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Critical point and percolation probability in a long range site percolation model on $\mathbb{Z}^d$

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Abstract

Consider an independent site percolation model with parameter $p \in (0, 1)$ on $\mathbb{Z}^d$, $d \geq 2$, where there are only nearest neighbor bonds and long range bonds of length k parallel to each coordinate axis. We show that the percolation threshold of such a model converges to $p_c(\mathbb{Z}^{2d})$ when k goes to infinity, the percolation threshold for ordinary (nearest neighbor) percolation on $\mathbb{Z}^{2d}$. We also generalize this result for models whose long range bonds have several lengths.

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1. Introduction and notation

Let $G = (\mathbb{V}, \mathbb{E})$ be a graph with a countably infinite vertex set $\mathbb{V}$. Consider the Bernoulli site percolation model on G; with each site $v \in \mathbb{V}$ we associate a Bernoulli random variable $X(v)$, which takes the values 1 and 0 with probability p and $1 - p$ respectively. This can be done considering the probability space $(\Omega, \mathcal{F}, \mathbb{P}_p)$, where $\Omega = \{0, 1\}^{\mathbb{V}}$, $\mathcal{F}$ is the σ-algebra

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generated by the cylinder sets in Ω and $\mathbb{P}_p = \prod_{v\in\mathbb{V}} \mu(v)$ is the product of Bernoulli measures with parameter p, in which the configurations $\{X(v), v \in \mathbb{V}\}$ occur. We denote a typical element of Ω by ω. When $X(v) = 1$ (respectively, $X(v) = 0$) we say that v is "open" (respectively, "closed").

Given two vertices v and u, we say that v and u are connected in the configuration ω if there exists a finite path $\langle v = v_0, v_1, \ldots, v_n = u\rangle$ of open vertices in $\mathbb{V}$ such that $v_i \neq v_j$, $\forall i \neq j$ and $\langle v_i, v_{i+1}\rangle$ belongs to $\mathbb{E}$ for all $i = 0, 1, \ldots, n-1$. We will use the short notation $\{v \leftrightarrow u\}$ to denote the set of configurations where u and v are connected.

Given the vertex v, the cluster of v in the configuration ω is the set $C_v(\omega) = \{u \in \mathbb{V}; v \leftrightarrow u \text{ on } \omega\}$. We say that the vertex v percolates when the cardinality of $C_v(\omega)$ is infinite; we will use the following standard notation: $\{v \leftrightarrow \infty\} \equiv \{\omega \in \Omega; \#C_v(\omega) = \infty\}$. Fixing some vertex v, we define the percolation probability of the vertex v as the function $\theta_v(p) : [0, 1] \mapsto [0, 1]$ with $\theta_v(p) = \mathbb{P}_p(v \leftrightarrow \infty)$.

From now on, the vertex set $\mathbb{V}$ will be $\mathbb{Z}^d$, $d \geq 2$, and for each positive integer k we define

$$\mathbb{E}_k = \{\langle (v_1, \ldots, v_d), (u_1, \ldots, u_d)\rangle \in \mathbb{V} \times \mathbb{V}; \exists! i \in \{1, \ldots, d\} \text{ such that } |v_i - u_i| = k \text{ and } v_j = u_j, \forall j \neq i\}.$$

Let us define the graph $G^k = (\mathbb{V}, \mathbb{E}_1 \cup \mathbb{E}_k)$, that is, G^k is $\mathbb{Z}^d$ equipped with nearest neighbor bonds and long range bonds with length k parallel to each coordinate axis. Observe that G^k is a transitive graph; hence the function $\theta_v(p)$ does not depend on v and we write only $\theta^k(p)$ to denote $\mathbb{P}_p(0 \leftrightarrow \infty)$ for any transitive graph.

The simplest version of our main result (see Theorem 1) states that $p_c(G^k)$ converges to $p_c(\mathbb{Z}^{2d})$ when k goes to infinity. The main motivation for studying this question is that we believe that the Conjecture 1 stated below can shed some light on the truncation problem for long range percolation. This problem, proposed by Andjel, is the following:

On $\mathbb{Z}^d$, $d \geq 2$, consider the complete graph $G = (\mathbb{Z}^d, \mathbb{E})$, that is for all $v, u \in \mathbb{Z}^d$ we have that $\langle u, v\rangle \in \mathbb{E}$. For each bond $\langle u, v\rangle \in \mathbb{E}$ we define its length as $\|u - v\|_1$. Given a sequence $(p_n \in [0, 1), n \in \mathbb{N})$ consider an independent bond percolation model where each bond whose length is n will be open with probability p_n. Assume that $\sum_{n\in\mathbb{N}} p_n = \infty$; by the Borel–Cantelli lemma, the origin will percolate to infinity with probability 1. The general and still open truncation question is the following: is it true that there exists some sufficiently large but finite integer K such that the origin in the truncated processes, obtained by deleting (or closing) all long range bonds whose length are bigger than K, still percolates to infinity with positive probability?

The general question is still open; see [4] for a more detailed discussion.

2. The main result

Given a positive integer n, define the n-vector $\overrightarrow{k} = (k_1, \ldots, k_n)$, where $k_i \in \{2, 3, \ldots\}$, $\forall i = 1, \ldots, n$. We define the graph $G^{\overrightarrow{k}}$ as $(\mathbb{Z}^d, \mathbb{E}_1 \cup (\cup_{i=1}^n \mathbb{E}_{k_1\times\cdots\times k_i}))$. Observe that when $n = 1$ and $k_1 = k$, the graph $G^{\overrightarrow{k}}$ is the graph G^k defined above. That is, $G^{\overrightarrow{k}}$ is $\mathbb{Z}^d$ decorated with all bonds parallel to each coordinate axis with lengths $1, k_1, k_1 \times k_2, \ldots, k_1 \times k_2 \times \cdots \times k_n$.

From now on, we will use the notation $S^{\overrightarrow{k}}$ to denote the $d(n+1)$-dimensional slab graph where the vertex set is $(\mathbb{Z} \times \prod_{i=1}^n \{0, 1, \ldots, k_{n-i+1} - 1\})^d$ and $S^{\overrightarrow{k}}$ is equipped with only nearest neighbor bonds.

The aim of this note is to prove that the percolation function of the graph $G^{\overrightarrow{k}}$ is bounded between the percolation functions of the slab $S^{\overrightarrow{k}}$ and of $\mathbb{Z}^{d(n+1)}$. More precisely, we have the following lemmas:

Lemma 1. *For any* $p \in [0,1]$, $\theta_v^{S^{\vec{k}}}(p) \le \theta^{G^{\vec{k}}}(p), \forall v \in \mathbb{V}(S^{\vec{k}})$.

Lemma 2. *For any* $p \in [0,1]$, $\theta^{G^{\vec{k}}}(p) \le \theta^{\mathbb{Z}^{d(n+1)}}(p)$.

The proof of these lemmas will be given in the next section. Combining Lemmas 1 and 2 it is straightforward to see that

$$\theta_v^{S^{\vec{k}}}(p) \le \theta^{G^{\vec{k}}}(p) \le \theta^{\mathbb{Z}^{d(n+1)}}(p), \quad \forall v \in \mathbb{V}(S^{\vec{k}}).$$

Moreover,

$$p_c(\mathbb{Z}^{d(n+1)}) \le p_c(G^{\vec{k}}) \le p_c(S^{\vec{k}}). \tag{2.1}$$

Using Theorem A of [6], we have that

$$\lim_{k_i \to \infty, \forall i} p_c(S^{\vec{k}}) = p_c(\mathbb{Z}^{d(n+1)}). \tag{2.2}$$

Then combining (2.1) and (2.2), we can conclude that

$$\lim_{k_i \to \infty, \forall i} p_c(G^{\vec{k}}) = p_c(\mathbb{Z}^{d(n+1)}).$$

We have just proved that main result of this paper.

Theorem 1. *Let* $p_c(\mathbb{Z}^{d(n+1)})$ *be the ordinary site percolation threshold for* $\mathbb{Z}^{d(n+1)}$ *with nearest neighbor connections; then*

$$\lim_{k_i \to \infty, \forall i} p_c(G^{\vec{k}}) = p_c(\mathbb{Z}^{d(n+1)}). \tag{2.3}$$

3. Proofs of the lemmas

Proof of Lemma 1. From the graph $G^{\vec{k}}$ we define the graph $F^{\vec{k}}$, deleting some bonds in $G^{\vec{k}}$. More precisely, $F^{\vec{k}}$ is the graph $(\mathbb{Z}^d, (\mathbb{E}_1 \cup (\cup_{i=1}^n \mathbb{E}_{k_1 \times \cdots \times k_i})) - \cup_{i=1}^n R_i)$, where $R_i = R_i(\vec{k})$ is the set of deleted bonds with length $k_1 \times \cdots \times k_{i-1}$ defined by

$$\begin{aligned} R_i(\vec{k}) = \{&\langle (v_1,\ldots,v_d),(u_1,\ldots,u_d)\rangle \in \mathbb{V}\times\mathbb{V}; \exists! r \in \{1,\ldots,d\}, \exists l \in \mathbb{Z},\\ &\exists j \in \{0,\ldots,k_1\times\cdots\times k_{i-1}-1\}\\ &\text{such that } u_r = lk_1\ldots k_i + j,\ v_r = k_1\ldots k_{i-1}(lk_i - 1) + j \text{ and } v_s = u_s, \forall s \ne r\}. \end{aligned} \tag{3.1}$$

Observe that in the simplest case, when $n = 1$ and $k_1 = k$, the set of deleted bonds is precisely all nearest neighbors bond in the rth direction where one of the endpoints has the rth coordinate a multiple of k and the other endpoints have the rth coordinate one unit below, for each $r = 1,\ldots,d$.

Since $F^{\vec{k}}$ is subgraph of $G^{\vec{k}}$, we have the inequality $\theta_v^{F^{\vec{k}}}(p) \le \theta^{G^{\vec{k}}}(p), \forall v \in \mathbb{V}(F^{\vec{k}})$.

Now, we claim that the graphs $F^{\vec{k}}$ and $S^{\vec{k}}$ are isomorphic. Consider the function

$$\psi : \mathbb{Z} \to \mathbb{Z} \times \prod_{i=1}^n \{0,1,\ldots,k_{n-i+1}-1\}, \tag{3.2}$$

where

$$\psi(v) = \left(\left\lfloor \frac{v}{k_1 \dots k_n} \right\rfloor, \left\lfloor \frac{v \bmod k_1 \dots k_n}{k_1 \dots k_{n-1}} \right\rfloor, \dots, \left\lfloor \frac{v \bmod k_1 k_2}{k_1} \right\rfloor, v \bmod k_1 \right).$$

Indeed the function

$$\Psi : \mathbb{Z}^d \to \left(\mathbb{Z} \times \left(\prod_{i=1}^{n} \{0, 1, \dots, k_{n-i+1} - 1\} \right) \right)^d, \tag{3.3}$$

where $\Psi(v_1, \dots, v_d) = (\psi(v_1), \dots, \psi(v_d))$, is a graph isomorphism between $F^{\vec{k}}$ and $S^{\vec{k}}$. Then, $\theta_v^{S^{\vec{k}}}(p) = \theta_{\Psi^{-1}(v)}^{F^{\vec{k}}}(p)$ for all $v \in \mathbb{V}(S^{\vec{k}})$, proving this lemma. $\square$

Remark. Since $\theta_v^{S^{\vec{k}}}(p) \le \theta^{G^{\vec{k}}}(p), \forall v \in \mathbb{V}(S^{\vec{k}}), \forall p \in [0, 1]$, it holds that $p_c(G^{\vec{k}}) \le p_c(S^{\vec{k}})$ (the second inequality in Eq. (2.1)). Indeed, we have that the strict inequality is also true, observing that there exists a periodic class of edges of $G^{\vec{k}}$, which do not belong to $S^{\vec{k}}$. (See example B in Section 3.3 of [5].)

Proof of Lemma 2. In this proof we will use Theorem 1 of [2], which is based on the original idea of Campanino and Russo in [3], to prove that the percolation threshold of the cubic lattice is bounded above by the percolation threshold of the triangular lattice.

In the proof of Lemma 1, we were able to show that the graphs $F^{\vec{k}}$ and $S^{\vec{k}}$ are isomorphic according the function Ψ defined in (3.3), where $F^{\vec{k}}$ is obtained by deleting some specific edges of $G^{\vec{k}}$. If we insert the respective edges again in $S^{\vec{k}}$, we obtain a new graph, denoted by $\widetilde{S}^{\vec{k}}$, and this latter graph is isomorphic to $G^{\vec{k}}$. Formally, we have

$$\widetilde{S}^{\vec{k}} = (\mathbb{V}(S^{\vec{k}}), \mathbb{E}(S^{\vec{k}}) \cup (\cup_{i=1}^{n} \widetilde{R}_i(\vec{k}))). \tag{3.4}$$

Here $\widetilde{e} = \langle \widetilde{u}, \widetilde{v} \rangle \in \widetilde{R}_i(\vec{k})$ if and only if $\widetilde{u} = \Psi(u), \widetilde{v} = \Psi(v)$ and $e = \langle u, v \rangle \in R_i(\vec{k})$, where $R_i(\vec{k})$ is defined in Eq. (3.1).

Thus, it is enough to prove that

$$\theta^{\widetilde{S}^{\vec{k}}}(p) \le \theta^{\mathbb{Z}^{d(n+1)}}(p), \quad \forall p \in [0, 1].$$

For this purpose, we will show that $\widetilde{S}^{\vec{k}}$ is a quotient graph of $\mathbb{Z}^{d(n+1)}$ by an automorphism group and apply Theorem 1 of [2].

We can write each vertex $v \in \mathbb{Z}^{d(n+1)}$ as $v = (v_1, \dots, v_d)$, with $v_i \in \mathbb{Z}^{n+1}, \forall i = 1, \dots, d$. For each

$$v_j = (v_{j,1}, \dots, v_{j,n+1}),$$

we will define the surjective function

$$\gamma : \mathbb{Z}^{n+1} \to \mathbb{Z} \times \prod_{i=1}^{n} \{0, 1, \dots, k_{n-i+1} - 1\}, \tag{3.5}$$

in a recursive manner. To simplify the notation, define $(v_{j,1}, \dots, v_{j,n+1}) = (y_1, \dots, y_{n+1})$.

We will define $\gamma(y_1, \dots, y_{n+1}) = (z_1, \dots, z_{n+1}) \in \mathbb{Z} \times \prod_{i=1}^{n} \{0, 1, \dots, k_{n-i+1} - 1\}$ (here $z_1 \in \mathbb{Z}, z_2 \in \{0, \dots, k_n - 1\}, \dots, z_{n+1} \in \{0, \dots, k_1 - 1\}$), where the sequence $(z_k)_{k=1}^{n+1}$ is obtained recursively in the following way:

First, define $z_{n+1} = y_{n+1} \bmod k_1$ and $t_{n+1} = \lfloor \frac{y_{n+1}}{k_1} \rfloor$.

Given t_{i+1} and z_{i+1} for $i = 2, \ldots, n$, we define z_i and t_i as

$$z_i = (y_i + t_{i+1}) \bmod k_{n+2-i} \quad \text{and} \quad t_i = \left\lfloor \frac{y_i + t_{i+1}}{k_{n+2-i}} \right\rfloor.$$

Finally, define $z_1 = y_1 + t_2$.

Now, we define the surjection

$$\Gamma : \mathbb{Z}^{d(n+1)} \to \left(\mathbb{Z} \times \prod_{i=1}^{n} \{0, 1, \ldots, k_{n-i+1} - 1\} \right)^d, \tag{3.6}$$

where

$$\Gamma(v) = (\gamma(v_1), \gamma(v_2), \ldots, \gamma(v_d)). \tag{3.7}$$

In words, considering the simplest case $d = 1$ and $n = 1$, the function Γ wraps $\mathbb{Z}^2$ onto the strip $\mathbb{Z} \times \{1, \ldots, k-1\}$ shifting one unit in the first coordinate in each wind around $\mathbb{Z} \times \{1, \ldots, k-1\}$.

Given $(z_1, \ldots, z_n, z_{n+1}) \in \mathbb{Z}^{n+1}$, for each $j = 1, \ldots, n$ define the functions $\delta_j : \mathbb{Z}^{n+1} \to \mathbb{Z}^{n+1}$ where

$$\delta_1(z_1, \ldots, z_n, z_{n+1}) = (z_1, \ldots, z_{n-1}, z_n - 1, z_{n+1} + k_1),$$
$$\delta_2(z_1, \ldots, z_n, z_{n+1}) = (z_1, \ldots, z_{n-2}, z_{n-1} - 1, z_n + k_2, z_{n+1}), \ldots$$
$$\delta_n(z_1, \ldots, z_n, z_{n+1}) = (z_1 - 1, z_2 + k_n, z_3, \ldots, z_{n+1}).$$

For each $i = 1, \ldots, d$ and $j = 1, \ldots, n$ define the group automorphism function $\Delta_{i,j} : \mathbb{Z}^{d(n+1)} \to \mathbb{Z}^{d(n+1)}$ where $\Delta_{i,j}(v_1, \ldots, v_d) = (v_1, \ldots, v_{i-1}, \delta_j(v_i), v_{i+1}, \ldots, v_d)$.

Using Theorem 1 of [2] and observing that $\widetilde{S}^{\vec{k}}$ is the quotient graph $\mathbb{Z}^{d(n+1)}/\Delta$, where $\Delta = \langle \Delta_{i,j}; i = 1, \ldots, d, \ j = 1, \ldots, n \rangle$ is the automorphism group of $\mathbb{Z}^{d(n+1)}$ generated by the set of automorphisms $\{\Delta_{i,j}; i = 1, \ldots, d, \ j = 1, \ldots, n\}$, we have that $\theta^{\widetilde{S}^{\vec{k}}}(p) \leq \theta^{\mathbb{Z}^{d(n+1)}}(p), \ \forall p \in [0, 1]$, concluding the proof of Lemma 2. Observe that the function Γ defined in (3.7) is precisely the quotient map (the function f in the proof of Theorem 1 of [2]) between $\mathbb{Z}^{d(n+1)}$ and $\widetilde{S}^{\vec{k}}$. □

4. Final remarks

(A) All these results remain the same if we consider bond percolation (where each bond is open with probability p) instead of site percolation, since Theorem A of [6] and Theorem 1 of [2] can also be used for bond percolation.

(B) The main result, Theorem 1, can be generalized without difficulty if we consider different sequences $\vec{k_i} = (k_{i,1}, \ldots, k_{i,n_i})$ for each direction i (here $n_i = 0$ means that there is only a nearest neighbor bond in the ith direction). In this case Eq. (2.3) is equivalent to

$$\lim_{k_{i,j} \to \infty, \forall i,j} p_c(G^{\vec{k}}) = p_c(\mathbb{Z}^{\sum_{i=1}^{d}(1+n_i)}).$$

(C) Computational simulations (see [1]) show that when $n = 1$ and $d = 2$, the function $\theta^k(p)$ is non-decreasing in k; then $p_c(G^{k+1}) \leq p_c(G^k)$ and $\lim_{k\to\infty} p_c(G^k) = p_c(\mathbb{Z}^4)$, confirming

Theorem 1. On the basis of these simulations and the shape of the graphs $\widetilde{S}^{\vec{k}}$, we think that the following conjecture is true.

Conjecture 1. *For any* $p \in [0, 1]$ *and for any* $\vec{k}$, *we have* $\theta^{G^{\vec{k}}}(p) \leq \theta^{G^{\vec{k'}}}(p)$, *where* $\vec{k'} = (k_1 + 1, \ldots, k_n + 1)$.

Indeed, it is possible to see that $G^{\vec{k}}$ is isomorphic to a quotient graph of $G^{\vec{k'}}$; nevertheless $G^{\vec{k}}$ is not a quotient graph by an automorphism group, and thus Theorem 1 of [2] cannot be used.

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