NORTH-HOLLAND

On Monge Sequences in d-Dimensional Arrays*

Rüdiger Rudolf
Technical University Graz
Institute for Mathematics B
Steyrerposse 30
A-8010 Graz, Austria

Submitted by Richard A. Brualdi

ABSTRACT

Let C be an $n \times m$ matrix. Then the sequence $\mathcal{S} := ((i_1, j_1), (i_2, j_2), \dots, (i_{nm}, j_{nm}))$ of pairs of indices is called a *Monge sequence* with respect to the given matrix C if and only if, whenever (i, j) precedes both (i, s) and (r, j) in $\mathcal{S}$, then $c[i, j] + c[r, s] \leqslant c[i, s] + c[r, j]$. Monge sequences play an important role in greedily solvable transportation problems. Hoffman showed that the greedy algorithm which maximizes all variables along a sequence $\mathcal{S}$ in turn solves the classical Hitchcock transportation problem for all supply and demand vectors if and only if $\mathcal{S}$ is a Monge sequence with respect to the cost matrix C. In this paper we generalize Hoffman's approach to higher dimensions. We first introduce the concept of a d-dimensional Monge sequence. Then we show that the d-dimensional axial transportation problem is solved to optimality for arbitrary right-hand sides if and only if the sequence $\mathcal{S}$ applied in the greedy algorithm is a d-dimensional Monge sequence. Finally we present an algorithm for obtaining a d-dimensional Monge sequence which runs in polynomial time for fixed d. © 1998 Elsevier Science Inc.

*The author acknowledges financial support by the Fonds zur Förderung der wissenschaftlichen Forschung, Project P8971-PHY.

655 Avenue of the Americas, New York, NY 10010
0024-3795/98/$19.00
PII S0024-3795(97)00007-4

1. INTRODUCTION

The well-known Hitchcock transportation problem (TP) can be formulated as a linear program in the following way:

$$\text{(TP)} \qquad \min \sum_{i=1}^{n} \sum_{j=1}^{m} c[i,j] x_{ij}$$

$$\text{such that} \sum_{j=1}^{m} x_{ij} = a_i^1 \quad \forall i = 1,\dots,n,$$

$$\sum_{i=1}^{n} x_{ij} = a_j^2 \quad \forall j = 1,\dots,m,$$

$$x_{ij} \geqslant 0 \quad \forall i,j.$$

It is well known that due to the special structure of (TP) an initial feasible solution of (TP) can be obtained as follows: Take an arbitrary order of the variables, say a sequence $\mathcal{S} := ((i_1, j_1), (i_2, j_2), \dots, (i_{nm}, j_{nm}))$, and perform the subsequent greedy algorithm $G_{\mathcal{S}}$:

ALGORITHM $G_{\mathcal{S}}$. For $k := 1$ to nm do
Set $x_{i_k j_k} := \min\{a_{i_k}^1, a_{j_k}^2\}$
$a_{i_k}^1 := a_{i_k}^1 - x_{i_k j_k}$
$a_{j_k}^2 := a_{j_k}^2 - x_{i_k j_k}$

This algorithm proceeds along the sequence $\mathcal{S}$ and maximizes each variable in turn. Thus its running time is $O(nm)$. By choosing special sequences $\mathcal{S}$, the greedy algorithm $G_{\mathcal{S}}$ turns e.g. into the *northwest-corner rule* or the *minimum-c_{ij} rule* (see e.g. Hadley [8]).

It is easy to see that in general $G_{\mathcal{S}}$ constructs only a feasible but not an optimal solution of (TP). Hoffman [9] gives, however, a sufficient and necessary condition on $\mathcal{S}$ such that $G_{\mathcal{S}}$ always constructs an optimal solution of (TP) for arbitrary demand and supply vectors a^1 and a^2. Hoffman's condition looks as follows:

For every $1 \leqslant i, r \leqslant n$, $1 \leqslant j, s \leqslant m$, whenever (i,j) precedes both (i,s) and (r,j), the corresponding entries in the matrix C are such that

$$c[i,j] + c[r,s] \leqslant c[i,s] + c[r,j].$$

Sequences $\mathcal{S}$ which fulfill the property above are called *Monge sequences*, after a similar condition found by G. Monge [10].

Closely related with matrices C for which there exists a Monge sequence are *Monge matrices*, i.e. matrices fulfilling the *Monge property*

$$c[i,j] + c[r,s] \leqslant c[i,s] + c[r,j]$$

$$\text{for all} \quad 1 \leqslant i < r \leqslant n, \quad 1 \leqslant j \leqslant s \leqslant m.$$

Note that for each Monge matrix there exists a Monge sequence; e.g., choose as sequence the lexicographical ordering of all pairs of indices of C. In this special case $G_{\mathcal{S}}$ degenerates to the northwest-corner rule, and therefore the transportation problem can be solved in $O(n + m)$ time.

A generalization of the classical transportation problem (TP) which occurs in some applications (see e.g. [7]) is the so-called d-dimensional axial transportation problem. This problem, (dTP) for short, can be formulated as follows:

(dTP)

$$\begin{array}{lll}
\min & \displaystyle\sum_{i_1=1}^{n_1}\sum_{i_2=1}^{n_2}\cdots\sum_{i_d=1}^{n_d} & c[i_1, i_2, \ldots, i_d]\, x_{i_1 i_2 \cdots i_d} \\
\text{such that} & \displaystyle\sum_{\substack{i_1, i_2, \ldots, i_d \\ i_k = q}} & x_{i_1 i_2 \cdots i_d} = a_q^k \\
 & & \forall k = 1, \ldots, d \quad \forall q = 1, \ldots, n_k, \\
 & & x_{i_1 i_2 \cdots i_d} \geqslant 0 \quad \forall i_1, i_2, \ldots, i_d.
\end{array}$$

Obviously, a similar greedy algorithm to that used for the classical problem (TP) can be applied to (dTP) to obtain a feasible solution. Given a sequence $\mathcal{S}$ of d-tuples of indices, this greedy algorithm maximizes each variable of $\mathcal{S}$ in turn. As in the two-dimensional case, the question raises on a necessary and sufficient condition on $\mathcal{S}$ which guarantees the optimality of $G_{\mathcal{S}}$.

Our Results

We first introduce the notion of a d-dimensional Monge sequence which generalizes Hoffman's Monge sequences. Then we show that being a d-

dimensional Monge sequence is a necessary and sufficient condition for a sequence $\mathcal{S}$ in order to guarantee the optimality of a greedy approach to the d-dimensional axial transportation problem (dTP). Furthermore, we deal with the problem of deciding whether there exists a d-dimensional Monge sequence for a given d-dimensional array C. Finally, we present an algorithm for constructing d-dimensional Monge sequences which runs in polynomial time for fixed d.

Related Results

In [2] Aggarwal and Park generalize the concept of a Monge matrix to arbitrary d-dimensional arrays. Bein, Brucker, Park, and Pathak [3] then prove that a generalized lexicographical greedy algorithm solves the d-dimensional axial transportation problem (dTP) if and only if the cost array is a d-dimensional Monge array.

In Rudolf [11] a polynomial time algorithm for recognizing permuted d-dimensional Monge arrays is given which generalizes the recognition algorithm of Deineko and Filonenko [4] for $n \times n$ matrices C.

Alon, Cosares, Hochbaum, and Shamir [1] deal with the problems of detecting and constructing a Monge sequence for an $n \times m$ matrix C. Their results were generalized to matrices with infinite entries by Dietrich [5] and Shamir [12] as well as by Dietrich and Shamir [6].

2. d-DIMENSIONAL MONGE SEQUENCES

We start with introducing some basic notation and definitions. Let C be an $n_1 \times n_2 \times \cdots \times n_d$ array, and let $N_k := \{1, 2, \ldots, n_k\}$ be the set of feasible indices for dimension k, $k = 1, \ldots, d$. Let $\mathcal{F}$ be a set of q d-tuples of feasible indices, say $\mathcal{F} := \{(i_k^1, i_k^2, \ldots, i_k^d) \mid 1 \leqslant k \leqslant q, i_k^l \in N_l\}$. Then $I_l(\mathcal{F}) := \{i_k^l \mid 1 \leqslant k \leqslant q\}$ is defined to be the corresponding list of all indices occurring at position l of a d-tuple in $\mathcal{F}$. Note that in contrast to a set, the lists $I_l(\mathcal{F})$ need not contain pairwise distinct elements.

We call a set $\mathcal{F}$ of d-tuples *feasible* with respect to the d-tuple (i_1, i_2, i_d) iff each index i_k is contained at least once in the list $I_k(\mathcal{F})$, $k = 1, \ldots, d$. A set $\mathcal{F}$ is said to be *minimal* with respect to $(i_1, i_2, \ldots, i_d)$ iff $\mathcal{F}$ is feasible and no proper subset $\mathcal{F}_1 \subset \mathcal{F}$ is feasible. Furthermore, denote by $M(\mathcal{F}) := \{(i_1, i_2, \ldots, i_d) \mid i_k \in I_k(\mathcal{F})\}$ the set of all d-tuples which can be composed using the indices occurring in $\mathcal{F}$.

Now we are prepared to introduce the notion of a d-dimensional Monge sequence. For the ease of exposition, let us first state the condition for a three-dimensional $n_1 \times n_2 \times n_3$ array C.

DEFINITION 2.1. $\mathcal{S}$ is said to be a *three-dimensional Monge sequence* if and only if the following conditions are satisfied:

1. (a) Whenever (i, j, k) precedes (i, s, k) and (i, j, t) in $\mathcal{S}$, then

$$c[i,j,k] + c[i,s,t] \leqslant c[i,j,t] + c[i,s,k].$$

(b) Whenever (i, j, k) precedes (i, j, t) and (r, j, k) in $\mathcal{S}$, then

$$c[i,j,k] + c[r,j,t] \leqslant c[i,j,t] + c[r,j,k].$$

(c) Whenever (i, j, k) precedes (i, s, k) and (r, j, k) in $\mathcal{S}$, then

$$c[i,j,k] + c[r,s,k] \leqslant c[i,s,k] + c[r,j,k].$$

2. Whenever (i, j, k) precedes (i, j, t), (r, s, k), (i, s, t), (r, j, k), (i, s, k), *and* (r, j, t) in $\mathcal{S}$, then
   (a) $c[i,j,k] + c[r,s,t] \leqslant c[i,j,t] + c[r,s,k]$,
   (b) $c[i,j,k] + c[r,s,t] \leqslant c[i,s,t] + c[r,j,k]$, and
   (c) $c[i,j,k] + c[r,s,t] \leqslant c[i,s,k] + c[r,j,t]$.
3. Let $(i, j, k) \in \mathcal{S}$. Then for each set $\mathcal{F} := \{(i, s_1, t_1), (r_1, j, t_2), (r_2, s_2, k)\}$ which is minimal with respect to (i, j, k), the following has to be satisfied: Whenever (i, j, k) is the element which occurs first in $\mathcal{S}$ among all elements in $M(\mathcal{F})$ then

$$c[i,j,k] + \min_{\phi,\psi} \left\{ c\left[r_1, s_{\phi(1)}, t_{\psi(1)}\right] + c\left[r_2, s_{\phi(2)}, t_{\psi(2)}\right] \right\}$$

$$\leqslant c[i, s_1, t_1] + c[r_1, j, t_2] + c[r_2, s_2, k],$$

where ϕ and ψ are mappings from $\{1,2\} \to \{1,2\}$.

If we look at the definition above in more detail, we see that conditions 1(a)–(c) coincide with the classical definition of a Monge sequence in two-dimensional matrices. Conditions 2(a)–(c) derive from the Monge condition for three-dimensional arrays (see [2] for details), whereas condition (3) has to be added to ensure the optimality of a greedy approach.

To generalize Definition 2.1 to d dimensions we introduce a more compact notation. Let $(i_1^1, i_1^2, \ldots, i_1^d)$ be a d-tuple and let $\mathcal{F}$ be a set of q d-tuples which is minimal with respect to $(i_1^1, i_1^2, \ldots, i_1^d)$. Let again $I_l(\mathcal{F})$

denote the list of all indices which occur at the lth position of a d-tuple in $\mathscr{F}$. Then we define $U_l := I_l(\mathscr{F}) \setminus \{i_1^l\}$ to be the list obtained from $I_l(\mathscr{F})$ by removing the element i_1^l. Furthermore, we define

$$W(U_1, U_2, \dots, U_d) := \min_{\phi_2, \dots, \phi_d} \sum_{k=2}^{q} c\left[i_k^1, i_{\phi_2(k)}^2, \dots, i_{\phi_d(k)}^d\right],$$

where ϕ_l, $l = 2, \dots, d$, are mappings from $\{2, \dots, q\}$ onto $\{2, \dots, q\}$.

Now we are able to describe the property for a d-dimensional Monge sequence $\mathscr{S}$ in a very compact form.

DEFINITION 2.2. $\mathscr{S}$ is called a *d-dimensional Monge sequence* if and only if the subsequent conditon is satisfied: Let $(i_1^1, i_1^2, \dots, i_1^d) \in \mathscr{S}$. Then for each set $\mathscr{F}$ of d-tuples which is minimal with respect to $(i_1^1, i_1^2, \dots, i_1^d)$, whenever $(i_1^1, i_1^2, \dots, i_1^d)$ is the element which occurs first in $\mathscr{S}$ among all elements of $M(\mathscr{F})$, then

$$c\left[i_1^1, i_1^2, \dots, i_1^d\right] + \min_{\phi_2, \dots, \phi_d} \sum_{k=2}^{q} c\left[i_k^1, i_{\phi_2(k)}^2, \dots, i_{\phi_d(k)}^d\right] \leqslant \sum_{\alpha \in \mathscr{F}} c_\alpha, \quad (1)$$

where again $\phi_2, \dots, \phi_d$ are mappings from $\{2, \dots, q\}$ onto $\{2, \dots, q\}$.

Note that in the case of $d = 3$ we arrive at Definition 2.1 by choosing in Definition 2.2 minimal feasible sets $\mathscr{F}$ with two or three triples of indices.

REMARK. For ease of exposition, we henceforth adopt the following convention: When referring to the terms in an equality of the type (1), we assume that the minimum on the left-hand side has already been evaluated. So on each side we have exactly q terms.

3. A GENERALIZATION OF HOFFMAN'S THEOREM

We are now prepared to prove the main result of this paper, the generalization of the theorem of Hoffman to $d \geqslant 3$ dimensions. For ease of exposition we only show the three-dimensional case; the proof for arbitrary dimensions can easily be established using the same techniques.

THEOREM 3.1. *The greedy algorithm $G_{\mathscr{S}}$ solves the three-dimensional axial transportation problem (3TP) for all right-hand-side vectors a^1, a^2, and a^3 if and only if $\mathscr{S}$ is a three-dimensional Monge sequence.*

Proof. $\Rightarrow$: Let an instance of (3TP) with cost array C be given. Assume that the greedy algorithm $G_{\mathscr{S}}$ solves this instance to optimality for arbitrary right-hand-side vectors a^1, a^2, and a^3. Suppose that $\mathscr{S}$ is not a three-dimensional Monge sequence. Then there must exist a triple (i_1^1, i_1^2, i_1^3) and a minimal set $\mathscr{F}$ of triples with respect to (i_1^1, i_1^2, i_1^3) such that (i_1^1, i_1^2, i_1^3) precedes all other elements of $M(\mathscr{F})$ in $\mathscr{S}$ and

$$c\left[i_1^1, i_1^2, i_1^3\right] + W(U_1, U_2, U_3) > \sum_{\alpha \in \mathscr{F}} c_\alpha \tag{2}$$

In order to construct a contradiction we generate the following special instance of (3TP): Denote by α_i^l the number of occurrences of an index i in the list $I_l(\mathscr{F})$. Now choose the right-hand-side vectors a^1, a^2, and a^3 in such a way that $a_i^l := \alpha_i^l$, for all $i \in I_l(\mathscr{F})$, $l = 1, 2, 3$ and all other coefficients of the vectors a^1, a^2, and a^3 are zero. Then $G_{\mathscr{S}}$ applied to this particular instance of (3TP) generates a solution with an objective-function value bounded from below by

$$c\left[i_1^1, i_1^2, i_1^3\right] + W(U_1, U_2, U_3).$$

This solution, however, is not optimal [cf. (2)].

$\Leftarrow$: Let $\mathscr{S}$ be a three-dimensional Monge sequence, and let x be the corresponding solution produced by algorithm $G_{\mathscr{S}}$. Assume x is not optimal. Choose the optimal solution y which maximizes p such that $x_{p_1} = y_{p_1}$ $\forall p_1 < p$ and whose y_p is largest among the set of all optimal solutions maximizing p. Let (i_1, j_1, k_1) be the triple of indices corresponding to p. Since $x_p \neq y_p$ and $G_{\mathscr{S}}$ maximizes each variable in turn, we have $x_p > y_p$. To compensate the deficits in $a_{i_1}^1$, $a_{j_1}^2$, and $a_{k_1}^3$, some positive variables with indices $(i_1, \cdot, \cdot)$, $(\cdot, j_1, \cdot)$ and $(\cdot, \cdot, k_1)$ have to succeed (i_1, j_1, k_1) in $\mathscr{S}$. The triples of indices corresponding to these variables form a feasible set $\mathscr{F}'$. Next we choose a set $\mathscr{F} \subseteq \mathscr{F}'$ such that $\mathscr{F}$ is minimal with respect to (i_1, j_1, k_1). Note that $\mathscr{F}$ contains $q \leqslant d$ triples. Let the lists of indices $I_1(\mathscr{F})$, $I_2(\mathscr{F})$, and $I_3(\mathscr{F})$ be as defined before.

We now have to show that (i_1, j_1, k_1) is the element of $M(\mathscr{F})$ which occurs first in $\mathscr{S}$. Assume the contrary, i.e. that there exists a triple $(i, j, k) \in M(\mathscr{F})$ which precedes (i_1, j_1, k_1) in $\mathscr{S}$. Since $G_{\mathscr{S}}$ maximizes y_{ijk}, either we have $y_{ivw} = 0$ for all (i, v, w) which are successors of (i, j, k) in $\mathscr{S}$, $y_{ujw} = 0$

for all (u, j, w) following (i, j, k) in $\mathscr{S}$, or $y_{uvk} = 0$ for all (u, v, k) later in $\mathscr{S}$. This leads, however, to a contradiction to the choice of $\mathscr{F}$, since this would imply that at least one variable indexed by a triple of indices of $\mathscr{F}$ has to be zero, a contradiction.

Thus (i_1, j_1, k_1) is the first element of $M(\mathscr{F})$ in $\mathscr{S}$. Now $\mathscr{F}$ is feasible and minimal with respect to (i_1, j_1, k_1), and since $\mathscr{S}$ is a Monge sequence, we get that

$$c[i_1, j_1, k_1] + \min_{\phi, \psi} \sum_{l=2}^{q} c[i_l, j_{\phi(l)}, k_{\psi(l)}] \leqslant \sum_{\alpha \in \mathscr{F}} c_\alpha, \tag{3}$$

where ϕ and ψ are mappings from $\{2, \dots, q\}$ onto $\{2, \dots, q\}$.

According to our convention there are exactly q variables on both sides of (3). Next we construct from the solution y a new solution $\hat{y}$ as follows: First note that since $\mathscr{F}$ is minimal with respect to (i_1, j_1, k_1), a triple of indices can occur at most once on each side of (3). Next we fix $\varepsilon_1 := \min_{\alpha \in \mathscr{F}} y_\alpha$ and set $\varepsilon := \min\{x_{i_1 j_1 k_1} - y_{i_1 j_1 k_1}, \varepsilon_1\} > 0$. Now we can obtain a new feasible solution $\hat{y}$ by setting

$$\hat{y}_\beta := \begin{cases} y_\beta - \varepsilon & \text{if } \beta \text{ occurs only on the right side of } (3), \\ y_\beta + \varepsilon & \text{if } \beta \text{ occurs only on the left side of } (3), \\ y_\beta & \text{otherwise.} \end{cases}$$

It is easy to verify that $\hat{y}$ is a feasible solution and because of (3) $\hat{y}$ is also optimal. Furthermore we have $\hat{y}_{p_1} = x_{p_1}$ $\forall p_1 < p$ and $\hat{y}_{i_1 j_1 k_1} = y_{i_1 j_1 k_1} + \varepsilon$, which leads to a contradiction to the selection of y. Thus the theorem is proven. ■

4. CONSTRUCTION OF d-DIMENSIONAL MONGE SEQUENCES

We start with some negative results, which on one hand point out the differences from d-dimensional Monge arrays and from classical Monge sequences, and on the other hand show that the construction of d-dimensional Monge sequences is not as easy as the recognition of d-dimensional Monge arrays.

d-dimensional Monge arrays have the nice property that each subarray is again Monge, and moreover, if each k-dimensional subarray of a d-dimensional array, $k \leqslant d$, fulfills the Monge property, then the d-dimensional array itself is a Monge array [2]. In the case of Monge sequences no analogous result holds. If an array C has a d-dimensional Monge sequence, clearly all subarrays of C have a lower-dimensional Monge sequence, but in constrast to Monge arrays the reverse is not true (consider the example given below).

As a second main difference we point out the following: In an $n \times 2$ matrix C there always exists a Monge sequence which can be constructed in $O(n \log n)$ time by sorting the rows of C such that $c_{11} - c_{12} \leqslant \cdots \leqslant c_{n1} - c_{n2}$. Unfortunately, this property does not generalize to higher dimensions as can be seen from the following example: Consider the $2 \times 2 \times 2$ array given by

$$\begin{pmatrix} 4 & 5 \\ 1 & 6 \end{pmatrix} \quad \begin{pmatrix} 8 & 6 \\ 3 & 2 \end{pmatrix}.$$

This array has no three-dimensional Monge sequence, although each 2×2 submatrix clearly has one.

The observations above demonstrate that an approach as used for the recognition of d-dimensional Monge arrays does not work for the construction of d-dimensional Monge sequences.

Fortunately, the construction of d-dimensional Monge sequences can be done using a generalization of a simple algorithm for two dimensions which was proposed by Alon et al. [1]. For the ease of exposition and to facilitate the complexity analysis we assume henceforth that our array is an $n \times n \times \cdots \times n$ array. An extension to $n_1 \times n_2 \times \cdots \times n_d$ arrays is straightforward.

The idea of our algorithm for constructing d-dimensional Monge sequences is based on a directed graph G which has the following properties. The nodes of G correspond to the entries in the array C, therefore, we have n^d nodes. Furthermore we have an arc (u, v) from node u to node v, iff $u \neq v$ and u has to precede v in the Monge sequence, i.e. the index corresponding to u occurs on the left side of (1), whereas v lies on the right side of (1) and (1) is indeed an inequality. Next we number the inequalities in (1) consecutively starting from 1. Each arc (u, v) in G is then labeled by the number of the unique inequality which induces this arc.

LEMMA 4.1. *The directed graph G described above can be constructed in $O(d^2(d!)^{d-1}n^{d^2})$ time.*

Proof. A fixed d-tuple $(i_1^1, i_1^2, \ldots, i_1^d)$ occurs on the left side of at most $O(n^{d(d-1)})$ inequalities of type (1). Thus we have to consider $O(n^{d_2})$ inequalities. To calculate the minimum in (1) we need $O((d!)^{d-1})$ time per inequality. Since each inequality induces at most d^2 arcs in G, G can be constructed in overall $O(d^2(d!)^{d-1}n^{d^2})$ time. ■

After the initialization of G described above, our algorithm proceeds as follows: We construct a d-dimensional Monge sequence step by step. In each step we have to find a node $v \in G$ with indegree equal to zero. If no such node exists, no d-dimensional Monge sequence exists, and we stop; otherwise the corresponding d-tuple of indices can be chosen as next element in the Monge sequence. If the algorithm stops with an empty graph, we have obtained a d-dimensional Monge sequence; otherwise no such sequence exists.

In each step, after the selection of a node v with indegree 0, we have to update the graph G. More precisely, we need to delete the node v and all arcs adjacent to v together with all arcs which are assigned a label of an arc with tail v. This is equivalent to canceling all conditions in which v occurs.

LEMMA 4.2. *The update of G as described above can be performed in $O(d^2(d!)^{d-1}n^{d^2-d})$ time.*

Proof. Since there are almost $O((d!)^{d-1}n^{d(d-1)})$ arcs emanating from node v which have distinct labels, and since there are at most d^2 arcs with the same label, we can perform the necessary updates in $O(d^2(d!)^{d-1}n^{d^2-d})$ time. ■

THEOREM 4.3. *The algorithm above constructs a d-dimensional Monge sequence in case one exists in $O(d^2(d!)^{d-1}n^{d^2})$ time.*

Proof. First we have to show the correctness of our algorithm. Suppose the algorithm stops while G is nonempty. Then each node in G has indegree greater than zero. But this means that all nodes in G occur at least once on the right side of an inequality which is still to be considered, i.e. an inequality all of whose terms correspond to nodes still in G. But this implies that no Monge sequence exists.

Now only the running-time complexity remains to be proven. From Lemma 4.1 we have that the initial graph G can be constructed in the claimed time. Each update needs $O(d^2(d!)^{d-1}n^{d^2-d})$ time per step, and since we have n^d steps (G has n^d nodes), we are finished. ■

It seems not worth trying to derive an improved algorithm for constructing d-dimensional Monge sequences which proceeds along the lines of the second algorithm of Alon, Cosares, Hochbaum, and Shamir [1], since using these techniques would lead to a very large increase of needed space while gaining only a minor improvement in running time.

We close this section by mentioning that the results described in this paper directly lead to a new class of polynomially solvable d-dimensional axial assignment problems. The d-dimensional axial assignment problem, (dAP), is obtained from the d-dimensional axial transportation problem by setting all right-hand-side coefficients a_j^i equal to 1 and by requiring additionally that all variables $x_{i_1 i_2 \cdots i_d}$ are integer and thus either 0 or 1. (dAP) is NP-hard in general, but in case that the cost array C has a d-dimensional Monge sequence, the problem becomes polynomially time solvable.

We mention that for the (dAP) the condition for a d-dimensional Monge sequence can be slightly relaxed. It is sufficient to require only those inequalities in (1) which are induced by minimal sets $\mathscr{F}$ for which the lists $I_l(\mathscr{F})$ are indeed sets and no lists, i.e. contain no multiple entries. For the greedy algorithm applied to (dAP) Theorem 3.1 can then be strengthened to hold for this relaxed notion of a d-dimensional Monge sequence. Hence (dAP)'s whose cost array has a d-dimensional Monge sequence can be solved in $O(n^{d^2})$ time. [If the Monge sequence is already at hand, $O(n^d)$ time suffices.]

5. CONCLUSION

In this paper we have investigated d-dimensional Monge sequences and generalized a known theorem of Hoffman for $d = 2$ to higher dimensions. Furthermore, we presented a polynomial-time algorithm for the construction and detection of a d-dimensional Monge sequence in a d-dimensional array for a fixed value of d.

We would like to thank Bettina Klinz for pointing out some inconsistencies in a first version.

REFERENCES

1 N. Alon, S. Cosares, D. S. Hochbaum, and R. Shamir, An algorithm for the detection and construction of Monge sequences, *Linear Algebra Appl.* 114/115:669–680 (1989).

2 A. Aggarwal and J. K. Park, Notes on searching in multidimensional monotone arrays, in *Proceedings of the 29th Annual IEEE Symposium on Foundations of Computer Science*, 1988, pp. 497–512.

3 W. Bein, P. Brucker, J. K. Park, and P. K. Pathak, A Monge property for the d-dimensional transportation problem, *Discrete Appl. Math.* 58:97–104 (1995).
4 V. G. Deineko and V. L. Filonenko, On the reconstruction of specially structured matrices (in Russian), in *Aktualnyye Problemy EVM, Programmirovaniye*, DGU, Dnepropetrovsk, 1979.
5 B. L. Dietrich, Monge sequences, antimatroids and the transportation problem with forbiden arcs, *Linear Algebra Appl.* 139:133–145 (1990).
6 B. L. Dietrich and R. Shamir, Characterization and algorithms for greedily solvable transportation problems, in *Proceedings of the First ACM–SIAM Symposium on Discrete Algorithms* Jan. 1990, pp. 358–366.
7 M. Fahmi and P. K. Pathak, Applications of a size reduction technique for transportation problems in sampling, in *Proceedings of the First International Symposium on Optimization and Statistics*, Aligarh, India, Dec. 1989.
8 G. Hadley, *Linear Programming*, Addison-Wesley, Reading, Mass., 1962.
9 A. J. Hoffman, On simple linear programming problems, in *Convexity*, Proc. Symp. Pure Math., Amer. Math. Soc., Prividence, pp. 317–327 (1961).
10 G. Monge, Déblai et remblai, *Mém. Acad. Sci.*, Paris, 1781.
11 R. Rudolf, Recognition of d-dimensional Monge arrays, *Discrete Appl. Math.* 52:71–82 (1994).
12 R. Shamir, A fast algorithm for constructing Monge sequences in transportation problems with forbidden arcs, *Discrete Math.* 114:435–444 (1993).

Received 12 May 1993; final manuscript accepted 14 November 1997