyond t_0, say, we must remember all the past values in the interval $[\phi(t_0), t_0]$, which is bounded in the case of (1.2) but unbounded in the case of (1.3) as $t_0 \to \infty$. As far as the numerical method is concerned, we need to store at least all the data in the set $S(t_0) = \{y_n \colon n \in \mathbb{Z}^+ \text{ such that } t_n \in [\phi(t_0), t_0]\}$. If the stepsizes are uniformly bounded then the number of data in the set $S(t_0)$ tends to infinity as $t_0 \to \infty$ in the case of infinite lag. This will inevitably bring serious storage problems even to the largest supercomputers. An example has been given by Iserles [11].

While (1.2) is being used as a test equation for assessing the stability of the θ-method for delay differential equations with finite lag, we will use (1.3) for equations with infinite time lag. The stability of the θ-method for the numerical solution of (1.2) has been studied extensively in [6, 14, 19, 22–24] and other papers. The analysis therein is mostly based on the fact that the resulting difference equations are of fixed order and have constant coefficients in the case of uniform grid. The stability of the θ-method is then proved by showing that the corresponding characteristic polynomials of the difference equations are of Schur type, i.e., polynomials whose roots lie inside the unit circle. Unfortunately, this approach is not suited to the θ-method for (1.3). As noted by Jackiewicz [14], the difference equation (1.4) is not of fixed order (in the case of uniform grid). Even if (1.4) can be converted into equations of fixed order by choosing the grid carefully (see the next section), its coefficients are not constants. We observe that besides being a test equation, (1.3) has many interesting applications (a comprehensive list features in [10]). Numerical methods for (1.3) have been studied by Fox et al. [8] and Bakke and Jackiewicz [1], and for a more general case, namely, the neutral equation

$$y'(t) = ay(t) + by(pt) + cy'(qt), \quad t > 0, \qquad y(0) = y_0,$$

by Buhmann and Iserles [2–4] and Buhmann et al. [5]. In these papers the grid is uniform, hence the storage problem remains problematic. We shall see in Section 5 of this paper that the long time behaviour of the solutions in some cases has not been predicted correctly by numerical methods in [3] due to insufficient computer (random access) memory. None of these papers has fully recovered the stability (asymptotic stability) condition, i.e., prove that the numerical solution is uniformly bounded (tends to zero) subject to the condition $\operatorname{Re} b < 0$, $|b| \geqslant |a|$ ($\operatorname{Re} b < 0, |b| > |a|$), and no global discretisation error estimate has been given in these papers. This brings doubt about the correct approximation of the numerical solution $y^h(t)$ to the exact solution $y(t)$ when t is large.

This paper is structured as follows. In Section 2, we prove the existence and uniqueness of the numerical solution of (1.3) by θ-methods subject to the condition $\operatorname{Re} b \leqslant 0$, $|b| \geqslant |a|$. In consideration of the storage problem, we formulate a kind of grid whose stepsizes increase geometrically after an initial stage. In Section 3, we discuss the monotonicity, uniform boundedness and asymptotic stability of the numerical solution. Our result shows that the solution of the backward method retains certain properties such as monotonicity, uniform boundedness (for all a and b satisfying $\operatorname{Re} b < 0$, $|b| \geqslant |a|$), asymptotic stability and algebraic decay (for all a and b satisfying $\operatorname{Re} b < 0$, $|b| > |a|$), as possessed by the exact solution. In Section 4, we discuss the global discretisation error of the θ-method. Our result reveals that the increased stepsizes are balanced by the algebraic decay of the solution. In Section 5, we present some numerical examples. We emphasise that our analysis can be easily applied to linear delay differential equations with variable coefficients and variable delays.

2. The algorithm

Throughout this paper, we use $\{y_n\}_{n=0}^{\infty}$ to denote the numerical solution of (1.3) by the θ-method (1.4).

Theorem 1. *If* $\operatorname{Re} b \leqslant 0$, $|b| \geqslant |a|$, *then the solution* $\{y_n\}_{n=0}^{\infty}$ *of* (1.4) *exists and is unique.*

Proof. If $\lambda t_{n+1} \leqslant t_n$, which is true for at least all $n \geqslant n_0$, we have the explicit recurrence relation

$$y_{n+1} = \frac{1}{1-\theta b h_n}((1+(1-\theta)bh_n)y_n + (1-\theta)ah_n y^h(\lambda t_n) + \theta a h_n y^h(\lambda t_{n+1})), \tag{2.1}$$

and if $\lambda t_{n+1} > t_n$, we have

$$y_{n+1} = \frac{1}{1 - \theta b h_n - \theta a(\lambda t_{n+1} - t_n)}((1 + (1-\theta) b h_n + \theta a(1-\lambda) t_{n+1})) y_n \\ +(1-\theta) a h_n y^h(\lambda t_n)). \tag{2.2}$$

This proves the existence and uniqueness of the numerical solution. □

To deal with the storage problem, we divide $[0,\infty)$ into a union of an infinite number of bounded intervals as follows:

$$[0,\infty) = [0,r] \bigcup_{k=0}^{\infty} I_k,$$

where r is a fixed positive number and

$$I_k = (\lambda^{-k} r, \lambda^{-k-1} r], \qquad k \geqslant 0.$$

For given integers $n_0, n_1 \geqslant 2$, we formulate the grid as follows:

$$0 = t_0 < t_1 < t_2 < \cdots < t_{n_0} = r,$$
$$\lambda^{-k} r = t_{n_0+kn_1} < t_{n_0+kn_1+1} < \cdots < t_{n_0+(k+1)n_1} = \lambda^{-k-1} r, \qquad k = 0,1,\ldots.$$

The advantage of the preceding grid is that the lag function $\phi(t) := \lambda t$ has the property

$$\phi : I_{k+1} \mapsto I_k, \qquad k \geqslant 0, \tag{2.3}$$

which implies that no storage problem arises for the θ-method. The magnitude of r, n_0 and n_1 depends on the parameters of Eq. (1.3). We will discuss this in Sections 4 and 5.

With the property (2.3), we need only to set up two finite-dimensional arrays in practical calculation. More specifically, we can do the calculation in the following way:

Step 1: Calculate $\{y_n\}_{n=1}^{n_0}$ by (2.1), (2.2) and store it in an n_0-dimensional array U.

Step 2: Calculate $\{y_{n_0+n}\}_{n=1}^{n_1}$ by (2.1) and store it in an n_1-dimensional array V, then free the storage occupied by U and set $k=0$.

Step 3: Calculate $\{y_{n_0+(k+1)n_1+n}\}_{n=1}^{n_1}$ by (2.1) and store it in an n_1-dimensional array U, then update V by U, increase k by 1, and repeat this step.

Commonly used θ-methods include the trapezoidal rule and midpoint rule ($\theta = \frac{1}{2}$ for both cases) and the backward Euler method ($\theta = 1$). We suggest that the Trapezoidal Rule be used to calculate $\{y_n\}_{n=1}^{n_0}$, when Eq. (1.3) is not stiff, i.e., $|b|$ is not large, otherwise the backward Euler method should be used. The solution $\{y_n\}_{n=n_0+1}^{\infty}$ should be calculated through the backward Euler method. The reasons for doing so will be explained in the next two sections.

Remark 1. If the grid is chosen as follows,

$$0 = t_0 < t_1 < t_2 < \cdots < t_{n_0} = r < t_{n_0+1} < \cdots < t_{n_0+n_1} = \lambda^{-1} r,$$
$$t_{n_0+kn_1+n} = \lambda^{-k} t_{n_0+n}, \qquad n = 0,1,\ldots,n_1, \;\; k = 1,2,\ldots,$$

the recurrence relation (1.4) then becomes

$$y_{n+1} = \frac{1+(1-\theta)bh_n}{1-\theta bh_n} y_n + \frac{ah_n}{1-\theta bh_n}((1-\theta)y_{n-n_1} + \theta y_{n-n_1+1}), \quad n \geqslant n_0 + n_1, \tag{2.4}$$

which is a difference equation of fixed order, but of variable coefficients.

Remark 2. We may solve the storage problem by reducing the delay differential equation with infinite lag into an equation with finite lag. In the case of (1.3), by letting $x(t) = y(\mathrm{e}^t)$, we get

$$x'(t) = a\mathrm{e}^t x(t + \log\lambda) + b\mathrm{e}^t x(t), \quad t \in \mathbb{R}, \qquad x(-\infty) = y_0,$$

which is equivalent to (1.3). However, the preceding equation is not a proper model for implementing numerical methods since the initial point is $-\infty$. To deal with this, we can apply numerical methods to Eq. (1.3) for $t \in [0, \mathrm{e}^{t_0}]$ and then to

$$x'(t) = a\mathrm{e}^t x(t + \log\lambda) + b\mathrm{e}^t x(t), \quad t > t_0, \qquad x(t) = y(\mathrm{e}^t), \quad t \leqslant t_0.$$

This approach has been investigated in Liu [17].

Remark 3. Consider the general equation (1.1), where $\phi(t)$ is an eventually strict monotonic increasing function, i.e., there exists $r_0 \geqslant 0$ such that $\phi(t)$ is a strictly monotonic increasing function on $[r_0, \infty)$, and $\lim_{t\to\infty} \phi(t) = \infty$. Given a fixed real number $r \geqslant r_0$, we can divide $[0, \infty)$ into a union of an infinite number of bounded intervals as follows:

$$[0, \infty) = [0, r] \bigcup_{k=0}^{\infty} I_k,$$

where

$$I_k = (\psi^k(r), \psi^{k+1}(r)], \qquad k \geqslant 0,$$

where $\psi(t)$ is the inverse function of $\phi(t)$ and $\psi^k(t)$ is the kth iterate of ψ. For given positive integers n_0, n_1, we formulate the grid as follows:

$$\begin{aligned} &0 = t_0 < t_1 < t_2 < \cdots < t_{n_0} = r, \\ &\psi^k(r) = t_{n_0+kn_1} < t_{n_0+kn_1+1} < \cdots < t_{n_0+(k+1)n_1} = \psi^{k+1}(r), \qquad k = 0, 1, \ldots. \end{aligned}$$

The advantage of the preceding grid is that the lag function $\phi(t)$ satisfies (2.3).

3. Stability analysis

The stability problem concerning (1.3) has been studied by Kato and MacLeod [16] and by Kato [15]. Equations of general forms have been investigated by Iserles [10], Iserles and Terjéki [13], Iserles and Liu [12], Liu [18], etc. The basic result is that (see [15]) when $\operatorname{Re} b < 0$ and $|b| \geqslant |a|$ the exact solution of (1.3) decays algebraically, i.e.,

$$y(t) = \mathcal{O}(t^{\kappa}), \qquad t \to \infty, \tag{3.1}$$

for $\kappa = \ln|b/a|/\ln\lambda$, and that no solution except the identical zero one is of $o(t^\kappa)$ as $t \to \infty$. Hence, the exact solution is uniformly bounded when $\operatorname{Re} b < 0$ and $|b| = |a|$, and tends to zero as $t \to \infty$ when $\operatorname{Re} b < 0$ and $|b| > |a|$. Another important feature is that when b is a real constant and $b + |a| \leqslant 0$ the exact solution of (1.3) is uniformly bounded by $|y_0|$ [13]. As far as the numerical method is concerned, we need to choose an appropriate numerical method so that the numerical solution can retain as many of these properties as possible.

Firstly, we consider the case where b is real constant.

Theorem 2. *If $b+|a| \leqslant 0$ then the solution $\{y_n\}_{n=0}^m$ of (1.4) is uniformly bounded by $|y_0|$ provided that*

$$((2\theta - 1)b + |a|)h_n \leqslant 2 \quad \textit{whenever} \quad 1 + (1-\theta)bh_n < 0, \qquad 0 \leqslant n \leqslant m-1. \tag{3.2}$$

Proof. It is easy to see from (2.1), (2.2) that

$$|y_{n+1}| \leqslant \max_{0 \leqslant k \leqslant n} |y_k|, \qquad 0 \leqslant n \leqslant m-1,$$

which implies that $\{y_n\}_{n=0}^m$ is uniformly bounded by $|y_0|$. □

Suppose that $a < 0$ and $y_k = y_0$ for all $k \leqslant n$. In either case of (2.1) and (2.2), we have $|y_{n+1}| > |y_0|$ if $1 + (1-\theta)bh_n < 0$ and $((2\theta-1)b + |a|)h_n > 2$. This implies that the condition (3.2) is necessary. A direct consequence of Theorem 2 is the following corollary.

Corollary 3. *If $b < 0$ and $(2\theta-1)|b| > |a|$ then the solution $\{y_n\}_{n=0}^\infty$ of (1.4) is uniformly bounded by $|y_0|$.*

Theorem 4. *If $b < 0$, $(2\theta - 1)|b| > |a|$ and $\liminf_{n\to\infty} h_n > 0$ then the solution y_n of (1.4) tends to zero as $n \to \infty$.*

Proof. It follows from Corollary 3 that $y^* := \limsup_{n\to\infty} |y_n| \leqslant |y_0|$. In order to prove that $y^* > 0$ leads to contradiction, we let

$$\gamma := \sup_{n \geqslant 0} \frac{|1 + (1-\theta)bh_n| + |a|h_n}{1 - \theta bh_n} < 1, \qquad \delta := \frac{1-\gamma}{1+\gamma} y^* > 0.$$

Noting that there are integers $m_1 > n_0$ and $m_2 > m_1$ such that

$$|y_n| < y^* + \delta, \quad n > m_1$$

and

$$\lambda t_n > t_{m_1}, \quad n > m_2,$$

we deduce from (2.1) that

$$|y_{n+1}| \leqslant \gamma(y^* + \delta) = y^* - \delta, \quad n > m_2,$$

which contradicts the definition of y^*. Hence, $\lim_{n\to\infty} y_n = 0$. □

Secondly, we consider the case where b is complex constant.

Theorem 5. *If* $\operatorname{Re} b < 0$ *then the solution* y_n *of* (1.4)
(i) *tends to zero as* $n \to \infty$ *provided that* $(2\theta - 1)|b| > |a|$ *and* $\lim_{n\to\infty} h_n = \infty$;
(ii) *is uniformly bounded provided that* $(2\theta - 1)|b| = |a|$ *and* $\sum_{n=0}^{\infty} h_n^{-1} < \infty$.

Proof. The proof of the first part is similar to that of Theorem 4 except that in this case we choose $m \gg 1$ such that

$$\gamma := \sup_{n \geqslant m} \frac{|1 + (1-\theta)bh_n| + |a|h_n}{1 - \theta bh_n} < 1.$$

To prove the second part, we obtain from (2.1) that

$$|y_{n+1}| \leqslant \gamma_n \max_{0 \leqslant k \leqslant n} |y_k|, \qquad n \geqslant n_0,$$

where $\gamma_n = (|1 + (1-\theta)bh_n| + |a|h_n)/(|1 - \theta bh_n|)$. Since $\sum_{n=0}^{\infty} h_n^{-1} < \infty$ implies that $\prod_{n=n_0}^{\infty} \gamma_n < \infty$, we see that the solution of (1.4) is uniformly bounded. □

Theorem 6. *If* $\operatorname{Re} b < 0$, $(2\theta - 1)|b| > |a|$ *and* $\lim_{n\to\infty} h_n = \infty$, *then there exists a positive integer* k_0 *and a sequence* $\{\alpha_k\}_{k=k_0}^{\infty}$ *with the property*

$$\lim_{k\to\infty} \alpha_k = -\frac{1}{\ln \lambda} \ln \frac{(1-\theta)|b| + |a|}{\theta |b|} \tag{3.3}$$

such that the solution y_n *of* (1.4) *satisfies the following decay estimate*:

$$|y_{n_0+kn_1+n}| \leqslant (r^{-1} t_{n_0+kn_1+n})^{\alpha_k} \max_{n_0+(k_0-1)n_1 < m \leqslant n_0+k_0 n_1} |y_m|, \qquad k \geqslant k_0, \;\; 0 < n \leqslant n_1. \tag{3.4}$$

Proof. Let $k_0 \in \mathbb{Z}^+$ such that

$$\gamma_k = \sup_{0 \leqslant n < n_1} \frac{|1 + (1-\theta)bh_{n_0+kn_1+n}| + |a|h_{n_0+kn_1+n}}{|1 - \theta bh_{n_0+kn_1+n}|} < 1, \qquad k \geqslant k_0.$$

It follows from (2.1) that

$$|y_{n_0+kn_1+n}| \leqslant \gamma_k \max_{n_0+(k-1)n_1 < m \leqslant n_0+kn_1+n-1} |y_m|, \qquad 0 < n \leqslant n_1, \quad k \geqslant k_0,$$

which implies that

$$|y_{n_0+kn_1+n}| \leqslant \left(\prod_{j=k_0}^{k} \gamma_j\right) \max_{n_0+(k_0-1)n_1 < m \leqslant n_0+k_0 n_1} |y_m|, \qquad 0 < n \leqslant n_1, \quad k \geqslant k_0.$$

Let

$$\alpha_k = -\frac{1}{(k+1)\ln \lambda} \sum_{j=k_0}^{k} \ln \gamma_j, \qquad k \geqslant k_0.$$

It is easy to see that

$$\prod_{j=k_0}^{k} \gamma_j = (\lambda^{-k-1})^{\alpha_k} \leqslant (r^{-1} t_{n_0+kn_1+n})^{\alpha_k}, \qquad k \geqslant k_0.$$

By L'Hopital's rule, we see that (3.3) holds. □

By comparing (3.3) with (3.1) we see that the numerical solution of (1.3) by the backward Euler method has asymptotically the same decay rate as the exact solution.

4. Global error estimate

Recall that the solution $y(t)$ of Eq. (1.3) is an analytic function on $[0,\infty)$ (see, e.g., [16]). It is easy to verify that the approximation oder of the Trapezoidal Rule is 2 at least and the orders of all other θ-methods are 1 at least. However, a global discretisation error estimate is needed to guarantee the approximation of the numerical solution to the exact solution globally even when both the numerical solution and the exact solution tend to zero at infinity. Let $e_n = y(t_n) - y_n$, $n \geqslant 0$. It follows from (2.1) that

$$e_{n+1} = \frac{1}{1-\theta bh_n}((1+(1-\theta)bh_n)e_n + (1-\theta)ah_n e^h(\lambda t_n) + \theta ah_n e^h(\lambda t_{n+1})) \\ +E_n, \quad n \geqslant n_0, \tag{4.1}$$

where e^h is the piecewise linear interpolation of $\{e_n\}_{n=0}^{\infty}$, and

$$E_n = y(t_{n+1}) - \frac{1}{1-\theta bh_n}((1+(1-\theta)bh_n)y(t_n) + (1-\theta)ah_n((1-\psi_n)y(t_{m(n)}) + \psi_n y(t_{m(n)+1})) \\ +\theta ah_n((1-\psi_{n+1})y(t_{m(n+1)}) + \psi_{n+1} y(t_{m(n+1)+1})))$$

is the local discretisation error, where $m(n): \mathbb{Z}^+ \mapsto \mathbb{Z}^+$ denotes the function that satisfies the following inequalities,

$$t_{m(n)} \leqslant \lambda t_n < t_{m(n)+1},$$

and $\psi_n = (\lambda t_n - t_{m(n)})/h_{m(n)}$. In order to give a reasonably good estimation of the global error, we need the following result (see, e.g., [15, 18]) which concerns the decay rate of the second-order derivative of the exact solution.

Lemma 7. *There exists a constant $M > 0$ such that the second-order derivative of the solution $y(t)$ of (1.3) satisfies the following estimate:*

$$|y''(t)| \leqslant M(t+1)^{\kappa-2}, \qquad t \geqslant 0,$$

where $\kappa = \ln|b/a|/\ln\lambda$.

Theorem 8. *If* $\operatorname{Re} b < 0$, $(2\theta - 1)|b| > |a|$ *and* $\lim_{n\to\infty} h_n = \infty$ *then the global discretisation error of the* θ*-method* (1.4) *for* (1.3) *is bounded by* $M_1 h + M_2 \max_{m\geqslant n_0} h_m^2(t_m + 1)^{\kappa-2}$, *where* M_1 *and* M_2 *are positive constants that depend on the coefficients* a, b *and* λ *only, and* $h = \max_{0\leqslant m\leqslant n_0+n_1} h_m$.

Proof. For simplicity, we prove this theorem in the case that b is negative constant. It follows from (4.1) that

$$|e_{n+1}| \leqslant \rho \max_{0\leqslant m\leqslant n} |e_m| + |E_n|, \qquad n \geqslant n_0, \tag{4.2}$$

where

$$\rho := \sup_{n\geqslant n_0} \frac{|1 + (1-\theta)bh_n| + |a|h_n}{|1 - \theta bh_n|} < 1.$$

By induction, we obtain from (4.2) that

$$\max_{n > n_0+n_1} |e_n| \leqslant \max\left\{ \max_{0\leqslant n\leqslant n_0+n_1} |e_n|, \frac{1}{1-\rho} \max_{n\geqslant n_0+n_0} |E_n| \right\}. \tag{4.3}$$

By standard procedure we can prove that

$$\max_{0\leqslant n\leqslant n_0+n_1} |e_n| \leqslant M_1 h \tag{4.4}$$

for some positive constant M_1. Using the estimate in Lemma 7 we see that the following estimate,

$$\max_{n\geqslant n_0+n_1} |E_n| \leqslant (1-\rho) M_2 \max_{m\geqslant n_0} h_m^2 (t_m + 1)^{\kappa-2}, \tag{4.5}$$

holds for some positive constant M_2. The desired error bound follows from the estimates (4.3)–(4.5). □

Remark 4. Our results about the scalar case (1.3) are mostly subject to the conditions that $\operatorname{Re} b < 0$ and $(2\theta - 1)|b| > |a|$. In the case of the initial value problem of the system of delay differential equations

$$\vec{y}'(t) = A\vec{y}(\lambda t) + B\vec{y}(t), \quad t > 0, \qquad \vec{y}(0) = \vec{y}_0,$$

those two conditions need only be replaced by $\alpha(B) < 0$ and $\rho(B^{-1}A) < 2\theta - 1$, where $\alpha(\cdot)$ is the maximal real part of the eigenvalues of the matrix (the *spectral abscissa*) and $\rho(\cdot)$ the spectral radius.

5. Numerical examples

In this section, we present some numerical examples. We are mainly interested in the long time behaviour of the numerical solution of Eq. (1.3) on the stability boundary $\operatorname{Re} b < 0$, $|b| = |a|$. For the convenience of our computation, we choose the grid points as follows:

$$t_n = nh, \qquad n = 0, 1, \ldots, n_0,$$

$$t_{n_0+kn_1+n} = p^n \lambda^{-k} r, \qquad n = 0, 1, \ldots, n_1, \quad k = 0, 1, \ldots,$$

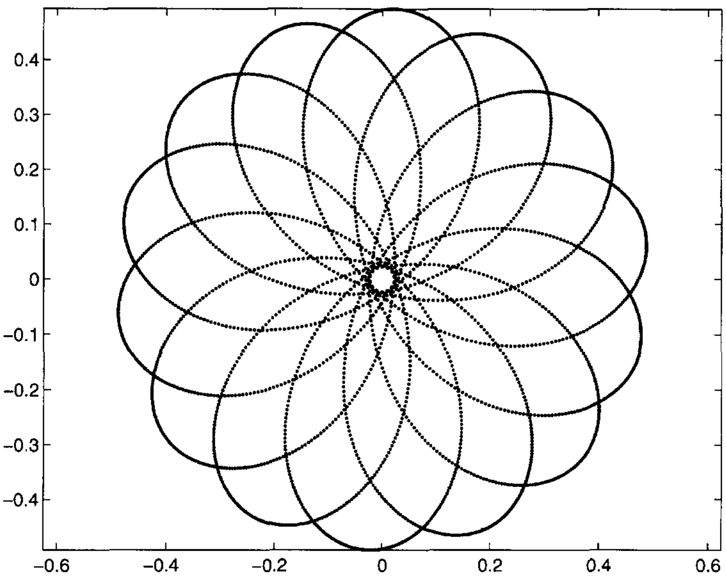

Fig. 1. $a = -e^{5\pi i/21}$, $b = e^{2\pi i/3}$, $\lambda = \frac{1}{4}$. $r = 10$, $n_0 = 100$, $n_1 = 200$, $n_2 = 100$, $n_3 = 42$. This one is the "same" as Fig. 2 of Iserles [11], the latter was produced from the exact solution (see [12]), whereas Fig. 4 of Iserles [11], produced by the Trapezoidal Rule with constant stepsize $h = \frac{1}{2}$, is used as an example to show the storage problem.

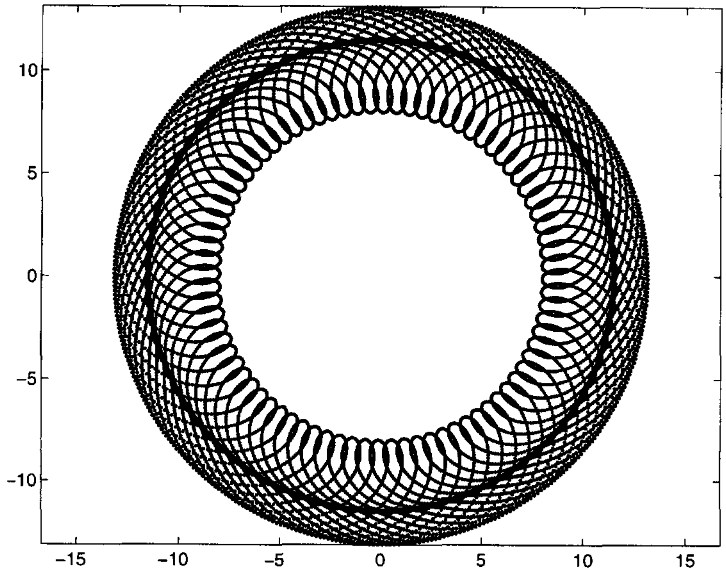

Fig. 2. $a = 1$, $b = e^{21\pi i/40}$, $\lambda = \frac{1}{2}$. $r = 10$, $n_0 = 100$, $n_1 = 200$, $n_2 = 200$, $n_3 = 80$. This figure can be verified by using the exact solution, whereas Fig. 7 (and some others as well) of Buhmann and Iserles [3] failed to predict the correct long time behaviour of the exact solution due to insufficient computer memory.

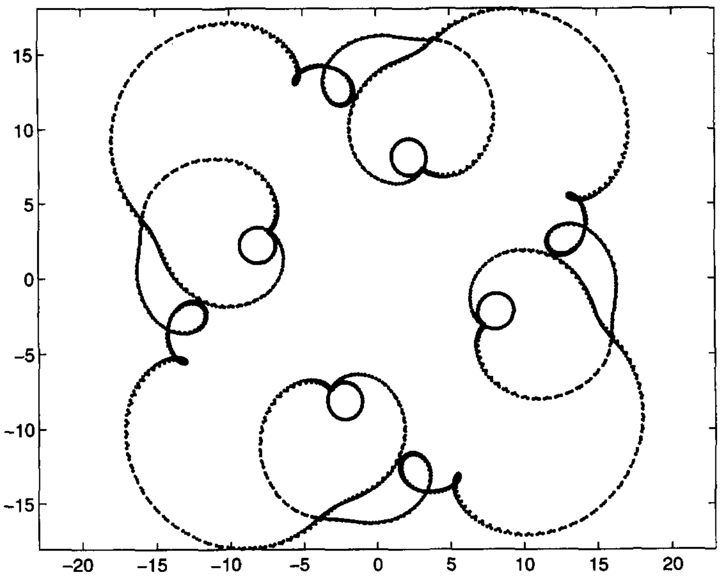

Fig. 3.

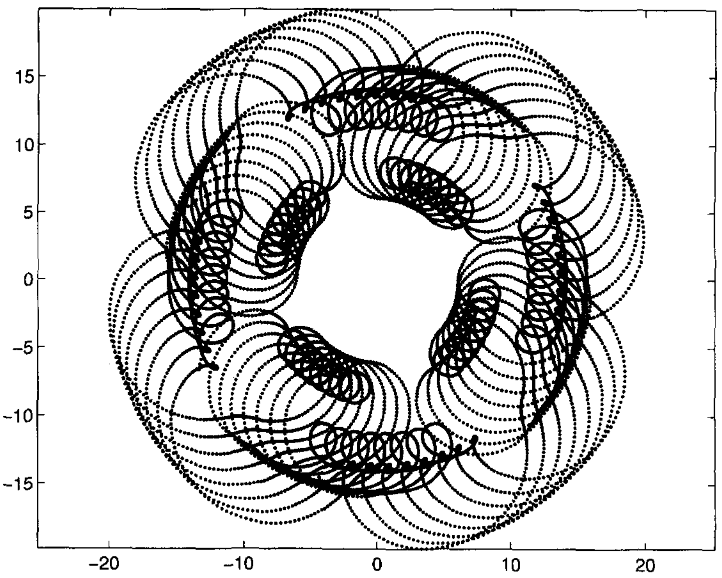

Fig. 4.

Figs. 3 and 4. In Fig. 3, $a = 4 + 0.1\mathrm{i}$, $b = -0.1 + 4\mathrm{i}$, $\lambda = \frac{1}{2}$. $r = 10$, $n_0 = 100$, $n_1 = 100$, $n_2 = 50$, $n_3 = 8$. The solution is uniformly bounded. In Fig. 4, $a = 4$, $b = -0.1 + 4i$, $\lambda = \frac{1}{2}$. $r = 10$, $n_0 = 100$, $n_1 = 100$, $n_2 = 50$, $n_3 = 32$. The solution tends to zero, but its geometric pattern is similar to that of Fig. 3 in finite time interval except that in Fig. 3 the curve is "closed" and in this figure the curve is not closed and actually consists of about 8 similar portions (or about k portions if we set $n_3 = 4k$).

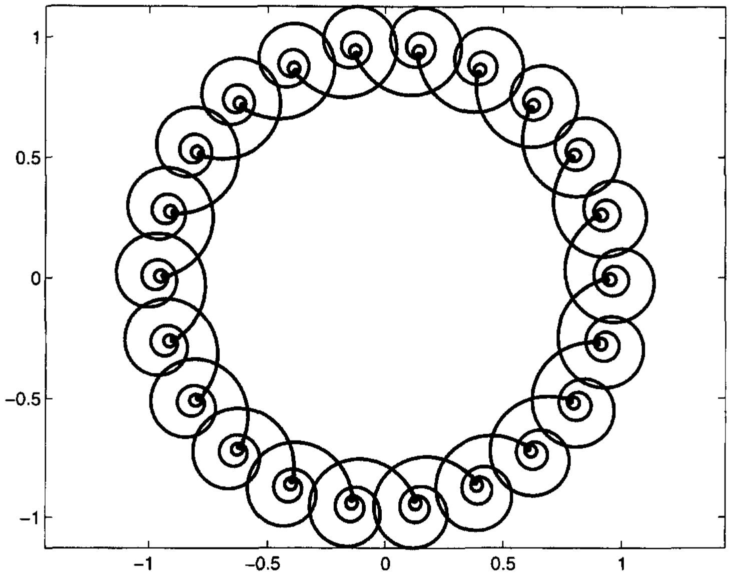

Fig. 5.

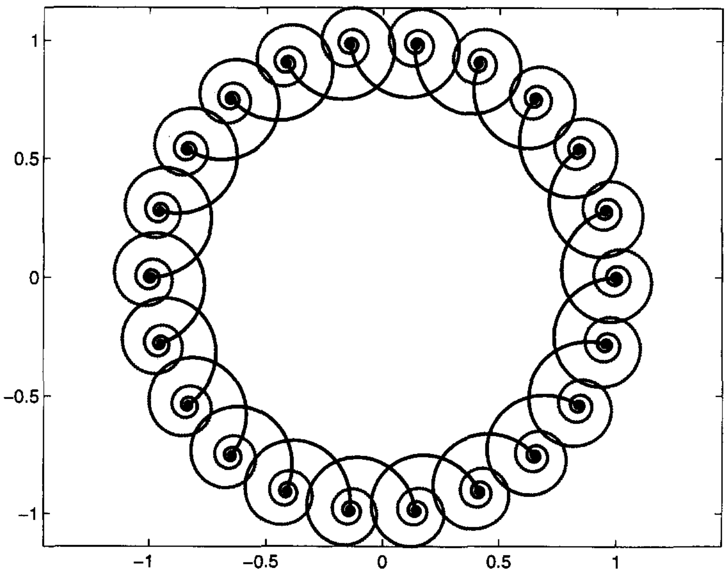

Fig. 6.

Figs. 5 and 6. In both figures, $a = e^{16\pi i/11}$, $b = e^{6\pi i/11}$, $r = 10$, $n_0 = 100$, $n_1 = 400$, $n_2 = 60$, $n_3 = 30$, except that $\lambda = 10^{-2}$ in Fig. 5 and $\lambda = 10^{-3}$ in Fig. 6. These two figures show that as $\lambda \to 0+$, the orbit of the solutions seemingly tends to a limit. They also reveal how different the long time dynamical behaviour of the solution in the case of $\lambda \ll 1$ is from that of $\lambda = 0$.

where $h = r/n_0$, $p = \lambda^{-1/n_1}$. Due to its higher accuracy, the Trapezoidal Rule is used in the interval $[0, r]$. Noting from previous sections that the backward Euler methods retain many long time dynamical behaviour such as uniform boundedness, asymptotic stability and algebraic decay for all pairs of (a, b) satisfying $\operatorname{Re} b < 0$, $|b| > |a|$, we use the backward Euler method to perform the numerical experiment outside $[0, r]$. In Figs. 1–6, we let $y_0 = 1$ and display $(\operatorname{Re} y_n, \operatorname{Im} y_n)$ for $n_0 + n_1 n_2 < n \leqslant n_0 + n_1 n_3$, where n_0, n_1 and $n_2 \gg 1$.

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