A spatially homogeneous Boltzmann equation for elastic, inelastic and coalescing collisions

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Abstract

Existence, uniqueness and qualitative behavior of the solution to a spatially homogeneous Boltzmann equation for particles undergoing elastic, inelastic and coalescing collisions are studied. Under general assumptions on the collision rates, we prove existence and uniqueness of an $L^1$ solution. This shows in particular that the cooling effect (due to inelastic collisions) does not occur in finite time. In the long time asymptotic, we prove that the solution converges to a mass-dependent Maxwellian function (when only elastic collisions are considered), to a velocity Dirac mass (when elastic and inelastic collisions are considered) and to 0 (when elastic, inelastic and coalescing collisions are taken into account). We thus show in the latter case that the effect of coalescence is dominating in large time. Our proofs gather deterministic and stochastic arguments.

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Résumé

Nous étudions les questions de l’existence, de l’unicité et du comportement en temps grand des solutions pour une équation de Boltzmann homogène en espace modélisant des chocs élastiques, inélastiques et des coalescences entre particules. Sous des hypothèses très générales sur les taux de collision, nous démontrons un théorème d’existence et d’unicité dans un cadre $L^1$. Notre résultat montre en particulier que l’effet de refroidissement (dû aux collisions inélastiques) n’aboutit pas en temps fini. Concernant le comportement en temps grand, nous démontrons que la solution converge vers un équilibre Maxwellien (lorsqu’on considère uniquement des chocs élastiques), vers une masse

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1. Introduction and notation

We consider the Cauchy problem for a spatially homogeneous kinetic equation modeling (at a mesoscopic level) the dynamics of a system of particles characterized by their mass and impulsion. These particles are supposed to undergo collisions. Each collision results in an elastic rebound, in an inelastic rebound or in a coalescence. These different kinds of collision are taken into account through a classical Boltzmann collision operator, a Granular collision operator (of inelastic interactions) and a Smoluchowski coalescence operator respectively. More precisely, describing the gas by the concentration density \( f(t, m, p) \geq 0 \) of particles with mass \( m \in (0, +\infty) \) and impulsion \( p \in \mathbb{R}^3 \) at time \( t \geq 0 \), we study existence, uniqueness and long time behavior of a solution to the Boltzmann-like equation

\[
\frac{\partial f}{\partial t} = Q(f) = Q_B(f) + Q_G(f) + Q_S(f),
\]

in \( (0, \infty) \times (0, \infty) \times \mathbb{R}^3 \),

\[
f(0) = f_0 \quad \text{in } (0, \infty) \times \mathbb{R}^3. \tag{1.1}
\]

In this introduction, we first describe the collision operators \( Q_B, Q_G, \) and \( Q_S \). We then deal with possible assumptions on the rates of collision and on the initial condition. Finally, we give the main ideas of the results, some references, and the plan of the paper.

1.1. Collision operators

Let us introduce some notation that will be of constant use in the sequel. We define the phase space of mass-momentum variable \( y := (m, p) \in Y := (0, \infty) \times \mathbb{R}^3 \), the velocity variable \( v = p/m \), the radius variable \( r = m^{1/3} \) and the energy variable \( E = |p|^2/m \). Then, for \( y^\sharp \in Y \), we will denote by \( \mu, \mu^\sharp, m^\sharp, r^\sharp, E^\sharp \) the associated mass, momentum, velocity, radius and energy respectively. We also denote by \( [y^\sharp] \) a particle which is characterized by \( y^\sharp \in Y \) and we write \( \varphi^\sharp = \varphi(y^\sharp) \) for any function \( \varphi : Y \to \mathbb{R} \). Finally, for a pair of particles \( \{y, y^*\} \), we define some reduced mass variables, the velocity of the center of mass and the relative velocity by

\[
m_{aa} = m + m^*, \quad \mu = \frac{m}{m_{aa}}, \quad \mu^* = \frac{m^*}{m_{aa}}, \quad \bar{\mu} = \frac{mm^*}{m_{aa}},
\]

\[
v_{aa} = \mu v + \mu^* v^* \quad \text{and} \quad w = |v_{aa} - v|.
\]

For any function \( T : Y^2 \to \mathbb{R} \), we will write \( T = T(y, y^*) \) and \( T_* = T(y^*, y) \).
We now describe the collision terms which are responsible of the changes in the density function due to creation and annihilation of particles with given phase space variable because of the interaction of particles by binary collisions. First, the Boltzmann collision operator $Q_B(f)$ models reversible elastic binary collisions, that is collisions which preserve masses, total momentum and kinetic energy. These collisions occur with symmetric rate $a_B$. In other words, denoting by $\{y, y_a\}$ the pre-collisional particles and by $\{y', y'_a\}$ the resulting post-collisional particles, $\{y\} + \{y_a\} \rightarrow \{y'\} + \{y'_a\}$ with
\[
\begin{aligned}
m' &= m, \quad m'_a = m_a, \\
p' + p'_a &= p + p_a, \\
E' + E'_a &= E + E_a.
\end{aligned}
\] (1.2)

The rate of elastic collision $a_B = a_B(y, y_a; y', y'_a)$ satisfies
\[
a_B(y, y_a; y', y'_a) = a_B(y'_a, y; y', y'_a) = a_B(y, y'_a; y', y) \geq 0.
\] (1.3)

The first equality expresses that collisions concern pairs of particles. The second one expresses the reversibility of elastic collisions: the inverse collision $\{y', y'_a\} \rightarrow \{y, y_a\}$ arises with the same probability than the direct one (1.2). The Boltzmann operator reads
\[
Q_B(f)(y) = \int Y \int S_2 a_B(f'_{*} f' - ff_{*}) d\nu dy_{*}.
\] (1.4)

Here, for every pair of post-collisional particles $\{y, y_a\}$ and every solid angle $\nu \in S_2$, the pair of pre-collisional particles $\{y', y'_a\}$ are given by $y' = (m, mv')$, $y'_a = (m_a, m_a v'_a)$ with
\[
\begin{aligned}
v' &= v + 2\mu_a (v_a - v, v) v, \\
v'_a &= v_a - 2\mu (v_a - v, v) v,
\end{aligned}
\] (1.5)

where $(.,.)$ stands for the usual scalar product on $\mathbb{R}^3$. Let us explain the meaning of the Boltzmann term $Q_B(f)(y)$ for any given particle $\{y\}$. The nonnegative part, the so-called gain term $Q^+_B(f)$, accounts for all the pairs of particles $\{y', y'_a\}$ which collide and give rise to the particle $\{y\}$ as one of the resulting particles. It is worth mentioning that, for any post-collisional particles $\{y, y_a\}$, Eq. (1.5) is nothing but a parameterization (thanks to the solid angle $\nu \in S_2$) of all possible pre-collisional velocities $(v', v'_a)$, that is pairs of velocities which satisfy the conservations (1.2). The nonpositive part, the loss term $Q^-_B(f)$, counts all possible collisions of the particle $\{y\}$ with another particle $\{y_a\}$.

Next, the Granular collision operator $Q_G(f)$ models inelastic binary collisions (preserving masses and total momentum but dissipating kinetic energy), which occur with rate $a_G$:
\[
\{y\} + \{y_a\} \rightarrow \{y''\} + \{y''_a\} \quad \text{with} \quad \begin{aligned}
m'' &= m, \quad m''_a = m_a, \\
p'' + p''_a &= p + p_a, \\
E'' + E''_a &= E + E_a.
\end{aligned}
\] (1.6)
In order to quantify the in-elasticity effect and make precise (1.6), it is convenient to parameterize, for any fixed pre-collisional particles \( \{ y, y_* \} \), the resulting post-collisional particles \( \{ y'', y_*'' \} \) in the following way:

\[
\begin{align*}
  v'' &= v + (1 + e)\mu_2 (v_* - v) v, \\
  v_*'' &= v_* - (1 + e)\mu_1 (v_* - v) v.
\end{align*}
\]  

(1.7)

The deflection solid angle \( \nu \) goes all over \( S^2 \) and where the restitution coefficient \( e \) goes all over \((0, 1)\). The coefficient \( e \) measures the loss of normal relative velocity during the collision, since

\[
(v'' - v_*, \nu) = e (v_* - v, \nu). 
\]  

(1.8)

The case where \( e = 1 \) corresponds to an elastic collision while \( e = 0 \) and \( v = (v_* - v)/|v_* - v| \) indicate a completely inelastic (or sticky) collision. The rate of inelastic collision \( a_G(y, y_*; y'', y_*'') \) satisfies the relation

\[
a_G(y, y_*; y'', y_*'') = a_G(y, y_*; y_*, y_*'') \geq 0, 
\]  

(1.9)

which expresses again the fact that (1.6) is an event concerning a pair of particles. The Granular operator reads

\[
Q_G(f)(y) = \int_{\tilde{y}} \int_{S^2} \int_{0}^{1} \left( \frac{\tilde{a}_G}{e} \tilde{f}_\nu - a_G f_* \right) d\nu dv dy_*.
\]  

(1.10)

For any given particle \( \{ y \} \), the gain term \( Q_G^+(f)(y) \) in \( Q_G(f)(y) \) accounts for all the pairs of pre-collisional particles \( \{ \tilde{y}, \tilde{y}_* \} \) which collide and give rise to the particle \( \{ y \} \). Inverting (1.7), the pre-collisional particles \( \{ \tilde{y}, \tilde{y}_* \} \) can be parameterized in the following way: \( \tilde{y} = (m, m\tilde{v}) \), \( \tilde{y}_* = (m_*, m_*\tilde{v}_*) \) with

\[
\tilde{v} = v + \frac{1 + e}{e} \mu_2 (v_* - v) v, \\
\tilde{v}_* = v_* - \frac{1 + e}{e} \mu_1 (v_* - v) v.
\]  

(1.11)

We have set \( \tilde{a}_G = a_G(\tilde{y}, \tilde{y}_*; y, y_*). \) Note that \( 1/e \) stands for the Jacobian function of the substitution \( (y, y_*) \mapsto (\tilde{y}, \tilde{y}_*). \) The loss term \( Q_G^-(f)(y) \) counts again all the possible collisions of the particle \( \{ y \} \) with another particle \( \{ y_* \} \).

Finally, the Smoluchowski coalescence operator models the following microscopic collision: two pre-collision particles \( \{ y \} \) and \( \{ y_* \} \) aggregate and lead to the formation of a single particle \( \{ y_*\} \), the mass and momentum being conserved during the collision. In other words,

\[
\{ y \} + \{ y_* \} \xrightarrow{as} \{ y_*\} \quad \text{with} \quad \begin{cases} m_* + m = m_*, \\
p_* + p = p + p_*, \end{cases}
\]  

(1.12)
The coalescence being again a pair of particles event, it results that the coalescence rate
\[ a_S = a_S(y, y_*) \] is symmetric
\[ a_S(y, y_*) = a_S(y_*, y) \geq 0. \] (1.13)

The Smoluchowski coalescence operator is thus given by
\[
Q_S(f)(y) = \frac{1}{2} \int_{\mathbb{R}^3} \int_0^m a_S(y_*, y - y_*) f(y_*) f(y - y_*) \, dm_* \, dp_* \\
- \int_{\mathbb{R}^3} \int_0^\infty a_S(y, y_*) f(y) f(y_*) \, dm_* \, dp_*.
\] (1.14)

The gain term \( Q^+_S(f)(y) \) accounts for the formation of particles \( y \) by coalescence of smaller ones, the factor \( 1/2 \) avoiding to count twice each pair \( \{y_*, y - y_*\} \). The loss term \( Q^-_S(f)(y) \) describes the depletion of particles \( y \) by coalescence with other particles.

Let us emphasize that the effect of these three kinds of collision are very different as it can be observed comparing (1.2), (1.6) and (1.12). First, elastic and inelastic collisions leave invariant the mass distribution, while coalescing collisions make grow the mean mass. Next, kinetic energy is conserved during a Boltzmann collision while it decreases during a Granular or a Smoluchowski collision. Finally, elastic collisions are reversible at the microscopic level while inelastic and coalescing collisions are irreversible microscopic processes.

### 1.2. On the collision rates

We want to address now the question of the assumptions we have to make on the collision rates \( a_B, a_G \) and \( a_S \). To that purpose, we need to describe a little the physical background of the collision events. For a more detailed physical discussion we refer to [3,58,59].

There exists many physical situations where particles evolve according to (at least one of) the above rules of collisions: ideal gases in kinetic physics for elastic collisions [14], granular materials for inelastic collisions [15], astrophysical bodies for coalescence collisions [12], to quote a few of them. We shall rather consider the case of liquid droplets carried out by a gaseous phase and undergoing collisions where, as we will see, the three above rules of collision arise together. The modeling of such liquid sprays is of major importance because of the numerous industrial processes in which they occur. It includes combustion-reaction in motor chambers and physics of aerosols. It also appears in meteorology science in order to predict the rain drop formation.

The Boltzmann formalism we adopt here (description of droplets by the density function) has been introduced by Williams [60] and then developed in [45,57,58,5,59]. There have been a lot of fundamental studies to improve the understanding of the complex physical effects that play a role in such a two-phase flow. The essential of this research focused
on the gas-droplets interactions (turbulent dispersion, burning rate, secondary break-up, ...) see [45,49,55,16,39,59]. But in dense sprays, the effect of droplet collisions is of great importance and has to be taken into account. Experimental and theoretical studies (Brazier-Smith et al. [8], Ashgriz and Poo [3] and Estrade et al. [28]) have shown that the interaction between two drops with moderate value of Weber number $W_e$ (see (1.19) for the definition of $W_e$) may basically result in:

(a) a grazing collision in which they just touch slightly without coalescence,
(b) a permanent coalescence,
(c) a temporary coalescence followed in a separation in which few satellite droplets are created.

Since the dynamics of such collisions are very complicated, the available expressions for predicting their outcomes are at the moment mostly empirical. Anyway, the collisions (a) may be well modeled by an elastic collisions (1.2) or by a stretching collision in which velocities are unchanged:

\[
\{y\} + \{y_\ast\} \xrightarrow{a_{\text{st}}} \{y\} + \{y_\ast\}. \tag{1.15}
\]

While stretching collisions are often considered in the physical literature, there is no need to take them into account from the mathematical point of view, since the corresponding operator $Q_{\text{st}}$ vanishes identically. Collisions of type (b) are naturally modeled by a coalescence collision (1.12). It is more delicate to model collisions (c), because of the many situations in which it can result. Nevertheless, it can be roughly modeled by an inelastic collision (1.6), where satellization is responsible of the inelasticity of the collision. Here, possible transfer of mass between the two particles as well as loss of mass (due to satellization) are neglected.

Therefore, at the level of the distribution function, the dynamics of a spray of droplets may be described by the Boltzmann equation (1.1). Of course, such an equation only takes into account binary collisions and neglects the fragmentation of droplets due to the action of the gas, as well as condensation/evaporation of droplets. It also neglects the fluid interaction, in particular the velocity correlation in the collision (see [59]), as well as collisions giving rise to two or more particles with different masses than the initial ones. Nevertheless, Eq. (1.1) is the most complete spatially homogeneous Boltzmann collision model we have found in the literature.

We now split into two parts the discussion about the collision rates. First, we address the question of what is the rate that two particles encounter and do collide. Next, we address the question of what is the outcome of the collision event.

It is well known in the Boltzmann theory, that for two free particles interacting by contact collision (hard spheres), the associated total collision frequency $a$ is given by

\[
a(y, \ y_\ast) = a_{\text{HS}}(y, \ y_\ast) := (r + r_\ast)^2|v - v_\ast|.
\]

(1.16)

Roughly speaking, $a$ is the rate that two particles $\{y\}$ and $\{y_\ast\}$ meet. Such a rate is deduced by solving the scattering problem for one free particle in a hard sphere potential.
Here the situation is much more intricate since the droplets are not moving in an empty space, but they are rather surrounded by the flows of an ambient gas. Even if the flow is not explicitly taken into account in the model (1.1), drops can not be realistically considered as going in straight line between two collisions. When a small droplet \( \{y_\ast\} \) approaches a larger one \( \{y\} \), it may be deflected of \( \{y\} \) due to its interaction with the surrounding gas. It is thus possible that \( \{y_\ast\} \) circumvent \( \{y\} \), so that the collision does not occur. The function \( a \) may also take into account the fact that the collision between two droplets does not necessarily result in significant change of trajectory (stretching collision (1.15)).

The effect of the deviation of the trajectory in the collision efficiency has been first studied by Langmuir and addressed then by many researchers both from theoretical, numerical and experimental points of view, in particular, in view to the application to meteorology sciences. Langmuir [35] and Beard and Grover [6] have considered the case when \( m/m_\ast \ll 1 \) or \( m_\ast/m \ll 1 \). The case \( m \sim m_\ast \) is much more complicated, and we refer to Davis and Sartor [18] and Neiburger et al. [43] for an analytic expression. See also Pigeonneaux [46] and the numerous references therein for a recent state of the art on that subject. Let us finally quote the experimental study of Brazier-Smith et al. [8]. In any cases, the total collision rate obtained in those works may be written as a modified hard sphere collision rate

\[
a(y, y_\ast) = E(y, y_\ast) a_{HS}(y, y_\ast) \quad \text{with } 0 \leq E \leq 1,
\]

where \( E \) is a symmetric function: \( E(y, y_\ast) = E(y_\ast, y) \) for all \( y, y_\ast \in Y \). From a mathematical point of view, we will always assume that the total collision efficiency \( a(y, y_\ast) \), i.e., the rate that two particles \( \{y\}, \{y_\ast\} \) do collide, is a measurable function on \( Y^2 \) and satisfies

\[
\forall y, y_\ast \in Y, \quad 0 \leq a(y, y_\ast) = a(y_\ast, y) \leq A(1 + m + m_\ast)(1 + |v| + |v_\ast|),
\]

for some constant \( A > 0 \). Note that such an assumption is always satisfied by a total collision efficiency \( a \) given by (1.17).

To fix the ideas, one can take for instance the following expression of \( a \) given by Beard and Grover in [6]

\[
E_{BG}(y, y_\ast) = E(\Delta, We) = \left( \frac{2}{\pi} \arctan[\max(a_0 + a_1 Z - a_2 Z^2 + a_3 Z^3, 0)] \right)^2,
\]

where the Weber number \( We \) and the mass quotient \( \Delta \) are defined by (recall that \( w = |v - v_\ast| \)).

\[
\Delta := \frac{\min(r, r_\ast)}{\max(r, r_\ast)}, \quad We := \min(r, r_\ast) w,
\]

and with

\[
Z = \ln(\Delta^2 We/K_0), \quad K_0 = \exp(-\beta_0 - \beta_1 \ln We + \beta_2 (\ln We)^2),
\]
\( \alpha_i, \beta_i \) being numerical positive real numbers. In contrast to the model of Langmuir [35,58] for which \( E \) vanishes for small value of \( \Delta^2 W_e \), observe that

\[
\forall \Delta \in (0, 1], \quad E_{BG}(\Delta, W_e) \to 1 \quad \text{when} \quad W_e \to 0. \tag{1.20}
\]

Once two particles have collided, one has to determine what is the outcome of the collision event. This question has been addressed in several physical works, and we refer to Brazier-Smith et al. [8], Ashgriz and Poo [3] and Estrade et al. [28] to quote few of the most significant works. More precisely these authors have mainly proposed an equation for the border line between the region of coalescing collisions (one output particle) and the region of other type of collisions (more than one output) in the plane of deflection angle \( \Theta \in [0, \pi/2] \) (or impact parameter \( b \))—Weber number \( W_e \) (for not too large values, typically \( W_e \leq 100 \)) for different values of \( \Delta \in (0, 1] \). It is worth mentioning that the authors do not quantify the in-elasticity of the rebound (when it occurs) that is the value of the restitution parameter \( e \in (0, 1] \). As a consequence, we have not been able to find in the physical literature explicit values of the kinetic coefficients \( a_B, a_G \) and \( a_S \). Moreover, they show that the number of particles after the collision increases when the Weber number increases and that for large values of \( W_e \) satellization really occurs. From this point of view, the validity of the Boltzmann model (1.1) is very contestable since satellization is not taken into account and that particles with large velocity (and therefore pairs of particles with large Weber number) will be created by elastic and inelastic rebounds even if we start with compactly supported initial datum. Once again we refer to [8,3,28] for more precise physical description and to the survey articles by Villedieu and Simon [59] and by Post and Abraham [47] and the references therein.

Therefore, the kinetic coefficients \( a_B, a_G \) and \( a_S \) take into account both the rate of occurrence of collision and the probability that this one results in an elastic, inelastic, or coalescing collision. Abusing notation, we assume that

\[
\begin{align*}
    a_B &= a_B(y, y^*, v) = E_B(y, y^*, \cos \Theta) a(y, y^*), \\
    a_G &= a_G(y, y^*, v, e) = E_G(y, y^*, \cos \Theta, e) a(y, y^*), \\
    a_S &= a_S(y, y^*) = E_S(y, y^*) a(y, y^*), \tag{1.21}
\end{align*}
\]

where \( \Theta \in [0, \pi/2] \) is the deflection angle (of \( v' \) or \( v'' \) with respect to \( v \)) defined by

\[
\Theta \in [0, \pi/2], \quad \cos \Theta := \left| \frac{v - v^*}{|v - v^*|} \cdot v \right|. \tag{1.22}
\]

The probability of elastic collision \( E_B \geq 0 \), of inelastic collision \( E_G \geq 0 \) and of coalescing collision \( E_S \geq 0 \) are measurable functions of their arguments, they are symmetric in \( y \) and \( y^* \), and they satisfy, for all \( y, y^* \) in \( Y \),

\[
\bar{E}_B + \bar{E}_G + \bar{E}_S \leq 1 \quad \text{with} \quad \bar{E}_B := \int_{S^2} E_B \, dv, \quad \bar{E}_G := \int_{S^2} \int_0^1 E_G \, de \, dv. \tag{1.23}
\]
With such a structure assumption, the symmetry conditions (1.3), (1.9) and (1.13) clearly hold. For future references we also define the total elastic and inelastic collision rates (for all \(y, y_\ast\) in \(Y\))

\[
\bar{a}_B = \int_{S^2} a_B \, d\nu = \bar{E}_B, \quad \bar{a}_G = \int_0^1 \int_{S^2} a_G \, dv \, de = \bar{E}_G. \tag{1.24}
\]

At last, thanks to the first inequality in (1.23), the collision efficiencies \(\bar{a}_B, \bar{a}_G, a_S\) and \(a\) satisfy, for all \(y, y_\ast\) in \(Y\),

\[
\bar{a}_B + \bar{a}_G + a_S \leq a. \tag{1.25}
\]

Let us finally emphasize that we implicitly take into account the stretching collisions (1.15), setting \(a_U := a - \bar{a}_B - \bar{a}_G - a_S \geq 0\).

Let us give an idea of possible shapes for \(\bar{E}_B, \bar{E}_G, \bar{E}_S\). As discussed in [59], the efficiency coefficients depend only on \(\text{We}, \Delta\) (see (1.19)) and on the impact parameter \(b = (r + r_\ast)\cos\Theta\). The collision efficiencies are then given by

\[
\begin{align*}
E_B(y, y_\ast, v) &= \kappa \cos\Theta \, 1_{\Theta \in \Lambda_B}, \\
E_G(y, y_\ast, v, e) &= \kappa \cos\Theta \, 1_{\Theta \in \Lambda_G}, \\
E_S(y, y_\ast) &= \kappa \int_{S^2} \cos\Theta \, 1_{\Theta \in \Lambda_S} \, dv,
\end{align*}
\]

where \(\Lambda_B, \Lambda_S\) and \(\Lambda_G\) are disjoint subsets (unions of intervals) of \([0, \pi/2]\) which are continuously depending of \(y, y_\ast \in Y, e \in [0, 1]\), and \(\kappa^{-1} := \int_{S^2} \cos\Theta \, dv\). For example, Brazier-Smith et al. [8] propose \(\Lambda_B = \Lambda_G = \emptyset\) and \(\Lambda_S = [0, \Theta_{ct}]\) where the critical impact parameter \(b_{ct}\) (and thus the corresponding \(\Theta_{ct}\)) is defined by

\[
b_{ct} = b_{ct}(\text{We}, \Delta) := \min \left(1, \frac{\beta(\Delta)}{\sqrt{\text{We}}} \right), \tag{1.26}
\]

for some continuous and decreasing function \(\beta : (0, 1] \to (0, \infty)\). Other examples are due to Ashgriz and Poo [3] and Estrade et al. [28]. In any of them, coalescence collisions are dominating for small value of the Weber number when \(\Delta = 1\), and this preponderance increases when \(\Delta\) decrease. Therefore the following bound holds: there exists \(\text{We}_0 > 0\) and \(\kappa_0 > 0\) such that

\[
\forall \Delta > 0, \forall \text{We} \in [0, \text{We}_0], \quad E_S(y, y_\ast) \geq \kappa_0. \tag{1.27}
\]

In these models, the coalescence efficiency may vanish for large values of the Weber number.
1.3. On the initial condition

The initial datum \( f_{\text{in}} \) is supposed to satisfy

\[
0 \leq f_{\text{in}} \in L_{k}^{1}(Y), \quad \int_{Y} f_{\text{in}} m \, dy = 1, \quad \int_{Y} f_{\text{in}} p \, dy = 0, \quad (1.28)
\]

for the weight functions \( k : Y \to \mathbb{R}_{+} \) defined by

\[
k = k_{S} := 1 + m + |\rho| + |v| \quad \text{or} \quad k = k_{B} := \frac{1}{m} + m + \mathcal{E}. \quad (1.29)
\]

Here and below, we denote, for any nonnegative measurable function \( \ell \) on \( Y \), the Banach space

\[
L_{\ell}^{1} = \left\{ f : Y \mapsto \mathbb{R} \text{ measurable}; \| f \|_{L_{\ell}^{1}} := \int_{Y} |f(y)| \ell(y) \, dy < \infty \right\}. \quad (1.30)
\]

Let us remark that \( k_{S} \leq k_{B} \). We always work with the weight \( k_{B} \), except when we consider the pure Smoluchowski coalescence equation. Let us finally notice that we do not loose generality assuming the two last moment conditions in (1.28), since we may always reduce to that case by a scaling and translation argument.

1.4. Aims and references

Our main aim in the present paper is to give results about existence, uniqueness, and long time behavior of a solution to (1.1). Roughly speaking, we shall establish the following two results.

Existence and uniqueness. Under the structure assumption (1.21), (1.23) on the collision rates \( a_{B}, a_{G} \) and \( a_{S} \) and the boundness assumption (1.18) on the total collision rate \( a \), there exists a unique solution \( f \in C([0, \infty), L_{1}^{1}(Y)) \) to the Boltzmann equation (1.1) with initial condition \( f_{\text{in}} \) satisfying (1.28).

Long time behavior. Under further suitable assumptions of positivity on \( a_{B}, a_{G} \) and \( a_{S} \) the long time behavior is the following

\( f(t) \to \Gamma \) when \( t \to \infty \),

where

- \( \Gamma \) is a centered mass-dependent Maxwellian with same mass distribution and temperature than \( f_{\text{in}} \) when \( a_{G} = a_{S} = 0 \) and \( a_{B} > 0 \);
Γ is a centered degenerated mass-dependent Maxwellian (Dirac mass) with same mass
distribution than the initial datum when \( a_S = 0, a_B \geq 0 \) and \( a_G > 0 \);
Γ = 0 when \( a_G \geq 0, a_B \geq 0 \) and \( a_S > 0 \).

This last result, the main of the paper, establishes that each particle’s mass tends to in-
finity in large time. We will give two proofs of it: a deterministic one (based on moment
arguments) which allows us to deal only with the pure coalescence equation and a prob-
abilistic one (based on a stochastic interpretation of (1.1)) which is valid in the general
case.

Concerning the existence and uniqueness theory, the main difficulty is that when we are
concerned by physical unbounded rates, the kernels \( Q_B, Q_G \) and \( Q_S \) do not map \( L^1 \)
into \( L^1 \). Thus a classical Banach fixed point theorem fails. Basically there are two strategies to
overcome this difficulty.

The robustness one (which some time extends to spatially nonhomogeneous context) is
to argue by compactness/stability and only provides existence of solutions. See the pioneer
work by Arkeryd [2] for the case of elastic collisions. In order to apply this method, one
has to prove super-linear estimates on the density function \( f \). For the elastic Boltzmann
equation this key information is given by the so-called H-theorem of Boltzmann, which
in particular implies that the entropy is bounded. For the kinetic coalescence equation one
may do more or less the same, but under a structure hypothesis on the coalescence rate
[41,25].

The second strategy, which we will adopt here, is based on a suitable modification of
the proof of a uniqueness result as introduced in [40] for the Boltzmann equation and then
taken up again in [26,27] for a Boltzmann equation for a gas of Bose particles. This method
makes possible to prove existence and uniqueness of an \( L^1 \) solution when no estimate of
superlinear functional of the density is available. We also refer to [20,40] for uniqueness
results on the Boltzmann equation and to [4,50,44,48] for uniqueness results on the Smolu-
chowski equation.

Finally, our long time asymptotic behavior result is based on an entropy dissipation
method (as introduced in [22]) and also on a stochastic interpretation of the solution.

The most studied operator and equation is undoubtedly the Boltzmann equation for
elastic collision since the pioneer works of Carleman [13] and the famous contribution of
DiPerna and Lions [21]. For a mathematical and physical presentation of the Boltzmann
equation we refer to [14,56] and the references therein.

The mathematical study of Granular media, which involves inelastic collisions, has re-
ceived much attention very recently. We refer to [15,10,54] and the references therein for
further discussions about modeling and physical meaning of that operator, see also [7,9]
for related models. Let us emphasize that more or less stochasticity can be introduce
in the inelastic collision. One may assume that the restitution coefficient is determined
by the other parameters \( \varepsilon = \bar{\varepsilon}(y, y_s, v) \). In this case, the rate of inelastic collision writes
\( a_G = \gamma G(y, y_s, v)\delta e = \bar{\varepsilon} \). Here for commodity and simplicity we make the opposite assump-
tion that \( a_G \) has a density in the \( \varepsilon \) variable. To our knowledge, existence proofs have been
handled only in two cases: the one-dimensional case, see [54], and the case of Maxwellian
rate and fixed restitution coefficient, that is, \( a_G = \delta e = e_0 \) for some \( e_0 \in (0, 1) \), see [9,10].
See also [11,31] for recent results on modified Boltzmann equations with inelastic collision
and [42] for some extensions of the present work to the Boltzmann equation for Granular media. Let us emphasize that it was conjectured that finite time collapse occurs for a class of collisional rates. Our existence result shows that it is not the case.

Less has been done concerning the kinetic coalescence equation (i.e., Eq. (1.1) with $a_B = a_S = 0$). We may only quote the recent works [48,25]. See also the paper by Slemrod for coagulation models with discrete velocities [51]. It is however closely related to the Smoluchowski coagulation equation encountered in colloid chemistry, physics of the atmosphere or astrophysics (see, for example, [23]), where only the mass is taken into consideration. In fact, the Smoluchowski coagulation model may be seen and obtained as a simplified model of the coalescence model (1.1)–(1.14) eliminating the $v$ variable if one knows the shape of the velocity distribution. In many applications involving dense sprays of droplets, no information is known a priori on the shape of the velocity distribution and therefore the dependency on $v$ must be kept in the model. A lot of mathematical work has been devoted to the coagulation equation such as existence, uniqueness, conservation of mass and gelation phenomena, long time behavior including convergence to an equilibrium state or self-similarity asymptotic. For further references and results on the coagulation model we refer to [23] and the monograph of Dubowskii [24], as well as the recent surveys [1,37].

1.5. Plan of the paper

In Section 2, we first give the main physical properties of the collision operators, and we state our main results. Then we study the three operators separately. Section 3 is devoted to the study (existence, uniqueness, a priori estimates, long time behavior) of the kinetic Smoluchowski equation $\partial_t f = Q_S(f)$. We study the mass-dependent Boltzmann equation $\partial_t f = Q_B(f)$ in Section 4, while Section 5 concerns the Granular media equation $\partial_t f = Q_G(f)$. Gathering all the arguments, we give an existence and uniqueness proof for the full equation (1.1) in Section 6. Introducing a stochastic interpretation of the solution, we also study the long time behavior of the solution. We finally present some more or less explicit solutions concerning specific rates in Section 6.

2. Main results

In this section we first describe the main physical properties of the collision operators. We then give the definition of solutions we will deal with in this paper. We will finally list the main results concerning the Boltzmann equation (1.1) when some or all the rules of collisions are considered.

2.1. Some properties of collision operators

We want to address now a very simple discussion about the weak and strong representation of the collision kernels. In the whole subsection, $g$ and $\varphi$ stand for sufficiently integrable functions on $Y$, and $g$ is supposed to be nonnegative.
First, using the substitution \((y', y'_*) \mapsto (y, y_*)\) (respectively, \((\tilde{y}, \tilde{y}_*) \mapsto (y, y_*)\)) in the gain term of \(QB\) (respectively, \(QG\)), we deduce, using the symmetry of collisions (1.3) and (1.9), that

\[
\int_{\mathcal{Y}} QB(g) \psi \, dy = \frac{1}{2} \int_{\mathcal{Y}} \int_{\mathcal{Y}} \int_{\mathbb{S}^2} a_B g g_*(\varphi' + \varphi'_* - \varphi - \varphi_*) \, dy \, dy_* \, dv, \tag{2.1}
\]

\[
\int_{\mathcal{Y}} QG(g) \psi \, dy = \frac{1}{2} \int_{\mathcal{Y}} \int_{\mathcal{Y}} \int_{\mathbb{S}^2} a_G g g_*(\varphi'' + \varphi'_* - \varphi - \varphi_*) \, dy \, dy_* \, dv \tag{2.2}
\]

The reversibility condition on the rate \(a_B\) (second equality in (1.3)) makes possible to perform one more substitution \((y, y_*) \mapsto (y', y'_*)\) to obtain

\[
\int_{\mathcal{Y}} QB(g) \psi \, dy = \frac{1}{4} \int_{\mathcal{Y}} \int_{\mathcal{Y}} \int_{\mathbb{S}^2} a_B (g g_* - g' g'_*)(\varphi' + \varphi'_* - \varphi - \varphi_*) \, dy \, dy_* \, dv. \tag{2.3}
\]

Performing the substitution \((y - y_*, y_*) \mapsto (y, y_*)\) in the gain term of \(QS(g)\), we also deduce that

\[
\int_{\mathcal{Y}} QS(g) \psi \, dy = \frac{1}{2} \int_{\mathcal{Y}} \int a_S g g_*(\varphi_* - \varphi - \varphi_*) \, dy \, dy_* \tag{2.4}
\]

These identities provide fundamental physical informations on the operators, well-choosing the test function \(\psi\) in (2.1)–(2.4). We want to list some of them now. First, mass and momentum are collisional invariant for the three operators:

\[
\int_{\mathcal{Y}} QB(g) \begin{pmatrix} m \\ p \end{pmatrix} \, dy = \int_{\mathcal{Y}} QG(g) \begin{pmatrix} m \\ p \end{pmatrix} \, dy = \int_{\mathcal{Y}} QS(g) \begin{pmatrix} m \\ p \end{pmatrix} \, dy = 0. \tag{2.5}
\]

In fact, since elastic and inelastic collisions are mass preserving, we also have, for all \(\psi : (0, \infty) \mapsto \mathbb{R}\)

\[
\int_{\mathcal{Y}} QB(g) \psi(m) \, dy = \int_{\mathcal{Y}} QG(g) \psi(m) \, dy = 0. \tag{2.6}
\]

Coalescence makes decrease the number of particles: for instance,

\[
D_{1,S}(g) := -\int_{\mathcal{Y}} QS(g) \, dy = \frac{1}{2} \int_{\mathcal{Y}} a_S g g_* \, dy_* \geq 0. \tag{2.7}
\]
The Boltzmann operator conserves energy
\[
\int Y Q_B(g) \mathcal{E} \, dy = 0, \tag{2.8}
\]
while Granular and Smoluchowski operators satisfy
\[
D_{\mathcal{E},G}(g) := -\int Y Q_G(g) \mathcal{E} \, dy = \frac{1}{2} \int Y \int S^2 \int_0^1 a_0 g g_s \delta_{\mathcal{E},G} \, dy \, dv \, dv \, de \geq 0, \tag{2.9}
\]
\[
D_{\mathcal{E},S}(g) := -\int Y Q_S(g) \mathcal{E} \, dy = \frac{1}{2} \int Y \int a_0 g g_s \delta_{\mathcal{E},S} \, dy \, dv \geq 0, \tag{2.10}
\]
with
\[
\delta_{\mathcal{E},G} := \mathcal{E} + \mathcal{E}_s - \mathcal{E}_s'' - (1 - e^2) \tilde{\mu} (v - v_s, v)^2 \quad \text{and} \quad \delta_{\mathcal{E},S} := \mathcal{E} + \mathcal{E}_s - \mathcal{E}_s'' = \tilde{\mu} w^2. \tag{2.11}
\]
Observe that coalescence has a stronger cooling effect than inelastic collisions, since \(\delta_{\mathcal{E},G} < \delta_{\mathcal{E},S}\). Finally, defining \(h(g) = g \log g\) and using (2.3), we get
\[
D_{h,B}(g) := -\int Y Q_B(g) h'(g) \, dy
\]
\[
= \frac{1}{4} \int Y \int S^2 \int a_B(g' g_*' - g g_*) \log \frac{g' g_*'}{g g_*} \, dy \, dv \geq 0, \tag{2.12}
\]
which is the key information for the H-theorem: the irreversibility of Boltzmann equation.

2.2. Definition of solutions

Let us now define the notion of solutions we deal with in this paper.

**Definition 2.1.** Assume (1.18), (1.21), (1.23). Recall that \(k_S\) is defined in (1.29). Consider an initial condition satisfying (1.28) with \(k = k_S\). A nonnegative function \(f\) on \([0, \infty) \times Y\) is said to be a solution to the Boltzmann equation (1.1) if
\[
f \in C([0, \infty); L^1_{k_s}(Y)), \tag{2.13}
\]
and if (1.1) holds in the sense of distributions, that is,
\[
\int_0^T \int_Y \left\{ f \frac{\partial \phi}{\partial t} + Q(f) \phi \right\} \, dy \, dt = \int_Y f_{in} \phi(0, \cdot) \, dy,
\]
(2.14)
for any \( t > 0 \) and any \( \phi \in C^1_c([0, T) \times Y) \).

It is worth mentioning that (2.13) and (1.18) ensure that the collision term \( Q(f) \) is well defined as a function of \( L^1(Y) \), so that (2.14) always makes sense. It turns out that a solution \( f \), defined as above, is also a solution of (1.1) in the mild sense:
\[
f(t, \cdot) = f_{in} + \int_0^t Q(f(s, \cdot)) \, ds \quad \text{a.e. in } Y.
\]
(2.15)
Another consequence is that if \( f \in L^\infty([0, T), L^1_{k^2}) \) and if the total collision efficiency satisfies \( a \leq kk_* \) for some weight function \( k: Y \to \mathbb{R}_+ \), then \( f \) satisfies the chain rule
\[
\frac{d}{dt} \int_Y \beta(f) \phi \, dy = \int_Y Q(f) \beta'(f) \phi \, dy \quad \text{in } D'([0, T]),
\]
(2.16)
for any \( \beta \in C^1(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R}) \) and any measurable function \( \phi \) such that \( \phi/k \in L^\infty(Y) \), see [32, Lemma 7.5, p. 151] and [36].

2.3. Elastic collisions

We first consider the mass-dependent Boltzmann equation for elastic collisions.

**Theorem 2.2.** Assume (1.18), (1.21), (1.23), with \( a_G = a_S = 0 \). Consider an initial condition satisfying (1.28) with \( k = k_B \) defined in (1.29). Then there exists a unique solution \( f \) to (1.1) such that for all \( T \geq 0 \), \( f \in C([0, T), L^1_{k_B}) \cap L^\infty([0, T), L^1_{k^2}) \). This solution conserves momentum, mass distribution, and kinetic energy: for all bounded measurable maps \( \phi: Y \to \mathbb{R} \) and for all \( t \geq 0 \),
\[
\int_Y pf(t, y) \, dy = \int_Y p_{in}(y) \, dy; \quad \int_Y \phi(m) f(t, y) \, dy = \int_Y \phi(m) f_{in}(y) \, dy; \quad \int_Y E f(t, y) \, dy = \int_Y E f_{in}(y) \, dy.
\]
(2.17)
(2.18)
Concerning the long time behavior of the solution, we have the following result.
Theorem 2.3. In addition to the assumptions of Theorem 2.2, suppose that for some \( \delta \in [0, 1/2] \), some function \( \psi : [0, \pi/2] \to \mathbb{R}_+ \), and some \( m_0 > 0 \),

\[
a_B \geq (mm_0)^\delta |v_\ast - v| \psi(\Theta) > 0 \quad \text{a.e. on } Y^2 \times S^2, \tag{2.19}
\]

\[
f_{\text{in}}(\log f_{\text{in}} + m^{6-4\delta}) \in L^1(Y), \tag{2.20}
\]

\[
f_{\text{in}} = 0 \quad \text{for a.e. } p \in \mathbb{R}^3, \ m \in (0, m_0). \tag{2.21}
\]

Then the solution \( f \) to (1.1) satisfies the following weak version of the H-theorem

\[
\forall t \geq 0, \quad H(f(t,.)) + \int_0^t D_{h,B}(f(s,.)) \, ds \leq H(f_{\text{in}}), \tag{2.22}
\]

with \( D_{h,B} \) defined by (2.12) and \( H(f) := \int_Y f \log f \, dy \). Furthermore, there holds, as \( t \) tends to infinity,

\[
f(t,.) \rightharpoonup M \text{ in } L^1(Y)\text{-weak}, \tag{2.23}
\]

where \( M \) is the unique mass-dependent Maxwellian function defined by

\[
M(y) := \frac{\rho(m)}{(2\pi m \Sigma)^{3/2}} \exp \left( -\frac{E}{2\Sigma} \right), \tag{2.24}
\]

with same mass distribution, momentum and kinetic energy as \( f_{\text{in}} \), that is,

\[
\rho(m) := \int_{\mathbb{R}^3} f_{\text{in}}(m, p) \, dp, \quad \Sigma := \left( 3 \int_{(0,\infty)} \rho \, dm \right)^{-1} \left( \int_Y f_{\text{in}} E \, dy \right). \tag{2.25}
\]

These theorems just extend some previous known results on the classical (without mass dependence) Boltzmann equation, see [56]. As for the classical Boltzmann equation, they are based on a Povzner inequality (which makes possible to bound weight \( L^1 \) norms of the solution) and on the H-theorem (which expresses the mesoscopic irreversibility of microscopic reversible elastic collisions). Following classical stability/compactness methods, one may also prove existence of solution for initial data satisfying \( f_{\text{in}} k_B + |f_{\text{in}}| \log f_{\text{in}}| \in L^1(Y) \). We thus believe that the strong weight \( L^1 \) bound \( f_{\text{in}} k_B^2 \in L^1 \) as well as (2.20) and (2.21) are technical hypothesis, but we have not be able to prove uniqueness and to study the long time asymptotic without these assumptions.

2.4. Elastic and inelastic collisions

We next consider the Boltzmann equation with elastic and inelastic collisions.
Theorem 2.4. Assume (1.18), (1.21), (1.23) with \( a_S = 0 \). Consider an initial condition satisfying (1.28) with \( k = k_B \) defined in (1.29). Then there exists a unique solution \( f \) to (1.1) such that for all \( T \geq 0 \), \( f \in C([0, T), L^1_{k_B}) \cap L^\infty([0, T), L^1_{k_B}) \). This solution furthermore conserves momentum and mass distribution (i.e., (2.17) holds), while the total kinetic energy satisfies

\[
\frac{d}{dt} \int_Y f E \, dy = -D_{E,G}(f) \geq 0,
\]

where the term of dissipation \( D_{E,G} \) is defined by (2.9). In particular, \( t \mapsto \int_Y f(t, y)E \, dy \) is nonincreasing and \( t \mapsto D_{E,G}(f(t, \cdot)) \in L^1([0, \infty)) \).

Under a suitable lower-bound of the inelastic collision rate \( a_G \), we also have the following result.

Theorem 2.5. In addition to the assumptions of Theorem 2.4, assume that the total inelastic collision rate \( \bar{a}_G \) (see (1.24)) is continuous and satisfies \( \bar{a}_G(y, y_\ast) > 0 \) for all \((y, y_\ast) \in Y^2 \) such that \( v \neq v_\ast \). Then the kinetic energy is strictly decreasing, and, as \( t \) tends to infinity,

\[
f(t, \cdot) \rightharpoonup \rho(m)\delta_{m=0} \text{ in } M^1(Y)\text{-weak}
\]

where \( \rho \) is defined by (2.25). The velocity distribution

\[
j(t, v) := \int_0^\infty f(t, m, mv)m^4 \, dm
\]

satisfies, as \( t \) tends to infinity,

\[
j(t, \cdot) \rightharpoonup \delta_{v=0} \text{ in } M^1(\mathbb{R}^3)\text{-weak}.
\]

If furthermore there exist some constants \( \kappa > 0 \) and \( \delta \in [0, 1/2] \) such that, for all \( y, y_\ast \) in \( Y \),

\[
\hat{a}_G := \int_{S^2} \int_0^1 (1 - e^2)(v - v_\ast, v)^2 \bar{a}_G(y, y_\ast, \cos \Theta, e) \, dv \, de \\
\geq \kappa (mm_\ast)^\delta |v - v_\ast|^3,
\]

the following rate of convergence holds, for some constant \( C \in (0, \infty) \),

\[
\forall t \geq 1, \quad \int_{\mathbb{R}^3} |v|^2 j(t, v) \, dv = \int_Y f(t, y)E \, dy \leq \frac{C}{t^2}.
\]
For the mass independent inelastic Boltzmann equation, existence of $L^1$ solutions is known in dimension 1 [54] and in all dimensions for the pseudo Maxwell molecules cross-section [9]. To our knowledge, Theorem 2.4 is thus the first existence (and uniqueness) result of $L^1$ solutions to the inelastic Boltzmann equation for the hard spheres cross-section in dimension $N > 1$, see also [31] for the Cauchy theory of a modified inelastic Boltzmann equation. We also answer by the negative to the question of finite time cooling, see [54]. Theorem 2.5 shows that the cooling effect occurs asymptotically in large time and thus extends to that context previous known results.

2.5. Elastic, inelastic and coalescing collisions

We finally treat the case of the full Boltzmann equation (1.1).

**Theorem 2.6.** Assume (1.18), (1.21), (1.23), and consider an initial condition satisfying (1.28) with $k = kB$ defined in (1.29). Then there exists a unique solution $f$ to (1.1) such that for all $T \geq 0$, $f \in C([0, T), L^1_{kB}) \cap L^\infty([0, T), L^1_{kB})$. This solution furthermore conserves mass and momentum

$$\int_Y f(t, y)m \, dy = \int_Y f_{in}(y)m \, dy, \quad \int_Y pf(t, y) \, dy = \int_Y pf_{in}(y) \, dy,$$

while kinetic energy and total number of particles density decrease, more precisely

$$\frac{d}{dt} \int_Y f \, dy = -D_{1,S}(f), \quad \frac{d}{dt} \int_Y fE \, dy = -D_{E,G}(f) - D_{E,S}(f),$$

where $D_{1,S}$, $D_{E,G}$ and $D_{E,S}$ were defined by (2.7), (2.9) and (2.10). In particular,

$$D_{1,S}(f), D_{E,G}(f), D_{E,S}(f) \in L^1([0, \infty)).$$

When $a_B = a_G = 0$ the same results holds replacing $kB$ by $kS$ defined in (1.29).

Existence results for the pure kinetic coalescence equation (that is $a_B = a_G = 0$) have been previously obtained in [48,25]. In [48], measure solutions have been built for general kernels, but $L^1$ solution have been obtained for more restrictive kernels. The authors have also proved a stabilization result to a family of stationary solutions but they were not able to identify that limit to be 0. In [25], an additional structure assumption (see (3.30)) on the coalescence kernel has been made, which permits to prove that any $L^p$ norm is a Lyapunov function. This assumption is satisfied by the hard sphere collisional efficiency $a_{HS}$ defined by (1.16) but not for any general coalescence rate $a_S$ of the form (1.17). The method used in [25] does anyway not extend to the case where $a_B \neq 0$ or $a_G \neq 0$ but it applies to spatially inhomogeneous model.

The next result shows that coalescence dominates other phenomena for large times.
Theorem 2.7. In addition to the assumptions of Theorem 2.6, suppose that for any \( m_0 \in (0, \infty) \), there exists \( A_0 > 0 \) such that,

\[
m \cdot \mathbf{1}_{m \leq m_0} \left[ E_B + E_G \right] \leq A_0 \tilde{E}_{\text{inel}} \text{ on } Y^2,
\]

where

\[
\tilde{E}_{\text{inel}}(y, y^*) := (1 + \bar{\mu}w^2)E_S(y, y^*) + \bar{\mu} \int \int (1 - e^2)(v - v_*, v)^2E_G(y, y^*, v, e) \, dv \, de.
\]

Also assume that for any \( y \in Y \), there exists \( \varepsilon > 0 \) such that

\[
a_S(y, y^*) > 0 \text{ for a.e. } y^* \in B_Y(y, \varepsilon).
\]

Then

\[
f(t, .) \to 0 \text{ in } L^1(Y) \text{ when } t \to \infty.
\]

Here the hypothesis (2.37) on \( a_S \) seems to be very general, and we believe that it is not restrictive for a physical application. It is in particular achieved for a coalescence rate \( a_S \) given by (1.21) with \( a \) and \( E_S \) satisfying (1.17), (1.20) and (1.27). It is satisfied by the collision kernel proposed in Brazier-Smith et al. (1.26). Of course, hypothesis (2.37) is fundamental in order to coalescence process dominate, not making that assumption (taking for instance \( a_S(y, y^*) = 0 \) for any \( y, y^* \) with \( |y - y^*| \leq 1 \) and \( \bar{a}_G > 0 \)) the asymptotic behavior should be driven by the inelastic Granular operator and (2.27) should hold again.

The hypothesis (2.35) on \( a_B \) and \( a_G \) are less obviously satisfied by collision kernels discussed in the physical literature, mainly because the collision rates \( a_B \) and \( a_G \) are not explicitly written. Notice that (2.35) automatically holds when collision are not elastic, quasi-elastic nor grazing, that is when \( a_B = 0, E_G = E_G(\Delta, We, \Theta, e) = 0 \) for any \( \Theta \in (\Theta_0, \pi/2] \) and \( e \in (e_0, 1] \) with \( e_0 \in (0, 1) \) and \( \Theta_0 \in (0, \pi/2) \) and \( E_S \) satisfying (1.27).

Indeed, in that case, condition (2.35) may be reduced to

\[
m_a \cdot \mathbf{1}_{m_a \leq m_0} E_G \leq A_0 \left\{ E_S + \frac{\bar{\mu}}{\min(m, m_a)} We^2 E_G \right\}.
\]

Then condition (2.39) holds for \( We < We_0 \) because of (1.27) and for \( We \geq We_0 \) because of

\[
\frac{\bar{\mu}}{\min(m, m_a)} \geq \frac{1}{2 \min(m, m_a)} \geq \frac{1}{2m_0^2} m_a \cdot \mathbf{1}_{m_a \leq m_0}.
\]

Therefore, assumption (2.35) contains two conditions. On the one hand, it says that for moderate values of the Weber number (says \( We \leq 1 \)) elastic and inelastic collisions do not dominate coalescence, and that always holds when \( E_S \) satisfies (1.27). On the other hand,
it says that for large values of Weber number (says \( \text{We} \geq 1 \)) elastic and quasi-elastic collisions do not dominate (strong) inelastic collisions. We believe that this second condition is technical and should be removed. Finally, the assumption on \( f_{\text{in}} \) is not the most general (a condition such as \( f_{\text{in}}(1 + m + |p|) \in L^1(Y) \) would be more natural), but this is not really restrictive from a physical point of view.

The convergence result (2.38) means exactly that the total concentration \( \int_Y f(t, y) \, dy \) tends to 0 as time tends to infinity. In other words, the mass of each particle tends to infinity: coalescence is the dominating phenomenon in large time. The convergence (2.38) is not a priori obvious because, when the collision rate \( a \) vanishes on \( v = v_0 \) (which is the case for a collision rate given by (1.16)–(1.17)), the density function \( S(m, p) = \lambda(d\mu)\delta_{p = m\mu} \) is a stationary solution to (1.1) for any bounded measure \( \lambda \in M^1(0, +\infty) \) and any vector \( v_0 \in \mathbb{R}^3 \). In particular, Theorem 2.7 implies that the zero solution is the only stationary state which is reached in large time when starting from an \( L^1 \) initial data. It also means that the cooling process (due to coalescing and inelastic collisions) is dominated (under assumption (2.35)) by the mass growth process (due to coalescence). We thus identify more accurately the asymptotic state than in [48], and we do it without any structure condition as introduced in [25]. We extend to more realistic kernels the result presented in [30].

In the pure coalescence case, we may give another asymptotic behavior for solutions which are O-symmetric. We say that \( g \in L^1(Y) \) is O-symmetric if \( g \) is symmetric with respect to the origin 0 in the impulsion variable \( p \in \mathbb{R}^3 \), that is

\[
g(m, -p) = g(m, p) \quad \text{for a.e.} \ (m, p) \in Y. \tag{2.40}
\]

**Theorem 2.8.** In addition to the assumptions of Theorem 2.6, suppose that \( f_{\text{in}} \) is O-symmetric and that (2.37) holds. Assume also that \( a_B = a_G = 0 \), and that \( a_S \) satisfies the natural conditions

\[
a_S(m, -p, m_*, -p_*) = a_S(m, p, m_*, p_*) \quad \text{for a.e.} \ y, y_* \in Y, \tag{2.41}
\]

and

\[
a_S(m, p, m_*, p_*) \leq a_S(m, p, m_*, -p_*) \quad \text{for a.e.} \ y, y_* \in Y \text{ such that } (p, p_*) > 0. \tag{2.42}
\]

Then the solution \( f \) to (1.1) given by Theorem 2.6 is also O-symmetric and satisfies

\[
\int_{\mathbb{R}^3} f(t, \cdot)|p|^2 \, dy \leq \int_{\mathbb{R}^3} f_{\text{in}}|p|^2 \, dy \quad \forall t \geq 0. \tag{2.43}
\]

Moreover, the velocity distribution \( j \) defined by (2.28) satisfies

\[
\int_{\mathbb{R}^3} |v|j(t, v) \, dv = \int_Y |p|f(t, y) \, dy \to 0 \quad \text{when } t \to \infty, \tag{2.44}
\]
and therefore, (2.29) holds.

Note that under the very stringent (and not physical) condition that the coalescence rate satisfies $a_S \geq (m + m_*)|v - v_*|$ (and $a_B = a_G = 0$) we may also show that (2.31) holds.

We finally present without proof (since it follows from a straightforward computation) a class of more or less explicit solutions to the Boltzmann equation for elastic and coalescing collisions.

**Proposition 2.9.** Assume that $a_G \equiv 0$, that $a_B$ and $a_S$ meet (1.18), (1.21), (1.23). Assume also that $a_S$ depends only on the mass variables

$$a_S(y, y_*) = a_S(m, m_*)$$

(2.45)

Consider a solution $c(t, m) : [0, \infty) \times (0, \infty) \to (0, \infty)$ to the classical Smoluchowski equation

$$\frac{\partial}{\partial t} c(t, m) = \frac{1}{2} \int_0^m a_S(m_*, m - m_*)c(t, m_*)c(t, m - m_*) \, dm_*
- \int_0^\infty a_S(m, m_*) c(t, m_*)c(t, m_*) \, dm_*, \quad (t, m) \in (0, \infty) \times (0, \infty).$$

(2.46)

Then the function $f(t, m, p) : [0, \infty) \times (0, \infty) \times \mathbb{R}^3 \to (0, \infty)$ defined by

$$f(t, m, p) = c(t, m) e^{-|p|^2/(2m)} (2\pi m)^{3/2}$$

(2.47)

solves the Boltzmann equation for elastic and coalescing collisions

$$\frac{\partial}{\partial t} f = Q_B(f) + Q_S(f), \quad (t, m, p) \in (0, \infty) \times (0, \infty) \times \mathbb{R}^3.$$ 

(2.48)

Since in specific cases, explicit solutions to the Smoluchowski equations are known (see, e.g., Aldous [1]), we obtain in the next corollary some particular examples.

**Corollary 2.10.** Assume that $a_G \equiv 0$ and that $a_B$ and meets (1.18), (1.21), (1.23). Then

(i) if $a_S(y, y_*) \equiv 1$, then

$$f(t, m, p) = \frac{4}{t^2} e^{-2m/t} e^{-|p|^2/(2m)} (2\pi m)^{3/2}$$

(2.49)

solves the Boltzmann equation for elastic and coalescing collisions (2.48);
(ii) if $a_S(y, y^*) = m + m_*$, then
\[
f(t, m, p) = \frac{1}{\sqrt{2\pi}m} e^{-\frac{3}{2}
} e^{-\frac{1}{2}m_*/m} e^{-|p|^2/2m} (2\pi m)^{3/2}
\]
solves the Boltzmann equation for elastic and coalescing collisions (2.48).

Besides its own interest, Proposition 2.9 might allow to go one step further in the long
time behavior study of general solutions to (2.48). One would expect that any solution to
(2.48) behaves as (2.47) for large times. We are far from being able to show such a result.

Note that expression (2.47) is quite unsurprising: since $a_S$ does not depend on the veloc-
ty variables and since elastic collisions do not act on masses, it is clear that a solution $f$ to
(2.48) satisfies $\int f(t, m, p) dp = c(t, m)$. Then the fact that given its mass $m$, a particle
has a Gaussian (or Maxwellian) momentum with variance $m$ is reasonable. On one hand,
Gaussian distributions are stationary along elastic collisions. On the other hand, Gaussian
distributions are stable under coalescence: adding two Gaussian random variables with
variances $m$ and $m^*$ produces a new Gaussian random variable with variance $m + m^*$.

3. The kinetic coalescence equation

In this section we focus on the sole kinetic Smoluchowski equation
\[
\frac{\partial f}{\partial t} = Q_S(f) \text{ on } (0, \infty) \times Y, \quad f(0, \cdot) = f_m \text{ on } Y,
\]
where $Q_S$ is given by (1.14). Our aim is to prove Theorems 2.6, 2.7 and 2.8 in this particular
situation. We first present a simple computation leading to a uniqueness result, then we
gather some a priori estimates. Next we prove an existence result, and we conclude with
some proofs concerning the long time asymptotic.

We assume in the whole section that $a_B \equiv 0$, $a_G \equiv 0$, (1.18), (1.21), (1.23), and consider
an initial condition satisfying (1.28) with $k = k_S$ or $k = k_B$ defined in (1.29).

3.1. Uniqueness

We start with an abstract uniqueness lemma.

**Lemma 3.1.** Let us assume that $a_S$ and $k$ are two measurable nonnegative functions on $Y^2$
and $Y$ respectively such that for any $y, y_0 \in Y$ there holds
\[
0 \leq a_S(y, y_0) = a_S(y_0, y) \leq k k_* \quad \text{and} \quad k_* \leq k + k_*.
\]
Then there exists at most one weak solution to the kinetic Smoluchowski equation (3.1)
such that for all $T > 0$,
\[
f \in C([0, T); L^1_k) \cap L^\infty([0, T); L^2_k).
\]
Note that under (1.18), one may choose $k = C_A k_S$ or $k = C_A k_B$ (for $C_A$ a constant).

The uniqueness part of Theorem 2.6 thus follows immediately when $(a_B \equiv 0$ and $a_G \equiv 0$).

**Proof.** We consider two weak solutions $f$ and $g$ associated to the same initial datum $f_{in}$ and which satisfy (3.3). We write the equation satisfied by $f - g$ that we multiply by $\phi(t, y) = \text{sign}(f(t, y) - g(t, y))k$. Using the chain rule (2.16) and the weak formulation (2.4) of the kinetic Smoluchowski operator, we get for all $t \geq 0$,

$$
\frac{d}{dt} \int_Y |f - g| k \, dy = \frac{1}{2} \int_Y \int_Y a_S(y, y_*) \left((f - g) g_*) (\phi_* - \phi) \, dy_*, dy
\leq \frac{1}{2} \int_Y \int_Y a_S(y, y_*) |f - g|(f_* + g_*) (k_* - k) \, dy_*, dy,
$$

where we have just used the symmetry hypothesis (3.2) on $a_S$. Then, thanks to the bounds (3.2), we deduce

$$
\frac{d}{dt} \int_Y |f - g| k \, dy \leq \int_Y \int_Y k_* |f - g|(f_* + g_*) \, dy_*, dy = \|f + g\|_{L^1} \|f - g\|_{L^1}.
$$

One easily concludes, by using the Gronwall lemma, that for all $T \geq 0$,

$$
\sup_{[0, T]} \|f(t, .) - g(t, .)\|_{L^1} \leq \|f(0, .) - g(0, .)\|_{L^1} \exp \left( \sup_{[0, T]} \|f(t, .) + g(t, .)\|_{L^1} T \right)
$$

(3.4)

which is identically null, since $f(0, .) = g(0, .)$. □

We deduce that O-symmetry propagates.

**Corollary 3.2.** Assume that $a_S$ and $k$ satisfy the assumptions of Lemma 3.1. Suppose also that $f_{in}$ satisfies the O-symmetry condition (2.40) while $a_S$ meets (2.41). Then a solution $f$ to (3.1) satisfying (3.3) is also O-symmetric.

**Proof.** Introduce the notation $f^\circ = f^\circ(m, p) := f(m, -p)$. Clearly $f^\circ$ also satisfies (3.3). A simple computation shows that

$$
Q_x^\circ(f)(m, -p) = \int_0^m \int_{\mathbb{R}^3} a_S(m - m_*, -p - p_*, m_*, p_*) f(m - m_*, -p - p_*) f(m_*, p_*) \, dm_* \, dp_*
$$
\[
\begin{align*}
&= \int_0^m \int_{\mathbb{R}^3} a_S(m - m_*, -p + q_*, m_*, -q_*) f(m - m_*, -p + q_*) f(m_*, -q_*) \, dm_* \, dq_* \\
&= \int_0^m \int_{\mathbb{R}^3} a_S(m - m_*, p - q_*, m_*, q_*) \, f^*(m - m_*, p - q_*) \, f^*(m_*, q_*) \, dm_* \, dq_* \\
&= Q^*_S(f^*)(m, p).
\end{align*}
\]

We have made the substitution \( q_* = -p_* \) and then used the symmetry (2.41) of \( a_S \). By the same way, one may prove \( Q^*_S(f)(m, -p) = Q^*_S(f^*)(m, p) \) for any \((m, p) \in Y\). In other words, the function \( f^*(t, m, p) := f(t, m, -p) \) is a solution to the Smoluchowski equation, and by hypothesis, \( f^2(0, \cdot) = f_m \). Lemma 3.1 ensures that \( f^2 = f \) and the claim is proved. \( \square \)

### 3.2. A priori estimates

We begin by some physical and formal a priori estimates.

**Lemma 3.3.** Let \( f \) be a solution to the kinetic Smoluchowski equation (3.1). Then mass and momentum conservation (2.32) hold (at least formally), and the dissipation of energy and of number of particles (2.33) also hold (with \( D_{\mathcal{E}, \mathcal{G}} \equiv 0 \)). As a matter of fact, for any sub-additive function \( \psi : Y \to (0, \infty) \), that is \( \psi^{**} \leq \psi + \psi_* \), the map

\[
t \mapsto \int_Y f(t, y) \psi(y) \, dy
\]

is nonincreasing. As an illustration of this fact, there holds

\[
t \mapsto \int_Y f(t, y) |p| \, dy \text{ is nonincreasing,}
\]

\[
t \mapsto \int_Y f(t, y) \xi(m) \, dy \text{ is nonincreasing,}
\]

\[
t \mapsto \int_Y f(t, y) \xi(|v - v_0|) \, dy \text{ is nonincreasing,}
\]

for any nonincreasing function \( \xi \) on \((0, \infty)\), any nondecreasing function \( \xi \) on \((0, \infty)\) and any \( v_0 \in \mathbb{R}^3 \). Another consequence is

\[
\int_Y f(T, y) m^a \, dy + \frac{1}{2} \int_0^T \int_Y a_S m^a f f_* \, dy \, dy_* \, dt \leq \int_Y m^a f_m \, dy,
\]

for any nonincreasing function \( \xi \) on \((0, \infty)\), any nondecreasing function \( \xi \) on \((0, \infty)\) and any \( v_0 \in \mathbb{R}^3 \). Another consequence is
for any $\alpha \in (-\infty, 0]$ and $T > 0$.

**Proof.** These results may be formal when the solution only satisfy (2.16) for bounded functions $\phi$. It become rigorous when $f$ satisfies an extra moment condition or when we deal with approximated solutions (to equations with cutoff rates). We assume in this proof that we may apply (2.16) and (2.4) without questioning.

First note that (2.32) is an immediate consequence of (2.16) applied with $\phi(y)$ and $\phi(y)$ = $m$ and $\beta(x) = x$ thanks to (2.5) (this last following from (2.4)). Next, the map defined by (3.5) is nonincreasing thanks to (2.16) and (2.4) applied with $\phi = \psi$ (and $\beta(x) = x$). We next deduce (3.6) choosing $\psi = |p|$ in (3.5), (3.7) choosing $\psi = \xi(m)$ and (3.8) choosing $\psi = \xi(|v - v_0|)$ (note that $|v_{aa} - v_0| \leq \max(|v - v_0|, |v_a - v_0|)$). We finally obtain (3.9) applying (2.16) and (2.4) with $\phi = m^\alpha$ (and $\beta(x) = x$) and remarking that $m + m^\alpha - m^\alpha \geq m^\alpha$ when $\alpha \leq 0$. □

The next lemma gives some estimations on $L^1$ norms with weight of the Smoluchowski term $Q_S$.

**Lemma 3.4.** There exists a constant $C_A$, depending only on $A$ (see (1.18)), such that for any measurable function $g : Y \mapsto (0, \infty)$ and any $z \in (1, 2]$,

\[
\int_Y Q_S(g)(m^z + |p|^z)\, dy \leq C_A \left(1 + m + |v| + |p| + E\right)g\, dy \\
\times \int_Y (1 + m^z + |p|^z)g\, dy,
\]

(3.10)

\[
\int_Y Q_S(g)(m^2 + E^2)\, dy \leq C_A \int_Y (m^{-1} + m + E)g\, dy \int_Y (m^{-2} + m^2 + E^2)g\, dy,
\]

(3.11)

\[
\int_Y Q_S(g)(m^3 + |p|^3)\, dy \leq C_A \int_Y (m + m^2 + |p|^2 + E)g\, dy \\
\times \int_Y (m^2 + |p|^2 + E^2 + m^3 + |p|^3)g\, dy,
\]

(3.12)

\[
\int_Y Q_S(g)(m^3 + E^3)\, dy \leq C_A \int_Y (m^{-1} + m^2 + E^2)g\, dy \int_Y (m^{-2} + m^3 + E^3)g\, dy.
\]

(3.13)

**Proof.** These results follow from tedious but straightforward computations. We only show the two first inequalities, the two last ones being proved similarly.

**Proof of (3.10).** Defining $\Phi(\zeta, \zeta_a) := (\zeta + \zeta_a)^z - \zeta^z - (\zeta_a)^z$, we observe that

\[
\Phi(\zeta, \zeta_a) \leq 2 \min((\zeta_a)^z, \zeta^z), \quad (\zeta + \zeta_a)\Phi(\zeta, \zeta_a) \leq 4[\zeta\zeta_a + \zeta^z\zeta_a].
\]

(3.14)
First, using (1.18) and the obvious fact that \( \Phi(m, m_\ast) \leq 2m^2 + 2m_\ast^2 \),

\[
\alpha_5 \Phi(m, m_\ast) \leq A(1 + m + m_\ast)(1 + |v| + |v_\ast|)\Phi(m, m_\ast)
\]

\[
\leq A(1 + |v| + |v_\ast|)(m + m_\ast)\Phi(m, m_\ast) + \Phi(m, m_\ast)
\]

\[
\leq CA(1 + |v| + |v_\ast|)mm_\ast^2 + m^2 + m^2 + m_\ast^2
\]

\[
\leq CA(T + T_\ast)
\]

(3.15)

where \( T = T(y, y_\ast) = (1 + m + |v| + |p|)(m_\ast^2 + m_\ast^2|v_\ast|) \) and \( T_\ast = T(y_\ast, y) \).

Next, using the first inequality in (3.14),

\[
\alpha_5 \Phi(|p|, |p_\ast|) \leq A(1 + m + m_\ast)(1 + |v_\ast| + |v_\ast|)\Phi(|p|, |p_\ast|)
\]

\[
\leq CA(1 + m + |v_\ast| + |p| + m|v_\ast|)|p|^{|-1|}p_\ast|
\]

\[
+ CA(1 + m_\ast + |v| + |p_\ast| + m_\ast|v|)|p_\ast|^{|-1|}|p|
\]

\[
\leq CA(S + S_\ast),
\]

(3.16)

where \( S = (|p| + \mathcal{E})(|p_\ast|^{-1} + m_\ast|p_\ast|^{-1} + |p_\ast|) \). Since furthermore \( z \in (1, 2) \), we deduce that \( |p_\ast|^{-1} \leq 1 + |p_\ast| \), that \( m_\ast|p_\ast|^{-1} = m_\ast^2|v_\ast|^{-1} \leq m_\ast^2 + |p_\ast| \), and that \( m_\ast^2|v_\ast| \leq m_\ast^2 + |p_\ast| \). Hence, for some numerical constant \( C \),

\[
S + T \leq C(1 + m + |v| + |p| + \mathcal{E})(1 + m_\ast^2 + |p_\ast|). \]

(3.17)

Applying finally (2.4) with \( \varphi = m^2 + |p|^2 \), we obtain

\[
\int_y \mathcal{Q}_S(g)(m^2 + |p|^2) \mathrm{d}y = \int_y \int \alpha_5 \left[ \Phi(m, m_\ast) + \Phi(|p|, |p_\ast|) \right] gg_\ast \mathrm{d}y_\ast \mathrm{d}y
\]

\[
\leq CA \int_y \int \alpha_5 [S + T + S_\ast + T_\ast] gg_\ast \mathrm{d}y_\ast \mathrm{d}y,
\]

(3.18)

which leads to (3.10).

**Proof of (3.11).** Observe now that

\[
\mathcal{E}_{\ast\ast}^2 - \mathcal{E}^2 - \mathcal{E}_{\ast}^2 = m_{\ast\ast}^{-2}\{mv + m_\ast v_\ast\}^4 - m^2|v|^4 - m_\ast^2|v_\ast|^4
\]

\[
= m_{\ast\ast}^{-2}\{m_\ast^3 m_\ast(4|v|^3|m_\ast| - 2|v|^4) + m^2 m_\ast^2(6|v|^2|v_\ast|^2 - |v|^4 - |v_\ast|^4)
\]

\[
+ 4m_\ast^2|v|^3|v_\ast|1_{|v|\leq 2|v_\ast|} + 6m_\ast^2|v|^2 m_\ast^2|v_\ast|^2 1_{|v_\ast|\leq 3|v|} 1_{|v|\leq 2|v_\ast|} \}
\]

\[
\leq m_{\ast\ast}^{-2}\{4m^3|v|^3 m_\ast|v_\ast|1_{|v|\leq 2|v_\ast|} + 6m^2|v|^2 m_\ast^2|v_\ast|^2 1_{|v_\ast|\leq 3|v|} 1_{|v|\leq 2|v_\ast|} \}
\]

\[
+ 4m|v|m_\ast^3|v_\ast|^3 1_{|v|\leq 2|v_\ast|} \}.
\]
Therefore, we get, after some tedious but straightforward computations,

\[ a_S \left[ E^2_{\phi} - E^2 - E^2 \right] \leq CA(U + U_*) \]  

(3.19)

with \( U = [m^{-1} + m + \mathcal{E}] [m^{-2} + m^2 + \mathcal{E}_a + \mathcal{E}^2] \). We have used here the inequalities \( m|v|^3 \leq m^{-2} + \mathcal{E}^2, m^2|v|^3 \leq m^{-2} + \mathcal{E}^2, |p| \leq m + \mathcal{E}, \) and \( |v| \leq m^{-1} + \mathcal{E} \).

Applying (2.4) with the choice \( \phi = \mathcal{E}^2 \), and using (3.15) with \( z = 2 \), we obtain

\[
\int_Y Q_S(g)(m^2 + \mathcal{E}^2) \, dy \leq CA \int_Y \int_Y (T + T_u + U + U_*) g g_* \, dy \, dy_*
\]

\[
\leq CA \int_Y (1 + m + |v| + |p| + m^{-1} + \mathcal{E}) g \, dy
\]

\[
\times \int_Y (|p_*| + m_*|p_*| + m_*^{-2} + \mathcal{E}_a + \mathcal{E}^2 + m_*^2) g_* \, dy
\]

\[
\leq CA \int_Y (m^{-1} + m + \mathcal{E}) g \, dy \int_Y (m^{-2} + m^2 + \mathcal{E}^2) g \, dy.
\]  

(3.20)

For the last inequality, we used \( 1 \leq m^{-1} + m, 1 \leq m^{-2} + m^2, \) and \( \mathcal{E} + \mathcal{E}^2 \leq 1 + 2\mathcal{E}^2 \leq m^{-2} + m^2 + 2\mathcal{E}^2 \). This concludes the proof of (3.11). \( \square \)

An immediate and fundamental consequence of Lemmas 3.3 and 3.4 is the following.

**Corollary 3.5. A solution \( f \) to (3.1) satisfies, at least formally, for any \( T \),**

\[
\text{for } z \in (1, 2], \quad f_{\in}(k^z_S + \mathcal{E}) \in L^1(Y) \text{ implies } \sup_{[0,T]} \int_Y f(t,y)(k^z_S + \mathcal{E}) \, dy \leq C_T,
\]  

(3.21)

\[
\text{for } z = 2 \text{ and } 3, \quad f_{\in}(k^z_S + m^{-2}) \in L^1(Y) \text{ implies } \sup_{[0,T]} \int_Y f(t,y)(k^3_S + m^{-2}) \, dy \leq C_T,
\]  

(3.22)

\[
\text{for } z = 2 \text{ and } 3, \quad f_{\in}k^z_B \in L^1(Y) \text{ implies } \forall T > 0 \sup_{[0,T]} \int_Y f(t,y)k^z_B \, dy \leq C_T,
\]  

(3.23)

**where the constant \( C_T \) depends on \( T, f_{\in} \) and \( A \) (see (1.18)).**

**Proof.** We only show (3.21), the other claims being proved similarly. Assume thus that \( (k^z_S + \mathcal{E}) f_{\in} \in L^1 \) with \( z \) fixed in \((1, 2]\). This assumption is equivalent to \((1 + m^z + |p|^z + |v|^z + \mathcal{E}) f_{\in} \in L^1 \). First, from Lemma 3.3 (or more precisely from (2.32), (2.33), (3.6), (3.8)), we have \((1 + m + |p| + |v| + \mathcal{E}) f \in L^\infty([0, \infty), L^1) \). Next, applying (2.16) with \( \phi = m^z + |p|^z \) (and \( \beta(x) = x \)) and using (3.10) in Lemma 3.3 we conclude (3.21) thanks to the Gronwall lemma. \( \square \)
3.3. Existence

We shall deduce from the previous estimates in Corollary 3.5 and a modification of the proof of the uniqueness Lemma 3.1 the existence part of Theorem 2.6 (in the case where $a_B = a_G = 0$ and replacing $k_B$ by $k_S$). First of all note that

$$0 \leq a_S \leq A(1 + m + m*) (1 + |v| + |v_*|) \leq Ak_S(y)k_S(y_*),$$

so that (3.2) holds with the choice $k = k_S$. We split the proof in several steps.

First step. We will first assume in this step that

$$\int_Y f_{in} [k_S^3 + m^{-2}] \, dy < \infty,$$

and we introduce the coalescence equation with cutoff

$$\frac{\partial g^n}{\partial t} = Q_{S,n}(g^n) \text{ on } (0, \infty) \times Y, \quad g^n(0, .) = f_{in} \text{ on } Y,$$

where $Q_{S,n}$ is the coalescence kernel associated to the coalescence rate $a_{S,n}(y, y_*) := a_S(y, y_*) \wedge n$. The coalescence rate being bounded it is a classical application of Banach fixed point theorem to prove that there exists a unique solution $0 \leq g^n \in C([0, \infty); L^1_{k_S^3 + m^{-2}}(Y))$ to (3.26) associated to the initial datum $f_{in}$ satisfying (3.25). We refer to [25] Section 6 where we may consider the Banach space $X := L^1_{k_S^3 + m^{-2}}$. Because of the estimates on $g^n$, it is possible to establish rigorously that $g^n$ satisfies for each $n \in \mathbb{N}$ the (a priori formal) properties stated in Lemma 3.3 and Corollary 3.5. In particular, for any $T > 0$, there holds

$$\sup_{[0, T]} \|g^n(t, .)\|_{L^1_{k_S^3}} \leq C_T,$$

where $C_T \in (0, \infty)$ may depends of $T$ and $f_{in}$, but not on the truncation parameter $n \in \mathbb{N}^*$. We now repeat the proof of the uniqueness Lemma 3.1. For $l \geq n$, we write the equation satisfied by $g^n - g^l$, we multiply it by $\phi = \text{sign}(g^n - g^l)k_S$, and we use (2.16) and (2.4) to obtain

$$\frac{d}{dt} \int_Y |g^n - g^l| \, k_S \, dy = \frac{1}{2} \int_Y \int_Y a_{S,n} (g^n g^n_* - g^l g^l_*) (\phi_* - \phi_*) \, dy_* \, dy$$

$$+ \frac{1}{2} \int_Y \int_Y (a_{S,n} - a_{S,l}) g^l g^l_* (\phi_* - \phi_*) \, dy_* \, dy$$

$$\leq A \int_Y \int_Y (g^n + g^l) k_S^3 |g^n_* - g^l_*| k_S \, dy_* \, dy$$
\[
\begin{align*}
&\int \int_Y a_S 1_{\{a_S \geq n\}} g' \, g_{\ast} (k_S + k_{S\ast}) \, dy_x \, dy \\
&\leq A \| g^n + g' \|_{L^1_{k_S}} \| g^n - g' \|_{L^1_{k_S}} \\
&+ 2A \int_Y g' (k_S + k_{S}\ast) \, dy \left( \int_Y g' (k_S + k_{S}\ast) 1_{k_S \geq \sqrt{n}/ \sqrt{A}} \, dy \right),
\end{align*}
\]

where we have used the fact \(a_S 1_{\{a_S \geq n\}} \leq A k_S k_{S\ast} (1_{\{k_S \geq \sqrt{n}/ \sqrt{A}\}} + 1_{\{k_{S\ast} \geq \sqrt{n}/ \sqrt{A}\}}).\) With the notation

\[
u_{l,n} = \| g^n - g' \|_{L^1_{k_S}}, \quad B_T = \sup_{n \in \mathbb{N}^*} \sup_{t \in [0,T]} \| g^n (t) \|_{L^1_{k_S}}, \quad (3.28)
\]

we end up with the differential inequality

\[
\frac{d}{dt} \nu_{l,n} \leq (2AB_T) \nu_{l,n} + (8AB_T) B_T \sqrt{A/\sqrt{n}}. \quad (3.29)
\]

The Gronwall lemma implies \(\sup_{[0,T]} \nu_{l,n} (t) \to 0\) when \(n, l \to \infty.\) Hence \((\nu_{l,n})\) is a Cauchy sequence in \(C([0, \infty), L^1_{k_S})\), and there exists \(f \in C([0, \infty), L^1_{k_S})\) such that \(g^n \to f\) in \(C([0, T), L^1_{k_S})\) for any \(T > 0.\) Of course, (3.27) allows to deduce that \(f \in L^\infty ([0, T), L^1_{k_S}).\) There is no difficulty to pass to the limit in the weak formulation (2.14) of (3.26), and that proves the existence of a solution \(f\) to (3.1) with initial datum \(f_{\in}\) satisfying (3.25).

**Second step.** When just assuming that \(f_{\in} \in L^1_{k_S}\) we consider the sequence of solutions \((f_n)\) to (3.1) associated to the rate \(a_S\) and the initial data \(f_n (0, .) = f_{\in} 1_{\{k_S \leq n\}} 1_{\{|m| \geq 1/n\}}\) (which satisfies (3.25)) for which existence has been established just above. Then one easily deduces from Corollary 3.5 that for each \(T \geq 0,\)

\[
\sup_{n} \sup_{[0,T]} \| f_n \|_{L^1_{k_S}} < \infty.
\]

We may use directly the estimate (3.4) for the difference \(f_m - f_n,\) and prove that \((f_n)\) is a Cauchy sequence in \(C([0, T); L^1_{k_S}) \cap L^\infty ([0, T); L^1_{k_S})\) for any \(T > 0.\) We conclude just like before. Let us emphasize that the information \(f \in L^\infty ([0, T); L^1_{k_S + k_{S}\ast})\) for any \(T \geq 0\) is sufficient to deduce that the statements of Lemma 3.3 rigorously hold. The uniqueness of the solution has yet been shown in Lemma 3.1.

We conclude this subsection by a slight improvement of the existence result established in [25].

**Proposition 3.6.** Assume that \(a_S\) satisfies the following structure assumption

\[
\forall y, y_{\ast} \in Y.\]

\[
as(y, y_{\ast}) \leq a_S (y, y_{\ast}) + a_S (y_{\ast}, y_{\ast}) \leq \| a_S \|_{L^\infty (\Omega)} (\| y \|_{L^1})^2 \quad (3.30)
\]
For any \( f_n \) such that there exists \( z \in (1, 2] \),

\[
0 \leq f_n \left\{ \left( 1 + m + |p| + |v| \right)^z + \mathcal{E} \right\} \in L^1(Y),
\]

(3.31)

there exists at least a solution \( f = f(t, y) \in C([0, \infty); L^1(Y)) \) to the kinetic Smoluchowski equation which satisfies (3.6), (3.9), (3.8) and the solution conserves mass and momentum (2.32).

**Proof.** The only new claims with respect to the existence result in [25] are the conservations (2.32). As usually, it is a straightforward consequence of (3.21) since \( z > 1 \).

\[ \square \]

### 3.4. Long time behavior

The aim of this subsection is to give a first (and deterministic) proof to Theorem 2.7 which is only valid when \( a_B = a_G = 0 \) and under the additional condition

\[ a_S \geq \underline{a}_S, \text{ with } \underline{a}_S \text{ continuous on } Y^2 \text{ and satisfying (2.37).} \]  

(3.32)

We will give the general proof in Section 6.

**Proof of Theorem 2.7 under strong assumptions.** We split the proof into three parts. In the first one we prove that \( f \) stabilizes around a solution of the shape \( \lambda(t, dm)\delta_{p=mv} \) using a dissipative entropy argument. In the second step, using a rich enough class of Liapunov functionals, we establish that \( \lambda \) and \( v \) are unique and not time-depending. We actually prove there exists \( \lambda \in M^1(\mathbb{R}_+) \) and \( v_0 \in \mathbb{R}^3 \) such that

\[
f(t, y) \rightarrow \lambda(dm)\delta_{p=mv_0} \text{ in } \mathcal{D}'(Y) \text{ when } t \rightarrow \infty.
\]

(3.33)

This was proved (by a different method) in [48] under less general assumptions on \( a_S \) and \( f \). In the last step we prove, arguing by contradiction, that \( \lambda = 0 \).

**Step 1.** Let us consider an increasing sequence \( (t_n)_{n \geq 1} \), \( t_n \rightarrow \infty \), and put \( f_n(t, .) := f(t + t_n, .) \) for \( t \in [0, T] \) and \( n \geq 1 \). We realize, thanks to Lemma 3.3 (recall the expression of \( k_S \)), that

\[
f_n \text{ is bounded in } L^\infty([0, T); L^{1}_{k_S}) \quad \text{and}
\]

\[
\int_Y f_n(t, y)\psi(y) \, dy \text{ is bounded in } BV(0, T)
\]

(3.34)

for any \( \psi \in L^\infty(Y) \). Therefore, up to the extraction of a subsequence, there exists \( \Gamma \in C([0, T); M^1(Y)\text{-weak}) \) such that
\[ f_n \rightharpoonup \Gamma \in \mathcal{D}'([0, T] \times Y), \]
\[ \int_Y \psi(y) f_n(t, y) \, dy \rightharpoonup \int_Y \psi(y) \Gamma(t, \cdot) \, C([0, T]) \]  
\[ (3.35) \]
for any \( \psi \in C_b(Y) \). Then, for any \( \chi \in C_b(Y^2) \), the above convergence is strong enough in order to pass to the semi-inferior limit as follows
\[ \int_0^T \int_Y \int_Y \chi(y, y_a) \Gamma(t, \cdot) \, \Gamma(t, \cdot) \, dy \, dy_a \, dt \leq \liminf_{n \to \infty} \int_0^T \int_Y \int_Y \chi(y, y_a) f_{n,n} \, dy \, dy_a \, dt. \]  
\[ (3.36) \]
Using standard truncation arguments (see [25]) and the fact that \( a_S \) is continuous (see (3.32)), we deduce from (3.36)
\[ \int_0^T \mathcal{D}_{1.5}(\Gamma(t, \cdot)) \, dt \leq \liminf_{n \to \infty} \int_0^T \mathcal{D}_{1.5}(f(t, \cdot)) \, dt, \]  
\[ (3.37) \]
where \( \mathcal{D}_{1.5} \) is defined by (2.7) with \( a_S \) instead of \( a_S \). Using now (2.34), we deduce that the RHS of (3.37) vanishes. Thus
\[ \forall t \in [0, T] \quad a_S(y, y_a) \Gamma(t, \cdot) \Gamma(t, \cdot) = 0 \quad \text{in } \mathcal{D}'(Y^2). \]  
\[ (3.38) \]
Thanks to (2.37), we deduce that for any \( t \in [0, T] \),
\[ \mathbf{1}_{\{y \neq y_a\}} \Gamma(t, \cdot) \Gamma(t, \cdot) = 0 \quad \text{in } \mathcal{D}'(Y^2). \]
We finally deduce that for any \( t \in [0, T] \) fixed, the support of \( \Gamma(t, \cdot) \) is contained in the subset \( \{(m, mvt), m \in \mathbb{R}_+\} \) for some \( v_t \in \mathbb{R}^3 \). Thus, we have \( \Gamma(t, \cdot) = \lambda(t, \cdot) \delta_{p=mv_t} \) with \( \lambda \in C([0, T], M^1(\mathbb{R}_+)-\text{weak}) \).

Step 2. We now prove that \( \lambda(t, dm) \) and \( v_t \) are unique and not depending on time, nor on the sequence \((t_n)\), so that (3.33) holds. First, we claim that for any \( R \in (0, \infty) \), there exists a real number \( \alpha(R) \geq 0 \), which does not depend on the sequence \((t_n)\), such that
\[ \forall t \in [0, T] \quad \int_0^R \lambda(t, dm) = \alpha(R). \]  
\[ (3.39) \]
That allows to identify for any \( t \in [0, T] \) the measure \( \lambda(t, \cdot) \) which is therefore not a function of time: \( \lambda(t, dm) = \lambda(dm) \). In order to prove (3.39) we argue as follows. We fix
\( R, \varepsilon > 0 \) and we define \( \zeta_\varepsilon \in C_c(\mathbb{R}^3) \) by \( \zeta_\varepsilon = 1 \) on \([0, R]\), \( \zeta_\varepsilon(m) = 1 - e^{-1}(m - R) \) for any \( m \in [R, R + \varepsilon) \) and \( \zeta_\varepsilon = 0 \) on \([R + \varepsilon, \infty)\). Gathering (3.35) and (3.7), there exists a real number \( \alpha(R, \varepsilon) \geq 0 \) such that for any \( t \in [0, T] \) there holds

\[
\int_{\mathbb{R}^3} \zeta_\varepsilon(m) \lambda(t, \cdot) dm = \lim_{n \to \infty} \int_{\mathbb{R}^3} \zeta_\varepsilon(m) f_n(t, y) dy = \lim_{n \to \infty} \int_{\mathbb{R}^3} \zeta_\varepsilon(m) f_n(t, y) dy.
\]

But \( \zeta_\varepsilon(m) \searrow 1 \) on \([0, R]\) for any \( m > 0 \). Hence \( (\alpha(R, \varepsilon)) \) is decreasing when \( \varepsilon \searrow 0 \). We then may pass to the limit \( \varepsilon \to 0 \) in (3.40) and we obtain (3.39) with \( \alpha(R) := \lim_{\varepsilon \to 0} \alpha(R, \varepsilon) \).

Next, we claim that for any \( u \in \mathbb{R}^3 \), there exists \( \beta(u) \in \{0, 1\} \), which does not depend on the sequence \((t_n)\), such that

\[
\forall t \in [0, T] \quad 1_{\{v_t \neq u\}} = \beta(u).
\]

This of course uniquely determines \( v_0 \in \mathbb{R}^3 \) such that \( v_t = v_0 \) on \([0, T]\) and then (3.33) holds. Let us establish (3.41). We fix \( u \in \mathbb{R}^3 \) and, for any \( \varepsilon > 0 \), we define \( \phi_\varepsilon = 0 \) on \([0, \varepsilon/2]\), \( \phi_\varepsilon(s) = 2s/\varepsilon - 1 \) for any \( s \in [\varepsilon/2, \varepsilon] \) and \( \phi_\varepsilon = 1 \) on \([\varepsilon, \infty)\). Then, we set \( \zeta_\varepsilon(m) = \phi_\varepsilon(m) \) and \( \xi_\varepsilon(v) = \phi_\varepsilon(|v - u|) \). From (3.7) we have, for any \( t \in [0, T] \),

\[
\left| \int_{\mathbb{R}^3} \xi_\varepsilon f_n dy - \int_{\mathbb{R}^3} \xi_\varepsilon \xi_\varepsilon f_n dy \right| = \int_{\mathbb{R}^3} \xi_\varepsilon (1 - \xi_\varepsilon) f_n dy \leq \int_{\mathbb{R}^3} 1_{m \in [0, \varepsilon]} f_n dy.
\]

From (3.8) and since \( \phi_\varepsilon \) is increasing, there exists a real number \( \gamma_\varepsilon(u) \geq 0 \) such that

\[
\gamma_\varepsilon(u) := \lim_{s \to \infty} \int_{\mathbb{R}^3} \xi_\varepsilon(v) f(s, y) dy = \lim_{n \to \infty} \int_{\mathbb{R}^3} \xi_\varepsilon(v) f_n(t, y) dy \quad \forall t \in [0, T].
\]

From (3.35) and since \( y \to \zeta_\varepsilon(m) \xi_\varepsilon(v) \) belongs to \( C_b(Y) \), we obtain

\[
\int_{0}^{\infty} \zeta_\varepsilon(m) \lambda(dm) \xi_\varepsilon(v_t) = \int_{\mathbb{R}^3} \zeta_\varepsilon \xi_\varepsilon \Gamma(t, \cdot) dy = \lim_{n \to \infty} \int_{\mathbb{R}^3} \zeta_\varepsilon \xi_\varepsilon f_n(t, y) dy \quad \forall t \in [0, T].
\]

Therefore, passing first to the limit \( n \to \infty \) in (3.42), we have for any \( \varepsilon > 0 \).
\[
\gamma_{\varepsilon}(u) - \int_0^\infty \zeta_{\varepsilon}(m) \lambda(d)m \xi_{\varepsilon}(v_t) \leq \int \mathbf{1}_{m \in [0, \varepsilon]} f_m \, dy \quad \forall t \in [0, T]. \quad (3.43)
\]

In the limit \(\varepsilon \searrow 0\) we have \(\zeta_{\varepsilon} \to 1\) pointwise on \((0, \infty)\) and \(\xi_{\varepsilon}(v) \to \mathbf{1}_{v \neq u}\) for any \(v \in \mathbb{R}^3\). In particular \(\gamma_{\varepsilon}(u)\) is increasing and thus converges as \(\varepsilon \searrow 0\). We deduce from (3.43), since \(f_m \in L^1(Y)\), that

\[
\int_0^\infty \lambda(d)m \mathbf{1}_{v \neq u} = \lim_{\varepsilon \to 0} \gamma_{\varepsilon}(u) \quad \forall t \in [0, T],
\]

from which (3.41) follows.

**Step 3.** We now prove that \(\lambda \equiv 0\). We argue by contradiction assuming that \(\lambda \not\equiv 0\). For any \(R \in (0, \infty]\), we define (thanks to (3.7))

\[
\alpha(R) := \lim_{t \to \infty} \int_0^R \int f(t, y) \, dy.
\]

We first remark that there exists \(R > 0\) such that \(\alpha(R/2) < \alpha(R)\). If not, we would have for any \(R > 0\)

\[
\alpha(R) = \alpha(R/2) = \cdots = \alpha(R/2^n) \leq \int_0^{R/2^n} f_m(y) \, dy \to 0,
\]

and that contradicts with the fact that \(\|\lambda\| = \lim_{R \to \infty} \alpha(R) > 0\).

Let us thus fix \(R > 0\) and \(\varepsilon > 0\) such that \(\alpha(R/2) + 2\varepsilon < \alpha(R)\). Thanks to (3.33) there exists \(T > 0\) and then \(\delta > 0\) such that

\[
\int_0^{R/2} \int f(T, y) \, dy \leq \alpha(R/2) + \varepsilon, \quad \int_{R/2}^R \int f(T, y) \, dy \leq \varepsilon,
\]

since \(f(T, \cdot) \in L^1(Y)\). We define \(A := \{y; (m \leq R/2) \text{ or } (m \in [R/2, R], |v - v_0| \leq \delta)\}\) and we observe that \(y_{ss} \in A\) implies \(y \in A\) or \(y_s \in A\), so that \(y \mapsto \mathbf{1}_A(y)\) is a sub-additive function. On the one hand, thanks to (3.5) and (3.44), we have for any \(t \geq T\)

\[
\int_Y f(t, y) \mathbf{1}_{y \in A} \, dy \leq \int_Y f(T, y) \mathbf{1}_{y \in A} \, dy \leq \alpha(R/2) + 2\varepsilon. \quad (3.45)
\]
On the other hand, thanks to (3.33), we have
\[ \alpha(R) \leq \lim_{t \to \infty} \int_Y f(t, y) 1_{y \in \Lambda} \, dy. \] (3.46)

Therefore, gathering (3.45) and (3.46) we obtain \( \alpha(R) \leq \alpha(R/2) + 2\varepsilon \). This contradicts our choice for \( R \) and \( \varepsilon \).

Let us remark that we cannot extend this deterministic proof to the full Boltzmann equation (1.1), because of the less rich class of Lyapunov functionals available in that general case. For the full Boltzmann equation (1.1) we then could only prove the following result: there exists \( \lambda \in M^1(0, \infty) \) such that for any increasing sequence \( (t_n) \) which converges to infinity, there exists a subsequence \( (t'_n) \) and \( u \in \mathbb{R}^3 \) such that
\[ f(t_n + \ldots) \rightharpoonup \lambda(dm)\delta_{p=mu} \text{ weakly in } C\left( \{0, T\}; M^1(Y) \right), \]
where \( \lambda \) does not depend on the subsequence \( (t'_n) \) but \( u \in \mathbb{R}^3 \) may depend on it.

**Proof of Theorem 2.8.** From Corollary 3.2 we already know that \( f \) satisfies the symmetry property (2.40). Then, we just compute, using (2.4), (2.41) and (2.42),
\[
\frac{d}{dt} \int_Y f |p|^2 \, dy = \int_Y \int_Y a_S f [p, p_*] \left( 1_{\{\langle p, p_* \rangle > 0 \}} + 1_{\{\langle p, p_* \rangle < 0 \}} \right) \, dy \, dy_* \\
= \int_Y \int_Y a_S (m, p, m_*, p_*) - a_S (m, p, m_*, -p_*) \times 1_{\{\langle p, p_* \rangle > 0 \}} \, dy \, dy_* \leq 0,
\]
and we obtain (2.43). Now, on the one hand, by definition (2.28) of \( j \), the moment condition (1.28), and the mass conservation (2.32), there holds
\[ \int_{\mathbb{R}^3} j \, dv = \int_Y f(t, y) m \, dy = 1 \] (3.47)
and
\[ \int_{\mathbb{R}^3} j |v| \, dv = \int_Y f(t, y) |p| \, dy \leq \left( \int_Y f(t, y) \, dy \right)^{1/2} \left( \int_Y |p|^2 f(t, y) \, dy \right)^{1/2}. \]

Thanks to (2.38) and (2.43), we deduce from the above estimate
\[ \int_{\mathbb{R}^3} j(t, v) |v| \, dv \to 0 \quad \text{when } t \to \infty. \] (3.48)
Gathering (3.48) and (3.47) allows us to conclude that \( j(t, v) \, dv \) tends to \( \delta_v \) in \( M^1(Y) \). □

4. The mass-dependent Boltzmann equation

In this section we focus on the sole Boltzmann equation for elastic collisions

\[
\frac{\partial f}{\partial t} = Q_B(f) \quad \text{on} \quad (0, \infty) \times Y, \quad f(0, \cdot) = f_{in} \quad \text{on} \quad Y, \tag{4.1}
\]

where \( Q_B \) is given by (1.4). We assume in the whole section (1.18), (1.21), (1.23), that \( a_G \equiv 0 \), \( a_S \equiv 0 \), and we consider an initial condition satisfying (1.28) with \( k = k_B \) defined in (1.29). We recall that \( \tilde{a}_B \) and \( \tilde{E}_B \) were defined in (1.24) and (1.23).

4.1. Uniqueness

We begin with a uniqueness result.

**Lemma 4.1.** Let us just assume that \( a_B \) and \( k \) are two nonnegative measurable functions on \( Y \times S^2 \) and \( Y \) respectively, such that the first symmetry condition on \( a_B \) in (1.3) holds and such that for any \( y, y_\ast \in Y \),

\[
0 \leq \tilde{a}_B(y, y_\ast) = \int_{S^2} a_B(y, y_\ast, v) \, dv \leq kk_\ast \quad \text{and} \quad k' + k'_\ast - k - k_\ast \leq 0. \tag{4.2}
\]

Then there exists at most one solution \( f \) to the mass dependent Boltzmann equation (4.1) such that for all \( T > 0 \), \( f \in C([0, T); L^1_k) \cap L^\infty([0, T); L^1_{k_\ast}) \).

Remark that under (1.18), one may choose \( k = C_A k_B \) (with \( C_A \) a constant) in the above lemma. The uniqueness part of Theorem 2.2 immediately follows.

**Proof.** We repeat the proof of Lemma 3.1. We multiply by \( \phi(t, y) = \text{sign}(f(t, y) - g(t, y))k \) the equation satisfied by \( f - g \). Using the weak formulations (2.16) and (2.1) of the Boltzmann equation and operator, we get for all \( t \geq 0 \),

\[
\frac{d}{dt} \int_Y |f - g| k \, dy = \frac{1}{2} \int_Y \int_{S^2} a_B((f - g)g_\ast + f(f_\ast - g_\ast))(\phi' + \phi'_\ast - \phi - \phi_\ast) \, dv \, dy_\ast dy
\]

\[
= \frac{1}{2} \int_Y \int_{S^2} a_B(f - g)(f_\ast + g_\ast)(\phi' + \phi'_\ast - \phi - \phi_\ast) \, dv \, dy_\ast dy
\]

\[
\leq \frac{1}{2} \int_Y \int_{S^2} a_B|f - g|(f_\ast + g_\ast)(k' + k'_\ast - k + k_\ast) \, dv \, dy_\ast dy.
\]
where we have just used the symmetry hypothesis (1.3) on $a_B$ and the substitution $(y, y_s) \rightarrow (y_s, y)$. Then, thanks to the bounds (4.2), we deduce
\[
\frac{d}{dt} \int_Y |f - g| k \, dy \leq \int_Y \int_Y kk^* |f - g|(f_s + g_s)k_s \, dy_s \, dy
\]
\[
= \|f + g\|_{L^1_k} \|f - g\|_{L^1_k},
\]
(4.3)
and we conclude as in the proof of Lemma 3.1. □

4.2. A priori estimates and existence

We begin by gathering some information satisfied (at least formally) by a solution to (4.1).

**Lemma 4.2.** A solution $f$ to the mass-dependent Boltzmann equation (4.1) conserves, at least formally, momentum, mass distribution and energy, (2.17), (2.18). In particular, if $f_{in}$ satisfies (2.21), then
\[
\forall t \geq 0, \quad f(t, y) = 0 \quad \text{for a.e. } p \in \mathbb{R}^3, m \in (0, m_0).
\]
(4.4)

**Proof.** This is an immediate consequence of (2.16) and (2.1) with the choices $\phi(y) = \phi(m), \phi(y) = p, \phi(y) = E$ (and $\beta(x) = x$). Moreover, we prove (4.4) making the choice $\phi(m) = 1_{0 \leq m \leq m_0}$ in the second identity of (2.17). □

We next give some estimates on $L^1$ norms with weight of the Boltzmann term $Q_B(f)$. It is based on a Povzner lemma, adapted to the mass-dependent case. We use here (and only here) the structure condition (1.21) on $a_B$.

**Lemma 4.3.** There exists a constant $C_A$, depending only on $A$ (see (1.18)), such that for any nonnegative measurable function $h$ on $Y$,
\[
\int_Y Q_B(h)E^2 \, dy \leq C_A \int_Y (m^{-1} + m + E)h \, dy \int_Y (m^{-2} + m^2 + E^2)h \, dy,
\]
(4.5)
\[
\int_Y Q_B(h)E^3 \, dy \leq C_A \int_Y (m^{-1} + m^2 + E^2)h \, dy \int_Y (m^{-3} + m^3 + E^3)h \, dy.
\]
(4.6)

**Proof.** We split the proof into several steps.

*Step 1. Preliminaries.* Writing the fundamental identity (2.1) with $\varphi = E^n$ for $n = 2$ or $3$, we get
\[
\int_Y Q_B(h)E^n \, dy = \int_Y hh_s K_n \, dy_s,
\]
(4.7)
where

\[ K_n := \int_{S^2} a_B(y, y_*, \cos \Theta) \left\{ (E')^n + (E_*)^n - E^n - E_*^n \right\} dv. \quad (4.8) \]

Here \( v' \) and \( v'_* \) are defined from \( v, v_* \) and \( v \) with the help of (1.5) and \( \Theta \) has been defined by (1.22). It is convenient to introduce another parameterization of post collisional velocities in order to make the computation more tractable. One easily deduces from (1.5) that, for any \( \nu \in S^2 \),

\begin{align*}
|v' - v_*| &= |\mu v - v_*|; \\
|v'_* - v_*| &= |\mu v - v_*|.
\end{align*}

(4.9)

We can then define the following alternative parameterization of \( v', v'_* \),

\begin{align*}
v' &= v_* + \mu [(v - v_*) + 2(v_* - v, v)\nu] = v_* + \mu w \sigma, \\
v'_* &= v_* - \mu [(v - v_*) + 2(v_* - v, v)\nu] = v_* - \mu w \sigma,
\end{align*}

(4.10)

with \( \sigma \in S^2 \). In other words, for any \( v \in S^2 \), we set

\[
\sigma = \frac{v - v_*}{w} + 2 \left( \frac{v_* - v}{w}, v \right) = (i_1 \cos \phi + i_2 \sin \phi) \sin \theta + \frac{v - v_*}{w} \cos \theta,
\]

(4.11)

and that indeed defines \( \sigma \in S^2 \) and next \( \phi \in [0, 2\pi], \theta \in [0, \pi] \). Here, \( (i_1, i_2, \frac{v - v_*}{w}) \) is the direct orthonormal basis of \( \mathbb{R}^3 \) such that \( (v_* , i_2) = 0 \).

Note also that \( \cos \Theta = \sin(\theta/2) \). Indeed, on one hand \( \Theta \in [0, \pi/2] \) is the angle between \( v' - v \) and \( v - v_* \) (or \( v'_* - v' \) and \( v_* - v_* \)). On the other hand, \( \theta \in [0, \pi] \) is that between \( v - v_* \) and \( v' - v_* \) (or between \( v_* - v_* \) and \( v'_* - v_* \)).

We now perform first the change of variables \( v \to \sigma \) in the integral expression (4.8) of \( K_n \), observing that \( d\sigma = 2\cos \Theta \, dv \) and next, the substitution \( \sigma \to (\theta, \phi) \), observing that \( d\sigma = \sin \theta \\, d\theta \\, d\phi \), and we obtain

\[
K_n(y, y_*) = \int_{S^2} a_B(y, y_*; \cos \Theta) \frac{1}{2 \cos \Theta} \left\{ (E')^n + (E'_*)^n - E^n - E_*^n \right\} d\sigma
\]

\[
= \int_0^\pi a_B(y, y_*; \sin(\theta/2)) \cos(\theta/2)
\]

\[
\times \left[ \int_0^{2\pi} \left\{ (E')^n + (E'_*)^n - E^n - E_*^n \right\} d\phi \right] d\theta,
\]

(4.12)

where now \( v' \) and \( v'_* \) are defined with the help of the new parameterization (4.10), (4.11).
Step 2. The Povzner lemma. Our aim is now to check that for any \( y, y^* \in Y \) and \( \theta \in [0, \pi] \),

\[
\frac{1}{4\pi} \int_0^{2\pi} \left\{ (E')^2 + (E')^2 - E^2 - E^2 \right\} d\varphi \\
\leq -\mu \mu^* \sin^2 \theta (E^2 + E^2) + 8\bar{\mu}|v||v^*|(E + E^*) + 26\mu \mu^* E E^*.
\] (4.13)

while

\[
\frac{1}{12\pi} \int_0^{2\pi} \left\{ (E')^3 + (E')^3 - E^3 - E^3 \right\} d\varphi \\
\leq 4m^2 \bar{\mu} |v|^5 |v^*| + 15m^2 \bar{\mu}^2 |v|^4 |v^*|^2 + 15m^2 \bar{\mu}^2 |v^*|^4 |v|^2 + 4m^2 \bar{\mu} |v^*|^5 |v|.
\] (4.14)

We will only prove (4.13), because the proof of (4.14) uses exactly the same arguments. In this whole step, we fix \( y \) and \( y^* \) and \( \theta \in [0, \pi] \), and we define \( \alpha \) to be the angle between the vectors \( v \) and \( v^* \). We also introduce the coordinates \((\xi, \eta, \zeta)\) in the orthonormal basis \( (\overrightarrow{\iota_1}, \overrightarrow{\iota_2}, (v-v^*)/w)\) of \( \mathbb{R}^3 \). Hence the coordinates of \( v, v^* \) and \( v^** \) are

\[
v^** = (\xi_0, 0, \zeta_0), \quad v = (\xi_0, 0, \zeta_0 + \mu w), \quad v^* = (\xi_0, 0, \zeta_0 - \mu w).
\] (4.15)

so that

\[
|v|^2 = |v^**|^2 + (\mu^* w)^2 + 2\xi_0 \mu^* w, \quad |v^*_a|^2 = |v^**|^2 + (\mu w)^2 - 2\xi_0 \mu w.
\] (4.15)

Since \( \sigma = (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta) \), we deduce from (4.10), (4.11)

\[
|v'|^2 = |v^**|^2 + (\mu^* w)^2 + 2\mu^* w (\xi_0 \cos \theta + \xi_0 \sin \theta \cos \varphi), \\
|v^*_a'|^2 = |v^**|^2 + (\mu w)^2 - 2\mu w (\xi_0 \cos \theta + \xi_0 \sin \theta \cos \varphi).
\] (4.16)

Gathering (4.15) and (4.16), we get

\[
|v'|^2 = |v|^2 + 2\mu^* (w\xi_0) (\cos \theta - 1) + 2\mu^* (w\xi_0) \sin \theta \cos \varphi, \\
|v^*_a'|^2 = |v^*_a|^2 - 2\mu (w\xi_0) (\cos \theta - 1) - 2\mu (w\xi_0) \sin \theta \cos \varphi.
\]

and then

\[
E' = E + \bar{\mu} (2w\xi_0) (\cos \theta - 1) + 2\bar{\mu} (w\xi_0) \sin \theta \cos \varphi =: E + A, \\
E'_a = E_a - \bar{\mu} (2w\xi_0) (\cos \theta - 1) - 2\bar{\mu} (w\xi_0) \sin \theta \cos \varphi =: E_a - A.
\] (4.17)

We remark that
\[ w_{\xi_0} = |v||v_\ast| \sin \alpha, \quad (4.18) \]
since both quantities equal twice the area of the triangle \((Ovv_\ast)\), and
\[ w_{\zeta_0} = \frac{1}{2} (|v|^2 - |v_\ast|^2 + (\mu^2 - \mu_\ast^2)w^2) \]
\[ = \mu |v|^2 - \mu_\ast |v_\ast|^2 - (\mu - \mu_\ast)|v||v_\ast| \cos \alpha, \quad (4.19) \]
since \(\mu^2 - \mu_\ast^2 = (\mu + \mu_\ast)(\mu - \mu_\ast) = \mu - \mu_\ast, 1 + \mu - \mu_\ast = 2\mu, 1 + \mu_\ast - \mu = 2\mu_\ast\), and \(\langle v, v_\ast \rangle = |v||v_\ast| \cos \alpha\). We also remark that, with the notation introduced in (4.17),
\[ (E_\ast'')^2 + (E')^2 - E^2 - E_\ast^2 = 2A^2 + 2(E - E_\ast)A. \quad (4.20) \]
Using that \(\int_0^{2\pi} 2 \cos^2 \phi d\phi = \int_0^{2\pi} d\phi = 2\pi\) while \(\int_0^{2\pi} \cos \phi d\phi = 0\), we obtain
\[ \int_0^{2\pi} \left\{(E')^2 + (E_\ast')^2 - E^2 - E_\ast^2\right\} \frac{d\phi}{4\pi} = 4(\bar{\mu})(w_{\xi_0})^2(1 - \cos \theta)^2 \]
\[ - 2\bar{\mu}(w_{\xi_0})(E - E_\ast)(1 - \cos \theta) + 2\bar{\mu}^2(w_{\xi_0})^2 \sin^2 \theta. \quad (4.21) \]
Using now (4.18) and (4.19), we deduce that
\[ \int_0^{2\pi} \left\{(E')^2 + (E_\ast')^2 - E^2 - E_\ast^2\right\} \frac{d\phi}{4\pi} = S(y, y_\ast, \theta) + S(y_\ast, y, \theta), \quad (4.22) \]
where
\[ S(y, y_\ast, \theta) = -2|v|^4 \mu \bar{\mu}(1 - \cos \theta)\left[1 - 2\mu_\ast(1 - \cos \theta)\right] \]
\[ + 2|v|^3 |v_\ast| m \bar{\mu} (\mu - \mu_\ast)(1 - \cos \theta) \cos \alpha \left[1 - 4\mu_\ast(1 - \cos \theta)\right] \]
\[ + |v|^2 |v_\ast|^2 \bar{\mu}^2 [2(1 - \cos \theta) + \sin^2 \theta \sin^2 \alpha] \]
\[ + 2(1 - \cos \theta)^2 \left[(\mu - \mu_\ast)^2 \cos \alpha - 2\mu_\ast \right]. \quad (4.23) \]
We observe that \(4\mu_\ast \leq 1\), so that
\[ \left[1 - 2\mu_\ast(1 - \cos \theta)\right] \geq 1 - \frac{1}{2}(1 - \cos \theta) = \frac{1}{2}(1 + \cos \theta). \quad (4.24) \]
We thus deduce that
\[ S(y, y_\ast, \theta) \leq -|v|^4 \mu \bar{\mu}(1 - \cos \theta)(1 + \cos \theta) \]
\[ + 8|v|^3 |v_\ast| m \bar{\mu} + 13|v|^2 |v_\ast|^2 \bar{\mu}^2. \quad (4.25) \]
Finally, (4.13) follows from (5.8) and (4.22).

**Step 3. Conclusion.** First note that with our new parameterization (see Step 1),

\[
\begin{align*}
\bar{a}_B(y, y^*) &= \int_{S_2} a_B(y, y^*, \cos \Theta) \, dv \\
&= \int_0^{\pi} d\theta \int_0^{2\pi} d\varphi \, a_B(y, y^*, \sin(\theta/2) \cos(\theta/2)) \\
&= 2\pi \int_0^{\pi} a_B(y, y^*, \sin(\theta/2)) \cos(\theta/2) \, d\theta.
\end{align*}
\]

Gathering now (4.13) (where we neglect the nonpositive term), (4.12), and using the bound (1.18), we deduce that

\[
K_2(y, y^*) \leq C_A \left( 1 + m + m_* \right) \left( 1 + \|v\| + \|v^*\| \right) \\
\times \left\{ \frac{m^2 m_*^2}{(m + m_*)^2} |v|^2 |v^*_|^2 + \frac{m^2 m_*^2}{m + m_*} |v| |v^*| |v^*_| + \frac{mm_*^2}{m + m_*} |v||v^*|^3 \right\},
\]

(4.26)

for some the constant $C_A$ depending only on $A$. Recalling (4.7), a tedious but straightforward computation allows us to conclude that (4.5) holds, using essentially some symmetry arguments and the facts that $mm_*/(m + m_*) \leq m$ and $mm_*/(m + m_*) \leq m_*$.

We omit the proof of (4.6), since it uses the same arguments, making use of (4.14) instead of (4.13). \qed

As an immediate consequence of Lemmas 4.2 and 4.3, we obtain the following a priori bounds for the solutions of (4.1). We omit the proof since it follows the same line as that of Corollary 3.5.

**Corollary 4.4.** A solution $f$ to (4.1) satisfies, at least formally, for any $T$,

\[
\text{for } z = 2 \text{ and } 3, \quad f_{\text{in}} k_B^z \in L^1(Y) \implies \sup_{[0,T]} \int_Y f(t, y) k_B^z \, dy \leq C_{T,z},
\]

(4.27)

where the constant $C_{T,z}$ depends only on $T$, $\|f_{\text{in}}\|_{L^1_k}$, and on $A$.

**Proof of Theorem 2.2.** It follows line by line Section 3.3. It suffices to use of Corollary 4.4 instead of Corollary 3.5, and to use the computation of Lemma 4.1 instead of that of Lemma 3.1. \qed
4.3. Long time behavior

The aim of this subsection is to prove Theorem 2.3. We start giving a more accurate version of the first estimate on the weight integral of the collision term $Q_B$ stated in Lemma 4.2.

Lemma 4.5. In addition to the current assumptions on $a_B$, suppose the structure hypothesis (2.19), and fix $m_0 > 0$ and $m_1 > 0$. For any measurable function $h : Y \to \mathbb{R}^+$ such that $h = 0$ for a.e. $m \in (0, m_0)$, $p \in \mathbb{R}^3$ and $\int_Y 1_{[m_0 < m < m_1]} h(y) \, dy \geq \kappa_1 > 0$, there holds

$$\int_Y Q_B(h) E^2 \, dy \leq C_1 - C_2 \int_Y h E^2 \, dy,$$

(4.28)

where $C_1, C_2$ are positive constants depending only on $a_B$, on $\int_Y h(1 + m^2 + m^6 - 4\delta + E) \, dy$ and on $\kappa_1, m_0, m_1$.

Proof. In the whole proof, an adapted constant is a constant depending only on the quantities allowed in the statement. Its value of it may change from one line to another.

We come back to the proof of Lemma 4.3. Using (4.7) with $n = 2$, taking into account the negative contribution in the Povzner inequality (4.13), and using finally (2.19), we easily obtain, for some positive constant $C_A$ depending only on $A$ (see (1.18)),

$$\int_Y Q_B(h) E^2 \, dy \leq C_A \int_Y (m^{-1} + m + E) h \, dy \int_Y (m^{-2} + m^2 + E^2) h \, dy$$

$$- \frac{\pi^2}{2} \int_0^{\theta} \sin^2 \theta \cos(\theta/2) \psi(\theta) \, d\theta \int_Y \mu \mu_r \mathcal{E}^2 (mm_r) \delta |v - v_*| h h_* \, dy \, dy_*$$

$$=: I_1 - I_2.$$

(4.29)

First, one easily obtains the existence of an adapted constant $C$ such that

$$I_1 \leq C \left\{ 1 + \int_Y E^2 h \, dy \right\}.$$

(4.30)

Next, since $|v - v_*| \geq |v| - |v_*|$, we get, for some adapted constants $C > 0, c > 0$,\n
$$I_2 \geq c \int_Y \frac{m^{3+\delta} m_1^{1+\delta}}{(m + m_*)^2} |v|^5 h h_* \, dy \, dy_* - C \int_Y \frac{m^{3+\delta} m_1^{1+\delta}}{(m + m_*)^2} |v|^4 |v_*| h h_* \, dy \, dy_*$$

$$=: J_1 - J_2.$$
\[ J_2 \leq C \| \varepsilon^2 h \|_y \int m_+^2 |v_+| h_+ dy_+ \]
\[ \leq C \int (m_0^{-1} + m + \varepsilon^2) h \, dy \int \varepsilon^2 h \, dy = C \int \varepsilon^2 h \, dy. \] (4.32)

Since for all \( m > m_0 \), all \( m_0 < m_+ < m_1 \), \( m + m_+ \leq m(1 + m_1/m_0) \),
\[ J_1 \geq \frac{c}{(1 + m_1/m_0)^2} \int m_+^{1+\delta} |v|^5 h \, dy \int m_+^{1+\delta} 1_{m_0 < m_+ < m_1} h_+ dy_+ \]
\[ \geq c \int m_+^{1+\delta} |v|^5 h \, dy. \] (4.33)

Finally, by the Young inequality, we deduce that for all \( \varepsilon > 0 \),
\[ m^2 |v|^4 \leq \varepsilon^{4/5} m_+^{1+\delta} |v|^5 + \frac{1}{\varepsilon^3} m^{6-4\delta}, \] (4.34)
so that
\[ J_1 \geq \frac{c}{\varepsilon^{4/5}} \int \varepsilon^2 h \, dy - \frac{c}{\varepsilon^{3+4/5}} \int m^{6-4\delta} h \, dy. \] (4.35)

Gathering all the above inequalities and choosing \( \varepsilon \) small enough allows us to conclude the proof. \( \square \)

In the following statement, we gather all the estimates we are able to obtain for the solution \( f \) to the Boltzmann equation and which are relevant to study the long time asymptotic.

**Lemma 4.6.** Under the hypothesis of Theorem 2.3, the solution \( f \) to the Boltzmann equation (4.1) associated to \( f_0 \) satisfies (2.22) and
\[ \sup_{t \geq 0} \int f(t, y) (m^{-1} + m + \varepsilon^2) \, dy < \infty, \] (4.36)
from which we deduce
\[ \int_0^\infty D_h, B(f(t, \cdot)) \, dt < \infty. \] (4.37)
Proof. We start proving that (2.22) holds. We just sketch the proof, and we refer to [2] for details. We first consider the solution $f_\ell$ to the Boltzmann equation (4.1) with rate $a_\ell = a_B \wedge \ell$ and initial condition $f_\ell^0 = f_{in} + \ell^{-1} M$ where $M$ is the Maxwellian (2.24). On the truncated equation we may prove that for some constant $C_\ell$, $f_\ell(t,. ) \geq \ell^{-1} M \exp(-C_\ell T)$ for any $t \in (0, T)$. Thus $|\log f_\ell| \leq C_\ell T$ on $[0, T]$, and we may choose $\beta(x) = h(x) = x \log x$ in (2.16), and deduce the following strong H-theorem

$$H (f_\ell(t,. ) + \int_0^t \int \int S^2 \frac{a_B \wedge \ell}{4} (f^\ell_\ell f^\ell_\ell - f_\ell f_\ell) \log \frac{f^\ell_\ell f^\ell_\ell}{f_\ell f_\ell} \, dy \, dy \, dv \, ds = H (f_{in})$$

(4.38)

Passing to the limit when $\ell \to \infty$ we get that the resulting limit $f$ satisfies the weak version (2.22) of the H-theorem.

Next, we recall that from Lemma 4.2 we have yet $(m^{-1} + m^2 + \mathcal{E}) f \in L^\infty([0, \infty), L^1)$ and that $f$ satisfies (4.4). Using the conservation of mass distribution and of energy (see (2.17) and (2.18)), we deduce that one may apply Lemma 4.6 to $h = f(t,. )$, the constants $C_1 > 0$ and $C_2 > 0$ being time-independent. Applying (2.16) with $\phi = \mathcal{E}^2$ (and $\beta(x) = x$) and using (4.28), we deduce that

$$\frac{d}{dt} \int \mathcal{E}^2 f \, dy \leq C_1 - C_2 \int \mathcal{E}^2 f \, dy,$$

(4.39)

from which (4.36) follows. We finally prove (4.37). From the weak H-theorem (2.22), we have for any $T > 0$

$$\int_0^T D_{h,B} (f(t,. ) ) \, dt \leq H (f_{in}) - H (f(T,. ) ) \leq H (f_{in}) - \int \int f(T,y) \ln f(T,y) \mathbf{1}_{f(T,y) \leq 1} \, dy.$$

(4.40)

It thus suffices to check that $-\int f(T,y) \ln f(T,y) \mathbf{1}_{f(T,y) \leq 1} \, dy$ is bounded by a constant not depending on $T$. The set $\{ f(T,y) \leq 1 \}$ may be decomposed into two parts, namely $\{ f(T,y) \leq 1 \} = \{ f(T,y) \leq \exp(-2/m - 2m - 2\mathcal{E}) \} \cup \{ \exp(-2/m - 2m - 2\mathcal{E}) \leq f(T,y) \leq 1 \}$. Using the elementary inequality $-s \ln s \leq 4\sqrt{s}$ on $[0, 1]$ for the first subset and just that $s \mapsto -\ln s$ is a decreasing function for the second subset, we obtain

$$-\int \int f(T,y) \ln f(T,y) \mathbf{1}_{f(T,y) \leq 1} \, dy$$
\[
\leq - \int_Y f(T, y) \ln f(T, y) 1_{f(T, y) \leq e^{-2/m - 2E}} dy
\]
\[
- \int_Y f(T, y) \ln f(T, y) 1_{e^{-2/m - 2E} \leq f(T, y) \leq 1} dy
\]
\[
\leq 4 \int_Y e^{-1/m - m - E} dy + \int_Y f(T, y) \left( \frac{2}{m} + 2m + 2E \right) dy. \tag{4.41}
\]
We conclude gathering (4.36), (4.40) and (4.41).

We end the preliminary steps for the proof of Theorem 2.3 by the following functional characterization of Maxwellian functions.

**Lemma 4.7.** (1) Consider a nonnegative function \( g \in L^1(\mathbb{R}^3; (1 + |v|^2) dv) \) such that

\[
g' g^* = gg^* \quad \text{for a.e. } v, v^* \in \mathbb{R}^3, v \in S^2, \tag{4.42}
\]
where \( v' \) and \( v^* \) are defined by (1.5) with \( m = m_* \). Then \( g \) is a Maxwellian, i.e., there exist some constants \( v_0 \in \mathbb{R}^3 \), \( \sigma \in (0, \infty) \), and \( \gamma \in [0, \infty) \) such that for a.e. \( v \in \mathbb{R}^3 \),

\[
g(v) = \gamma \left( \frac{2}{2\pi\sigma^2} \right)^{3/2} e^{-\frac{|v-v_0|^2}{2\sigma^2}}. \tag{4.43}
\]

(2) Consider a nonnegative function \( f \in L^1(Y; (1 + E) dy) \) such that

\[
f' f^* = ff^* \quad \text{for a.e. } y, y^* \in Y, v \in S^2, \tag{4.44}
\]
where \( y' = (m, mv') \) and \( y^* = (m_*, m_* v^*_{y'}) \) with \( v' \) and \( v^* \) defined by (1.5). Then \( f \) is a mass-dependent Maxwellian, i.e., there exist a function \( 0 \leq \gamma \in L^1((0, \infty)) \), a constant \( v_0 \in \mathbb{R}^3 \), and a constant \( \sigma \in (0, \infty) \) such that for a.e. \( y \in Y \),

\[
f(y) = \frac{\gamma(m)}{(2\pi m\sigma)^{3/2}} e^{-\frac{|p-m v_0|^2}{2\sigma^2}}. \tag{4.45}
\]

**Proof.** Since the proof of (1) is classical, see [17,38,2], we just present the proof of (2). Let us set \( \Omega = \{ m, f f(m, p) \, dp > 0 \} \). Note that for \( m \notin \Omega \), (4.45) holds, choosing \( \gamma(m) = 0 \). The functional equation (4.44) with \( m = m_* \in \Omega \) and (4.43) imply that for any \( m \in \Omega \), there exist \( \tilde{\gamma}(m), \tilde{\sigma}(m) \) and \( \tilde{v}_0(m) \) such that

\[
f(m, m v) = \frac{\tilde{\gamma}(m)}{(2\pi \tilde{\sigma}(m))^{3/2}} e^{-\frac{|p-m_v|^2}{2\tilde{\sigma}(m)^2}}.
\]
which can be written, using other functions \( \gamma(m), \sigma(m) \) and \( v_0(m) \),

\[
f(m, p) = \frac{\gamma(m)}{(2\pi m \sigma(m))^{3/2}} e^{-\frac{|p-m_v|^2}{2\sigma(m)^2}}.
\]
Using (4.44) with \( v_* = 0 \) and using the parameterization (1.5) for \( v', v_*' \), we deduce that for all \( \nu \in S^2 \),
\[
|p - mv_0(m)|^2 + |m_*v_0(m_*)|^2 = \frac{|p - mv_0(m) - 2\overline{\mu}(v, v)v|^2}{m\sigma(m)} + \frac{|2\overline{\mu}(v, v)v - m_*v_0(m_*)|^2}{m_*\sigma(m_*)}.
\]

A straightforward computation shows that
\[
4(\nu, v)^2\overline{\mu}\left(-\frac{1}{\sigma(m)} + \frac{1}{\sigma(m_*)}\right) + 4(\nu, v)\overline{\mu}\left\{\frac{v_0(m)}{\sigma(m)} - \frac{v_0(m_*)}{\sigma(m_*)}, \nu \right\} = 0.
\]

We thus deduce that \( \sigma(m) = \sigma(m_*) \) and then that \( v_0(m) = v_0(m_*) \) for a.e. \( m, m_* \in \mathcal{O} \). This implies that \( v_0 \) and \( \sigma \) are constant on \( \mathcal{O} \).

We are now able to present the

**Proof of Theorem 2.3.** Let us consider an increasing sequence \( (t_n)_{n \geq 1} \), \( t_n \to \infty \), and put \( f_n(t, .) := f(t + t_n, .) \) for \( t \in [0, T] \) and \( n \geq 1 \). We realize, from (2.17), (2.22) and (4.36), that for all \( T \),
\[
f_n \text{ is bounded in } L^\infty([0, T); L^1_{k_B} \cap L^1 \log L^1)
\]
and then, using the fact that \( f \) solves the Boltzmann equation, that (4.1),
\[
\int_Y f_n(t, y)\psi(y) \, dy \text{ is bounded in } BV(0, T)
\]
for any \( \psi \in L^\infty(Y) \). Therefore, up to the extraction of a subsequence, there exists \( \Gamma \in C([0, T); L^1_{k_B}) \) such that
\[
f_n \rightharpoonup \Gamma \text{ weakly in } L^1((0, T) \times Y) \quad \text{and} \quad \int_Y f_n(t, y)\psi(y) \, dy \to \int_Y \Gamma(t, y)\psi(y) \, dy \text{ in } C([0, T)), \quad (4.46)
\]
for any function \( \psi \) on \( Y \) such that \( |\psi|k_B^{-1} \in L^\infty(Y) \). On the one hand, the above convergence is strong enough in order to pass to the semi-inferior limit as follows
\[
\int_0^T D_{h,B}(\Gamma(t, .)) \, dt \leq \liminf_{n \to \infty} \int_0^T D_{h,B}(f_n(t, .)) \, dt \leq \lim_{n \to \infty} \int_0^\infty D_{h,B}(f(s, .)) \, ds = 0,
\]
where, for the last equality, we have used (4.37). We deduce that for each \( t \), \( \Gamma(t,. \) ) satisfies the functional relation (4.44) and then, Lemma 4.7 implies that \( \Gamma(t,. \) ) is a mass-dependent Maxwellian function.

On the other hand, from (4.46) and (2.17) we have

\[
\int_Y \Gamma(t,y)\phi(m) \, dy = \int_Y f_m \phi(m) \, dy,
\]

for any bounded measurable function \( \phi : (0, \infty) \mapsto \mathbb{R} \), so that, with the notation of (2.25),

\[
\int_{\mathbb{R}^3} \Gamma(t,y) \, dp = \rho(m) \quad \forall m > 0 \text{ and } \forall t \in (0,T).
\]

We also have, from (4.46), (2.17) and (2.18), that for any \( t \in (0,T) \)

\[
\int_Y \Gamma(t,y) \rho \, dy = \int_Y f_m \rho \, dy = 0, \quad \int_Y \Gamma(t,y) \epsilon \, dy = \int_Y f_m \epsilon \, dy = 3 \left( \int_0^\infty \rho(m) \, dm \right) \Sigma.
\]

One easily concludes that for all \( t \), \( \Gamma(t) = M \), where \( M \) is the Maxwellian defined in (2.24). Hence the \( \Gamma \) does not depend on \( t \) nor on the sequence \( (t_n) \), so that we may conclude (2.23).

\[
\square
\]

5. The mass-dependent Granular media equation

In this section we sketch the proof of Theorems 2.4 and 2.5. The main difficulty, compared to the Boltzmann elastic equation, is to extend the Povzner inequality to this inelastic context. We thus consider the inelastic Boltzmann equation

\[
\frac{d}{dt} f = Q_G(f) \text{ on } (0, \infty) \times Y, \quad f(0,. \) = f_m \text{ on } Y, \quad (5.1)
\]

where \( Q_G \) is given by (1.10). We assume in the whole section that \( a_S = 0, a_B = 0, (1.18), (1.21), (1.23) \), and we consider an initial condition satisfying (1.28) with \( k = k_B \) defined in (1.29). We recall that \( a_G \) and \( E_G \) were defined in (1.24) and (1.23).

All the statements we present below also hold replacing \( Q_G(f) \) by \( Q_B(f) + Q_G(f) \) in the RHS of (5.1). This follows without difficulty gathering the arguments of the preceding section with those introduced below.

5.1. Uniqueness

We start with uniqueness.
**Lemma 5.1.** Assume that $a_G$ and $k$ are two nonnegative measurable functions on $Y^2 \times S^2 \times (0, 1)$ and $Y$ respectively, such that the symmetry condition (1.9) holds, and such that for any $y, y_\ast \in Y$, any $\nu \in S^2$, $e \in (0, 1)$,

\[
0 \leq \tilde{a}_G = \int_{S^2} \int_0^1 a_G \, de \, dv \leq kk_\ast \quad \text{and} \quad k'' + k''_\ast - k - k_\ast \leq 0.
\]  

(5.2)

Then there exists at most one solution $f$ to (5.1) such that for all $T > 0$, $f \in C([0, T); L^1_k) \cap L^\infty([0, T); L^2_k)$.

The proof is a fair copy of that of Lemma 4.1. Note that here again, one may choose $k = k_B$ defined in (1.29) under (1.18), so that the uniqueness part of Theorem 2.4 follows.

### 5.2. A priori estimates and existence

We next state the conservations for such an equation.

**Lemma 5.2.** A solution $f$ to the mass-dependent Granular equation (5.1) conserves, at least formally, momentum, mass distribution (2.17) while kinetic energy decreases (2.26).

The proof is again identical to the corresponding result (Lemma 4.2) for equation (4.1).

We next present some a priori bounds for the Granular operator, which rely on a Povzner lemma.

**Lemma 5.3.** There exists a constant $C_A$, depending only on $A$ (see (1.18)), such that for any nonnegative measurable function $h$ on $Y$,

\[
\int_Y Q_G(h)E^2 \, dy \leq C_A \int_Y (m^{-1} + m + E^2)h \, dy \int_Y (m^{-2} + m^2 + E^2)h \, dy, \tag{5.3}
\]

\[
\int_Y Q_G(h)E^3 \, dy \leq C_A \int_Y (m^{-1} + m^2 + E^2)h \, dy \int_Y (m^{-3} + m^3 + E^3)h \, dy. \tag{5.4}
\]

**Proof.** We follow here the line of the proof of Lemma 4.3. We will only check (5.3), the other case being treated similarly. We split the proof in several steps.

**Step 1. Preliminaries.** Using (2.2) with $\phi(y) = E^2$, we deduce that

\[
\int_Y Q_G(h)E^2 \, dy = \int_Y \int_Y hh_\ast \int_0^1 K_2 \, dy \, dy_\ast \, de, \tag{5.5}
\]
\[ K_2(y, y_*, e) := \int_{S^2} a G(y, y_*, \cos \Theta, e) \left\{ (E'')^2 + (E'')^2 - E^2 - E^2 \right\} \, d\nu. \quad (5.6) \]

Here \( v'' \) and \( v''_* \) are defined from \( v, v_* \) and \( \nu, e \) with the help of (1.7) and \( \Theta \) has been defined by (1.22). We introduce a new parameterization of post collisional velocities, in the same spirit as in the elastic case. An easy computation shows that one may write

\[ v'' = v^* + \lambda \mu w \sigma, \quad v''_* = v^* - \lambda \mu w \sigma, \quad (5.7) \]

with

\[ \lambda = \left(1 - (1 - e^2) \cos^2 \Theta \right)^{1/2} \in (0, 1), \quad \sigma = \lambda^{-1} \left(\frac{v - v_*}{w} + (1 + e) \cos \Theta v\right) \in S^2. \quad (5.8) \]

Considering the direct orthonormal basis \((\mathbf{i}_1, \mathbf{i}_2, \frac{v - v_*}{w})\) of \(\mathbb{R}^3\) such that \(\langle v^*_*, \mathbf{i}_2 \rangle = 0\), we may parameterize \(\sigma\) in this basis, writing

\[ \sigma = (\mathbf{i}_1 \cos \phi + \mathbf{i}_2 \sin \phi) \sin \theta + \frac{v - v_*}{w} \cos \theta. \quad (5.9) \]

Performing the substitution \(v \mapsto (\theta, \phi)\), we get

\[ K_2(y, y_*, e) = \int_0^\pi \int_0^{2\pi} a G(y, y_*, \cos \Theta, e) \frac{\lambda \sin \theta \, d\theta}{(1 + e) \cos \Theta} \, d\phi \left\{ (E'')^2 + (E'')^2 - E^2 - E^2 \right\} \]

\[ (5.10) \]

where now \( v'' \) and \( v''_* \) are defined by (5.7).

**Step 2. The Povzner lemma.** As in the proof of Lemma 4.3, we denote by \(v^* = (\xi_0, 0, \xi_0)\) the coordinates of \(v^*\) in the basis \((\mathbf{i}_1, \mathbf{i}_2, \frac{v - v_*}{w})\). Then one easily checks that

\[ v = (\xi_0, 0, \xi_0 + \mu w), \quad v_* = (\xi_0, 0, \xi_0 - \mu w), \]

\[ v'' = (\xi_0 + \lambda \mu w \cos \phi \sin \theta, \lambda \mu w \sin \phi \sin \theta, \xi_0 + \lambda \mu w \cos \theta), \]

\[ v''_* = (\xi_0 - \lambda \mu w \cos \phi \sin \theta, -\lambda \mu w \sin \phi \sin \theta, \xi_0 - \lambda \mu w \cos \theta). \quad (5.11) \]

Then, a straightforward computation shows that

\[ E'' = E - \mu B + A, \quad E''_* = E_* - \mu B - A, \]

where, \(\alpha\) standing the angle between \(v\) and \(v_*\), and noting that (4.18) and (4.19) still hold,
\[ B := (1 - \lambda^2) \bar{\mu} w^2 = (1 - \lambda^2) \bar{\mu} [|v|^2 + |v_u|^2 - 2|v||v_u| \cos \alpha], \quad (5.12) \]

and

\[ A := -2\bar{\mu}(1 - \lambda \cos \theta)(w \xi_0) + 2\bar{\mu}\lambda(w \xi_0) \cos \phi \sin \theta \\
= -2\bar{\mu}[\mu |v|^2 - \mu_u |v_u|^2 - (\mu - \mu_u)|v||v_u| \cos \alpha](1 - \lambda \cos \theta) \\
+ 2\lambda \bar{\mu}|v||v_u| \sin \alpha \sin \theta \cos \phi. \quad (5.13) \]

We deduce that

\[ (E'^2) + (E'')^2 - \mathcal{E}^2 = B^2(\mu^2 + \mu_u^2) - 2B(\mu \mathcal{E} + \mu \mathcal{E}_u) \\
+ 2A^2 + 2A(\mathcal{E} - \mu_u B - \mathcal{E}_u + \mu B). \]

A tedious computation using that \[ \int_0^{2\pi} \cos \phi \, d\phi = 0, \text{ while } \int_0^{2\pi} \cos^2 \phi \, d\phi = \pi \] shows that

\[ \frac{1}{2\pi} \int_0^{2\pi} ((E'^2) + (E'')^2 - \mathcal{E}^2) \, d\phi = S(y, y_u, \theta, \lambda) + S(y_u, y, \theta, \lambda) \quad (5.14) \]

where

\[ S(y, y_u, \theta, \lambda) = \alpha_1 |v|^4 + \alpha_2 |v|^4 |v_u|^2 + \alpha_3 |v|^2 |v_u|^2, \quad (5.15) \]

with

\[ \alpha_1 = \bar{\mu}^2(\mu^2 + \mu_u^2)(1 - \lambda^2)^2 + 8\bar{\mu}^2(1 - \lambda \cos \theta)^2 \mu^2 - 2\bar{\mu}^2(1 - \lambda^2) \\
- 4\bar{\mu}m \mu (1 - \lambda \cos \theta) - 4\bar{\mu}^2 \mu (1 - \lambda \cos \theta)(\mu - \mu_u)(1 - \lambda^2), \quad (5.16) \]

\[ \alpha_2 = -4\bar{\mu}^2(\mu^2 + \mu_u^2)(1 - \lambda^2)^2 \cos \alpha - 16\bar{\mu}^2(1 - \lambda \cos \theta)^2 \mu(\mu - \mu_u) \cos \alpha \\
+ 4\bar{\mu}^2(1 - \lambda^2) \cos \alpha + 8\bar{\mu}^2(1 - \lambda \cos \theta)\mu(\mu - \mu_u)(1 - \lambda^2) \cos \alpha \\
+ 4\bar{\mu}m(1 - \lambda \cos \theta)(\mu - \mu_u) \cos \alpha \\
+ 4\bar{\mu}^2(1 - \lambda \cos \theta)(\mu - \mu_u)^2(1 - \lambda^2), \quad (5.17) \]

\[ \alpha_3 = \bar{\mu}^2(\mu^2 + \mu_u^2)(1 - \lambda^2)^2 [1 + 2 \cos^2 \alpha] + 2\bar{\mu}^2 \lambda^2 \sin^2 \alpha \sin^2 \theta \\
+ 4\bar{\mu}^2(1 - \lambda \cos \theta)^2 [2\mu \mu_u + (\mu - \mu_u)^2 \cos^2 \alpha] - 2\bar{\mu}^2(1 - \lambda^2) \\
- 2\bar{\mu}(1 - \lambda \cos \theta)[-2\bar{\mu} + \bar{\mu}\mu_u(\mu - \mu_u)(1 - \lambda^2) + \mu_u \bar{\mu}(\mu - \mu_u)(1 - \lambda^2) \\
+ 2\bar{\mu}(\mu - \mu_u)^2 \cos^2 \alpha(1 - \lambda^2)]. \quad (5.18) \]
First, a straightforward computation shows that

\[ \alpha_2 \leq 104m\bar{\mu} \quad \text{and} \quad \alpha_3 \leq 57\bar{\mu}^2. \]  

(5.19)

Next, using that \((1 - \lambda^2)^2 - 2(1 - \lambda^2) \leq 0\), and that \(|\cos \theta| \leq 1\), we get

\[ \alpha_1 \leq 4 \frac{\mu\bar{\mu}}{(m + m_\ast)^2} (1 - \lambda\cos \theta) P(\lambda), \]  

(5.20)

with \(P(\lambda) := \frac{m_\ast \lambda^2(m - m_\ast) + 2\lambda mm_\ast - m(m + m_\ast)}{m_\ast m - m}\). But \(P\) is nonpositive on \([0, 1]\).

Indeed, \(P'(\lambda) = 0\) only for \(\lambda = \lambda_0 := \frac{m}{m_\ast - m}\). Thus if \(m_\ast < 2m\), \(P'\) does not vanish on \([0, 1]\), so that \(P(\lambda) \leq \max\{P(0), P(1)\} = \max\{-\frac{m(m + m_\ast), -(m - m_\ast)^2}{m_\ast m - m}\} \leq 0\).

Next if \(m_\ast \geq 2m\), then \(P(\lambda) \leq P(\lambda_0) = \frac{mm_\ast^2}{m_\ast m - m} (\frac{m}{m(m_\ast - m)} + (\frac{m}{m_\ast}) - 1) \leq 0\) since \(m/m_\ast \in [0, 1/2]\).

We finally deduce that \(\alpha_1\) is nonpositive, so that

\[ \frac{1}{2\pi} \int_0^{2\pi} (\mathcal{E}_\ast'^\prime)' + (\mathcal{E}_\ast'^\prime)^2 - \mathcal{E}_\ast'^2 \, d\phi \]  

\[ \leq 104m\bar{\mu}|v|^3|v_\ast| + 114\bar{\mu}^2|v|^2|v_\ast|^2 + 104m_\ast\bar{\mu}|v_\ast|^3|v| \]  

\[ \leq 104\bar{\mu}|v_\ast|v_\ast|\mathcal{E}_\ast + 114\mu_\ast\mathcal{E}\mathcal{E}_\ast. \]  

(5.21)

**Step 3. Conclusion.** First note that, with the notation of Step 1,

\[ \tilde{a}_G(y, y_\ast) = \frac{1}{2\pi} \int_0^{2\pi} \int_0^\pi \int_0^\pi \frac{\lambda \sin \theta \, d\theta}{(1 + e) \cos \theta} \, d\phi. \]  

(5.22)

Thus, gathering (1.18), (5.21), and (5.10), we get

\[ \int_0^1 K_2(y, y_\ast, e) \, de \leq C_A(1 + m + m_\ast)(1 + |v| + |v_\ast|)(\bar{\mu}|v||v_\ast|\mathcal{E}_\ast + \mu_\ast\mathcal{E}\mathcal{E}_\ast) + \mu_\ast\mathcal{E}\mathcal{E}_\ast. \]  

(5.23)

As in the proof of Lemma 4.3, a tedious computation using finally (5.5) allows us to conclude that (5.3) holds. \(\square\)

As an immediate consequence of Lemmas 5.2 and 5.3, we obtain the following a priori bounds on the solutions of (5.1). We omit the proof since it follows the same line as that of Corollary 3.5.
Corollary 5.4. A solution $f$ to (5.1) satisfies, at least formally, for any $T$,

$$\text{for } z = 2 \text{ and } 3 \text{ } f_m k_B^z \in L^1(Y) \text{ implies } \sup_{[0,T]} \int_Y f(t,y) k_B^z dy \leq C_{T,z},$$

(5.24)

where the constant $C_{T,z}$ depends only on $T$, $\|f_m\|_{L^1(k_B^z)}$, and on $A$.

Proof. It follows line by line Section 3.3. It suffices to use of Corollary 5.4 instead of Corollary 3.5, and to use the computation of Lemma 5.1 instead of that of Lemma 3.1.

5.3. Long time behavior

We finally give the

Proof of Theorem 2.5. We split the proof into three parts.

Proof of (2.27). It is based on the dissipation of kinetic energy. Let us consider an increasing sequence $(t_n)_{n \geq 1}$, $t_n \to \infty$, and put $f_n(t,.) := f(t + t_n,.)$ for $t \in [0,T]$, and $n \geq 1$. We then proceed along the line of the proof of Theorem 2.7 to which we refer for details and notation. From (2.17) and (2.26), there holds

$$\sup_{n \in \mathbb{N}} \sup_{t \in [0,T]} \int_Y f_n \left(1 + m^2 + \mathcal{E}\right) dy < \infty.$$  

(5.25)

On the one hand, we deduce of (5.25) that, up to the extraction of a subsequence, there exists $\Gamma \in C([0,T), M^1(Y)$-weak) such that (3.35) holds. On the other hand, we know from Theorem 2.4 that $t \mapsto D_{\mathcal{E},G}(f(t,.)) \in L^1([0,\infty))$, so that

$$\int_0^T D_{\mathcal{E},G}(\Gamma(t,.)) dt \leq \liminf_{n \to \infty} \int_0^T D_{\mathcal{E},G}(f_n(t,.)) dt$$

$$\leq \liminf_{n \to \infty} \int_0^T D_{\mathcal{E},G}(f(s,.)) ds = 0.$$  

(5.26)

Gathering (3.35) and (5.26) we get that for all $t \in [0,T]$, all $y, y_\ast$ in $Y$,

$$\mu^2 \left( \int_0^1 \int_{S^2} \bar{a}_G(y, y_\ast, v, e) \left(1 - e^2\right)(v - v_\ast)^2 dv de \right) \Gamma(t, dy_\ast) \Gamma(t, dy) = 0.$$  

Since $\bar{a}_G(y, y_\ast) > 0$ as soon as $v \neq v_\ast$ by assumption, we deduce that

$$1_{\{v \neq v_\ast\}} \Gamma(t, dy_\ast) \Gamma(t, dy) = 0,$$
and therefore $\Gamma(t, dy)$ is of the shape $\Gamma(t, dy) = \lambda(t, m) \delta_{p = m v_t}$, for some $v_t \in \mathbb{R}^3$. Next, observing that $|p|^{4/3} \leq \mathcal{E} + m^2$ by the Young inequality, we may pass to the limit in the conservation laws (2.18) thanks to (5.25), and we obtain for any $t \in [0, T]$,

$$v_t \int_0^\infty m \lambda(t, dm) = \int_Y p \Gamma(t, dy) = \lim_{n \to \infty} \int_Y p f_n(t, y) dy = 0$$  \hspace{1cm} (5.27)

and

$$\int_0^\infty \phi(m) \lambda(t, dm) = \int_Y \phi(m) \Gamma(t, dy) = \lim_{n \to \infty} \int_Y \phi(m) f(t_n, y) dy = \int_0^\infty \phi(m) \rho(m) dm,$$  \hspace{1cm} (5.28)

for any $\phi \in C_c(0, \infty)$, where $\rho$ is defined in (2.25). We first deduce of (5.28) that $\lambda(t, dm) = \rho(m)$ for any $t \in [0, T]$ and then from (5.27), since $\int_0^\infty m \rho(m) dm > 0$, that $v_t \equiv 0$ for any $t \in [0, T]$. We then easily conclude the proof of (2.27).

Proof of (2.29). For any $\varphi \in C_b(\mathbb{R}^3)$, we get from (2.28) and (2.27)

$$\int_{\mathbb{R}^3} j(t, v) \varphi(v) dv = \int_Y \varphi\left(\frac{p}{m}\right) mf(t, y) dy \longrightarrow \varphi(0) \int_0^\infty m \rho(m) dm,$$

and we conclude recalling that $\int_0^\infty m \rho(m) dm = 1$.

Proof of (2.31). Using (2.30), the dissipation of kinetic energy (2.26) reads

$$\frac{d}{dt} \int_Y f E \ dy = -\kappa \int_Y \int_Y \frac{(mm_a)^{1+\delta}}{m + m_a} |v - v_a|^3 ff_a dy dy_a.$$  \hspace{1cm} (5.29)

Using that $mm_a |v - v_a|^2 = m|v|^2 m_a + mm_a |v_a|^2 - 2(p \cdot p_a)$, and that for all $t$, $\int_Y mf dy = 1$ while $\int_Y pf dy = 0$, we observe that

$$2 \int_Y f E \ dy = \int_Y \int_Y mm_a |v - v_a|^2 ff_a dy dy_a$$

$$\leq \left( \int_Y \int_Y (m + m_a)^{2 \delta} mm_a dy dy_a \right)^{1/3}$$

$$\times \left( \int_Y \int_Y (m + m_a)^{1+\delta} |v - v_a|^3 ff_a dy dy_a \right)^{2/3}.$$
Thanks to the conservation of mass distribution,

$$C = \left( \int_Y \int_Y (m + m_\ast)^2 (mm_\ast)^{1-2\delta} f f_\ast \, dy \, dy_\ast \right)^{1/3}$$

does not depend on time. We finally deduce that for some $K > 0$,

$$\frac{d}{dt} \int_Y fE \, dy \leq -K \left( \int_Y fE \, dy \right)^{3/2},$$

and we easily conclude. \(\square\)

6. The full Boltzmann equation

We now study the full equation (1.1). We thus assume in the whole section that (1.18), (1.21) and (1.23) hold. We consider an initial condition satisfying (1.28) with $k = k_B$ defined in (1.29).

6.1. Existence and uniqueness

All the lemmas below are obtained by gathering the arguments concerning the kinetic Smoluchowski equation, the mass dependent Boltzmann equation and the mass-dependent Granular equation.

**Lemma 6.1.** Assume that $a_B$, $a_G$ and $a_S$ satisfy the symmetry conditions (1.3), (1.9) and (1.13). Let $k$ be a measurable map on $Y$ such that for all $y, y_\ast$ in $Y$, all $v \in S^2$ and all $e \in (0, 1)$,

$$\tilde{a}_B(y, y_\ast) + \tilde{a}_G(y, y_\ast) + a_S(y_\ast) \leq kk_\ast,$$

$$k' + k'_\ast - k - k_\ast \leq 0, \quad k'' + k''_\ast - k - k_\ast \leq 0$$

and

$$k_{\ast\ast} - k - k_\ast \leq 0.$$  \hspace{1cm} (6.1)

Then there exists at most one solution $f$ to the Boltzmann equation (1.1) such that for all $T \geq 0$, $f \in C([0, T]; L^1(k) \cap L^\infty([0, T]; L^1(k)))$.

The proof is immediate using the arguments of Lemmas 3.1, 4.1 and 5.1. Since $C_A k_B$ satisfies all the required properties (for some constant $C_A$ depending on $A$), the uniqueness part of Theorem 2.6 follows.
Lemma 6.2. A solution $f$ to the Boltzmann equation (1.1) conserves (at least formally) mass and momentum (2.32). Moreover, the dissipation of total concentration and of kinetic energy (2.33) hold. Finally, there holds,
\[ \frac{d}{dt} \int_Y \psi(m) f \, dy \leq 0 \] (6.3)
for any sub-additive function $\psi : (0, \infty) \mapsto (0, \infty)$, that is $\psi(m^{**}) \leq \psi(m) + \psi(m_*)$.

The proof follows the line of that of Lemma 3.3, and relies on the use of (2.16) and (2.1), (2.2), and (2.4) with $\beta(x) = x$, and with suitable choices for $\phi$.

Gathering the estimates proved in Lemmas 3.4, 4.3 and 5.3 with those of Lemma 6.2, we obtain the following estimates.

Corollary 6.3. Recall that $k_B$ was defined in (1.29). A solution $f$ to (1.1) satisfies, at least formally, for any $T$,
\[ f_{in} k_B^z \in L^1 \] implies \[ \sup_{[0,T]} \int_Y f(t,y) k_B^z \, dy \leq C_{T,z}, \] (6.4)
where the constant $C_{T,z}$ depends only on $T$, $\| f_{in} \|_{L^1}$, and on $A$ (see (1.18)).

Proof of Theorem 2.6. It follows the line of Section 3.3, with the help of the bounds stated in Corollary 6.3 and a convenient modification of the proof of Lemma 6.1. □

6.2. A stochastic interpretation

We now introduce a stochastic version of Eq. (1.1), that contains more information about the particles, which will be useful to study the long time behavior of solutions.

Since it is more convenient here to work with the couple of variables $(m, v)$ rather than $(m, p)$. We introduce the phase space $Z := (0, \infty) \times \mathbb{R}^3$ of (mass, velocity) variables.

Definition 6.4. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$ be a (sufficiently large) probability space. Consider a càdlàg $Z$-valued adapted stochastic process $(Z_t)_{t \geq 0} = (M_t, V_t)_{t \geq 0}$. Denote, for each $t \geq 0$, by $Y_t = (M_t, M_t V_t)$, and by $Q_t$ the law of $Y_t$, which is a probability measure on $Y$.

Then $(Z_t)_{t \geq 0}$ is said to solve (SDE) if the following conditions hold.

(i) $M$ is a.s. nondecreasing, and is $(0, \infty)$-valued, while $V$ is $\mathbb{R}^3$-valued.
(ii) The law $Q_0$ of $Y_0$ is given by $mf_{in}(y) \, dy$.
(iii) For any $T < \infty$,
\[ E \left[ \sup_{[0,T]} (M_t + |V_t|) \right] < \infty \quad \text{and} \quad \sup_{[0,T]} E[|V_t|^2] < \infty. \] (6.5)
(iv) There exists three independent $(\mathcal{F}_t)_{t\geq 0}$-adapted Poisson measures

\[ N_S(ds, dy, du), \quad N_B(ds, dy, dv, du), \quad N_G(ds, dy, dv, de, du), \quad (6.6) \]

on $[0, \infty) \times Y \times [0, \infty), [0, \infty) \times Y \times S^2 \times [0, \infty), [0, \infty) \times Y \times S^2 \times (0, 1) \times [0, \infty)$ respectively, with intensity measures

\[ ds \mathcal{Q}_s(dy) du, \quad ds \mathcal{Q}_s(dy) dv du, \quad ds \mathcal{Q}_s(dy) dv de du \quad (6.7) \]

respectively, such that a.s., for all $t \geq 0$,

\[
M_t = M_0 + \int_0^t \int_0^\infty \int_Y m \mathbf{1}_{\{u \leq \frac{a_s(Y_s, y)}{m}\}} N_S(ds, dy, du), \\
V_t = V_0 + \int_0^t \int_0^\infty \int_Y \frac{m(v - V_s)}{m + M_s} \mathbf{1}_{\{u \leq \frac{a_s(Y_s, y)}{m}\}} N_S(ds, dy, du) \\
+ \int_0^t \int_0^\infty \int_Y \int_{S^2} \frac{2m}{m + M_s} (v - V_s, v) \mathbf{1}_{\{u \leq \frac{a_s(Y_s, y, e)}{m}\}} N_B(ds, dy, dv, du) \\
+ \int_0^t \int_0^\infty \int_Y \int_{S^2} \frac{(1 + e)m}{m + M_s} (v - V_s, v) \mathbf{1}_{\{u \leq \frac{a_s(Y_s, y, e)}{m}\}} N_G(ds, dy, dv, de, du). \quad (6.8)
\]

This process $(Z_t)_{t \geq 0}$ represents the evolution of the couple of characteristics (mass, velocity) of a typical particle. Of course, $(Y_t)_{t \geq 0}$ represents the evolution of the couple of characteristics (mass, momentum) of the same typical particle. We refer to Tanaka [53], Sznitman [52], Graham and Méléard [33] for similar stochastic interpretations of the Boltzmann equation for elastic collisions, and to Deaconu et al. [19] and Fournier and Giet [29] for the Smoluchowski coagulation equation.

**Theorem 6.5.** Assume that the conditions of Theorem 2.6 hold. Then there exists a solution $(Z_t)_{t \geq 0} = (M_t, V_t)_{t \geq 0}$ to (SDE). This solution furthermore satisfies that for each $t$, the law $Q_t$ of $Y_t = (M_t, M_t V_t)$ has a density $h(t, m, p)$. Then $f(t, y) = h(t, m, p)/m$ is the unique solution to (1.1) such that $f \in C([0, T), L^1_{k_B}) \cap L^\infty([0, T), L^1_{k_B})$. In other words, for all bounded measurable functions $\psi : Y \mapsto \mathbb{R}, \phi : Y \mapsto \mathbb{R},$ any $t \geq 0$,

\[
E[\psi(Y_t)] = \int_Y \psi(y) mf(t, y) dy \quad \text{and} \quad E[\phi(Z_t)] = \int_Y \phi(m, v) mf(t, y) dy. \quad (6.9)
\]
This result is completely unsurprising since existence holds for Eq. (1.1). We will only give the main steps of the proof, since it is quite standard and tedious. We refer to [53,52, 33,19,29] for detailed proofs of similar results.

**Sketch of proof.** We first assume in this proof that

\[ f_{\text{in}} \in L^{1}_{k^3}. \]  

(6.10)

**Step 1.** The result of Theorem 6.5 holds if the rates \(a_B, a_G, \) and \(a_S\) are bounded. Indeed, the existence of a solution \((Z_t)_{t \geq 0}\) can be obtained immediately by using the exact simulation technique of Fournier and Giet [29]. The obtained solution clearly satisfies the moment properties that for all \(T \geq 0\),

\[ E\left[ \sup_{0 \leq t \leq T} (M - 2t + M^2_t |V_t|^2) \right] < \infty, \]

since (6.10) ensures that

\[ E\left[ (M_0 - 2^1 + M^2_0 |V_0|^2) \right] < \infty. \]

Using such inequalities, one may prove that for each \(t \geq 0\), the law of \(Y_t\) has a density \(h(t,y)\). Setting \(f(t,y) = h(t,y)/m\), the above finite expectation ensures that \(f \in C([0,T), L^{1}_{k^3B}) \cap L^\infty([0,T), L^{1}_{k^2B}).\) Finally, the fact that \(f\) solves (1.1) (or rather its weak form (2.14)) follows from a fair computation involving the Itô formula for jump processes.

**Step 2.** We thus consider a sequence of solutions \((Z^l_t)_{t \geq 0}\) associated with the rates \(a^l_B = a_B \land l, a^l_G = a_G \land l, \) and \(a^l_S = a_S \land l,\) and with an initial condition \(f_{\text{in}}\) satisfying (6.10). We also denote by \(g^l\) the corresponding solution to (1.1) (that is, the law of \(Y^l_t\) is given, for each \(t\), by \(mg^l(t,y) \, dm \, dp\)). Using stochastic versions of the estimates obtained in Lemmas 3.4, 4.3, 5.3 and 6.2, one can check that the sequence \((Z^l_t)_{t \geq 0}\) satisfies the Aldous criterion for tightness (see Jacod Shiryaev [34]). Hence one may find a limiting process \((Z_t)_{t \geq 0}\). Martingale technique allows to show that process \((Z_t)_{t \geq 0}\) solves (SDE) with the rates (without cutoff) \(a_B, a_G, a_S\).

**Step 3.** The fact that for each \(t \geq 0\), the law \(Q_t\) of \(Z_t\) has a density can be obtained from the proof of Theorem 2.6. Indeed, we have built \((Z^l_t)_{t \geq 0}\) as the limit of \((Z^l_t)_{t \geq 0}\). Recall that for each \(t\), the density of \(Z^l_t\) is given by \(mg^l(t,y) \, dm \, dy\). Following the proof of Theorem 2.6, we realize that the sequence \(g^l(t,y)\) is Cauchy in \(C([0,T), L^{1}_{k^3B}) \cap L^\infty([0,T), L^{1}_{k^2B})\). Hence its limit \(g(t,.)\) is still a function. The law of \(Z_t\) is thus \(mg(t,y) \, dy, g\) being the unique solution to (1.1).

**Step 4.** Finally, the extension to initial conditions \(f_{\text{in}}\) satisfying only (1.28) can be obtained by using some approximations, as in the proof of Theorem 2.6. □

### 6.3. Long time behavior

We are finally able to prove that the solution \(f\) to (1.1) built in Theorem 2.6 tends to 0 in \(L^1\) under the assumptions of Theorem 2.7. To this aim, we will in fact prove that \(M_t\) tends a.s. to infinity where \((Z_t)_{t \geq 0} = (M_t, V_t)_{t \geq 0}\) is a solution to (SDE) associated to \(f\) thanks to Theorem 6.5.
The main tools of the proof are the dissipations of the total concentration and kinetic energy (2.34) which can be written, recalling (2.36) and the expressions (2.7), (2.9), (2.10) of $D_{1,S}(f) + D_{E,S}(f) + D_{E,G}(f)$,

$$
\int_0^\infty \int_Y \int_Y ff_s aS \ dy \ dy_s < \infty, \quad \int_0^\infty \int_Y \int_Y ff_s \tilde{a}_{\text{inel}} \ dy \ dy_s < \infty \quad (6.11)
$$

where $\tilde{a}_{\text{inel}} = a \tilde{E}_{\text{inel}}$. We next remark the following fact.

**Lemma 6.6.** Almost surely, $M_\infty = \lim_{t \to \infty} M_t$ exists as an element of $(0, \infty) \cup \{ \infty \}$.

The proof is obvious, since $M$ is a nondecreasing process.

We now introduce some notation. We denote by $J_s^S$ (respectively, $J_s^B$ and $J_s^G$) the number of coalescing (respectively, elastic and inelastic) collisions endured by our typical particle before $t$. In other words,

$$
J_s^S = \int_0^t \int_Y \int_0^\infty 1_{[u \leq \alpha_S(y, Y_s, \cdot)]/m} N_S(ds, dy, du),
$$

$$
J_s^B = \int_0^t \int_Y \int_0^{S^2} \int_Y 1_{[u \leq \alpha_B(y, Y_s, \nu)]/m} N_B(ds, dy, dv, du),
$$

$$
J_s^G = \int_0^t \int_Y \int_0^{S^2} \int_Y 1_{[u \leq \alpha_G(y, Y_s, \nu, e)]/m} N_G(ds, dy, dv, de, du).
$$

Note that $J_s^S + J_s^B + J_s^G$ counts the number of jumps of $\{Z_t\}_{t \geq 0}$, that is, $J_s^S + J_s^B + J_s^G = \sum_{s \leq t} 1_{[\Delta Z_s \neq 0]}$.

**Lemma 6.7.** The following estimates on the number of collisions hold: for any $m_0 \in (0, \infty)$,

$$
E\left[1_{[M_\infty \leq m_0]} \left[ J_\infty^S + J_\infty^B + J_\infty^G \right] \right] < \infty. \quad (6.12)
$$

Consequently, $\{M_\infty \leq m_0\} \subset \{ J_\infty^S + J_\infty^B + J_\infty^G < \infty \}$ a.s. for any $m_0 \in (0, \infty)$, and then

$$
P[J_\infty^S + J_\infty^B + J_\infty^G < \infty] \geq P[M_\infty \leq m_0]. \quad (6.13)
$$

**Proof.** Since $M$ is nonincreasing, and since the intensity measure of $N_S$ is given by $mf(s, y) \ du \ dy \ ds$,
\[ E[1_{\{M_\infty \leq m_0\}} J_\infty^S] = E \left[ 1_{\{M_\infty \leq m_0\}} \int_0^\infty \int_0^\infty 1_{\{u \leq aS(Y_t, y)/m\}} N_S(ds, dy, du) \right] \]
\[ \leq E \left[ \int_0^\infty \int_0^\infty E[1_{\{M_t \leq m_0\}} 1_{\{u \leq aS(Y_t, y)/m\}}] mf(s, y) du dy ds \right] \]
\[ = \int_0^\infty \int_0^\infty E[1_{\{M_t \leq m_0\}} aS(Y_t, y)] f(s, y) dy ds \]
\[ \leq m_0 E \left[ \int_0^\infty ds \int Y 1_{\{m_s \leq m_0\}} \frac{aS(y_s, y)}{m_s} f(s, y) dy \right] \]
\[ = m_0 \int ds \int Y 1_{\{m_s \leq m_0\}} \tilde{a}_B(y_s, y) f(s, y) f(s, y_s) dy dy_s < \infty. \]

We used here that the law of \( Y_t \) is \( mf(t, y) \) \( dy \), and the first dissipation inequality in (6.11). Next, using the same arguments,
\[ E[1_{\{M_\infty \leq m_0\}} J_\infty^G] \leq E \left[ \int_0^\infty ds \int Y 1_{\{M_t \leq m_0\}} \tilde{a}_B(Y_t, y) f(s, y) dy \right] \]
\[ = \int_0^\infty ds \int Y 1_{\{m_s \leq m_0\}} \tilde{a}_B(y_s, y) f(s, y) f(s, y_s) dy dy_s < \infty, \]
where we used (2.35) and (6.11). The same computation allows us to obtain the same bound for \( E[1_{\{M_\infty \leq m_0\}} J_\infty^G] \), and that concludes the proof of (6.12). Inequality (6.13) then directly follows from (6.12).

We are finally able to prove our main result.

**Proof of Theorem 2.7.** We argue by contradiction and we thus assume
\[ P[M_\infty < \infty] > 0. \]  
(6.14)
Step 1. The assumption (6.14) ensures that there exists $m_0 \in (0, \infty)$ such that $P[M_\infty \leq m_0] > 0$. Denoting by $\tau$ the last time of jump of $(Y_t)_{t \geq 0}$, we deduce from (6.13) that

$$P[\tau < \infty] = P[J_\infty^S + J_\infty^B + J_\infty^G < \infty] \geq P[M_\infty \leq m_0] > 0.$$ 

Therefore, we have proved that under assumption (6.14), there exists a time $t_0$, such that

$$P[\text{for all } t \geq t_0, Y_t = Y_{t_0}] > 0. \quad (6.15)$$

Step 2. We now deduce from (6.15) that there exists a nonnegative function $g_0$ on $Y$ such that

$$f(t, y) \geq g_0(y) \forall t \geq t_0, \text{ a.e. } y \in Y \quad \text{and} \quad \int_Y g_0(y) m\,dy > 0. \quad (6.16)$$

Let consider the nonnegative measure $\Gamma(dy)$ on $Y$ defined by

$$\Gamma(A) = P[Y_{t_0} \in A \text{ and } Y_t = Y_{t_0} t \geq t_0]. \quad (6.17)$$

On the one hand, $\Gamma(A) \leq P(Y_{t_0} \in A) = \int_A m f(t_0, y)\,dy$ for any measurable set $A \subset Y$, which means $\Gamma \ll m f(t_0, y)\,dy$, and the Radon–Nykodim theorem ensures that $\Gamma(dy) = m g_0(y)\,dy$ for some $g_0 \in L^1(Y; m\,dy)$. On the other hand, $\int_Y m g_0\,dy = \Gamma(Y) = P(Y_{t_0} = Y_{t_0} \text{ for all } t \geq t_0) > 0$ from (6.15). Finally, for any measurable set $A \subset Y$ and any $t \geq t_0$, there holds

$$\int_A m f(t, y)\,dy = P(Y_t \in A) \geq \Gamma(A) = \int_A m g_0(y)\,dy$$

and (6.16) follows.

Step 3. The lower bound (6.16) ensures that

$$\forall t \geq t_0 \quad D_{1,S}(f(t, \cdot)) \geq D_{1,S}(g_0) > 0, \quad (6.18)$$

the last strict inequality following from the fact that $g_0$ does not identically vanish and from the positivity condition (2.37). The lower bound (6.18) obviously contradicts the fact that $D_{1,S}(f) \in L^1([0, \infty))$. We then conclude that (6.14) does not hold and therefore $M_\infty = \infty$ a.s. or, equivalently, $1/M_t \to 0$ a.s. when $t$ goes to the infinity. Finally, since $M$ is a nondecreasing process and since $E[1/M_0] = \int_Y f_0\,m\,dy < \infty$, we deduce from the Lebesgue theorem that

$$\int_Y f(t, y)\,dy = E[1/M_t] \xrightarrow{t \to \infty} 0. \quad \Box$$
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References


