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Image Filtering, Mean Curvature, Dirichlet Problems

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Abstract—Numerical experiments of the model for image filtering proposed in [1] by Crandall, Lions and others show that, given the PDE, the differential problem which is correct to solve is the Dirichlet one, on a domain Ω convex with holes. The aim of this paper is to produce a result of existence and uniqueness of the viscosity solution for this problem. The result is stated for convex domains and for nonconvex ones, we propose an easy example in order to show that an existence theorem has not been expected.

Keywords—Fully-nonlinear PDEs, Viscosity solutions, Image smoothing, Barriers.

1. INTRODUCTION

In the last few years, a big effort has been given to the mathematical modeling of image processing. The problem of obtaining an automatic recognition of objects in a picture is growing important in the latest technological developments.

We study the model for image filtering proposed by Alvarez, Crandall, Lions and Morel in [1]. If we call $u_0 : \mathbf{R}^2 \to \mathbf{R}$ the grey levels of the image extended to $\mathbf{R}^2$ by reflection, they propose to solve the Cauchy problem

$$\frac{\partial u}{\partial t} = |Du|(\operatorname{curv}(u))^{1/3}, \qquad u(x,0) = u_0(x), \tag{1}$$

where $\operatorname{curv}(u) = \operatorname{div}(Du/|Du|)$ is the scalar mean curvature of the level sets of u and u is, at each time, a different filtered image. The parameter t plays the role of *scale* of microscopic details which are eliminated.

Later numerical experiments have shown (see [2]) that, on one hand, reflections cause false symmetries which damage the quality of the filtered image and, on the other hand, there are *special points* in a neighbourhood in which we do not want the filter to act. Such points are called X-junctions and T-junctions and take account of the superimposition of different objects.

Mainly because of these results, we have studied the Dirichlet problem which reads

$$\begin{aligned} \frac{\partial u}{\partial t} &= |Du| G(\operatorname{curv}(u)), && \Omega \times (0,T) = \Omega_t, \\ u(x,t) &= u_0(x), && \{\partial\Omega \times [0,T)\} \cup \{\Omega \times \{0\}\} = \partial_p \Omega_t, \end{aligned} \tag{2}$$

where $G : \mathbf{R} \to \mathbf{R}$ is a nondecreasing continuous function with $G(0) = 0$.

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Looking at the application, we would like to solve this problem on a Lipschitz convex (nonnecessarily strictly) domain with holes. Holes eliminate the neighbourhood of singular points from the domain of the filter.

From a mathematical point of view, the equations we are dealing with are fully nonlinear degenerate parabolic PDEs whose weak solutions are well defined in the theory of viscosity solutions introduced by Crandall, Ishii and Lions in [3]. This means we ask the initial data to be continuous and we look for continuous solutions. We study the existence and uniqueness of the viscosity solution of problem (2) on a bounded domain Ω under the following assumptions.

We analyse four different cases: smooth strictly and nonstrictly convex domain, a nonsmooth convex one, and finally, a nonconvex one. Indeed, in this last case, we present a negative result, that is, an existence theorem cannot be expected in the nonconvex case.

2. PRELIMINARIES

From now on, the set Ω is supposed to be an open set in $\mathbf{R}^2$ with C^2 boundary.

Let $a \in \mathbf{R}^+$; we set

$$N_a = \{x \in \Omega : 0 < \operatorname{dist}(x, \partial\Omega) < a\} \tag{3}$$

and we give some definitions.

DEFINITION 2.1. *Let $\Omega \subset \mathbf{R}^2$ be as above and $a \in \mathbf{R}^+$. A continuous function $f : \bar{N}_a \to \mathbf{R}$ is a static subbarrier for (2) if it is a viscosity subsolution of (2) in N_a and if*

$$\forall x \in \partial\Omega, \qquad f(x) = u_0(x) \wedge \forall x \in N_a, \quad f(x) < u_0(x).$$

A *static superbarrier* is obviously defined in a similar way, and we call *static barrier* a couple of a static superbarrier and static subbarrier.

DEFINITION 2.2. *A continuous nontime-increasing function $b^- : \Omega \times [O,T] \to \mathbf{R}$ is a local dynamic subbarrier for (2) if it is a viscosity subsolution of (2) and if*

$$\exists \bar{x} \in \Omega : b^-(\bar{x}, 0) = u_0(\bar{x}) \wedge b^-(x,t) < u_0(x), \qquad \forall x \neq \bar{x}, \quad \forall t \in [0,T].$$

A *local dynamic superbarrier* is obviously defined in a similar way, and we call *local dynamic barrier* a couple of a dynamic superbarrier and dynamic subbarrier which acts at the same point.

The attention is focused on the construction of a subsolution f and a supersolution g in the hypotheses

$$\begin{gathered} f(x,t) \leq g(x,t), \quad -\infty < f_*(x,t), \quad g^*(x,t) < +\infty, \qquad \forall\, (x,t) \in \Omega_t, \\ f_*(x,t) = g^*(x,t), \qquad \forall\, (x,t) \in \partial_p \Omega_t, \end{gathered} \tag{4}$$

(where f_* and g^* are the lower and upper semicontinuous relaxations of f and g, respectively) which allows us to use the Perron Method to obtain the existence result.

Such subsolution (supersolution) will be obtained by taking the supremum (infimum) of a suitable family of subbarriers (superbarriers). We shall construct one static barrier which is supposed to *keep* the solution equal to the data on the boundary $\partial\Omega$ for all $t \leq T$ and a family of dynamic barriers which must *keep* the solution equal to the data in Ω only for $t = 0$.

3. CONSTRUCTION OF SUBSOLUTIONS AND SUPERSOLUTIONS

3.1. The Case of Strictly Convex Domain

Let Ω be a strictly convex, C^2 open set in $\mathbf{R}^2$. We set $\mathcal{K} : \partial\Omega \to \mathbf{R}^+$ the function which gives the curvature of the manifold $\partial\Omega$. We choose a positive real number $\bar{a}$ such that $\{\bar{a} \max_{y \in \partial\Omega} \mathcal{K}(y)\} < 1/2$ and this condition ensures that $\operatorname{dist}(\cdot, \partial\Omega) \in C^2(\bar{N}_{\bar{a}})$.

We construct a subsolution under hypothesis (4) being the supersolution exactly symmetric.

Static barrier

The static subbarrier must be a subsolution for the associated elliptic problem

$$-G(\text{curv}\,(u)) = 0. \tag{5}$$

As in [4], we define the static barrier b in a set of the form (3) with $a \leq \bar{a}$ of the type

$$b(x) = u_0(x) - f(d(x)), \qquad x \in N_a, \tag{6}$$

for $d(x) = \text{dist}\,(x, \partial\Omega)$ and $f : [0, a) \to R$ defined as

$$f(\xi) = \frac{1}{\mu}(\ln(1 + k\xi)), \qquad \xi \in [0, a), \tag{7}$$

where $\mu > 0$, $k > 0$, and $a \leq \bar{a}$ are parameters to be fixed suitably.

We ask b to be a subsolution for (5) in N_a and, thanks to the monotonicity of G, it means we ask $\text{curv}\,(b(x)) \geq 0$, $\forall x \in N_a$. After some algebraic computations, we obtain that for every fixed μ, we can choose a value $k_1 = k_1(\mu)$ such that, for every $k \geq k_1$, b is a subsolution in N_a with $a \leq \min\{\bar{a}, 1/k_1\}$. For every μ, let $a = \min\{\bar{a}, 1/k_1\}$ and $k = 1/a$ be the values which define b (see [5] for details). We have, in this way, a family b_μ^- of static subbarriers.

Note that the symmetric family of static superbarriers is

$$b_\mu^+(x,t) = u_0(x) + f(d(x)), \qquad x \in N_a.$$

Dynamic barrier

Lemma 6.1 in [6] provides the existence of radial subsolutions (supersolutions) for the PDE in (2) which are decreasing (increasing, respectively) in time and space and verify $\frac{\partial u^\pm}{\partial t}(0,t) = \pm 1$ and $u^\pm(0,0) = 0$. We call $u^-(x,t)$ the subsolution and $u^+(x,t) = -u^-(x,t)$ the supersolution, respectively. If $h : \mathbf{R} \to \mathbf{R}$ is a nondecreasing function, then $h(u^-)(h(u^+))$ is a subsolution (super) provided that the differential operator is geometric (see [6]).

For every $\xi \in \Omega$, we consider the function $h_\xi(u^-(x-\xi,t))$, where h_ξ is chosen so that

$$\begin{gathered} h_\xi(0) = u_0(\xi), \\ h_\xi\left(u^-(x-\xi,t)\right) \leq u_0(x), \qquad \forall x \in \Omega, \quad \forall t \in [0,T]. \\ \exists\, mU > 0 : 0 < h'_\xi(l) < U, \qquad \forall l \in \mathbf{R}. \end{gathered}$$

The existence of these h_ξ is proved in [6] and now we have a family of local dynamic subbarriers for the PDE in (2).

As a compatibility condition with the static barrier, we ask

$$\forall x \in \Omega \cap \partial N_a, \quad \forall t \in (0,T), \qquad \exists\, \xi \in \Omega : h_\xi(u^-(x-\xi,t)) \geq b_\mu^-(x); \tag{8}$$

note that what we want is 'to hide' (when taking the supremum) the internal boundary of the set N_a.

Condition (8) is verified with the choice $\mu = \ln 2/2UT$.

The symmetric local dynamic superbarriers are obviously defined for each point $\xi \in \Omega$ as $h_\xi(u^+(x-\xi,t))$.

Finally, we consider the subsolution f^- defined as

$$f^-(x,t) = \begin{cases} \sup\left\{b_\mu^-(x)\,,\, h_\xi(u^-(x-\xi,t)),\, \forall \xi \in \Omega\right\}, & (x,t) \in N_a \times]0,T], \\ \sup\{h_\xi(u^-(x-\xi,t)),\, \forall \xi \in \Omega\}, & (x,t) \in \{\Omega \setminus N_a\} \times]0,T], \end{cases}$$

and its symmetric supersolution $f^+(x,t)$; thanks to (8), the functions f^- and f^+ are continuous both in time and space.

By using (8), for every x there exists a ball $B_\rho(x)$ ($\rho > 0$) such that, for every time t, the function f^- can be presented as the supremum, in B_ρ, of functions which are subsolutions of (2) in B_ρ. The same is valid for f^+ and this means they are a couple of sub and supersolutions which verifies conditions (4).

We can then conclude that the viscosity solution of (2) exists and it is unique for an initial data $u_0 \in C^{2,1}(\bar\Omega)$. Stability results under uniform convergence together with the uniqueness theorem for viscosity solution of (2) (see [6,7]) ensure the solvability of problem (2) for every data $u_0 \in C^{0,1}(\bar\Omega)$.

3.2. The Case of Nonstrictly Convex Domain

All results concerning dynamic barriers can be applied to the case of nonstrictly convex domain because only the construction of the static barrier deals with the shape of the domain.

Static barrier

In the proof of the previous result, the hypothesis $\mathcal{K}(y) > 0$, $\forall\, y \in \partial\Omega$ is a key argument. In [5], we are able to construct a uniform approximation of u_0, that we call w_0, which allows us to apply the same reasoning as before. We first approximate $u_0|_{\partial\Omega}$ with a function, say $\tilde u_0|_{\partial\Omega}$, whose first derivative has only isolated zeros with the following properties. They are only maximum or minimum points (we eliminate flexes) and the second derivative calculated at these points is always nonzero. We extend $\tilde u_0|_{\partial\Omega}$ in Ω as $\tilde u_0$ and we construct the following barrier:

$$\tilde b(x) = \tilde u_0(x) - f(d(x)), \qquad x \in N_a,$$

where f is now $f(\xi) = Mk\xi^{1/k}$, $\xi \in [0,a)$ and the parameters $M > 0$, $k > 0$, $0 < a \le \bar a$ are to be fixed. However, $\tilde b$ will not, in general, be a subsolution in neighbourhoods of maximum points of $u_0|_{\partial\Omega}$. We have then to construct suitable cut-offs of $\tilde b$ near those points. The truncated barrier b, obtained in that way, can be viewed as static barrier for a further approximation, say w_0, of u_0. The barrier b reads $b(x) = w_0(x) - f(d(x))$, $x \in N_a$.

Condition (8) now reads

$$\forall\, z \in \partial N_a \backslash \partial\Omega, \qquad w_0(z) - UT \ge b(z), \tag{9}$$

and there exists a choice of M, k, and a which let $\tilde b$ be a subsolution in N_a (see [5] for details).

3.3. The Case of Nonsmoothness

We assume now that Ω is a convex domain whose boundary $\partial\Omega$ is piecewise C^2 and that the number of corners is finite. As in the previous case, we have to construct only a suitable static barrier.

Let Γ_i for $i = 1 \dots n_\Omega$ be the n_Ω smooth *open* manifolds which are subsets of the boundary $\partial\Omega$ and we define a new distance function from Γ_i as

$$\mathrm{dist}_0(x,\Gamma_i) = \min_{y\in\Gamma_i} |x-y|, \qquad x \in \Omega.$$

When the minimum does not exist, our distance function is not defined.

Let us rename $d_i(x) = \mathrm{dist}_0(x,\Gamma_i)$, $\forall\, x \in \Omega$.

We set

$$N_a^i = \{x \in \Omega : d_i(x) \text{ exists and } 0 < d_i(x) < a\}\,.$$

roughly speaking, these sets are like rectangles inside Ω in front of Γ_i.

On each of these sets, we define a function b_i as

$$b_i(x) = w_0(x) - f(d_i(x)), \qquad \forall x \in \bar{N}_a^i,$$

where w_0 is a suitable approximation of the initial data u_0 (as we did in the previous case), f, as before, is given by $f(\xi) = Mk\xi^{1/k}$, $\xi \in [0, a)$ and the parameters are chosen to make each b_i a static barrier in N_a^i.

We define $b(x) = \sup\{b_i(x); i \text{ such that } x \in N_a^i\}, x \in N_a$ and in [5], we prove that b is a static barrier with a suitable choice of parameters.

This means we obtain existence and uniqueness in this case as well.

3.4. The Case of Nonconvex Domain

This is a negative result and with an easy example, we show that, in general, the existence of the viscosity solution was not to be expected.

We consider the equation given in [1] and we try to solve the Dirichlet problem (2) in a domain $\Omega = B_2(0) \setminus B_1(0)$ and with $u_0(\rho) = \max\{0, \ -2\rho + 3\}$ $1 \le \rho \le 2$ (in polar coordinate) as initial data. The solution of this problem does not exist. Actually, one can verify that there exists a unique solution of the evolution equation in Ω_t that satisfies the initial condition and the Dirichlet condition on $\partial B_2 \times [0, T)$, but this solution fails to match the Dirichlet datum on $\partial B_1 \times (0, T)$.

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