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Multiple constant sign and nodal solutions for nonlinear elliptic equations with the p-Laplacian

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Abstract

We consider a nonlinear elliptic equation driven by the p-Laplacian with Dirichlet boundary conditions. Using variational techniques combined with the method of upper–lower solutions and suitable truncation arguments, we establish the existence of at least five nontrivial solutions. Two positive, two negative and a nodal (sign-changing) solution. Our framework of analysis incorporates both coercive and p-superlinear problems. Also the result on multiple constant sign solutions incorporates the case of concave–convex nonlinearities.

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1. Introduction

Let $Z \subseteq \mathbb{R}^{\mathbb{N}}$ be a bounded domain with a C^2-boundary ∂Z. We consider the following nonlinear elliptic problem:

$$\begin{cases} -\operatorname{div}\left(\|Dx(z)\|^{p-2}Dx(z)\right) = f\left(z, x(z)\right) & \text{in } Z, \\ x|_{\partial Z} = 0, \quad 1 < p < \infty. \end{cases} \tag{1.1}$$

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The goal of this work is to prove multiplicity results for problem (1.1) without assuming any symmetry conditions on the right-hand side nonlinearity $x \to f(z, x)$ and also determine the sign of the solutions. Our results are very general and cover both problems with coercive and indefinite Euler functional and improve several works existing in the literature.

Recently multiplicity results for the p-Laplacian without any symmetry conditions on the right-hand side nonlinearity, were proved by Jiu and Su [15], Liu and Liu [17] and Liu [18]. Their approach uses variational methods combined with Morse theory (critical groups). However, the multiplicity results they prove, do not provide information about the sign of all solutions.

We also mention the very recent work of Motreanu, Motreanu and Papageorgiou [20], who consider a class of nonlinear eigenvalue problems and using variational and truncation techniques, prove the existence of three nontrivial solutions, one positive, the second negative, but they do not determine the sign of the third solution. However, in [20] the growth of the nonlinearity is a general polynomial growth, not necessarily subcritical.

The existence of multiple positive solutions was investigated by Ambrosetti, Garcia Azorero and Peral Alonso [1], Anello and Cordaro [3] and Garcia Azorero, Manfredi and Peral Alonso [11].

In [1] and [11], the right-hand side nonlinearity has the form $\lambda|x|^{q-2}x + |x|^{r-2}x$, $1 < q < p < r < p^*$, and $\lambda > 0$ is a parameter (problems with concave and convex nonlinearities). The authors prove the existence of a $\lambda_0 > 0$, such that for all $\lambda \in (0, \lambda_0)$, the equation has at least two positive solutions. In [1] they use the radial p-Laplacian and the main tool in the proof is the Leray–Schauder degree theory. In [11], $Z \subseteq \mathbb{R}^{\mathbb{N}}$ is an arbitrary bounded domain with a smooth boundary and their approach is variational based on the critical point theory. Anello and Cordaro [3] use a different set of technical hypotheses, which distinguish their nonlinearity from that in [1] and [11] and they prove the existence of a whole sequence of small positive solutions, which converge uniformly to zero. Their method of proof is completely different and is based on an abstract variational principle due to Ricceri [22].

The question of existence of nodal (sign-changing) solutions was investigated for the p-Laplacian only very recently. We have the works of Bartsch and Liu [4], Carl and Perera [6], Zhang and Li [25] and Zhang, Chen and Li [24]. Bartsch and Liu [4] use critical point theory for C^1-functionals on ordered Banach spaces. Carl and Perera [6] extend to the p-Laplacian, the method of Dancer and Du [9] (semilinear equations, i.e. $p = 2$), which is based on upper–lower solutions and variational arguments. Finally Zhang and Li [25] and Zhang, Chen and Li [24] carefully construct a pseudogradient vector field, whose descent flow has the appropriate invariance properties.

Our approach uses variational arguments based on critical point theory, the method of upper and lower solutions and suitable truncation techniques.

2. Mathematical background

In the analysis of problem (1.1), we use some basic facts about the spectrum of the negative p-Laplacian with Dirichlet boundary conditions. So let $m \in L^\infty(Z)_+$, $m \neq 0$ and consider the following nonlinear weighted (with weight m) eigenvalue problem:

$$\begin{cases} -\operatorname{div}\big(\|Dx(z)\|^{p-2}Dx(z)\big) = \hat{\lambda} m(z)\big|x(z)\big|^{p-2}x(z) & \text{in } Z, \\ x|_{\partial Z} = 0, \quad 1 < p < \infty, \quad \hat{\lambda} \in \mathbb{R}. \end{cases} \tag{2.1}$$

The least number $\hat{\lambda} \in \mathbb{R}$ for which problem (2.1) has a nontrivial solution, is the first eigenvalue of $(-\Delta_p, W_0^{1,p}(Z), m)$ and it is denoted by $\hat{\lambda}_1(m)$. We know that $\hat{\lambda}_1(m) > 0$, it is isolated and also simple (i.e., the associated eigenspace is one-dimensional). Moreover, we have the following variational characterization of $\hat{\lambda}_1(m) > 0$,

$$\hat{\lambda}_1(m) = \min\left[\frac{\|Dx\|_p^p}{\int_Z m|x|^p\,dz}: x \in W_0^{1,p}(Z),\ x \neq 0\right]. \tag{2.2}$$

The minimum in (2.2) is attained on the corresponding one-dimensional eigenspace. In what follows, by $u_1 \in W_0^{1,p}(Z)$ we denote the normalized eigenfunction (i.e. $\int_Z m|u_1|^p\,dz = 1$). Note that $|u_1|$ also realizes the minimum and so u_1 does not change sign and we may assume that $u_1(z) \geqslant 0$ a.e. on Z. Moreover, from nonlinear regularity theory (see Lieberman [16], Gasinski and Papageorgiou [12, p. 738] and the references therein), we have $u_1 \in C_0^1(\overline{Z}) = \{u \in C^1(\overline{Z}):\ u(z) = 0 \text{ for all } z \in \partial Z\}$. The Banach space $C_0^1(\overline{Z})$ is an ordered Banach space with order cone given by

$$C_+ = \left\{x \in C_0^1(\overline{Z}):\ x(z) \geqslant 0 \text{ for all } z \in \overline{Z}\right\}.$$

We know that this cone has a nonempty interior and in fact we have

$$\operatorname{int} C_+ = \left\{x \in C_+:\ x(z) > 0 \text{ for all } z \in Z \text{ and } \frac{\partial x}{\partial n}(z) < 0 \text{ for all } z \in \partial Z\right\}.$$

Here by $n(z)$ we denote the unit outward normal at $z \in \partial Z$. By virtue of the strong maximum principle of Vazquez [23], we have $u_1 \in \operatorname{int} C_+$.

The Lusternik–Schnirelmann theory, in addition to $\hat{\lambda}_1(m) > 0$, gives a whole strictly increasing sequence $\{\hat{\lambda}_k(m)\}_{k\geqslant 1} \subseteq \mathbb{R}_+$ of eigenvalues of (2.1) and $\hat{\lambda}_k(m) \to +\infty$ as $k \to \infty$. These are the so-called "LS (or variational) eigenvalues" of $(-\Delta_p, W_0^{1,p}(Z), m)$. When $p = 2$ (linear eigenvalue problem), these are all the eigenvalues of (2.1). If $p \neq 2$ (nonlinear eigenvalue problem), we do not know if this is true. However, since $\hat{\lambda}_1(m) > 0$ is isolated, we can define

$$\hat{\lambda}_2^*(m) = \inf\left\{\hat{\lambda}:\ \hat{\lambda} \text{ is an eigenvalue of (2.1)}, \lambda > \hat{\lambda}_1(m)\right\} > \hat{\lambda}_1(m).$$

Since the set of eigenvalues of (2.1) is closed, we infer that $\hat{\lambda}_2^*(m)$ is the second eigenvalue of $(-\Delta_p, W_0^{1,p}(Z), m)$. In fact we have

$$\hat{\lambda}_2^*(m) = \hat{\lambda}_2(m),$$

i.e., the second eigenvalue and the second LS-eigenvalue of $(-\Delta_p, W_0^{1,p}(Z), m)$ coincide. So, the second eigenvalue admits a variational characterization provided by the Lusternik–Schnirelmann theory. The eigenvalues $\hat{\lambda}_1(m)$ and $\hat{\lambda}_2(m)$ exhibit certain monotonicity properties with respect to the weight $m \in L^\infty(Z)_+$, namely:

- If $m(z) \leqslant m'(z)$ a.e. on Z, $m \neq m'$, then $\hat{\lambda}_1(m') < \hat{\lambda}_1(m)$ (see (2.2)).
- If $m(z) < m'(z)$ a.e. on Z, then $\hat{\lambda}_2(m') < \hat{\lambda}_2(m)$ (see Anane and Tsouli [2]).

If $m \equiv 1$, then we set $\hat{\lambda}_1(m) = \lambda_1$ $\hat{\lambda}_2(m) = \lambda_2$. For $\lambda_2 > 0$, Cuesta, De Figueiredo and Gossez [8], produced an alternative variational characterization. More precisely, let $\partial B_1^{L^p(Z)} = \{x \in L^p(Z)\colon \|x\|_p = 1\}$, $S = W_0^{1,p}(Z) \cap \partial B_1^{L^p(Z)}$ and $\Gamma_0 = \{\gamma_0 \in C([-1,1], S)\colon \gamma(-1) = -u_1, \gamma(1) = u_1\}$.

Then we have

$$\lambda_2 = \inf_{\gamma_0 \in \Gamma_0} \sup_{x \in \gamma_0([-1,1])} \|Dx\|_p^p. \tag{2.3}$$

This characterization of $\lambda_2 > 0$ will be useful in establishing the existence of nodal solutions. Another result that we will need in that direction, is the so-called "second deformation theorem." To state this theorem, we need to introduce some notions and some notation.

Definition 2.1. Let Y be a Hausdorff topological space and $A \subseteq Y$ nonempty.

(a) A deformation of A is a continuous map $h : [0,1] \times A \to A$ such that

$$h(0, y) = y \quad \text{for all } y \in A.$$

(b) If $C \subseteq A$ is nonempty, then we say that C is a "strong deformation retract" of A, there exists a deformation h of A, such that

$$h(1, A) \subseteq C \quad \text{and} \quad h(t, \cdot)|_C = \mathrm{id}|_C \quad \text{for all } t \in [0,1].$$

Now let X be a Banach space, $\varphi \in C^1(X)$ and $c \in \mathbb{R}$. We introduce the following sets:

$$\varphi^c = \{x \in X\colon \varphi(x) \leqslant c\} \quad (\text{sublevel set of } \varphi \text{ at } c),$$
$$K = \{x \in X\colon \varphi'(x) = 0\} \quad (\text{set of critical points of } \varphi), \quad \text{and}$$
$$K_c = \{x \in K\colon \varphi(x) = c\} \quad (\text{set of critical points of } \varphi \text{ at level } c).$$

Definition 2.2. Let $\varphi \in C^1(X)$. We say that φ satisfies the "Palais–Smale condition at level $c \in \mathbb{R}$" (the "PS_c-condition" for short), if every sequence $\{x_n\}_{n \geqslant 1} \subseteq X$ such that

$$\varphi(x_n) \to c \quad \text{and} \quad \varphi'(x_n) \to 0 \quad \text{in } X^* \text{ as } n \to \infty,$$

has a strongly convergent subsequence. We say that φ satisfies the "PS-condition," if it satisfies the "PS_c-condition" for every $c \in \mathbb{R}$.

The second deformation theorem, reads as follows (see Chang [7, p. 23] and Gasinski and Papageorgiou [12, p. 628]). Note that, if $b = +\infty$, the $\varphi^b \setminus K_b = X$.

Theorem 2.3. *If $\varphi \in C^1(X)$, $a \in \mathbb{R}$, $a < b \leqslant +\infty$, φ satisfies the PS_c-condition for every $c \in [a, b)$, φ has no critical values in (a, b) and $\varphi^{-1}(a)$ contains at most a finite number of critical points of φ, then there exists a deformation $h : [0,1] \times (\varphi^b \setminus K_b) \to \varphi^b$ such that*

(a) φ^a *is a strong deformation retract of* $\varphi^b \setminus K_b$;
(b) $\varphi(h(t,x)) \leqslant \varphi(h(s,x))$ *for all* $t,s \in [0,1]$, $s \leqslant t$ *and* $x \in \varphi^b \setminus K_b$ (*i.e. the deformation* h *is* φ*-decreasing*).

Finally, we recall the notions of upper and lower solutions for problem (1.1).

Definition 2.4. (a) A function $\overline{x} \in W^{1,p}(Z)$ with $\overline{x}|_{\partial Z} \geqslant 0$, is an "upper solution" for problem (1.1), if

$$\int_Z \|D\overline{x}\|^{p-2}(D\overline{x}, D\psi)_{\mathbb{R}^N}\,dz \geqslant \int_Z f\big(z,\overline{x}(z)\big)\psi(z)\,dz, \tag{2.4}$$

for all $\psi \in C_0^1(\overline{Z})$ with $\psi(z) \geqslant 0$, $z \in \overline{Z}$. If $\overline{x}$ is not a solution, then $\overline{x}$ is said to be a "strict upper solution."

(b) A function $\underline{x} \in W^{1,p}(Z)$ with $\underline{x}|_{\partial Z} \leqslant 0$, is a "lower solution" for problem (1.1), if

$$\int_Z \|D\underline{x}\|^{p-2}(D\underline{x}, D\psi)_{\mathbb{R}^N}\,dz \leqslant \int_Z f\big(z,\underline{x}(z)\big)\psi(z)\,dz, \tag{2.5}$$

for all $\psi \in C_0^1(\overline{Z})$ with $\psi(z) \geqslant 0$, $z \in \overline{Z}$. If $\underline{x}$ is not a solution, then $\underline{x}$ is said to be a "strict lower solution."

3. Solutions of constant sign

We start with the following hypotheses on the right-hand side nonlinearity $f(z,x)$:

$H(f)_1$ $f: Z \times \mathbb{R} \to \mathbb{R}$ is a function such that $f(z,0) = 0$ a.e. on Z and

(i) for all $x \in \mathbb{R}$, $z \to f(z,x)$ is measurable;

(ii) for a.a. $z \in Z$, $x \to f(z,x)$ is continuous;

(iii) for a.a. $z \in Z$ and all $x \in \mathbb{R}$, we have

$$\big|f(z,x)\big| \leqslant a(z) + c|x|^{r-1}$$

with $a \in L^\infty(Z)_+$, $c > 0$, $1 < r < p^* = \begin{cases} \frac{Np}{N-p} & \text{if } N > p, \\ 0 & \text{otherwise;} \end{cases}$

(iv) there exists $\eta \in L^\infty(Z)_+$ such that $\lambda_1 \leqslant \eta(z)$ a.e. on Z, $\lambda_1 \neq \eta$ and

$$\eta(z) \leqslant \liminf_{x\to 0} \frac{f(z,x)}{|x|^{p-2}x} \quad \text{uniformly for a.a. } z \in Z;$$

(v) there exist a strict upper solution $\overline{x} \in \operatorname{int} C_+$ and a strict lower solution $\underline{v} \in -\operatorname{int} C_+$ for problem (1.1), such that $-\Delta_p\overline{x}, -\Delta_p\underline{v} \in L^\infty(Z)$ and $f(z,x) < -\Delta_p\overline{x}(z)$ a.e. on Z for all $x \in [0,\overline{x}(z)]$, $f(z,x) > -\Delta_p\underline{v}(z)$ a.e. on Z for all $x \in [-\underline{v}(z), 0]$;

(vi) there exist $\delta_0, \sigma_0 > 0$ such that
for a.a. $z \in Z$, $f(z,\cdot)$ is increasing on $[-\delta_0, \delta_0]$ and
for a.a. $z \in Z$, $f(z,x) \geqslant \sigma_0$ when $x \geqslant \delta_0$ and $f(z,x) \leqslant -\sigma_0$ when $x \leqslant -\delta_0$.

First, we produce a strict lower solution $\underline{x} \in \operatorname{int} C_+$, $\overline{x} - \underline{x} \in \operatorname{int} C_+$ and a strict upper solution $\overline{v} \in -\operatorname{int} C_+$, $\overline{v} - \underline{v} \in \operatorname{int} C_+$, for problem (1.1). To this end, let u_1 be the L^p-normalized principal eigenfunction of $(-\Delta_p, W_0^{1,p}(Z))$. We consider the following auxiliary boundary value problem:

$$\begin{cases} -\operatorname{div}\big(\|Dx(z)\|^{p-2}Dx(z)\big) = \eta(z)\big|x(z)\big|^{p-2}x(z) - u_1(z)^{p-1} & \text{in } Z, \\ x|_{\partial Z} = 0. \end{cases} \tag{3.1}$$

Here $\eta \in L^\infty(Z)_+$ is as in hypothesis $H(f)_1$(iv). We solve problem (3.1). The solutions of (3.1), are the critical points of the C^1-functional $\varphi_0 : W_0^{1,p}(Z) \to \mathbb{R}$ defined by

$$\varphi_0(x) = \frac{1}{p}\|Dx\|_p^p - \frac{1}{p}\int_Z \eta|x|^p\,dz + \int_Z u_1^{p-1}x\,dz \quad \text{for all } x \in W_0^{1,p}(Z).$$

In what follows by $\langle\cdot,\cdot\rangle$ we denote the duality brackets for the pair $(W^{-1,p'}(Z), W_0^{1,p}(Z))$ $(\frac{1}{p} + \frac{1}{p'} = 1)$. Let $A : W_0^{1,p}(Z) \to W^{-1,p'}(Z)$ be the nonlinear operator defined by

$$\big\langle A(x), y\big\rangle = \int_Z \|Dx\|^{p-2}(Dx, Dy)_{\mathbb{R}^N}\,dz \quad \text{for all } x, y \in W_0^{1,p}(Z).$$

Note that for all $x \in W_0^{1,p}(Z)$

$$\varphi_0'(x) = A(x) - \eta|x|^{p-2}x + u_1^{p-1}. \tag{3.2}$$

Proposition 3.1. *$\varphi_0 : W_0^{1,p}(Z) \to \mathbb{R}$ satisfies the* PS*-condition.*

Proof. Let $\{x_n\}_{n\geqslant 1} \subseteq W_0^{1,p}(Z)$ be a sequence such that

$$\big|\varphi_0(x_n)\big| \leqslant M_1 \quad \text{for some } M_1 > 0, \text{ all } n \geqslant 1 \quad \text{and} \quad \varphi_0'(x_n) \to 0 \quad \text{in } W^{-1,p'}(Z).$$

We have (see (3.2))

$$\big|\big\langle\varphi_0'(x_n), v\big\rangle\big| = \left|\big\langle A(x_n), v\big\rangle - \int_Z \eta|x_n|^{p-2}x_n v\,dz + \int_Z u_1^{p-1}v\,dz\right| \leqslant \varepsilon_n\|v\|, \tag{3.3}$$

for all $v \in W_0^{1,p}(Z)$, with $\varepsilon_n \downarrow 0$. We claim that $\{x_n\}_{n\geqslant 1} \subseteq W_0^{1,p}(Z)$ is bounded. We argue indirectly. So suppose that $\{x_n\}_{n\geqslant 1} \subseteq W_0^{1,p}(Z)$ is unbounded. We may assume that

$$\|x_n\| \to \infty \quad \text{as } n \to \infty.$$

We set $y_n = \frac{x_n}{\|x_n\|}$, $n \geqslant 1$. By passing to a suitable subsequence if necessary, we may assume that

$$y_n \xrightarrow{w} y \quad \text{in } W_0^{1,p}(Z), \qquad y_n \to y \quad \text{in } L^p(Z), \qquad y_n(z) \to y(z) \quad \text{a.e. on } Z$$

and

$$\big|y_n(z)\big| \leqslant k(z) \quad \text{for a.a. } z \in Z,\ \text{all } n \geqslant 1,\ \text{with } k \in L^p(Z)_+.$$

In (3.3) we use the test function $v = y_n - y \in W_0^{1,p}(Z)$ and we divide with $\|x_n\|^{p-1}$. We obtain

$$\left|\big\langle A(y_n), y_n - y\big\rangle - \int_Z \eta|y_n|^{p-2}y_n(y_n - y)\,dz + \int_Z \frac{u_1}{\|x_n\|^{p-1}}(y_n - y)\,dz\right| \leqslant \varepsilon_n\|y_n - y\|. \tag{3.4}$$

Evidently

$$\int_Z \eta|y_n|^{p-2}y_n(y_n - y)\,dz \to 0 \quad \text{and} \quad \int_Z \frac{u_1}{\|x_n\|^{p-1}}(y_n - y)\,dz \to 0 \quad \text{as } n \to \infty.$$

So from (3.4), we have

$$\lim\big\langle A(y_n), y_n - y\big\rangle = 0. \tag{3.5}$$

But clearly A is demicontinuous, monotone (in fact strictly monotone), hence maximal monotone. A maximal monotone operator, is generalized pseudomonotone (see Gasinski and Papageorgiou [12, p. 230]). So from (3.5) it follows that

$$\|Dy_n\|_p^p = \big\langle A(y_n), y_n\big\rangle \to \big\langle A(y), y\big\rangle = \|Dy\|_p^p.$$

Recall that $Dy_n \xrightarrow{w} Dy$ in $L^p(Z, \mathbb{R}^{\mathbb{N}})$. The space $L^p(Z, \mathbb{R}^{\mathbb{N}})$ is uniformly convex. Therefore, from the Kadec–Klee property, we have

$$Dy_n \to Dy \quad \text{in } L^p\big(Z, \mathbb{R}^{\mathbb{N}}\big) \quad \Rightarrow \quad y_n \to y \quad \text{in } W_0^{1,p}(Z) \quad \text{and so} \quad \|y\| = 1.$$

Also from the choice of the sequence $\{x_n\}_{n\geqslant} \subseteq W_0^{1,p}(Z)$, we have

$$\begin{aligned} &A(y) = \eta|y|^{p-2}y \\ &\Rightarrow \quad \begin{cases} -\operatorname{div}\big(\|Dy(z)\|^{p-2}Dy(z)\big) = \eta(z)\big|y(z)\big|^{p-2}y(z) & \text{a.e. on } Z, \\ y|_{\partial Z} = 0, \quad y \neq 0. \end{cases} \end{aligned} \tag{3.6}$$

Without any loss of generality, we may assume that

$$\lambda_1 \leqslant \eta(z) < \lambda_2 \quad \text{a.e. on } Z, \quad \lambda_1 \neq \eta \tag{3.7}$$

(see hypothesis $H(f)_1$(v)). Then from the monotonicity properties of $\hat{\lambda}_1(m)$, $\hat{\lambda}_2(m)$ on the weight function m (see Section 2), we have

$$\hat{\lambda}_1(\eta) < \hat{\lambda}_1(\lambda_1) = 1 \tag{3.8}$$

and

$$\hat{\lambda}_2(\eta) > \hat{\lambda}_2(\lambda_2) = 1 \quad \big(\text{see (3.7)}\big). \tag{3.9}$$

From (3.6), (3.8) and (3.9), we infer that $y = 0$, a contradiction. So $\{x_n\}_{n\geqslant 1} \subseteq W_0^{1,p}(Z)$ is bounded and we may assume that

$$x_n \to x \quad \text{in } W_0^{1,p}(Z), \qquad x_n \to x \quad \text{in } L^p(Z), \qquad x_n(z) \to x(z) \quad \text{a.e. on } Z,$$

and

$$\big|x_n(z)\big| \leqslant k(z) \quad \text{for a.a. } z \in Z,\ \text{all } n \geqslant 1,\ \text{with } k \in L^p(Z)_+.$$

In (3.3) we set $v = x_n - x$ and pass to the limit as $n \to \infty$. We obtain

$$\lim_{n\to\infty} \big\langle A(x_n), x_n - x \big\rangle = 0.$$

From this limit, as above we deduce that $x_n \to x$ in $W_0^{1,p}(Z)$. This proves that φ_0 satisfies the PS-condition. □

Let $V = \{x \in W_0^{1,p}(Z)\colon \int_Z u_1^{p-1} x\, dz = 0\}$. We have the direct sum decomposition

$$W_0^{1,p}(Z) = \mathbb{R}u_1 \oplus V.$$

We define

$$\lambda_V = \inf\left[\frac{\|Dx\|_p^p}{\|x\|_p^p}\colon x \in V,\ x \neq 0\right] > \lambda_1.$$

Again without any loss of generality we may assume that

$$\lambda_1 \leqslant \eta(z) \leqslant \lambda_1 + \varepsilon < \lambda_V \leqslant \lambda_2 \quad \text{a.e. on } Z,\ \text{for } \varepsilon > 0 \text{ small} \tag{3.10}$$

(see hypothesis $H(f)$(iv)). Because of (3.10), it is clear that we have

Proposition 3.2. $\varphi_0|_V \geqslant 0$.

Proposition 3.3. *For $t > 0$ large, we have $\varphi_0(\pm t u_1) < 0$.*

Proof. For $t > 0$, we have

$$\begin{aligned}
\varphi_0(\pm t u_1) &= \frac{t^p}{p}\|Du_1\|_p^p - \frac{t^p}{p}\int_Z \eta u_1^p\, dz + t\|u_1\|_p^p \\
&\leqslant \frac{t^p}{p}\left[\int_Z \big(\lambda_1 - \eta(z)\big)u_1(z)^p\, dz\right] + t \quad \big(\text{since } \|u_1\|_p^p = 1,\ t > 0\big).
\end{aligned} \tag{3.11}$$

Since $u_1 \in \operatorname{int} C_+$ and $\lambda_1 \leqslant \eta(z)$ a.e. on Z with $\lambda_1 \neq \eta$, we have

$$\xi = \int_Z \big(\lambda_1 - \eta(z)\big) u_1(z)^{p-1}\, dz < 0 \quad \Rightarrow \quad \varphi_0(\pm t u_1) \leqslant \frac{t^p}{p}\xi + t \quad \big(\text{see (3.11)}\big). \tag{3.12}$$

Therefore, if in (3.12) $t > 0$ is large, then $\varphi_0(\pm t u_1) < 0$. □

Proposition 3.4. *The auxiliary problem* (3.1) *has a solution* $\underline{\hat{x}} \in \operatorname{int} C_+$.

Proof. Propositions 3.1–3.3 permit the application of the saddle point theorem. So we can find $\underline{\hat{x}} \in W_0^{1,p}(Z)$ such that

$$\varphi_0'(\underline{\hat{x}}) = 0 \quad \Rightarrow \quad A(\underline{\hat{x}}) = \eta |\underline{\hat{x}}|^{p-2}\underline{\hat{x}} - u_1^{p-1}, \quad \text{hence} \quad \underline{\hat{x}} \neq 0,$$

$$\begin{cases} -\operatorname{div}\big(\|D\underline{\hat{x}}(z)\|^{p-2} D\underline{\hat{x}}(z)\big) = \eta(z)\big|\underline{\hat{x}}(z)\big|^{p-2}\underline{\hat{x}} - u_1(z)^{p-1} & \text{a.e. on } Z, \\ \underline{\hat{x}}|_{\partial Z} = 0, \quad \underline{\hat{x}} \neq 0. \end{cases} \tag{3.13}$$

From nonlinear regularity theory, we have $\underline{\hat{x}} \in C_0^1(\overline{Z})$. By taking in (3.10) $\varepsilon > 0$ even smaller if necessary, we can apply Theorem 5.1 of Godoy, Gossez and Paczka [13] (the antimaximum principle) and conclude that $\underline{\hat{x}} \in \operatorname{int} C_+$. □

We also consider the auxiliary problem

$$\begin{cases} -\operatorname{div}\big(\|Dv(z)\|^{p-2} Dv(z)\big) = \eta(z)\big|v(z)\big|^{p-2} v(z) + u_1(z)^{p-1} & \text{a.e. on } Z, \\ v|_{\partial Z} = 0. \end{cases} \tag{3.14}$$

The corresponding Euler functional $\psi_0 : W_0^{1,p}(Z) \to \mathbb{R}$ is defined by

$$\psi_0(v) = \frac{1}{p}\|Dv\|_p^p - \frac{1}{p}\int_Z \eta |v|^p\, dz - \int_Z u_1^{p-1} v\, dz \quad \text{for all } v \in W_0^{1,p}(Z).$$

Working as for problem (3.1), using this time $\psi_0 \in C^1(W_0^{1,p}(Z))$, we obtain:

Proposition 3.5. *The auxiliary problem* (3.14) *has a solution* $\hat{\overline{v}} \in -\operatorname{int} C_+$.

Using $\underline{\hat{x}}$ and $\hat{\overline{v}}$, we will produce the desired lower and upper solutions for problem (1.1). We will need the following simple fact about ordered Banach spaces.

Lemma 3.6. *If X is an ordered Banach space, K is the order cone of X,* $\operatorname{int} K \neq \emptyset$ *and* $x_0 \in \operatorname{int} K$, *then for every $y \in X$, we can find $t = t(y) > 0$ such that $t x_0 - y \in \operatorname{int} K$.*

Proof. Since $x_0 \in \operatorname{int} K$, we can find $\delta > 0$ such that

$$\overline{B}_\delta(x_0) = \big\{x \in X: \ \|x - x_0\| \leqslant \delta\big\} \subseteq \operatorname{int} K.$$

Let $y \in X$ and assume that $y \neq 0$ (if $y = 0$, then clearly the lemma is true for all $t > 0$). We have

$$x_0 - \delta \frac{y}{\|y\|} \in \operatorname{int} K \quad \Rightarrow \quad \frac{\|y\|}{\delta} x_0 - y \in \operatorname{int} K.$$

So, if $t = t(y) = \frac{\|y\|}{\delta}$, then $tx_0 - y \in \operatorname{int} K$. □

Proposition 3.7. *If hypotheses $H(f)_1$ hold, then problem* (1.1) *has a strict lower solution $\underline{x} \in \operatorname{int} C_+$ with $\overline{x} - \underline{x} \in \operatorname{int} C_+$ and a strict upper solution $\overline{v} \in -\operatorname{int} C_+$ with $\overline{v} - \underline{v} \in \operatorname{int} C_+$.*

Proof. By virtue of hypothesis $H(f)_1$(iv), given $\varepsilon > 0$, we can find $\delta = \delta(\varepsilon) > 0$ such that

$$\big(\eta(z) - \varepsilon\big) x^{p-1} \leqslant f(z, x) \quad \text{for a.a. } z \in Z \text{ and all } x \in [0, \delta]. \tag{3.15}$$

We choose $\varepsilon > 0$ small as indicated in the proof of Proposition 3.4 and also such that by virtue of Lemma 3.6, we have

$$u_1^{p-1} - \varepsilon \hat{\underline{x}}^{p-1} \in \operatorname{int} C_+. \tag{3.16}$$

Having chosen $\varepsilon > 0$ this way and using Lemma 3.6 once more, we can find $\beta \in (0, 1]$ small such that

$$\overline{x} - \beta \hat{\underline{x}} \in \operatorname{int} C_+, \quad u_1 - \beta \hat{\underline{x}} \in \operatorname{int} C_+ \quad \text{and} \quad \beta \hat{\underline{x}}(z) \in [0, \delta] \quad \text{for all } z \in \overline{Z} \tag{3.17}$$

(recall that $\hat{\underline{x}} \in \operatorname{int} C_+$, see Proposition 3.4). We set $\underline{x} = \beta \hat{\underline{x}} \in \operatorname{int} C_+$. Then for a.a. $z \in Z$ we have

$$\begin{aligned} -\operatorname{div}\big(\big\|D\underline{x}(z)\big\|^{p-2} D\underline{x}(z)\big) &= -\beta^{p-1} \operatorname{div}\big(\big\|D\hat{\underline{x}}(z)\big\|^{p-1} D\hat{\underline{x}}(z)\big) \\ &= \beta^{p-1}\big(\eta(z)\hat{\underline{x}}(z)^{p-1} - u_1(z)^{p-1}\big) \quad \big(\text{see (3.13)}\big) \\ &= \eta(z)\underline{x}(z)^{p-1} - \beta^{p-1} u_1(z)^{p-1} \quad (\text{since } \underline{x} = \beta \hat{\underline{x}}) \\ &< \eta(z)\underline{x}(z)^{p-1} - \varepsilon \underline{x}(z)^{p-1} \quad \big(\text{see (3.16)}\big) \\ &\leqslant f\big(z, \underline{x}(z)\big) \quad \big(\text{see (3.15)}\big). \end{aligned}$$

Therefore $\underline{x} \in \operatorname{int} C_+$ is a strict lower solution for problem (1.1) (see Definition 2.4(b)). From (3.17) we have $\overline{x} - \underline{x} \in \operatorname{int} C_+$.

Similarly, using $\hat{\overline{v}} \in -\operatorname{int} C_+$ and since $(\eta(z) - \varepsilon)|x|^{p-2}x \geqslant f(z, x)$ for a.a. $z \in Z$ and all $x \in [-\delta, 0]$, we obtain $\overline{v} = \beta' \hat{\overline{v}} \in -\operatorname{int} C_+$ with $\beta' \in (0, 1]$, a strict upper solution for problem (1.1) such that $\overline{v} - \underline{v} \in \operatorname{int} C_+$. □

Now using the ordered pairs of upper–lower solutions $\{\overline{x}, \underline{x}\}$ and $\{\overline{v}, \underline{v}\}$, we will produce the first two solutions of constant sign.

Let $\varphi : W_0^{1,p}(Z) \to \mathbb{R}$ be the Euler functional for problem (1.1) defined by

$$\varphi(x) = \frac{1}{p}\|Dx\|_p^p - \int_Z F\big(z, x(z)\big)\, dz \quad \text{for all } x \in W_0^{1,p}(Z),$$

with $F(z, x) = \int_0^x f(z, s)\, ds$. Evidently $\varphi \in C^1(W_0^{1,p}(Z))$.

Proposition 3.8. *If hypotheses $H(f)_1$ hold, then problem* (1.1) *has two solutions $x_0 \in \operatorname{int} C_+$ and $v_0 \in -\operatorname{int} C_+$ both local minimizers of φ.*

Proof. We introduce the order interval

$$I_+ = [\underline{x}, \overline{x}] = \{x \in W_0^{1,p}(Z):\ \underline{x}(z) \leqslant x(z) \leqslant \overline{x}(z) \text{ a.e. on } Z\}$$

and the following truncation of the nonlinearity $f(z,x)$,

$$\widetilde{f_+}(z,x) = \begin{cases} f(z,\underline{x}(z)) & \text{if } x < \underline{x}(z), \\ f(z,x) & \text{if } \underline{x}(z) \leqslant x \leqslant \overline{x}(z), \\ f(z,\overline{x}(z)) & \text{if } \overline{x}(z) < x. \end{cases} \tag{3.18}$$

We set $\widetilde{F_+}(z,x) = \int_0^x \widetilde{f_+}(z,s)\,ds$ and we consider the functional $\widetilde{\varphi_+} : W_0^{1,p}(Z) \to \mathbb{R}$ defined by

$$\widetilde{\varphi_+}(x) = \frac{1}{p}\|Dx\|_p^p - \int_Z \widetilde{F_+}\big(z,x(z)\big)\,dz \quad \text{for all } x \in W_0^{1,p}(Z).$$

We have $\widetilde{\varphi_+} \in C^1(W_0^{1,p}(Z))$ and in addition, due to the compact embedding of $W_0^{1,p}(Z)$ into $L^p(Z)$, we can easily see that $\widetilde{\varphi_+}$ is w-lower semicontinuous. Moreover, from (3.18) and hypothesis $H(f)_1$(iii), we see that

$$\widetilde{\varphi_+}(x) \geqslant \frac{1}{p}\|Dx\|_p^p - c_1 \quad \text{for some } c_1 > 0, \text{ all } x \in W_0^{1,p}(Z) \quad \Rightarrow \quad \widetilde{\varphi_+} \text{ is coercive.}$$

So, by the Weierstrass theorem, we can find $x_0 \in I_+$ such that

$$\widetilde{\varphi_+}(x_0) = \inf_{I_+} \widetilde{\varphi_+} = \widetilde{m_+}. \tag{3.19}$$

For any $y \in I_+$, let $\xi_0(t) = \widetilde{\varphi_+}(ty + (1-t)x_0)$, $t \in [0,1]$. Then because of (3.19)

$$0 \leqslant \xi_0'(0) \quad \Rightarrow \quad 0 \leqslant \langle A(x_0), y - x_0\rangle - \int_Z \widetilde{f_+}\big(z,x_0(z)\big)(y-x_0)(z)\,dz. \tag{3.20}$$

Given $h \in W_0^{1,p}(Z)$ and $\varepsilon > 0$, we set

$$y(z) = \begin{cases} \underline{x}(z) & \text{if } z \in \{x_0 + \varepsilon h \leqslant \underline{x}\}, \\ x_0(z) + \varepsilon h(z) & \text{if } z \in \{\underline{x} < x_0 + \varepsilon h < \overline{x}\}, \\ \overline{x}(z) & \text{if } z \in \{\overline{x} \leqslant x_0 + \varepsilon h\}. \end{cases}$$

Clearly $y \in I_+$ and so using it in (3.20), we obtain

$$\begin{aligned}
0 \leqslant{}& \varepsilon \int\limits_{\{\underline{x}<x_0+\varepsilon h<\overline{x}\}} \|Dx_0\|^{p-2}(Dx_0, Dh)_{\mathbb{R}^{\mathbb{N}}}\,dz - \varepsilon \int\limits_{\{\underline{x}<x_0+\varepsilon h<\overline{x}\}} f(z,x_0)h\,dz \\
&+ \int\limits_{\{x_0+\varepsilon h\leqslant \underline{x}\}} \|Dx_0\|^{p-2}(Dx_0, D\underline{x}-Dx_0)_{\mathbb{R}^{\mathbb{N}}}\,dz - \int\limits_{\{x_0+\varepsilon h\leqslant \underline{x}\}} f(z,x_0)(\underline{x}-x_0)\,dz \\
&+ \int\limits_{\{\overline{x}\leqslant x_0+\varepsilon h\}} \|Dx_0\|^{p-2}(Dx_0, D\overline{x}-Dx_0)_{\mathbb{R}^{\mathbb{N}}}\,dz - \int\limits_{\{\overline{x}\leqslant x_0+\varepsilon h\}} f(z,x_0)(\overline{x}-x_0)\,dz \\
={}& \varepsilon \int\limits_Z \|Dx_0\|^{p-2}(Dx_0, Dh)_{\mathbb{R}^{\mathbb{N}}}\,dz - \varepsilon \int\limits_Z f(z,x_0)h\,dz \\
&+ \int\limits_{\{x_0+\varepsilon h\leqslant \underline{x}\}} \|D\underline{x}\|^{p-2}\big(D\underline{x}, D(\underline{x}-x_0-\varepsilon h)\big)_{\mathbb{R}^{\mathbb{N}}}\,dz \\
&- \int\limits_{\{x_0+\varepsilon h\leqslant \underline{x}\}} f(z,\underline{x})(\underline{x}-x_0-\varepsilon h)\,dz \\
&- \int\limits_{\{\overline{x}\leqslant x_0+\varepsilon h\}} \|D\overline{x}\|^{p-2}\big(D\overline{x}, D(x_0+\varepsilon h-\overline{x})\big)_{\mathbb{R}^{\mathbb{N}}}\,dz \\
&+ \int\limits_{\{\overline{x}\leqslant x_0+\varepsilon h\}} f(z,\overline{x})(x_0+\varepsilon h-\overline{x})\,dz \\
&- \int\limits_{\{x_0+\varepsilon h\leqslant \underline{x}\}} \big(f(z,x_0)-f(z,\underline{x})\big)(\underline{x}-x_0-\varepsilon h)\,dz \\
&+ \int\limits_{\{\overline{x}\leqslant x_0+\varepsilon h\}} \big(f(z,\overline{x})-f(z,x_0)\big)(\overline{x}-x_0-\varepsilon h)\,dz \\
&+ \int\limits_{\{x_0+\varepsilon h\leqslant \underline{x}\}} \big(\|Dx_0\|^{p-2}Dx_0-\|D\underline{x}\|^{p-2}D\underline{x}, D(\underline{x}-x_0)\big)_{\mathbb{R}^{\mathbb{N}}}\,dz \\
&- \varepsilon \int\limits_{\{x_0+\varepsilon h\leqslant \underline{x}\}} \big(\|Dx_0\|^{p-2}Dx_0-\|D\underline{x}\|^{p-2}D\underline{x}, Dh\big)_{\mathbb{R}^{\mathbb{N}}}\,dz \\
&+ \int\limits_{\{\overline{x}\leqslant x_0+\varepsilon h\}} \big(\|D\overline{x}\|^{p-2}D\overline{x}-\|Dx_0\|^{p-2}Dx_0, D(x_0-\overline{x})\big)_{\mathbb{R}^{\mathbb{N}}}\,dz \\
&+ \varepsilon \int\limits_{\{\overline{x}\leqslant x_0+\varepsilon h\}} \big(\|D\overline{x}\|^{p-2}D\overline{x}-\|Dx_0\|^{p-2}Dx_0, Dh\big)_{\mathbb{R}^{\mathbb{N}}}\,dz. \qquad (3.21)
\end{aligned}$$

Since $\underline{x} \in \operatorname{int} C_+$ is a strict lower solution for problem (1.1), we have (see Definition 2.4(b))

$$\int_{\{x_0+\varepsilon h \leqslant \underline{x}\}} \|D\underline{x}\|^{p-2}\big(D\underline{x}, D(\underline{x} - x_0 - \varepsilon h)\big)_{\mathbb{R}^N}\, dz - \int_{\{x_0+\varepsilon h \leqslant \underline{x}\}} f(z, \underline{x})(\underline{x} - x_0 - \varepsilon h)\, dz \leqslant 0. \tag{3.22}$$

Similarly, since $\overline{x} \in \operatorname{int} C_+$ is a strict upper solution for problem (1.1), we have (see Definition 2.4(a))

$$\int_{\{\overline{x} \leqslant x_0+\varepsilon h\}} \|D\overline{x}\|^{p-2}\big(D\overline{x}, D(x_0 + \varepsilon h - \overline{x})\big)_{\mathbb{R}^N}\, dz - \int_{\{\overline{x} \leqslant x_0+\varepsilon h\}} f(z, \overline{x})(x_0 + \varepsilon h - \overline{x})\, dz \geqslant 0. \tag{3.23}$$

From the monotonicity of the map $\theta_p : \mathbb{R}^N \to \mathbb{R}^N$ defined by $\theta_p(x) = \|x\|^{p-2}x$ for $x \neq 0$, $\theta_p(0) = 0$, we have

$$\int_{\{x_0+\varepsilon h \leqslant \underline{x}\}} \big(\|Dx_0\|^{p-2}Dx_0 - \|D\underline{x}\|^{p-2}D\underline{x}, D(\underline{x} - x_0)\big)_{\mathbb{R}^N}\, dz \leqslant 0 \tag{3.24}$$

and

$$\int_{\{\overline{x} \leqslant x_0+\varepsilon h\}} \big(\|D\overline{x}\|^{p-2}D\overline{x} - \|Dx_0\|^{p-2}Dx_0, D(x_0 - \overline{x})\big)_{\mathbb{R}^N}\, dz \leqslant 0. \tag{3.25}$$

Moreover, we have

$$\begin{aligned}
&- \int_{\{x_0+\varepsilon h \leqslant \underline{x}\}} \big(f(z, x_0) - f(z, \underline{x})\big)(\underline{x} - x_0 - \varepsilon h)\, dz \\
&= - \int_{\{x_0+\varepsilon h \leqslant \underline{x} < x_0\}} \big(f(z, x_0) - f(z, \underline{x})\big)(\underline{x} - x_0 - \varepsilon h)\, dz \\
&\leqslant c_2 \int_{\{x_0+\varepsilon h \leqslant \underline{x} < x_0\}} (\underline{x} - x_0 - \varepsilon h)\, dz \quad \text{for some } c_2 > 0 \quad \big(\text{see hypothesis } H(f)_1\text{(iii)}\big) \\
&\leqslant -\varepsilon c_2 \int_{\{x_0+\varepsilon h \leqslant \underline{x} < x_0\}} h\, dz \quad (\text{since } \underline{x} \leqslant x_0)
\end{aligned} \tag{3.26}$$

and

$$\int\limits_{\{\overline{x}\leqslant x_0+\varepsilon h\}} \big(f(z,\overline{x})-f(z,x_0)\big)(\overline{x}-x_0-\varepsilon h)\,dz$$

$$= -\int\limits_{\{x_0<\overline{x}\leqslant x_0+\varepsilon h\}} \big(f(z,\overline{x})-f(z,x_0)\big)(\overline{x}-x_0-\varepsilon h)\,dz$$

$$\leqslant c_3 \int\limits_{\{x_0<\overline{x}\leqslant x_0+\varepsilon h\}} (x_0+\varepsilon h-\overline{x})\,dz \quad \text{for some } c_3>0 \quad \big(\text{see hypothesis } H(f)_1(\text{iii})\big)$$

$$\leqslant \varepsilon c_3 \int\limits_{\{x_0<\overline{x}\leqslant x_0+\varepsilon h\}} h\,dz \quad (\text{since } x_0\leqslant \overline{x}). \tag{3.27}$$

Returning to (3.21) and using (3.22)–(3.27), we obtain

$$\begin{aligned}0\leqslant{}& \varepsilon\int\limits_Z \|Dx_0\|^{p-2}(Dx_0,Dh)_{\mathbb{R}^\mathbb{N}}\,dz-\varepsilon\int\limits_Z f(z,x_0)h\,dz\\ &-\varepsilon c_2\int\limits_{\{x_0+\varepsilon h\leqslant \underline{x}<x_0\}} h\,dz+\varepsilon c_3\int\limits_{\{x_0<\overline{x}\leqslant x_0+\varepsilon h\}} h\,dz\\ &-\varepsilon\int\limits_{\{x_0+\varepsilon h\leqslant \underline{x}\}}\big(\|Dx_0\|^{p-2}Dx_0-\|D\underline{x}\|^{p-2}D\underline{x},Dh\big)_{\mathbb{R}^\mathbb{N}}\,dz\\ &+\varepsilon\int\limits_{\{\overline{x}\leqslant x_0+\varepsilon h\}}\big(\|D\overline{x}\|^{p-2}D\overline{x}-\|Dx_0\|^{p-2}Dx_0,Dh\big)_{\mathbb{R}^\mathbb{N}}\,dz.\end{aligned} \tag{3.28}$$

If by $|\cdot|_N$ we denote the Lebesgue measure on $\mathbb{R}^\mathbb{N}$, then

$$\big|\{x_0+\varepsilon h\leqslant \underline{x}<x_0\}\big|_N\downarrow 0 \quad\text{and}\quad \big|\{x_0+\varepsilon h\geqslant \overline{x}>x_0\}\big|_N\downarrow 0 \quad\text{as } \varepsilon\downarrow 0. \tag{3.29}$$

Moreover, by Stampacchia's theorem (see for example Gasinski and Papageorgiou [12, pp. 195–196]), we have

$$Dx_0(z)=D\underline{x}(z) \quad\text{a.e. on } \{x_0=\underline{x}\} \quad\text{and}\quad Dx_0(z)=D\overline{x}(z) \quad\text{a.e. on } \{x_0=\overline{x}\}. \tag{3.30}$$

So, if we divide (3.28) by $\varepsilon>0$, then let $\varepsilon\downarrow 0$ and use (3.29), (3.30), we obtain

$$0\leqslant \int\limits_Z \|Dx_0\|^{p-2}(Dx_0,Dh)_{\mathbb{R}^\mathbb{N}}\,dz-\int\limits_Z f(z,x_0)h\,dz. \tag{3.31}$$

Since $h\in W_0^{1,p}(Z)$ was arbitrary, from (3.31) we conclude that

$$\begin{cases} -\operatorname{div}\big(\big\|Dx_0(z)\big\|^{p-2}Dx_0(z)\big)=f\big(z,x_0(z)\big) & \text{in } Z,\\ x_0|_{\partial Z}=0,\end{cases}$$

hence $x_0 \in W_0^{1,p}(Z)$ is a solution of (1.1) and from nonlinear regularity theory we have $x_0 \in \operatorname{int} C_+$.

From the proof of Proposition 3.7, we know that

$$-\operatorname{div}\left(\left\|D\underline{x}(z)\right\|^{p-2} D\underline{x}(z)\right) < f\left(z, \underline{x}(z)\right) \quad \text{a.e. on } Z. \tag{3.32}$$

By hypothesis $H(f)_1$(vi), we have

$$f\left(z, \underline{x}(z)\right) \leqslant f\left(z, x_0(z)\right) = -\operatorname{div}\left(\left\|Dx_0(z)\right\|^{p-2} Dx_0(z)\right) \quad \text{a.e. on } \{x_0 \leqslant \delta_0\}. \tag{3.33}$$

From (3.32) and (3.33), it follows that

$$-\operatorname{div}\left(\left\|D\underline{x}(z)\right\|^{p-2} D\underline{x}(z)\right) < -\operatorname{div}\left(\left\|Dx_0(z)\right\|^{p-2} Dx_0(z)\right) \quad \text{a.e. on } \{x_0 \leqslant \delta_0\}. \tag{3.34}$$

If $\sigma_0 > 0$ is as in hypothesis $H(f)_1$(vi), then in the definition of $\underline{x} \in \operatorname{int} C_+$ (see the proof of Proposition 3.7), we can always choose $\beta \in (0, 1]$ small enough such that

$$\eta(z)\underline{x}(z)^{p-1} = \eta(z)\left(\beta\hat{\underline{x}}(z)\right)^{p-1} \leqslant \sigma_0 \quad \text{a.e. on } \overline{Z}. \tag{3.35}$$

Hypothesis $H(f)_1$(vi) implies that

$$\begin{aligned} &\eta(z)\underline{x}(z)^{p-1} \leqslant f\left(z, x_0(z)\right) \quad \text{a.e. on } \{x_0 > \delta_0\} \quad \left(\text{see (3.35)}\right) \\ \Rightarrow \quad &-\operatorname{div}\left(\left\|D\underline{x}(z)\right\|^{p-2} D\underline{x}(z)\right) < -\operatorname{div}\left(\left\|Dx_0(z)\right\|^{p-1} Dx_0(z)\right) \quad \text{a.e. on } \{x_0 > \delta_0\}. \end{aligned} \tag{3.36}$$

From (3.34) and (3.36), we conclude that

$$-\operatorname{div}\left(\left\|D\underline{x}(z)\right\|^{p-2} D\underline{x}(z)\right) < -\operatorname{div}\left(\left\|Dx_0(z)\right\|^{p-2} Dx_0(z)\right) \quad \text{a.e. on } Z.$$

From this, by virtue of Proposition 2.2 of Guedda and Veron [14], we infer that

$$x_0 - \underline{x} \in \operatorname{int} C_+. \tag{3.37}$$

Also from hypothesis $H(f)_1$(v), we have

$$\begin{aligned} -\operatorname{div}\left(\left\|D\overline{x}(z)\right\|^{p-2} D\overline{x}(z)\right) &> f\left(z, x_0(z)\right) \\ &= -\operatorname{div}\left(\left\|Dx_0(z)\right\|^{p-2} Dx_0(z)\right) \quad \text{a.e. on } Z. \end{aligned}$$

Invoking once more Proposition 2.2 of Guedda and Veron [14], we conclude that

$$\overline{x} - x_0 \in \operatorname{int} C_+. \tag{3.38}$$

From (3.37) and (3.38), it follows that x_0 is a local $C_0^1(\overline{Z})$-minimizer of φ. Then by Theorem 1.1 of Garcia Azorero, Manfredi and Peral Alonso [11], we have that x_0 is also a local $W_0^{1,p}(Z)$-minimizer of φ.

Similarly, truncating $f(z,x)$ with respect to the ordered pair $\{\underline{v}, \overline{v}\}$ and working on $I_- = [\underline{v}, \overline{v}] = \{v \in W_0^{1,p}(Z)\colon\ \underline{v}(z) \leqslant v(z) \leqslant \overline{v}(z)$ a.e. on $Z\}$, we obtain another solution $v_0 \in -\operatorname{int} C_+$ of problem (1.1), which too is a local minimizer of φ. $\square$

Therefore, we have produced two solutions of (1.1), the first (the positive) in I_+ and the second (the negative) in I_-.

Now by imposing conditions concerning the behavior of the nonlinearity in a neighborhood of $\pm\infty$, we will present two broad classes of problems for which hypotheses $H(f)_1$ hold and so the multiplicity result in Proposition 3.8 is valid.

The first class of problems, are the coercive problems (namely the corresponding Euler functional is coercive). So the hypotheses on the nonlinearity $f(z,x)$ are the following:

$H(f)_2$ $f : Z \times \mathbb{R} \to \mathbb{R}$ is a function such that $f(z,0) = 0$ a.e. on Z and
- (i) for all $x \in \mathbb{R}$, $z \to f(z,x)$ is measurable;
- (ii) for a.a. $z \in Z$, $x \to f(z,x)$ is continuous;
- (iii) for a.a. $z \in Z$ and all $x \in \mathbb{R}$, we have
$$\left|f(z,x)\right| \leqslant a(z) + c|x|^{r-1}$$
with $a \in L^\infty(Z)_+$, $c > 0$, $1 < r < p^*$;
- (iv) there exists $\eta \in L^\infty(Z)_+$ such that $\lambda_1 \leqslant \eta(z)$ a.e. on Z, $\lambda_1 \neq \eta$ and
$$\eta(z) \leqslant \liminf_{x\to 0} \frac{f(z,x)}{|x|^{p-2}x} \quad \text{uniformly for a.a. } z \in Z;$$
- (v) there exists $\theta \in L^\infty(Z)_+$ such that $\theta(z) \leqslant \lambda_1$ a.e. on Z, $\theta \neq \lambda_1$ and
$$\limsup_{x\to\pm\infty} \frac{f(z,x)}{|x|^{p-2}x} \leqslant \theta(z) \quad \text{uniformly for a.a. } z \in Z;$$
- (vi) there exist $\delta_0, \sigma_0 > 0$ such that
for a.a. $z \in Z$, $f(z,\cdot)$ is increasing on $[-\delta_0, \delta_0]$ and
for a.a. $z \in Z$, $f(z,x) \geqslant \sigma_0$ when $x \geqslant \delta_0$ and $f(z,x) \leqslant -\sigma_0$ when $x \leqslant -\delta_0$.

We start a simple lemma, which is an easy consequence of the hypothesis on the function $\theta \in L^\infty(Z)_+$ and of the fact that $u_1 \in \operatorname{int} C_+$. So we omit its proof.

Lemma 3.9. *If $\theta \in L^\infty(Z)_+$, $\theta(z) \leqslant \lambda_1$ a.e. on Z and $\theta \neq \lambda_1$, then there exists $\xi_0 > 0$ such that*
$$\|Dx\|_p^p - \int_Z \theta|x|^p\,dz \geqslant \xi_0\|Dx\|_p^p \quad \textit{for all } x \in W_0^{1,p}(Z).$$

Using this lemma, we will be able to produce a strict upper solution and a strict lower solution for problem (1.1), under the new hypothesis $H(f)_2$(v). This way we will satisfy hypothesis $H(f)_1$(v) and so Proposition 3.8 will apply to coercive problems.

Proposition 3.10. *If hypotheses* $H(f)_2$(i)–(iii), (v) *hold, then we can find* $\overline{x} \in \operatorname{int} C_+$ *a strict upper solution for problem* (1.1) *and* $\underline{v} \in -\operatorname{int} C_+$ *a strict lower solution, both of which satisfy hypothesis* $H(f)_1$(v).

Proof. By virtue of hypotheses $H(f)_2$(iii), (v), given $\varepsilon > 0$, we can find $\gamma_\varepsilon \in L^\infty(Z)_+$, $\gamma_\varepsilon \neq 0$, such that

$$f(z,x) < \big(\theta(z) + \varepsilon\big)x^{p-1} + \gamma_\varepsilon(z) \quad \text{for a.a. } z \in Z \text{ and all } x \geqslant 0. \tag{3.39}$$

We consider the following auxiliary boundary value problem:

$$\begin{cases} -\operatorname{div}\big(\|Dx(z)\|^{p-2}Dx(z)\big) = \big(\theta(z)+\varepsilon\big)\big|x(z)\big|^{p-2}x(z) + \gamma_\varepsilon(z) & \text{in } Z, \\ x|_{\partial Z} = 0. \end{cases} \tag{3.40}$$

Let $K_\varepsilon : L^p(Z) \to L^{p'}(Z)$ ($\frac{1}{p} + \frac{1}{p'} = 1$) be the nonlinear operator defined by

$$K_\varepsilon(x)(\cdot) = \big(\theta(\cdot) + \varepsilon\big)\big|x(\cdot)\big|^{p-2}x(\cdot).$$

Clearly K_ε is bounded continuous and due to the compact embedding of $W_0^{1,p}(Z)$ into $L^p(Z)$, we have $K_\varepsilon|_{W_0^{1,p}(Z)}$ is completely continuous.

Recall that the operator $A : W_0^{1,p}(Z) \to W^{-1,p'}(Z)$ defined by

$$\big\langle A(x), y\big\rangle = \int_Z \|Dx\|^{p-2}(Dx, Dy)_{\mathbb{R}^N}\, dz \quad \text{for all } x, y \in W_0^{1,p}(Z),$$

is bounded, demicontinuous, monotone (in fact strictly monotone), hence maximal monotone. Therefore the operator $A - K_\varepsilon : W_0^{1,p}(Z) \to W^{-1,p'}(Z)$ is pseudomonotone. Moreover, for every $x \in W_0^{1,p}(Z)$, we have

$$\begin{aligned} \big\langle A(x) - K_\varepsilon(x), x\big\rangle &= \|Dx\|_p^p - \int_Z \theta|x|^p\, dz - \varepsilon\|x\|_p^p \\ &\geqslant \left(\xi_0 - \frac{\varepsilon}{\lambda_1}\right)\|Dx\|_p^p \quad \big(\text{see Lemma 3.7 and (2.2)}\big). \end{aligned} \tag{3.41}$$

So, if we choose $\varepsilon \in (0, \xi_0\lambda_1)$, then from (3.41) we infer that the operator $x \to A(x) - K_\varepsilon(x)$ is coercive. But a pseudomonotone, coercive operator is surjective (see Gasinski and Papageorgiou [12, p. 336]). Therefore, we can find $\overline{x} \in W_0^{1,p}(Z)$, $\overline{x} \neq 0$ such that

$$A(\overline{x}) - K_\varepsilon(\overline{x}) = \gamma_\varepsilon. \tag{3.42}$$

On (3.42), we act with the test function $-\overline{x}^- \in W_0^{1,p}(Z)$ and obtain

$$\|D\overline{x}^-\|_p^p - \int_Z \theta|\overline{x}^-|^p\, dz \leqslant \varepsilon\|\overline{x}^-\|_p^p \quad \Rightarrow \quad \xi_0\|D\overline{x}^-\|_p^p \leqslant \frac{\varepsilon}{\lambda_1}\|D\overline{x}^-\|_p^p. \tag{3.43}$$

Since $0 < \varepsilon < \xi_0\lambda_1$, from (3.43) we infer that $\overline{x}^- = 0$ and so $\overline{x} \geqslant 0$, $\overline{x} \neq 0$. From (3.42), we have

$$\begin{cases} -\operatorname{div}\bigl(\|D\overline{x}(z)\|^{p-2}D\overline{x}(z)\bigr) = \theta(z)\bigl|\overline{x}(z)\bigr|^{p-2}\overline{x}(z) + \gamma_\varepsilon(z) \quad \text{a.e. on } Z, \\ \overline{x}|_{\partial Z} = 0. \end{cases} \tag{3.44}$$

From nonlinear regularity theory, we have $\overline{x} \in C_+ \setminus \{0\}$. Also from (3.44) and since $\gamma_\varepsilon \geqslant 0$, we have

$$\operatorname{div}\bigl(\|D\overline{x}(z)\|^{p-2}D\overline{x}(z)\bigr) \leqslant 0 \quad \text{a.e. on } Z.$$

Invoking the nonlinear strong maximum principle of Vazquez [23], we have $\overline{x} \in \operatorname{int} C_+$.

Because of (3.39), we have that $\overline{x} \in \operatorname{int} C_+$ is a strict upper solution for problem (1.1). Also $-\Delta_p\overline{x} \in L^\infty(Z)_+$ and $f(z,x) < -\Delta_p\overline{x}(z)$ for a.a. $z \in Z$ and all $x \in [0, \overline{x}(z)]$ (see (3.39)). Hence $\overline{x} \in \operatorname{int} C_+$ satisfies hypothesis $H(f)_1$(v).

Hypotheses $H(f)_2$(iii), (vi) also imply that

$$f(z,x) > \bigl(\theta(z) + \varepsilon\bigr)|x|^{p-2}x - \gamma_\varepsilon(z) \quad \text{for a.a. } z \in Z \text{ and all } x \leqslant 0. \tag{3.45}$$

In this case, we consider the problem

$$\begin{cases} -\operatorname{div}\bigl(\|Dv(z)\|^{p-2}Dv(z)\bigr) = \bigl(\theta(z) + \varepsilon\bigr)\bigl|v(z)\bigr|^{p-2}v(z) - \gamma_\varepsilon(z) \quad \text{a.e. on } Z, \\ v|_{\partial Z} = 0. \end{cases} \tag{3.46}$$

As we did for problem (3.40), we can show that problem (3.46) has a solution $\underline{v} \in -\operatorname{int} C_+$, which is a strict lower solution for problem (1.1) and satisfies hypothesis $H(f)_1$(v). $\square$

Combining Propositions 3.8 and 3.10, we have the first multiplicity result for coercive problems.

Proposition 3.11. *If hypotheses $H(f)_2$ hold, then problem* (1.1) *has at least two solutions of constant sign, $x_0 \in \operatorname{int} C_+$, $v_0 \in -\operatorname{int} C_+$.*

Another important class of problems, which fit in the general framework of Proposition 3.8, are certain parametric p-superlinear problems. Namely we consider the following problems:

$$\begin{cases} -\operatorname{div}\bigl(\|Dx(z)\|^{p-2}Dx(z)\bigr) = f\bigl(z, x(z), \lambda\bigr) \quad \text{a.e. on } Z, \\ x|_{\partial Z} = 0, \quad 1 < p < \infty, \quad \lambda > 0. \end{cases} \tag{3.47}$$

$H(f)_3$ $f : Z \times \mathbb{R} \times (0, +\infty) \to \mathbb{R}$ is a function such that $f(z, 0, \lambda) = 0$ a.e. on Z, for all $\lambda > 0$ and

(i) for all $(x, \lambda) \in \mathbb{R} \times (0, +\infty)$, $z \to f(z, x, \lambda)$ is measurable;

(ii) for a.a. $z \in Z$ and all $\lambda \in (0, \infty)$, $x \to f(z, x, \lambda)$ is continuous;

(iii) for a.a. $z \in Z$, all $x \in \mathbb{R}$ and all $\lambda \in (0, \infty)$, we have

$$\bigl|f(z, x, \lambda)\bigr| \leqslant a(z, \lambda) + c|x|^{r-1}$$

with $a(\cdot, \lambda) \in L^\infty(Z)_+$, $\|a(\cdot, \lambda)\|_\infty \to 0$ as $\lambda \to 0^+$, $c > 0$, $p < r < p^*$;

(iv) for every $\lambda \in (0,\infty)$, there exists a function $\eta = \eta(\lambda) \in L^\infty(Z)_+$ such that $\lambda_1 \leqslant \eta(z)$ a.e. on Z, $\lambda_1 \neq \eta$ and

$$\eta(z) \leqslant \liminf_{x \to 0} \frac{f(z,x,\lambda)}{|x|^{p-2}x} \quad \text{uniformly for a.a. } z \in Z;$$

(v) for every $\lambda \in (0,\infty)$, there exist $M = M(\lambda) > 0$ and $\mu = \mu(\lambda) > p$ such that

$$0 < \mu F(z,x,\lambda) \leqslant f(z,x,\lambda)x \quad \text{for a.a. } z \in Z, \text{ all } |x| \geqslant M,$$

where $F(z,x,\lambda) = \int_0^x f(z,s,\lambda)\,ds$;

(vi) for every $\lambda \in (0,\infty)$, there exist $\delta_0 = \delta_0(\lambda) > 0$ and $\sigma_0 = \sigma_0(\lambda)$, such that for a.a. $z \in Z$, $f(z,\cdot,\lambda)$ is increasing on $[-\delta_0, \delta_0]$ and for a.a. $z \in Z$, $f(z,x,\lambda) \geqslant \sigma_0$ when $x \geqslant \delta_0$ and $f(z,x,\lambda) \leqslant -\sigma_0$ when $x \leqslant -\delta_0$.

Proposition 3.12. *If hypotheses $H(f)_3$(i)–(iii), (v) hold, then there exists $\lambda^* \in (0,\infty)$ such that for all $\lambda \in (0,\lambda^*)$, problem (3.47) has a strict upper solution $\overline{x} \in \operatorname{int} C_+$ and a strict lower solution $\underline{v} \in -\operatorname{int} C_+$, both of which satisfy hypothesis $H(f)_1$(v).*

Proof. Let $\varrho \in \operatorname{int} C_+$ be such that

$$-\operatorname{div}\big(\|D\varrho(z)\|^{p-2} D\varrho(z)\big) = 1 \quad \text{a.e. on } Z, \quad \varrho|_{\partial Z} = 0. \tag{3.48}$$

We claim that we can find $\lambda^* \in (0,\infty)$ such that, if $\lambda \in (0,\lambda^*)$, then we can choose $\xi_1 = \xi_1(\lambda) > 0$ satisfying

$$\big\|a(\cdot,\lambda)\big\|_\infty + c\big(\xi_1\|\varrho\|_\infty\big)^{r-1} < \xi_1^{p-1}. \tag{3.49}$$

We argue by contradiction. So suppose that we cannot find $\xi_1 > 0$ for which (3.49) holds. This means that there exists a sequence $\{\lambda_n\}_{n\geqslant 1} \subseteq (0,\overline{\lambda})$ such that $\lambda_n \to 0^+$ and

$$\xi^{p-1} \leqslant \big\|a(\cdot,\lambda_n)\big\|_\infty + c\big(\xi\|\varrho\|_\infty\big)^{r-1} \quad \text{for all } n \geqslant 1 \text{ and all } \xi > 0.$$

Passing to the limit as $n \to \infty$ and using hypothesis $H(f)_3$(iii), we have

$$\xi^{p-1} \leqslant c\big(\xi\|\varrho\|_\infty\big)^{r-1} \quad \Rightarrow \quad 1 \leqslant c\xi^{r-p}\|\varrho\|_\infty^{r-1} \quad \text{for all } \xi > 0.$$

Since $r > p$, letting $\xi \to 0^+$, we have a contradiction. Therefore, we can find $\xi_1 = \xi_1(\lambda) > 0$ for which (3.49) is true.

We fix $\lambda \in (0,\lambda^*)$ and we choose $\xi_1 = \xi_1(\lambda) > 0$ as in (3.49). We set $\overline{x} \in \xi_1\varrho \in \operatorname{int} C_+$. Then

$$\begin{aligned}
-\operatorname{div}\big(\|D\overline{x}(z)\|^{p-2} D\overline{x}(z)\big) &= -\xi_1^{p-1}\operatorname{div}\big(\|D\varrho(z)\|^{p-2} D\varrho(z)\big) \\
&= \xi_1^{p-1} \quad \big(\text{see (3.49)}\big) \\
&> \big\|a(\cdot,\lambda)\big\|_\infty + c\big(\xi_1\|\varrho\|_\infty\big)^{r-1} \quad \big(\text{see (3.49)}\big) \\
&\geqslant f\big(z,\overline{x}(z),\lambda\big) \quad \text{a.e. on } Z \quad \big(\text{see hypothesis } H(f)_3(\text{iii})\big).
\end{aligned} \tag{3.50}$$

From (3.50) we infer that $\overline{x} \in \operatorname{int} C_+$ is a strict upper solution for problem (1.1). Moreover, we have $-\Delta_p \overline{x} \in L^\infty(Z)_+$ and $f(z, x, \lambda) < -\Delta_p \overline{x}(z)$ for a.a. $z \in Z$, all $x \in [0, \overline{x}(z)]$ and all $\lambda \in (0, \lambda^*)$. Therefore $\overline{x} \in \operatorname{int} C_+$ satisfies hypothesis $H(f)_1$(v).

Similarly, let $\underline{v} = (-\xi_1)\varrho \in -\operatorname{int} C_+$, with $\xi_1 > 0$ as in (3.49). Then as above, using (3.49), we can verify that $\underline{v} \in -\operatorname{int} C_+$ is a strict lower solution for problem (3.47), which satisfies hypothesis $H(f)_1$(v). □

Combining Propositions 3.8 and 3.12, we can have the first multiplicity result for p-superlinear problems.

Proposition 3.13. *If hypotheses $H(f)_3$ hold, then there exists $\lambda^* \in (0, \infty)$ such that for all $\lambda \in (0, \lambda^*)$ problem* (3.47) *has at least two solutions of constant sign, $x_0 \in \operatorname{int} C_+$ and $v_0 \in -\operatorname{int} C_+$.*

In fact in the case of p-superlinear problems, we can have more solutions of constant sign. More precisely, we have the following multiplicity result.

Theorem 3.14. *If hypotheses $H(f)_3$ hold, then there exists $\lambda^* \in (0, \infty)$ such that for all $\lambda \in (0, \lambda^*)$ problem* (3.47) *has at least four solutions of constant sign $x_0, \hat{x} \in \operatorname{int} C_+$, $x_0 \leqslant \hat{x}$, $x_0 \neq \hat{x}$ and $v_0, \hat{v} \in -\operatorname{int} C_+$, $\hat{v} \leqslant v_0$, $\hat{v} \neq v_0$.*

Proof. Let $x_0 \in I_+$ be the positive solution obtained in Proposition 3.13 (see also Proposition 3.8). We may assume that this is the only solution of (3.47) in the order interval I_+ or otherwise we have already a second positive solution. We introduce the following truncation of the nonlinearity $f(z, x, \lambda)$:

$$\overline{f}_+(z, x, \lambda) = \begin{cases} f(z, x_0(z), \lambda) & \text{if } x \leqslant x_0(z), \\ f(z, x, \lambda) & \text{if } x > x_0(z), \end{cases} \tag{3.51}$$

for all $(z, x, \lambda) \in Z \times \mathbb{R} \times (0, \lambda^*)$.

Let $\overline{F}_+(z, x, \lambda) = \int_0^x \overline{f}_+(z, s, \lambda)\, ds$, the primitive of $\overline{f}_+$ and consider the functional $\overline{\varphi}_+^\lambda : W_0^{1,p}(Z) \to \mathbb{R}$ defined by

$$\overline{\varphi}_+^\lambda(x) = \frac{1}{p}\|Dx\|_p^p - \int_Z \overline{F}_+\big(z, x(z), \lambda\big)\, dz \quad \text{for all } x \in W_0^{1,p}(Z).$$

Clearly $\overline{\varphi}_+^\lambda \in C^1(W_0^{1,p}(Z))$. We consider the following auxiliary boundary value problem:

$$\begin{cases} -\operatorname{div}\big(\|Dx(z)\|^{p-2} Dx(z)\big) = \overline{f}_+\big(z, x(z), \lambda\big) & \text{in } Z, \\ x|_{\partial Z} = 0. \end{cases} \tag{3.52}$$

From the proof of Proposition 3.8, we know that

$$-\operatorname{div}\big(\|D\underline{x}(z)\|^{p-2} D\underline{x}(z)\big) < f\big(z, x_0(z), \lambda\big) \quad \text{a.e. on } Z.$$

But from (3.51) and since $\underline{x} \leqslant x_0$, we have

$$f\big(z, x_0(z), \lambda\big) = \overline{f}_+\big(z, \underline{x}(z), \lambda\big) \quad \text{a.e. on } Z \text{ for all } \lambda \in (0, \lambda^*).$$

Therefore,

$$-\operatorname{div}\big(\|D\underline{x}(z)\|^{p-2}D\underline{x}(z)\big) < \overline{f}_+\big(z,\underline{x}(z),\lambda\big) \quad \text{a.e. on } Z$$
$$\Rightarrow \quad \underline{x} \in \operatorname{int} C_+ \text{ is a strict lower solution for problem (3.52).}$$

On the other hand, from Proposition 3.12 and since $x_0 \leqslant \overline{x}$, we have

$$-\operatorname{div}\big(\|D\overline{x}(z)\|^{p-2}D\overline{x}(z)\big) > f\big(z,\overline{x}(z),\lambda\big)$$
$$= \overline{f}_+\big(z,\overline{x}(z),\lambda\big) \quad \text{a.e. on } Z,\ \lambda \in (0,\lambda^*)$$
$$\Rightarrow \quad \overline{x} \in \operatorname{int} C_+ \text{ is a strict upper solution for problem (3.47).}$$

So we have an ordered pair $\{\overline{x},\underline{x}\}$ of upper–lower solutions for problem (3.52).

Then since $\overline{\varphi}_+^\lambda$ is w-lower semicontinuous and $\overline{\varphi}_+^\lambda|_{I_+}$ is coercive, by the Weierstrass theorem we can find $\tilde{x} \in I_+$ such that $\overline{\varphi}_+^\lambda(\tilde{x}) = \inf_{I_+} \overline{\varphi}_+^\lambda$. As in the proof of Proposition 3.8, we have that $(\overline{\varphi}_+^\lambda)'(\tilde{x}) = 0$ and so

$$\begin{cases} -\operatorname{div}\big(\|D\tilde{x}(z)\|^{p-2}D\tilde{x}(z)\big) = \overline{f}_+\big(z,\tilde{x}(z),\lambda\big) & \text{in } Z, \\ \tilde{x}|_{\partial Z} = 0. \end{cases} \tag{3.53}$$

We multiply (3.53) with the test function $(x_0-\tilde{x})^+ \in W_0^{1,p}(Z)$, integrate over Z and use the nonlinear Green's identity (see Gasinski and Papageorgiou [12, p. 211]). We obtain

$$\int_{\{x_0>\tilde{x}\}} \|D\tilde{x}\|^{p-2}\big(D\tilde{x}, D(x_0-\tilde{x})\big)_{\mathbb{R}^N}\,dz$$
$$= \int_{\{x_0>\tilde{x}\}} f(z,x_0,\lambda)(x_0-\tilde{x})\,dz \quad \big(\text{see (3.47)}\big)$$
$$= \int_{\{x_0>\tilde{x}\}} \|Dx_0\|^{p-2}\big(Dx_0, D(x_0-\tilde{x})\big)_{\mathbb{R}^N}\,dz \quad \big(\text{since } x_0 \text{ solves (3.47)}\big)$$
$$\Rightarrow \quad \int_{\{x_0>\tilde{x}\}} \big(\|D\tilde{x}\|^{p-2}D\tilde{x} - \|Dx_0\|^{p-2}Dx_0, D(\tilde{x}-x_0)\big)_{\mathbb{R}^N}\,dz = 0. \tag{3.54}$$

Due to the strict monotonicity of the map $\theta_p:\mathbb{R}^N \to \mathbb{R}^N$ defined by $\theta_p(x) = \|x\|^{p-2}x$, $x \neq 0$, $\theta_p(0)=0$, from (3.54) we infer that $|\{x_0>\tilde{x}\}|_N = 0$, hence $x_0 \leqslant \tilde{x}$ and so we have $\overline{f}_+(z,\tilde{x}(z),\lambda) = f(z,\tilde{x}(z),\lambda)$ a.e. on Z, $\lambda \in (0,\lambda^*)$. Then from (3.53) and since we have assumed that $x_0 \in I_+$ is the only solution of problem (3.47) in the order interval I_+, we deduce that $\tilde{x} = x_0$. We have

$$x_0 - \underline{x} \in \operatorname{int} C_+ \quad \text{and} \quad \overline{x} - x_0 \in \operatorname{int} C_+.$$

Therefore, it follows that x_0 is a local $C_0^1(\overline{Z})$-minimizer of the functional $\overline{\varphi}_+^\lambda$. Hence, Theorem 1.1 of Garcia Azorero, Manfredi and Peral Alonso [11], implies that x_0 is a local

$W_0^{1,p}(Z)$-minimizer of $\overline{\varphi}_+^\lambda$. We may assume that x_0 is an isolated local minimizer of $\overline{\varphi}_+^\lambda$ or otherwise arguing as above, we can generate a whole sequence of distinct positive solutions for problem (3.47). Then as in Motreanu, Motreanu and Papageorgiou [21], we can find $\rho > 0$ small such that

$$\overline{\varphi}_+^\lambda(x_0) < \inf\bigl[\overline{\varphi}_+^\lambda(x)\colon\ \|x - x_0\| = \rho\bigr] = c_\rho. \tag{3.55}$$

Note that hypothesis $H(f)_3$(v) (the Ambrosetti–Rabinowitz condition) implies that

$$F(z,x,\lambda) \geqslant c_4|x|^\mu - c_5 \quad \text{for a.a. } z \in Z,\ \text{all } x \in \mathbb{R},\ \text{all } \lambda > 0,\ \text{with } c_4, c_5 > 0. \tag{3.56}$$

So using (3.56), we see that

$$\overline{\varphi}_+^\lambda(tu_1) \to -\infty \quad \text{as } t \to +\infty. \tag{3.57}$$

We also check that $\overline{\varphi}_+^\lambda$, $\lambda \in (0, \lambda^*)$, satisfies the PS-condition. To this end let $\{x_n\}_{n\geqslant 1} \subseteq W_0^{1,p}(Z)$ be the sequence such that

$$\bigl|\overline{\varphi}_+^\lambda(x_n)\bigr| \leqslant M_1 \quad \text{for some } M_1 > 0,\ \text{all } n \geqslant 1 \quad \text{and} \quad \bigl(\overline{\varphi}_+^\lambda\bigr)'(x_n) \to 0 \quad \text{as } n \to \infty.$$

We have

$$\bigl|\bigl\langle\bigl(\overline{\varphi}_+^\lambda\bigr)'(x_n), v\bigr\rangle\bigr| \leqslant \varepsilon_n\|v\| \quad \text{for all } v \in W_0^{1,p}(Z),\ \text{with } \varepsilon_n \downarrow 0.$$

Let $v = -x_n^- \in W_0^{1,p}(Z)$. Then

$$\begin{aligned}
&\left|\bigl\|Dx_n^-\bigr\|_p^p + \int_Z f(z,x_0,\lambda)x_n^-\,dz\right| \leqslant \varepsilon_n\bigl\|x_n^-\bigr\| \\
&\quad\Rightarrow\quad \bigl\|Dx_n^-\bigr\|_p^p \leqslant \varepsilon_n\bigl\|x_n^-\bigr\| \quad \bigl(\text{since } f(z,x_0,\lambda)x_n^- \geqslant 0\bigr) \\
&\quad\Rightarrow\quad \bigl\{x_n^-\bigr\}_{n\geqslant 1} \subseteq W_0^{1,p}(Z) \text{ is bounded.}
\end{aligned} \tag{3.58}$$

From the choice of the sequence $\{x_n\}_{n\geqslant 1} \subseteq W_0^{1,p}(Z)$ and (3.47), (3.58), we have

$$\frac{\mu}{p}\bigl\|Dx_n^+\bigr\|_p^p - \int_{\{x_0\leqslant x_n\}} \mu F(z,x_n,\lambda)\,dz \leqslant M_2 \quad \text{for some } M_2 > 0,\ \text{all } n \geqslant 1 \tag{3.59}$$

and

$$-\bigl\|Dx_n^+\bigr\|_p^p + \int_{\{x_0\leqslant x_n\}} f(z,x_n,\lambda)x_n^+\,dz \leqslant M_3 \quad \text{for some } M_3 > 0,\ \text{all } n \geqslant 1. \tag{3.60}$$

Adding (3.59) and (3.60) and using hypothesis $H(f)_3$(v), we obtain

$$\begin{aligned}&\left(\frac{\mu}{p}-1\right)\|Dx_n^+\|_p^p \leqslant M_4 \quad \text{for some } M_4>0,\ \text{all } n\geqslant 1\\ &\Rightarrow \quad \{x_n^+\}_{n\geqslant 1}\subseteq W_0^{1,p}(Z) \text{ is bounded}\\ &\Rightarrow \quad \{x_n\}_{n\geqslant 1}\subseteq W_0^{1,p}(Z) \text{ is bounded.}\end{aligned}$$

Therefore we may assume that

$$x_n \xrightarrow{w} x \quad \text{in } W_0^{1,p}(Z), \qquad x_n \to x \quad \text{in } L^r(Z), \qquad x_n(z)\to x(z) \quad \text{a.e. on } Z$$

and

$$|x_n(z)| \leqslant k(z) \quad \text{a.e. on } Z,\ \text{for all } n\geqslant 1,\ \text{with } k\in L^r(Z)_+.$$

We have

$$\begin{aligned}&\left|\left\langle (\overline{\varphi}_+^\lambda)'(x_n), x_n-x\right\rangle\right| \leqslant \varepsilon_n\|x_n-x\|\\ &\Rightarrow \quad \left|\left\langle A(x_n), x_n-x\right\rangle - \int_Z \overline{f}_+\big(z,x_n(z),\lambda\big)(x_n-x)(z)\,dz\right| \leqslant \varepsilon_n\|x_n-x\|.\end{aligned} \tag{3.61}$$

Note that

$$\int_Z \overline{f}_+\big(z,x_n(z),\lambda\big)(x_n-x)(z)\to 0.$$

So from (3.61) it follows that

$$\lim_{n\to\infty}\left\langle A(x_n), x_n-x\right\rangle = 0. \tag{3.62}$$

Then by virtue of the generalized pseudomonotonicity of A, from (3.62) we infer that

$$x_n \to x \quad \text{in } W_0^{1,p}(Z) \quad \text{(see the proof of Proposition 3.1).}$$

Therefore, $\overline{\varphi}_+^\lambda$ satisfies the PS-condition. Combining this fact with (3.55) and (3.57), we see that we can apply the mountain pass theorem and find $\hat{x}\in W_0^{1,p}(Z)$, $\hat{x}\neq x_0$ which is a critical point of the functional $\overline{\varphi}_+^\lambda$, $\lambda\in(0,\lambda^*)$. Then, as we did for $\tilde{x}$, we can show that $\hat{x}\geqslant x_0$ and so $\hat{x}\in \operatorname{int} C_+$ is a second positive solution of (3.47) distinct from x_0.

On the other hand, let $v_0\in I_-$ be the negative solution obtained in Proposition 3.13 (see also Proposition 3.8). We use $v_0\in I_-$ and the following modification of the nonlinearity $f(z,x,\lambda)$:

$$\overline{f}_-(z,x,\lambda)=\begin{cases} f(z,x,\lambda) & \text{if } x<v_0(z),\\ f(z,v_0(z),\lambda) & \text{if } x\geqslant v_0(z)\end{cases} \quad \text{for all } (z,x,\lambda)\in Z\times\mathbb{R}\times(0,\infty).$$

We set $\overline{F}_-(z,x,\lambda)=\int_0^x \overline{f}_-(z,s,\lambda)\,ds$ and consider the C^1-functional $\overline{\varphi}_-^\lambda : W_0^{1,p}(Z)\to\mathbb{R}$ defined by

$$\overline{\varphi}_-^\lambda(x)=\frac{1}{p}\|Dx\|_p^p-\int_Z \overline{F}_-\big(z,x(z),\lambda\big)\,dz \quad \text{for all } x\in W_0^{1,p}(Z),\ \lambda\in(0,\lambda^*).$$

Then arguing as above, through the mountain pass theorem, we obtain a second negative solution $\hat{v}\in -\operatorname{int}C_+$, distinct from v_0 and such that $\hat{v}\leqslant v_0$. So finally we have generated four nontrivial constant sign solutions, namely $x_0,\hat{x}\in\operatorname{int}C_+$ and $v_0,\hat{x}\in-\operatorname{int}C_+$. $\square$

Remark 3.15. Let $p<r<p^*$ and assume that $g:Z\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function (i.e. measurable in $z\in Z$ and continuous in $x\in\mathbb{R}$) with $g(z,0)=0$ a.e. on Z. Let $G(z,x)=\int_0^x g(z,s)\,ds$. We suppose that

- $g(z,x)x\geqslant 0$ for a.a. $z\in Z$ and all $x\in\mathbb{R}$;
- there exist $M>0$, $\hat{c}_0>0$ and $1<q<p$ such that

$$\big|g(z,x)\big|\leqslant \hat{c}_0|x|^{q-1} \quad \text{for a.a. } z\in Z,\ \text{all } |x|\geqslant M;$$

- there exist $\delta_0,\sigma_0>0$ such that $x\to g(z,x)$ is increasing for a.a. $z\in Z$, all $x\in[-\delta_0,\delta_0]$ and $g(z,x)\geqslant\sigma_0$ for a.a. $z\in Z$, all $x\geqslant\delta_0$ and $g(z,x)\leqslant-\sigma_0$ for a.a. $z\in Z$ and all $x\leqslant-\delta_0$.

Set $f(z,x,\lambda)=|x|^{r-2}x+\lambda g(z,x)$, $\lambda>0$. Then $f(z,x,\lambda)$ satisfies hypotheses $H(f)_3$. A particular case, is when $g(z,x)=g(x)=|x|^{q-1}x$ (convex–concave nonlinearity). This is the nonlinearity in the works of Ambrosetti, Garcia Azorero and Peral Alonso [1] and Garcia Azorero, Manfredi and Peral Alonso [11]. So Theorem 3.14, extends the aforementioned works. Also partially extends the result of Motreanu, Motreanu and Papageorgiou [20] and the existence result of Boccardo, Escobedo and Peral Alonso [5].

4. Nodal solutions and multiplicity results

In this section, we go beyond solutions of constant sign and look for nodal (sign-changing) solutions. Recall that every eigenfunction of (2.1) corresponding to an eigenvalue $\lambda\neq\hat{\lambda}_1(m)$, must change sign. So we expect that in general, the nodal solutions of (1.1) must be more than the constant sign solutions. Nevertheless, to produce a nodal solution for problem (1.1), is a highly nontrivial task which requires involved arguments using various tools from nonlinear analysis.

Here we follow the approach of Dancer and Du [9], where $p=2$ (semilinear problems) (see also Carl and Perera [6], for nonlinear problems). Roughly speaking the strategy is the following. Continuing the argument employed in Section 3, we generate a smallest positive solution $y_+\in\operatorname{int}C_+$ and a biggest negative solution $y_-\in-\operatorname{int}C_+$. We form the order interval $[y_-,y_+]$. Using variational techniques on certain appropriate truncations of the original Euler functional, we are able to produce a solution $y_0\in[y_-,y_+]$ of (1.1) different from y_- and y_+. Evidently, if $y_0\neq 0$, then y_0 is nodal. To show the nontriviality of y_0, we use (2.3) and Theorem 2.3.

We start with a lemma, which shows that the set of upper solutions for problem (1.1) is downward directed.

Lemma 4.1. *If $y_1, y_2 \in W^{1,p}(Z)$ are two upper solutions for problem* (1.1) *and* $y = \min\{y_1, y_2\} \in W^{1,p}(Z)$*, then y is also an upper solution for problem* (1.1).

Proof. Given $\varepsilon > 0$, we consider the truncation function $\xi_\varepsilon : \mathbb{R} \to \mathbb{R}$ defined by

$$\xi_\varepsilon(s) = \begin{cases} -\varepsilon & \text{if } s \leqslant \varepsilon, \\ s & \text{if } -\varepsilon \leqslant s \leqslant \varepsilon, \\ \varepsilon & \text{if } \varepsilon \leqslant s. \end{cases}$$

Clearly ξ_ε is Lipschitz continuous. So from Marcus and Mizel [19], we have

$$\xi_\varepsilon\big((y_1 - y_2)^-\big) \in W^{1,p}(Z)$$

and

$$D\xi_\varepsilon\big((y_1 - y_2)^-\big) = \xi_\varepsilon'\big((y_1 - y_2)^-\big) D(y_1 - y_2)^-.$$

Consider a test function $\psi \in C_c^1(Z)$ with $\psi \geqslant 0$. Then

$$\xi_\varepsilon\big((y_1 - y_2)^-\big)\psi \in W^{1,p}(Z) \cap L^\infty(Z)$$

and

$$D\big(\xi_\varepsilon\big((y_1 - y_2)^-\big)\psi\big) = \psi D\xi_\varepsilon\big((y_1 - y_2)^-\big) + \xi_\varepsilon\big((y_1 - y_2)^-\big) D\psi.$$

Since by hypothesis $y_1, y_2 \in W^{1,p}(Z)$ are upper solutions for problem, then from Definition 2.4(a), we have

$$\big\langle A(y_1), \xi_\varepsilon\big((y_1 - y_2)^-\big)\psi\big\rangle \geqslant \int\limits_Z f(z, y_1)\xi_\varepsilon\big((y_1 - y_2)^-\big)\, dz$$

and

$$\big\langle A(y_2), \big(\varepsilon - \xi_\varepsilon\big((y_1 - y_2)^-\big)\big)\psi\big\rangle \geqslant \int\limits_Z f(z, y_2)\big(\varepsilon - \xi_\varepsilon\big((y_1 - y_2)^-\big)\big)\psi\, dz.$$

Adding these two inequalities, we obtain

$$\begin{aligned} &\big\langle A(y_1), \xi_\varepsilon\big((y_1 - y_2)^-\big)\psi\big\rangle + \big\langle A(y_2), \big(\varepsilon - \xi_\varepsilon\big((y_1 - y_2)^-\big)\big)\psi\big\rangle \\ &\quad \geqslant \int\limits_Z f(z, y_1)\xi_\varepsilon\big((y_1 - y_2)^-\big)\, dz + \int\limits_Z f(z, y_2)\big(\varepsilon - \xi_\varepsilon\big((y_1 - y_2)^-\big)\big)\psi\, dz. \end{aligned} \tag{4.1}$$

Note that

$$\begin{aligned}
&\big\langle A(y_1), \xi_\varepsilon\big((y_1-y_2)^-\big)\psi\big\rangle \\
&\quad = \int_Z \|Dy_1\|^{p-2}\big(Dy_1, D(y_1-y_2)^-\big)_{\mathbb{R}^{\mathbb{N}}}\xi_\varepsilon'\big((y_1-y_2)^-\big)\psi\,dz \\
&\qquad + \int_Z \|Dy_1\|^{p-2}(Dy_1, D\psi)_{\mathbb{R}^{\mathbb{N}}}\xi_\varepsilon\big((y_1-y_2)^-\big)\,dz \\
&\quad = - \int_{\{-\varepsilon\leqslant y_1-y_2\leqslant 0\}} \|Dy_1\|^{p-2}\big(Dy_1, D(y_1-y_2)\big)_{\mathbb{R}^{\mathbb{N}}}\,dz \\
&\qquad + \int_Z \|Dy_1\|^{p-2}(Dy_1, D\psi)_{\mathbb{R}^{\mathbb{N}}}\xi_\varepsilon\big((y_1-y_2)^-\big)\,dz
\end{aligned} \tag{4.2}$$

and

$$\begin{aligned}
&\big\langle A(y_2), \big(\varepsilon - \xi_\varepsilon\big((y_1-y_2)^-\big)\big)\psi\big\rangle \\
&\quad = \int_{\{-\varepsilon\leqslant y_1-y_2\leqslant 0\}} \|Dy_2\|^{p-2}\big(Dy_2, D(y_1-y_2)\big)_{\mathbb{R}^{\mathbb{N}}}\psi\,dz \\
&\qquad + \int_Z \|Dy_2\|^{p-2}(Dy_2, D\psi)_{\mathbb{R}^{\mathbb{N}}}\big(\varepsilon - \xi_\varepsilon\big((y_1-y_2)^-\big)\big)\,dz.
\end{aligned} \tag{4.3}$$

Adding (4.2) and (4.3) and recalling that $\psi \geqslant 0$, we have

$$\begin{aligned}
&\big\langle A(y_1), \xi_\varepsilon\big((y_1-y_2)^-\big)\psi\big\rangle + \big\langle A(y_2), \big(\varepsilon - \xi_\varepsilon\big((y_1-y_2)^-\big)\big)\psi\big\rangle \\
&\quad = \int_{\{-\varepsilon\leqslant y_1-y_2\leqslant 0\}} \big(\|Dy_2\|^{p-2}Dy_2 - \|Dy_1\|^{p-2}Dy_1, D(y_1-y_2)\big)_{\mathbb{R}^{\mathbb{N}}}\psi\,dz \\
&\qquad + \int_Z \|Dy_1\|^{p-2}(Dy_1, D\psi)_{\mathbb{R}^{\mathbb{N}}}\xi_\varepsilon\big((y_1-y_2)^-\big)\,dz \\
&\qquad + \int_Z \|Dy_2\|^{p-2}(Dy_2, D\psi)_{\mathbb{R}^{\mathbb{N}}}\big(\varepsilon - \xi_\varepsilon\big((y_1-y_2)^-\big)\big)\,dz \\
&\quad \leqslant \int_Z \|Dy_1\|^{p-2}(Dy_1, D\psi)_{\mathbb{R}^{\mathbb{N}}}\xi_\varepsilon\big((y_1-y_2)^-\big)\,dz \\
&\qquad + \int_Z \|Dy_2\|^{p-2}(Dy, D\psi)_{\mathbb{R}^{\mathbb{N}}}\big(\varepsilon - \xi_\varepsilon\big((y_1-y_2)^-\big)\big)\,dz.
\end{aligned} \tag{4.4}$$

We return to (4.1), use (4.4) and then divide by $\varepsilon > 0$. We obtain

$$\int_Z \|Dy_1\|^{p-2}(Dy_1, D\psi)_{\mathbb{R}^N} \frac{1}{\varepsilon}\xi_\varepsilon\big((y_1 - y_2)^-\big)\,dz + \int_Z \|Dy_2\|^{p-2}(Dy_2, D\psi)_{\mathbb{R}^N}\left(1 - \frac{1}{\varepsilon}\xi_\varepsilon\big((y_1 - y_2)^-\big)\right)dz \geqslant \int_Z f(z, y_1)\xi_\varepsilon\big((y_1 - y_2)^-\big)\psi\,dz + \int_Z f(z, y_2)\big(\varepsilon - \xi_\varepsilon\big((y_1 - y_2)^-\big)\big)\psi\,dz. \tag{4.5}$$

Note that

$$\frac{1}{\varepsilon}\xi_\varepsilon\big((y_1 - y_2)^-(z)\big) \to \chi_{\{y_1<y_2\}}(z) \quad \text{a.e. on } Z \text{ as } \varepsilon \downarrow 0$$

and

$$\chi_{\{y_1 \geqslant y_2\}} = 1 - \chi_{\{y_1<y_2\}}.$$

Therefore, if we pass to the limit as $\varepsilon \to 0^+$ in (4.5), we obtain

$$\int_{\{y_1<y_2\}} \|Dy_1\|^{p-2}(Dy_1, D\psi)_{\mathbb{R}^N}\,dz + \int_{\{y_1\geqslant y_2\}} \|Dy_2\|^{p-2}(Dy_2, D\psi)_{\mathbb{R}^N}\,dz \geqslant \int_{\{y_1<y_2\}} f(z, y_1)\,dz + \int_{\{y_1\geqslant y_2\}} f(z, y_2)\psi\,dz. \tag{4.6}$$

Recall that $y = \min\{y_1, y_2\} \in W^{1,p}(Z)$ and

$$Dy(z) = \begin{cases} Dy_1(z) & \text{a.e. on } \{y_1 < y_2\}, \\ Dy_2(z) & \text{a.e. on } \{y_1 \geqslant y_2\}. \end{cases}$$

Using this in (4.6), we have

$$\int_Z \|Dy\|^{p-2}(Dy, D\psi)_{\mathbb{R}^N}\,dz \geqslant \int_Z f(z, x)\psi\,dz. \tag{4.7}$$

Since $\psi \in C_c^1(Z)_+$ was arbitrary and $C_c^1(Z)_+$ is dense in $W_0^{1,p}(Z)_+$, from (4.7) we conclude that $y = \min\{y_1, y_2\} \in W^{1,p}(Z)$ is an upper solution for problem (1.1) (see Definition 2.4(a)). □

Using a similar argument, we can show that the set of lower solutions for problem (1.1) is upward directed. So we have:

Lemma 4.2. *If $w_1, w_2 \in W^{1,p}(Z)$ are two lower solutions for problem* (1.1) *and $w = \max\{w_1, w_2\} \in W^{1,p}(Z)$, then w is also a lower solution for problem* (1.1).

Using the above two auxiliary results, we can show that problem (1.1) has a smallest solution in the order interval $I_+ = [\underline{x}, \overline{x}]$ and a biggest solution in the order interval $I_- = [\underline{v}, \overline{v}]$. By a smallest solution x_* of (1.1) in I_+ (resp. biggest solution v^* of (1.1) in I_-) we mean a solution $x_* \in I_+$ (resp. $v^* \in I_-$) such that if $x \in I_+$ (resp. $v \in I_-$) is any other solution of (1.1) in I_+ (resp. in I_-), then $x_* \leqslant x$ (resp. $v \leqslant v^*$).

Proposition 4.3. *If hypotheses $H(f)_1$ hold, then problem* (1.1) *admits a smallest solution x_* in I_+ and a biggest solution $v^* \in I_-$.*

Proof. Let S_+ be the set of solutions of (1.1) which belong in the order interval I_+. We claim that the set S_+ is downward directed. To this end let $x_1, x_2 \in S_+$. Both x_1 and x_2 are also upper solutions for problems (1.1). So by virtue of Lemma 4.1, $\hat{x} = \min\{x_1, x_2\} \in W_0^{1,p}(Z)$ is an upper solution for problem (1.1). We set

$$\hat{I}_+ = [\underline{x}, \hat{x}] = \bigl\{x \in W_0^{1,p}(Z): \ \underline{x}(z) \leqslant x(z) \leqslant \hat{x}(z) \text{ a.e. on } Z\bigr\}.$$

As before, truncating $f(z, \cdot)$ at the ordered pair $\{\underline{x}, \hat{x}\}$ and using the Weierstrass theorem, we can find $\hat{x}_0 \in I_+$ a solution of (1.1). Nonlinear regularity theory implies that $\hat{x} \in \operatorname{int} C_+$ and we have

$$\underline{x} \leqslant x_0 \leqslant \hat{x} = \min\{x_1, x_2\} \quad \Rightarrow \quad S_+ \text{ is downward directed.} \tag{4.8}$$

Let $C \subseteq S_+$ be a chain in S_+ (i.e. a totally ordered subset of S_+). From Corollary 7, p. 336 of Dunford and Schwartz [10], we can find $\{x_n\}_{n\geqslant 1} \subseteq C$ such that

$$\inf_{n\geqslant 1} x_n = \inf C.$$

Because of (4.8), we may assume that $\{x_n\}_{n\geqslant 1}$ is decreasing. Also because of hypothesis $H(f)_1$(iii) and since $A(x_n) = f(\cdot, x_n(\cdot))$, we infer that $\{x_n\}_{n\geqslant 1} \subseteq W_0^{1,p}(Z)$ is bounded. Hence we may assume that

$$x_n \xrightarrow{w} \hat{y} \quad \text{in } W_0^{1,p}(Z) \quad \text{and} \quad x_n \to \hat{y} \quad \text{in } L^p(Z).$$

Note that

$$\bigl\langle A(x_n), x_n - \hat{y}\bigr\rangle = \int_Z f\bigl(z, x_n(z)\bigr)(x_n - \hat{y})(z)\,dz \to 0 \quad \text{as } n \to \infty.$$

From this as before (see the proof of Proposition 3.1), we deduce that

$$x_n \to \hat{y} \quad \text{in } W_0^{1,p}(Z) \text{ as } n \to \infty.$$

So, in the limit as $n \to \infty$, we have

$$\begin{aligned} & A(\hat{y}) = f\big(\cdot, \hat{y}(\cdot)\big) \\ \Rightarrow\quad & -\operatorname{div}\big(\big\|D\hat{y}(z)\big\|^{p-2} D\hat{y}(z)\big) = f\big(z, \hat{y}(z)\big) \quad \text{a.e. on } Z, \quad \hat{y}|_{\partial Z} = 0 \\ \Rightarrow\quad & \hat{y} \in S_+ \quad \text{and} \quad \hat{y} = \inf C. \end{aligned}$$

Because C was an arbitrary chain in S_+, from Zorn's lemma, we obtain a minimal element $x_* \in S_+$. From (4.8), we infer that x_* is the smallest solution of (1.1) in I_+.

A similar argument in I_-, using this time Lemma 4.2, produces a greatest solution $v^* \in I_-$ of (1.1) in I_-. $\square$

Using this proposition and a strengthened version of hypothesis $H(f)_1$(iv), we will be able to produce a minimal positive solution $y_+ \in \operatorname{int} C_+$ and a maximal negative solution $y_- \in -\operatorname{int} C_+$ for problem (1.1). The new more restrictive version of hypothesis $H(f)_1$(iv) dictates a strictly p-linear behavior of the nonlinearity $f(z, \cdot)$ near the origin. More precisely, the new hypotheses on the nonlinearity $f(z, x)$, are the following:

$H(f)_4$ $f : Z \times \mathbb{R} \to \mathbb{R}$ is a function such that $f(z, 0) = 0$ a.e. on Z, hypotheses $H(f)_4$(i)–(iii), (v), (vi) are the same as hypotheses $H(f)_1$(i)–(iii), (v), (vi) and

(iv) there exist functions $\eta, \hat{\eta} \in L^\infty(Z)_+$ such that $\lambda_1 \leqslant \eta(z)$ a.e. on Z, $\lambda_1 \neq \eta$ and

$$\eta(z) \leqslant \liminf_{x \to 0} \frac{f(z, x)}{|x|^{p-2}x} \leqslant \limsup_{x \to 0} \frac{f(z, x)}{|x|^{p-2}x} \leqslant \hat{\eta}(z)$$

uniformly for a.a. $z \in Z$.

Proposition 4.4. *If hypotheses $H(f)_4$ hold, then problem* (1.1) *has a smallest positive solution $y_+ \in \operatorname{int} C_+$ and a biggest negative solution $y_- \in -\operatorname{int} C_+$.*

Proof. Let $\underline{x} \in \operatorname{int} C_+$ be the strict lower solution for problem (1.1) obtained in Proposition 3.7 and let $\underline{x}_n = \varepsilon_n \underline{x}$ with $\varepsilon_n \downarrow 0$ and $I_+^n = [\underline{x}_n, \overline{x}]$. From Proposition 4.3, we know that problem (1.1) admits a smallest solution x_*^n in the order interval I_+^n. We know that the sequence $\{x_*^n\}_{n \geqslant 1} \subseteq W_0^{1,p}(Z)$ is bounded and so we may assume that

$$x_*^n \xrightarrow{w} y_+ \quad \text{in } W_0^{1,p}(Z) \quad \text{and} \quad x_*^n \to y_+ \quad \text{in } L^p(Z) \text{ as } n \to \infty.$$

We have

$$A\big(x_*^n\big) = N_f\big(x_*^n\big), \quad n \geqslant 1, \tag{4.9}$$

where $N_f(x_*^n)(\cdot) = f(\cdot, x_*^n(\cdot))$, the Nemytskii operator corresponding to the nonlinearity f. Acting on (4.9) with the function $x_*^n - y_+$ and passing to the limit as $n \to \infty$, from the properties of A, as before, we obtain

$$x_*^n \to y_+ \quad \text{in } W_0^{1,p}(Z) \text{ as } n \to \infty. \tag{4.10}$$

Suppose that $y_+ = 0$. Then we have $\|x_*^n\| \to 0$ as $n \to \infty$ (see (4.10)). We set $w_n = \frac{x_*^n}{\|x_*^n\|}$, $n \geqslant 1$. We may assume that

$$w_n \xrightarrow{w} w \quad \text{in } W_0^{1,p}(Z) \quad \text{and} \quad w_n \to w \quad \text{in } L^p(Z) \text{ as } n \to \infty.$$

From (4.9) we have

$$A(w_n) = \frac{N_f(x_*^n)}{\|x_*^n\|^{p-1}} \quad \text{for all } n \geqslant 1$$

$$\Rightarrow \quad \langle A(w_n), w_n - w\rangle = \int_Z \frac{f(z, x_*^n(z))}{\|x_*^n\|^{p-1}} (w_n - w)(z)\, dz. \tag{4.11}$$

Hypotheses $H(f)_4$ imply that

$$\left|f(z,x)\right| \leqslant c_0|x|^{p-1} \quad \text{for a.a. } z \in Z,\ \text{all } x \in \mathbb{R},\ \text{for some } c_0 > 0. \tag{4.12}$$

From this growth relation we infer that

$$\left\{\frac{N_f(x_*^n)}{\|x_*^n\|^{p-1}}\right\}_{n\geqslant 1} \subseteq L^{p'}(Z) \text{ is bounded.} \tag{4.13}$$

Therefore

$$\int_Z \frac{N_f(x_*^n)}{\|x_*^n\|^{p-1}} (w_n - w)\, dz \to 0 \quad \text{as } n \to \infty$$

$$\Rightarrow \quad \lim_{n\to\infty} \langle A(w_n), w_n - w\rangle = 0 \quad \big(\text{see (4.11)}\big)$$

$$\Rightarrow \quad w_n \to w \quad \text{in } W_0^{1,p}(Z), \quad \text{hence} \quad \|w\| = 1 \quad \text{(see the proof of Proposition 3.1).}$$

Because of (4.13), we may assume that

$$h_n = \frac{N_f(x_*^n)}{\|x_*^n\|^{p-1}} \xrightarrow{w} h \quad \text{in } L^{p'}(Z) \text{ as } n \to \infty. \tag{4.14}$$

For every $\varepsilon > 0$ and $n \geqslant 1$, we introduce the following sets:

$$E_+^n = \left\{z \in Z\colon\ x_*^n(z) > 0,\ \eta(z) - \varepsilon \leqslant \frac{f(z, x_*^n(z))}{x_*^n(z)^{p-1}} \leqslant \hat{\eta}(z) + \varepsilon\right\}$$

and

$$E_-^n = \left\{z \in Z\colon\ x_*^n(z) < 0,\ \eta(z) - \varepsilon \leqslant \frac{f(z, x_*^n(z))}{|x_*^n(z)|^{p-2} x_*^n(z)} \leqslant \hat{\eta}(z) + \varepsilon\right\}.$$

Since $x_*^n \to 0$ in $W_0^{1,p}(Z)$, we may assume (at least for a subsequence) that $x_*^n(z) \to 0$ a.e. on Z. We have

$$x_*^n(z) \to 0^+ \quad \text{a.e. on } \{w > 0\} \quad \text{and} \quad x_*^n(z) \to 0^- \quad \text{a.e. on } \{w < 0\}.$$

Therefore, hypothesis $H(f)_4$(v) implies

$$\chi_{E_+^n}(z) \to 1 \quad \text{a.e. on } \{w > 0\} \quad \text{and} \quad \chi_{E_-^n}(z) \to 1 \quad \text{a.e. on } \{w < 0\}.$$

From this and (4.14) it follows that

$$\chi_{E_+^n} h_n \xrightarrow{w} h \quad \text{in } L^{p'}\big(\{w > 0\}\big) \quad \text{and} \quad \chi_{E_-^n} h_n \xrightarrow{w} h \quad \text{in } L^{p'}\big(\{w < 0\}\big).$$

From the definition of the set E_+^n, we have

$$\begin{aligned}\chi_{E_+^n}(z)\big(\eta(z) - \varepsilon\big) w_n(z)^{p-1} &\leqslant \chi_{E_+^n}(z) \frac{f(z, x_*^n(z))}{x_*^n(z)^{p-1}} w_n(z)^{p-1} = \chi_{E_+^n}(z) \frac{f(z, x_*^n(z))}{\|x_*^n\|^{p-1}} \\ &\leqslant \chi_{E_+^n}(z)\big(\hat{\eta}(z) + \varepsilon\big) w_n(z)^{p-1} \quad \text{a.e. on } Z.\end{aligned}$$

Taking weak limits in $L^{p'}(\{w > 0\})$, via Mazur's lemma and since $\varepsilon > 0$ was arbitrary, we obtain

$$\eta(z) w(z)^{p-1} \leqslant h(z) \leqslant \hat{\eta}(z) w(z)^{p-1} \quad \text{a.e. on } \{w > 0\}. \tag{4.15}$$

Working similarly with the set E_-^n, we obtain

$$\hat{\eta}(z)\big|w(z)\big|^{p-2} w(z) \leqslant h(z) \leqslant \eta(z)\big|w(z)\big|^{p-2} w(z) \quad \text{a.e. on } \{w < 0\}. \tag{4.16}$$

Finally from (4.12) it is clear that

$$h(z) = 0 \quad \text{a.e. on } \{w = 0\}. \tag{4.17}$$

Combining (4.15)–(4.17), we can say that

$$h(z) = k(z)\big|w(z)\big|^{p-2} w(z) \quad \text{a.e. on } Z,$$

with $k \in L^\infty(Z)_+$, $\eta(z) \leqslant k(z) \leqslant \hat{\eta}(z)$ a.e. on Z. Then, in the limit as $n \to \infty$, we have

$$\begin{aligned}&A(w) = k|w|^{p-2} w \\ &\quad \Rightarrow \quad -\operatorname{div}\big(\big\|Dw(z)\big\|^{p-2} Dw(z)\big) = k(z)\big|w(z)\big|^{p-2} w(z) \quad \text{a.e. on } Z, \\ &\quad w|_{\partial Z} = 0, \quad w \neq 0.\end{aligned} \tag{4.18}$$

We know that

$$\hat{\lambda}_1(k) < \hat{\lambda}_1(\lambda_1) = 1.$$

So from (4.18), we see that $w \in C_0^1(\overline{Z})$ (nonlinear regularity theory) cannot be the principal eigenfunction and so it must change sign. But $w_n = \frac{x_*^n}{\|x_*^n\|} \in \operatorname{int} C_+$ for all $n \geqslant 1$ and $w_n \to w$ in $W_0^{1,p}(Z)$. Therefore $w \geqslant 0$, a contradiction. This proves that $y_+ \neq 0$ and of course $y_+ \geqslant 0$. Moreover, since $x_*^n \to y_+$ in $W_0^{1,p}(Z)$ as $n \to \infty$ (see (4.10)), passing to the limit as $n \to \infty$ in (4.9), we obtain

$$\begin{aligned} &A(y_+) = N_f(y_+) \\ &\Rightarrow \quad -\operatorname{div}\big(\|Dy_+(z)\|^{p-2} Dy_+(z)\big) = f\big(z, y_+(z)\big) \quad \text{a.e. on } Z, \\ &y|_{\partial Z} = 0, \quad y_+ \geqslant 0, \quad y_+ \neq 0. \end{aligned}$$

From nonlinear regularity theory, we have $y_+ \in C_+ \setminus \{0\}$ and then the nonlinear strong maximum principle of Vazquez [23], implies that $y_+ \in \operatorname{int} C_+$.

We claim that y_+ is the smallest positive solution of (1.1). Indeed, let $\hat{y} \neq 0$ be another positive solution. As before, from nonlinear regularity and the nonlinear strong maximum principle of Vazquez [23], we have $\hat{y} \in \operatorname{int} C_+$. Using Lemma 3.6, we can find $\hat{\varepsilon} \in (0,1)$ such that $\hat{\varepsilon}\underline{x} \leqslant \hat{y}$. Then for $n \geqslant 1$ large we will have $\varepsilon_n < \hat{\varepsilon}$ and so $\underline{x}_n \leqslant \hat{\varepsilon}\underline{x} \leqslant \hat{y} \leqslant \overline{x}$. We fix such large $n \geqslant 1$ and work on the order interval $[\underline{x}_n, \overline{x}]$. On this interval we have $x_*^n \leqslant \hat{y}$ (see Proposition 4.3) and so $y_+ \leqslant \hat{y}$. This proves that y_+ is the smallest positive solution of (1.1).

In a similar fashion, if $\overline{v}_n = \varepsilon_n \overline{v}$, $\varepsilon_n \downarrow 0$ and working on the order interval $I_-^n = [\underline{v}, \overline{v}_n]$, we obtain $y_- \in -\operatorname{int} C_+$, the biggest negative solution of (1.1). □

Now we are ready to produce a nodal solution for problem (1.1). This requires a further strengthening of hypothesis $H(f)_1$(v).

$H(f)_5$ $f: Z \times \mathbb{R} \to \mathbb{R}$ is a function such that $f(z,0) = 0$ a.e. on Z, hypotheses $H(f)_5$(i)–(iii), (v) are the same as hypotheses $H(f)_1$(i)–(iii), (v), respectively, and
(iv) there exists $\hat{\eta} \in L^\infty(Z)_+$ such that

$$\lambda_2 < \liminf_{x \to 0} \frac{f(z,x)}{|x|^{p-2}x} \leqslant \limsup_{x \to 0} \frac{f(z,x)}{|x|^{p-2}x} \leqslant \hat{\eta}(z)$$

uniformly for a.a. $z \in Z$;
(vi) $f(z,x)x \geqslant 0$ for a.a. $z \in Z$ and all $x \in \mathbb{R}$.

Remark 4.5. The stronger version of hypothesis $H(f)_5$(v) (uniform nonresonance with respect to $\lambda_2 > 0$ at the origin) allows us to relax hypothesis $H(f)_5$(vi) to a simple sign condition. The reason for this is that now a positive strict lower solution can be obtained much easier, by just taking a small multiple of the principal eigenfunction $u_1 \in \operatorname{int} C_+$ and then the comparison with x_0 is straightforward.

Theorem 4.6. *If hypotheses $H(f)_5$ hold, then problem* (1.1) *has at least three nontrivial solutions, $x_0 \in \operatorname{int} C_+$, $v_0 \in -\operatorname{int} C_+$ and a nodal solution $y_0 \in C_0^1(\overline{Z})$, $y_0 \neq 0$.*

Proof. The two nontrivial smooth solutions of constant sign $x_0 \in \operatorname{int} C_+$, $v_0 \in -\operatorname{int} C_+$, were obtained in Proposition 3.8. So we need to produce the nodal solution.

Let $y_+ \in \operatorname{int} C_+$ and $y_- \in -\operatorname{int} C_+$ be the two extremal constant sign solutions of (1.1) obtained in Proposition 4.4. We introduce the following truncated versions of the nonlinearity $f(z,x)$:

$$\hat{f}_+(z,x) = \begin{cases} 0 & \text{if } x < 0, \\ f(z,x) & \text{if } 0 \leqslant x \leqslant y_+(z), \\ f(z, y_+(z)) & \text{if } y_+(z) < x, \end{cases}$$

$$\hat{f}_-(z,x) = \begin{cases} f(z, y_-(z)) & \text{if } x < y_-(z), \\ f(z,x) & \text{if } y_-(z) \leqslant x \leqslant 0, \\ 0 & \text{if } 0 < x, \end{cases}$$

$$\hat{f}(z,x) = \begin{cases} f(z, y_-(z)) & \text{if } x < y_-(z), \\ f(z,x) & \text{if } y_-(z) \leqslant x \leqslant y_+(z), \\ f(z, y_+(z)) & \text{if } y_+(z) < x. \end{cases}$$

Then we set

$$\widehat{F}_+(z,x) = \int_0^x \hat{f}_+(z,s)\,ds, \quad \widehat{F}_-(z,x) = \int_0^x \hat{f}_-(z,s)\,ds \quad \text{and} \quad \widehat{F}(z,x) = \int_0^x \hat{f}(z,s)\,ds.$$

Finally we introduce the following C^1-functionals defined on $W_0^{1,p}(Z)$:

$$\hat{\varphi}_+(x) = \frac{1}{p}\|Dx\|_p^p - \int_Z \widehat{F}_+\big(z, x(z)\big)\,dz,$$

$$\hat{\varphi}_-(x) = \frac{1}{p}\|Dx\|_p^p - \int_Z \widehat{F}_-\big(z, x(z)\big)\,dz \quad \text{and}$$

$$\hat{\varphi}(x) = \frac{1}{p}\|Dx\|_p^p - \int_Z \widehat{F}\big(z, x(z)\big)\,dz \quad \text{for all } x \in W_0^{1,p}(Z).$$

In what follows, we will use the following order intervals

$$T_+ = [0, y_+], \quad T_- = [y_-, 0] \quad \text{and} \quad T = [y_-, y_+].$$

The critical points of $\hat{\varphi}_+$ are located in T_+, the critical points of $\hat{\varphi}_-$ in T_- and the critical points of $\hat{\varphi}$ in $\widehat{T}$. We do the proof for $\hat{\varphi}_+$, the proof for the others being similar.

Suppose $x \in W_0^{1,p}(Z)$ is a critical point of $\hat{\varphi}_+$. Then

$$A(x) = \widehat{N}_+(x), \tag{4.19}$$

where $\widehat{N}_+(x)(\cdot) = \hat{f}_+(\cdot, x(\cdot))$ (the Nemytskii operator corresponding to the nonlinearity $\hat{f}_+(z,x)$). We take duality brackets of (4.19) with $(x - y_+)^+ \in W_0^{1,p}(Z)$. We obtain

$$\begin{aligned}
\langle A(x), (x-y_+)^+\rangle &= \int_Z \hat{f}_+(z,x)(x-y_+)^+\,dz \\
&= \int_Z f(z,y_+)(x-y_+)^+\,dz \quad (\text{recall the definition of } \hat{f}_+) \\
&= \langle A(y_+), (x-y_+)^+\rangle \quad \big(\text{since } y_+ \in \operatorname{int} C_+ \text{ is a solution of (1.1)}\big) \\
&\Rightarrow \quad \langle A(x)-A(y_+), (x-y_+)^+\rangle = 0 \\
&\Rightarrow \quad \int_{\{x>y_+\}} \big(\|Dx\|^{p-2}Dx - \|Dy_+\|^{p-2}Dy_+, Dx-Dy_+\big)_{\mathbb{R}^{\mathbb{N}}}\,dz = 0.
\end{aligned} \tag{4.20}$$

Since the map $\theta_p : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}$ defined by $\theta_p(u) = \|u\|^{p-2}u$ if $u \neq 0$, $\theta_p(0) = 0$, is strictly monotone, from (4.20) we infer that

$$\big|\{x>y_+\}\big|_N = 0 \quad \big(\text{recall } |\cdot|_N \text{ is the Lebesgue measure on } \mathbb{R}^{\mathbb{N}}\big) \quad \Rightarrow \quad x \leqslant y_+.$$

Similarly we show $0 \leqslant x$. Therefore $x \in T_+$.

Since the critical points of $\hat{\varphi}_+$ are in T_+, we see that $\{0, y_+\}$ are the only critical points of $\hat{\varphi}_+$. By virtue of hypothesis $H(f)_5$(v), we can find $\delta > 0$ small such that

$$\begin{aligned}
&\lambda_2 x^{p-1} < f(z,x) \quad \text{for a.a. } z \in Z \text{ and all } x \in [0,\delta] \\
&\Rightarrow \quad \frac{\lambda_2}{p} x^p < F(z,x) \quad \text{for a.a. } z \in Z \text{ and all } x \in [0,\delta].
\end{aligned} \tag{4.21}$$

We choose $\varepsilon > 0$ small, such that

$$\varepsilon u_1(z) \leqslant \min\big\{y_+(z), \delta\big\} \quad \text{for all } z \in \overline{Z}, \tag{4.22}$$

see Lemma 3.6. Then

$$\begin{aligned}
\hat{\varphi}_+(\varepsilon u_1) &= \frac{\varepsilon^p}{p}\|Du_1\|_p^p - \int_Z \widehat{F}_+\big(z, \varepsilon u_1(z)\big)\,dz \\
&< \frac{\varepsilon^p}{p} \int_Z (\lambda_1 - \lambda_2) u_1(z)^p\,dz \quad \big(\text{see (4.20), (4.21) and (2.2)}\big) \\
&< 0 \\
&\Rightarrow \quad \inf_{W_0^{1,p}(Z)} \hat{\varphi}_+ < 0 = \hat{\varphi}_+(0).
\end{aligned}$$

Clearly $\hat{\varphi}_+$ is coercive and w-lower semicontinuous. So by the Weierstrass theorem, we can find $\hat{y}_0 \in W_0^{1,p}(Z)$ such that

$$\hat{\varphi}_+(\hat{y}_0) = \inf_{W_0^{1,p}(Z)} \hat{\varphi}_+ = \hat{m}_+ < 0 = \hat{\varphi}_+(0) \quad \Rightarrow \quad \hat{y}_0 \neq 0.$$

Therefore $\hat{y}_0$ is a nonzero critical point of $\hat{\varphi}_+$, hence $\hat{y}_0 = y_+$. Recall that $y_+ \in \operatorname{int} C_+$. So y_+ is a local $C_0^1(\overline{Z})$-minimizer of $\hat{\varphi}$ and so y_+ is a local $W_0^{1,p}(Z)$-minimizer of $\hat{\varphi}$ (see Garcia Azorero, Manfredi and Peral Alonso [11]). We may assume that y_+ is an isolated local minimizer of $\hat{\varphi}_+$. Indeed, if this is not the case, we can find a sequence $\{x_n\}_{n\geqslant 1} \subseteq W_0^{1,p}(Z)$ such that

$$x_n \to y_+ \quad \text{in } W_0^{1,p}(Z), \quad x_n \neq 0, y_+, y_- \quad \text{and} \quad \hat{\varphi}'(x_n) = 0.$$

We have $x_n \in T$ and since $x_n \neq y_+$, $x_n \neq y_-$, $x_n \neq 0$, $n \geqslant 1$, we have produced a whole sequence of distinct nodal solutions for problem (1.1) and so we are done.

Similarly working with $\hat{\varphi}_-$ on T_-, we deduce that $y_- \in -\operatorname{int} C_+$ is a global minimizer of $\hat{\varphi}_-$, $\hat{\varphi}_-(y_-) = \hat{\varphi}(y_-) < 0 = \hat{\varphi}(0)$, it is a local minimizer of $\hat{\varphi}$ and we may also assume that it is an isolated local minimizer of $\hat{\varphi}$.

Then we can find $\delta > 0$ small such that

$$\hat{\varphi}(y_+) < \inf\bigl[\hat{\varphi}(x)\colon x \in \partial B_\delta(y_+)\bigr] \leqslant 0$$

and

$$\hat{\varphi}(y_-) < \inf\bigl[\hat{\varphi}(x)\colon x \in \partial B_\delta(y_-)\bigr] \leqslant 0$$

with $\partial B_\delta(y_\pm) = \{x \in W_0^{1,p}(Z)\colon \|x - y_\pm\| = \delta\}$ (see Motreanu, Motreanu and Papageorgiou [21]). We may assume without any loss of generality that $\hat{\varphi}(y_-) \leqslant \hat{\varphi}(y_+)$.

If we set $S_0 = \partial B_\delta(y_+)$ and $T = [y_-, y_+]$, $T_0 = \{y_-, y_+\}$, then we can easily see that the pair $\{T_0, T\}$ is linking with S_0 in $W_0^{1,p}(Z)$ (see for example Gasinski and Papageorgiou [12, p. 642]). Moreover, it is clear that $\hat{\varphi}$ is coercive and so it verifies the PS-condition. Therefore, we can apply the minimax theorem for linking sets (see for example Gasinski and Papageorgiou [12, p. 646]) and produce $y_0 \in W_0^{1,p}(Z)$, a critical point of $\hat{\varphi}$ such that

$$\hat{\varphi}(y_\pm) < \hat{\varphi}(y_0) = \inf_{\overline{\gamma} \in \overline{\Gamma}} \max_{t \in [-1,1]} \hat{\varphi}\bigl(\overline{\gamma}(t)\bigr), \tag{4.23}$$

where $\overline{\Gamma} = \{\overline{\gamma} \in C([-1,1], W_0^{1,p}(Z))\colon \overline{\gamma}(-1) = y_-, \overline{\gamma}(+1) = y_+\}$. From (4.23) we see that $y_0 \neq y_\pm$.

We will show that $\hat{\varphi}(y_0) < 0 = \hat{\varphi}(0)$ and so $y_0 \neq 0$ and of course is nodal since $y_0 \in T$.

According to (4.23), to show the nontriviality of y_0, it is enough to produce a path $\overline{\gamma}_0 \in \overline{\Gamma}$ such that

$$\hat{\varphi}\bigl(\overline{\gamma}_0(t)\bigr) < 0 \quad \text{for all } t \in [-1,1].$$

So, in what follows, we construct such a path $\overline{\gamma}_0$.

Recall that $\partial B_1^{L^p(Z)} = \{x \in L^p(Z)\colon \|x\|_p = 1\}$ and set $S = W_0^{1,p}(Z) \cap \partial B_1^{L^p(Z)}$ endowed with $W_0^{1,p}(Z)$-topology. We also set

$$S_c = W_0^{1,p}(Z) \cap \partial B_1^{L^p(Z)} \cap C_0^1(\overline{Z})$$

equipped with the $C_0^1(\overline{Z})$-topology. Evidently S_c is dense in S in the $W_0^{1,p}(Z)$-topology. Then, if

$$\Gamma_0 = \{\gamma \in C([-1,1], S):\ \gamma(-1) = -u_1,\ \gamma(+1) = u_1\}$$

and

$$\Gamma_{0,c} = \{\gamma \in C([-1,1], S_c):\ \gamma(-1) = -u_1,\ \gamma(+1) = u_1\},$$

we have that $\Gamma_{0,c}$ is dense in Γ_0. Because of (2.3), we can find $\hat{\gamma}_0 \in \Gamma_{0,c}$ such that

$$\max[\|Dx\|_p^p:\ x \in \hat{\gamma}_0([1,1])] \leqslant \lambda_2 + \delta, \quad \delta > 0. \tag{4.24}$$

We can always choose $\delta > 0$ small such that

$$\lambda_2 + 2\delta < \liminf_{x \to 0} \frac{f(z,x)}{|x|^{p-2}x} \quad \text{uniformly for a.a. } z \in Z$$

(see hypothesis $H(f)_5$(v)). We can find $\hat{\delta} > 0$ such that

$$\begin{aligned} &\lambda_2 + \delta < \frac{f(z,x)}{|x|^{p-2}x} \quad \text{for a.a. } z \in Z \text{ and all } 0 < |x| \leqslant \hat{\delta} \\ &\Rightarrow \quad \frac{1}{p}(\lambda_2 + \delta)|x|^p \leqslant F(z,x) \quad \text{for a.a. } z \in Z \text{ and all } 0 < |x| \leqslant \hat{\delta}. \end{aligned} \tag{4.25}$$

Because $\hat{\gamma}_0([-1,1]) \subseteq S_c$ and $-y_-, y_+ \in \operatorname{int} C_+$, we can find $\varepsilon > 0$ small such that

$$|\varepsilon x(z)| \leqslant \hat{\delta} \quad \text{for all } z \in \overline{Z},\ \text{all } x \in \hat{\gamma}_0([-1,1])$$

and

$$\varepsilon x \in [y_-, y_+] \quad \text{for all } x \in \hat{\gamma}_0([-1,1]).$$

Then, if $x \in \hat{\gamma}_0([-1,1])$, we have

$$\begin{aligned} \hat{\varphi}(\varepsilon x) = \varphi(\varepsilon x) &= \frac{\varepsilon^p}{p}\|Dx\|_p^p - \int_Z F(z, \varepsilon x(z))\,dz \\ &< \frac{\varepsilon^p}{p}\|Dx\|_p^p - \frac{\varepsilon^p}{p}(\lambda_2 + \delta)\|x\|_p^p \quad (\text{see (4.25)}) \\ &\leqslant 0 \quad (\text{see (4.24) and recall that } \|x\|_p = 1). \end{aligned} \tag{4.26}$$

So, if we consider the continuous path $\gamma_0 = \varepsilon\hat{\gamma}_0$ which joins $-\varepsilon u_1$ and εu_1, then

$$\hat{\varphi}|_{\gamma_0} < 0. \tag{4.27}$$

Next, with the help of Theorem 2.3 (the second deformation theorem), we will produce a continuous path joining εu_1 and y_+, along which the functional $\hat{\varphi}$ is strictly negative. Recall that $\{0, y_+\}$ are the only critical points of the functional $\hat{\varphi}_+$. Let $a_+ = \hat{\varphi}_+(y_+) = \inf \hat{\varphi}_+ < 0 = b_+$. The functional $\hat{\varphi}_+$ is coercive and so it satisfies the PS-condition. Therefore according to Theorem 2.3, we can find a deformation $h : [0,1] \times (\hat{\varphi}^{b_+} \setminus K_{b_+}) \to \hat{\varphi}^{b_+}$ such that

$$h(t,\cdot)|_{K_{a_+}} = \mathrm{id}|_{K_{a_+}} \quad \text{for all } t \in [0,1],$$
$$h\big(1, \hat{\varphi}^{b_+} \setminus K_{b_+}\big) \subseteq \hat{\varphi}^{a_+},$$
$$\varphi\big(h(t,x)\big) \leqslant \varphi\big(h(s,x)\big) \quad \text{for all } t,s \in [0,1],\ s \leqslant t \text{ and all } x \in \hat{\varphi}^{b_+} \setminus K_{b_+}.$$

We consider the path $\gamma_+ : [0,1] \to \hat{\varphi}^{b_+}$ defined by

$$\gamma_+(t) = h(t, \varepsilon u_1) \quad \text{for all } t \in [0,1].$$

Clearly γ_+ is a continuous path and

$$\gamma_+(0) = h(0, \varepsilon u_1) = u_1 \quad \text{(since } h \text{ is a deformation, see Definition 2.1)},$$
$$\gamma_+(1) = h(1, \varepsilon u_1) = y_+ \quad \big(\text{since } \hat{\varphi}^{a_+} = \{y_+\}\big) \quad \text{and}$$
$$\hat{\varphi}_+\big(\gamma_+(t)\big) = \hat{\varphi}_+\big(h(t,\varepsilon u_1)\big) \leqslant \hat{\varphi}_+(\varepsilon u_1) < 0 \quad \text{for all } t \in [0,1] \quad \big(\text{see (4.27)}\big).$$

Therefore the continuous path γ_+ joins εu_1 and γ_+ and

$$\hat{\varphi}_+|_{\gamma_+} < 0.$$

But note that $\hat{\varphi}_+ \geqslant \hat{\varphi}$ on γ_+ due to hypothesis $H(f)_5$(iv). Therefore

$$\hat{\varphi}|_{\gamma_+} < 0. \tag{4.28}$$

Similarly we construct a continuous path γ_- which joins $-\varepsilon u_1$ and y_-, such that

$$\hat{\varphi}|_{\gamma_-} < 0. \tag{4.29}$$

If we concatenate paths $\gamma_-, \gamma_0, \gamma_+$, we produce a path $\overline{\gamma}_0 \in \overline{\Gamma}$ such that

$$\hat{\varphi}|_{\overline{\gamma}} < 0 \quad \big(\text{see (4.27)–(4.29)}\big).$$

Therefore y_0 is a nontrivial critical point of φ in $T = [y_-, y_+]$, $y_0 \neq y_+$, $y_0 \neq y_-$. Hence y_0 is a nodal solution of (1.1). Nonlinear regularity theory guarantees that $y_0 \in C_0^1(\overline{Z})$. □

Now we can have the final multiplicity result for coercive problems (see Proposition 3.12). So we introduce the following hypotheses on the nonlinearity $f(z,x)$:

$H(f)_6$ $f : Z \times \mathbb{R} \to \mathbb{R}$ is a function such that $f(z,0) = 0$ a.e. on Z and
- (i) for all $x \in \mathbb{R}$, $z \to f(z,x)$ is measurable;
- (ii) for a.a. $z \in Z$, $x \to f(z,x)$ is continuous;

(iii) for a.a. $z \in Z$ and all $x \in \mathbb{R}$, we have

$$\left|f(z,x)\right| \leqslant a(z) + c|x|^{r-1}$$

with $a \in L^\infty(Z)_+$, $c > 0$, $1 < r < p^*$;

(iv) there exists $\hat{\eta} \in L^\infty(Z)_+$ such that

$$\lambda_2 < \liminf_{x\to 0} \frac{f(z,x)}{|x|^{p-2}x} \leqslant \limsup_{x\to 0} \frac{f(z,x)}{|x|^{p-2}x} \leqslant \hat{\eta}(z)$$

uniformly for a.a. $z \in Z$;

(v) there exists $\theta \in L^\infty(Z)_+$ such that $\theta(z) \leqslant \lambda_1$ a.e. on Z, $\lambda_1 \neq \theta$ and

$$\limsup_{x\to\pm\infty} \frac{f(z,x)}{|x|^{p-2}x} \leqslant \theta(z) \quad \text{uniformly for a.a. } z \in Z;$$

(vi) $f(z,x)x \geqslant 0$ for a.a. $z \in Z$, all $x \in \mathbb{R}$.

Combining Proposition 3.11 and Theorem 4.6, we have

Theorem 4.7. *If hypotheses $H(f)_6$ hold, then problem* (1.1) *has at least three nontrivial solutions, $x_0 \in \operatorname{int} C_+$, $v_0 \in -\operatorname{int} C_+$ and a nodal solution $y_0 \in C_0^1(\overline{Z})$, $y_0 \neq 0$.*

Remark 4.8. Liu and Liu [17] and Liu [18] prove multiplicity results for coercive problems. They obtain three nontrivial solutions, but they do not give any information about the sign of the third solution. So Theorem 4.9 improves the multiplicity results of [17] and [18].

Next we state the complete multiplicity result for the p-superlinear problem (see Theorem 3.14). For this purpose, we introduce the following hypotheses on the nonlinearity $f(z,x,\lambda)$.

$H(f)_7$ $f : Z \times \mathbb{R} \times (0,+\infty) \to \mathbb{R}$ is a function such that $f(z,0,\lambda) = 0$ a.e. on Z, for all $\lambda > 0$ and

(i) for all $(x,\lambda) \in \mathbb{R} \times (0,+\infty)$, $z \to f(z,x,\lambda)$ is measurable;

(ii) for a.a. $z \in Z$ and all $\lambda \in (0,+\infty)$, $x \to f(z,x,\lambda)$ is continuous;

(iii) for a.a. $z \in Z$, all $x \in \mathbb{R}$ and all $\lambda \in (0,+\infty)$, we have

$$\left|f(z,x,\lambda)\right| \leqslant a(z,\lambda) + c|x|^{r-1}$$

with $a(\cdot,\lambda) \in L^\infty(Z)_+$, $\|a(\cdot,\lambda)\|_\infty \to 0$ as $\lambda \to 0^+$, $c > 0$, $1 < r < p^*$;

(iv) for every $\lambda \in (0,+\infty)$, there exists a function $\hat{\eta} = \hat{\eta}(\lambda) \in L^\infty(Z)_+$ such that

$$\lambda_2 < \liminf_{x\to 0} \frac{f(z,x,\lambda)}{|x|^{p-2}x} \leqslant \limsup_{x\to 0} \frac{f(z,x,\lambda)}{|x|^{p-2}x} \leqslant \hat{\eta}(z)$$

uniformly for a.a. $z \in Z$;

(v) for every $\lambda \in (0,+\infty)$, there exist $M = M(\lambda) > 0$ and $\mu = \mu(\lambda) > p$ such that

$$0 < \mu F(z,x,\lambda) \leqslant f(z,x,\lambda)x \quad \text{for a.a. } z \in Z, \text{ all } |x| \geqslant M;$$

(vi) $f(z,x,\lambda)x \geqslant 0$ for a.a. $z \in Z$, all $x \in \mathbb{R}$ and all $\lambda \in (0,+\infty)$.

Combining Proposition 3.13 and Theorem 4.6, we have:

Theorem 4.9. *If hypotheses* $H(f)_7$ *hold, then there exists* $\lambda^* \in (0, +\infty)$ *such that for all* $\lambda \in (0, \lambda^*)$ *problem* (1.1) *has at least five nontrivial solutions* $x_0, \hat{x} \in \operatorname{int} C_+$, $x_0 \leqslant \hat{x}$, $x_0 \neq \hat{x}$, $v_0, \hat{v} \in -\operatorname{int} C_+$, $\hat{v} \leqslant v_0$, $\hat{v} \neq v_0$ *and a nodal solution* $y_0 \in C_0^1(\overline{Z})$, $y_0 \neq 0$.

Remark 4.10. In the works of Zhang and Li [25] and Zhang, Chen and Li [24], the quotients $\frac{f(x)}{|x|^{p-2}x}$ (f is independent of $z \in Z$ in both works) have finite limits as $x \to 0^{\pm}$ and as $x \to \pm\infty$. This is important in their analysis. Also f is locally Lipschitz and in Zhang and Li [25] $N < p$. This low dimensionality of the problems permits the authors to exploit the compact embedding of the Sobolev space $W_0^{1,p}$ into $C(\overline{Z})$. In both papers, the authors prove the existence of at least three nontrivial solutions, one positive, the second negative and the third nodal. As we already mentioned in the introduction, their approach is completely different and it is based on the invariance properties of the descent flow of a pseudogradient vector field. In Bartsch and Liu [4] the nonlinearity $f(z, x)$ is continuous, they employ the Ambrosetti–Rabinowitz growth condition (so their problem is p-superlinear), they assume that for some $m > 0$, $x \to f(z, x) + m|x|^{p-2}x$ is increasing and when $N \geqslant 6$, they require a technical condition on the exponent $p > 1$. Again they obtain three nontrivial solutions, one positive, one negative and the third nodal. Their approach uses critical point theory for C^1-functionals for ordered Banach spaces.

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