Some types of generalized fuzzy filters of $BL$-algebras

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**Abstract**

The aim of this paper is to introduce the notions of interval valued $(\epsilon, \in \lor q)$-fuzzy Boolean (implicative, positive implicative and fantastic) filters in $BL$-algebras and to investigate some related properties. Some characterizations of these generalized fuzzy filters are derived. Finally, the relations among these generalized fuzzy filters are discussed. It is proved that an interval valued $(\epsilon, \in \lor q)$-fuzzy filter in $BL$-algebras is an interval valued $(\epsilon, \epsilon \lor q)$-fuzzy positive implicative filter and an interval valued $(\epsilon, \epsilon \lor q)$-fuzzy fantastic filter.

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**1. Introduction**

$BL$-algebras are the algebraic structures for Hájek’s Basic Logic [1]. The main example of a $BL$-algebra is the interval $[0, 1]$ endowed with the structure induced by a continuous $t$-norm. $MV$-algebras, introduced by Chang [2] in 1958, are one of the most known classes of $BL$-algebras. In order to investigate the logic system whose semantic truth-value is given in a lattice, Xu [3] proposed the concept of lattice implication algebras, and studied the properties of filters in such algebras in [4]. Wang [5] proved that lattice implication algebras are categorically equivalent to $MV$-algebras. Furthermore, Wang [6] introduced the concept of $R_0$-algebras in order to provide an algebraic proof of the completeness theorem of a formal deductive system [7]. It is well known that $MV$-algebras, Gödel algebras and Product algebras are the most known class of $BL$-algebras. Filters theory play an important role in studying these logical algebras. From the logical point of view, various filters correspond to various sets of provable formulae. Hájek [1] introduced the concepts of filters and prime filters in $BL$-algebras. Using prime filters of $BL$-algebras, Hájek proved the completeness of Basic Logic $BL$. Turunen [8–11] studied some properties of the prime filters in $BL$-algebras. Havelshki et al. [12] further discussed some types of filters in $BL$-algebras. Kondo [13] also discussed the properties of filters of $BL$-algebras.

The theory of fuzzy sets which was introduced by Zadeh [14] has been applied to many research areas. In 2005, Liu and Li [15,16] introduced the concepts of fuzzy filters, fuzzy Boolean (implicative) and fuzzy positive implicative filters in $BL$-algebras and obtained some important properties of these fuzzy filters. Zhang et al. [17] introduced the concepts of fuzzy ultra filters and fuzzy obstinate filters in $BL$-algebras and gave some equivalent conditions among these filters. Among many theories, Zadeh [18] also introduced the concept of interval valued fuzzy subset, where the values of the membership functions are intervals of numbers instead of the numbers. Biswas [19] defined interval valued fuzzy subgroups of the same nature of Rosenfeld’s fuzzy subgroups. A new type of fuzzy subgroup (viz, $(\epsilon, \epsilon \lor q)$-fuzzy subgroup) was introduced in an earlier paper of Bhakat and Das [20,21] by using the combined notions of “belongingness ” and “quasi-coincidence” of fuzzy points and fuzzy sets, which was introduced by Pu and Liu [22]. In fact, $(\epsilon, \epsilon \lor q)$-fuzzy subgroup is an important and

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useful generalization of Rosenfeld’s fuzzy subgroup. Davvaz [23] applied this theory to near-rings and obtained some useful results. Zhan et al. [24] also discussed the properties of interval valued \((e, e \lor q)\)-fuzzy hyperideals in hypernear-rings.

Recently, Ma et al. [25] introduced the concept of interval valued \((e, e \lor q)\)-fuzzy filters in \(BL\)-algebras and discussed some properties. As a continuative work in [25], in this paper, we introduce the notions of interval valued \((e, e \lor q)\)-fuzzy Boolean (implicative, positive implicative and fantastic) filters in \(BL\)-algebras and investigate some related properties. Some characterizations of these generalized fuzzy filters are derived. Finally, we discuss the relations among these generalized fuzzy filters and prove that an interval valued \((e, e \lor q)\)-fuzzy filter in \(BL\)-algebras is Boolean (implicative) if and only if it is both positive implicative and fantastic.

2. Preliminaries

By a \(BL\)-algebra we mean a bounded lattice \(L = (L, \leq, \wedge, \lor, 0, 1)\) such that
(i) \((L, \lor, 1)\) is a commutative monoid,
(ii) \(\lor\) and \(\rightarrow\) form an adjoint pair, i.e., \(z \leq x \rightarrow y\) if and only if \(x \lor z \leq y\) for all \(x, y, z \in L\),
(iii) \(x \land y = x \land (x \rightarrow y)\),
(iv) \((x \rightarrow y) \lor (y \rightarrow x) = 1\).

In any \(BL\)-algebra \(L\), the following are true (see [26–29]):
(1) \(x \leq y \iff x \rightarrow y = 1\),
(2) \(x \rightarrow (y \rightarrow z) = (x \lor y) \rightarrow z = y \rightarrow (x \rightarrow z)\),
(3) \(x \land y \leq x \land y\),
(4) \(x \land y \leq (z \rightarrow x) \rightarrow (z \rightarrow y), x \rightarrow y \leq (y \rightarrow z) \rightarrow (x \rightarrow z)\),
(5) \(x \land x' = x\),
(6) \(x \lor x' = 1 \Rightarrow x \land x' = 0\),
(7) \(x \lor y = ((x \rightarrow y) \rightarrow y) \land ((y \rightarrow x) \rightarrow x)\),
where \(x' = x \rightarrow 0\).

A non-empty subset \(A\) of a \(BL\)-algebra \(L\) is called a filter of \(L\) if it satisfies: (i) \(1 \in A\), (ii) \(\forall x, y \in A, x \rightarrow y \in A \Rightarrow y \in A\).

It is easy to check that a non-empty subset \(A\) of a \(BL\)-algebra \(L\) is a filter of \(L\) if and only if it satisfies: (i) \(\forall x, y \in L, x \lor y \in A \Rightarrow y \in A\), (ii) \(\forall x \in A, x \leq y \Rightarrow y \in A\). A filter \(A\) of a \(BL\)-algebra \(L\) is called an implicative filter of \(L\) if it satisfies: \(x \rightarrow (z \rightarrow y) \in A, y \rightarrow z \in A \Rightarrow x \rightarrow z \in A\). A filter \(A\) of a \(BL\)-algebra \(L\) is called a Boolean filter of \(L\) if \(x \land x' \in A\), for all \(x \in L\). A filter \(A\) of a \(BL\)-algebra \(L\) is called a positive implicative filter of \(L\) if it satisfies: \(x \rightarrow (y \rightarrow z) \in A, x \rightarrow y \rightarrow z \in A \Rightarrow x \rightarrow z \in A\) (see [9,16]). A filter \(A\) of a \(BL\)-algebra \(L\) is called a fantastic filter of \(L\) if it satisfies: \(z \rightarrow (y \rightarrow x) \in A, z \in A \Rightarrow ((x \rightarrow y) \rightarrow y) \rightarrow x \in A\) (see [12]).

We now review some fuzzy logic concepts. A fuzzy set \(F\) of \(X\) is a function \(F : X \rightarrow [0, 1]\).

**Definition 2.1** ([15]). A fuzzy set \(F\) of a \(BL\)-algebra \(L\) is called a fuzzy filter of \(L\) if it satisfies:
(1) \(\forall x, y \in L, F(x \land y) \geq \min\{F(x), F(y)\}\),
(2) \(F\) is order-preserving, that is, \(\forall x, y \in L, x \leq y \Rightarrow F(x) \leq F(y)\).

**Definition 2.2** ([16]). A fuzzy filter \(F\) of \(L\) is called a fuzzy Boolean filter of \(L\) if it satisfies:
(3) \(\forall x \in L, F(x \lor x') = 1\).

**Definition 2.3** ([16]). A fuzzy filter \(F\) of \(L\) is called a fuzzy implicative filter of \(L\) if it satisfies:
(4) \(F(x \rightarrow z) \geq \min\{F(x \rightarrow (z \rightarrow y)), F(y \rightarrow z)\}\), for all \(x, y, z \in L\).

**Theorem 2.4** ([16]). A fuzzy set \(F\) of any \(BL\)-algebra is a fuzzy Boolean filter if and only if it is a fuzzy implicative filter.

A fuzzy set \(F\) of a \(BL\)-algebra \(L\) of the form

\[
F(y) = \begin{cases} 
1 & \text{if } y = x, \\
0 & \text{if } y \neq x.
\end{cases}
\]

is said to be a fuzzy point with support \(x\) and value \(t\) and is denoted by \(U(x; t)\). A fuzzy point \(U(x; t)\) is said to belong to (resp. be quasi-coincident with) a fuzzy set \(F\), written as \(U(x; t) \in F\) (resp. \(U(x; t) \in F\)) if \(F(x) \geq t\) (resp. \(F(x) + t > 1\)). If \(U(x; t) \in F\) or (resp. and) \(U(x; t) \in F\), then we write \(U(x; t) \in \bigvee q\). The symbol \(\bigvee q\) means \(\lor q\) does not hold. Using the notion of “belongingness (\(e\))” and “quasi-coincidence (\(q\))” of fuzzy points with fuzzy subsets, the concept of \((\alpha, \beta)\)-fuzzy subsemigroup, where \(\alpha, \beta\) are any two of \(\{e, q\}, e \lor q \in \bigvee q\) with \(\alpha \neq \alpha \lor q\), was introduced in [21]. It is noteworthy that the most viable generalization of Rosenfeld’s fuzzy subgroup is the notion of \((e, e \lor q)\)-fuzzy subgroup. The detailed study with \((e, e \lor q)\)-fuzzy subgroup has been considered in [20].

By an interval \(\tilde{a}\) we mean [cf. [18,19]] an interval \([a^-, a^+], 0 \leq a^- \leq a^+ \leq 1\). The set of all intervals is denoted by \(D[0, 1]\). The interval \([a, a]\) is identified with the number \(a \in [0, 1]\).

For interval \(\tilde{a}_i = [a_i^-, a_i^+] \in D[0, 1], \tilde{b}_i = [b_i^-, b_i^+] \in D[0, 1], i \in I,\) we define
\[
\max(\tilde{a}_i, \tilde{b}_i) = \{\max(a_i^-, b_i^-), \max(a_i^+, b_i^+)\},
\]
\[
\min(\tilde{a}_i, \tilde{b}_i) = \{\min(a_i^-, b_i^-), \min(a_i^+, b_i^+)\},
\]
\[ \text{rinf}_{\tilde{a}} = [\bigwedge_{i \in I} a_i^-, \bigwedge_{i \in I} a_i^+] \]
\[ \text{rsup}_{\tilde{a}} = [\bigvee_{i \in I} a_i^-, \bigvee_{i \in I} a_i^+] \]

and put

1. \( \tilde{a}_1 \leq \tilde{a}_2 \iff a_1^- \leq a_2^- \) and \( a_1^+ \leq a_2^+ \),
2. \( \tilde{a}_1 = \tilde{a}_2 \iff a_1^- = a_2^- \) and \( a_1^+ = a_2^+ \),
3. \( \tilde{a}_1 < \tilde{a}_2 \iff a_1^- \leq a_2^- \) and \( a_1^+ \neq a_2^+ \),
4. \( k\tilde{a} = [ka^-, ka^+] \), whenever \( 0 \leq k \leq 1 \).

It is clear that \( (D[0, 1], \leq, \lor, \land) \) is a complete lattice with \( 0 = \{0, 0\} \) as the least element and \( 1 = \{1, 1\} \) as the greatest element.

By an interval valued fuzzy set \( F \) on \( X \) we mean (cf. [18]) the set

\[ F = \{(x, [\mu^-_F(x), \mu^+_F(x)]) \mid x \in X\} \]

where \( \mu^-_F \) and \( \mu^+_F \) are two fuzzy subsets of \( X \) such that \( \mu^-_F(x) \leq \mu^+_F(x) \) for all \( x \in X \). Putting \( \bar{\mu}_F(x) = [\mu^-_F(x), \mu^+_F(x)] \), we see that \( F = \{(x, \bar{\mu}_F(x)) \mid x \in X\} \), where \( \bar{\mu}_F : X \rightarrow D[0, 1] \).

Based on the above kinds of fuzzy filters of \( BL \)-algebras, we can introduce the following concepts:

**Definition 2.5.** An interval valued fuzzy set \( F \) of a \( BL \)-algebra \( L \) is called an interval valued fuzzy filter of \( L \) if it satisfies:

1. \( \forall x, y \in L, \bar{\mu}_F(x \circ y) \geq \text{rmin}[\bar{\mu}_F(x), \bar{\mu}_F(y)] \),
2. \( \forall x, y \in L, x \leq y \Rightarrow \bar{\mu}_F(x) \leq \bar{\mu}_F(y) \).

**Definition 2.6.** An interval valued fuzzy filter \( F \) of \( L \) is called an interval valued fuzzy Boolean filter of \( L \) if it satisfies:

1. \( \forall x \in L, \bar{\mu}_F(x \lor x') = [1, 1] \).

**Definition 2.7.** An interval valued fuzzy filter \( F \) of \( L \) is called an interval valued fuzzy implicative filter of \( L \) if it satisfies:

1. \( \bar{\mu}_F(x \rightarrow z) \geq \text{rmin}[\bar{\mu}_F(x \rightarrow (z' \rightarrow y)), \bar{\mu}_F(y \rightarrow z)] \), for all \( x, y, z \in L \).

**Theorem 2.8.** An interval valued fuzzy set of any \( BL \)-algebra is an interval valued fuzzy Boolean filter if and only if it is an interval valued fuzzy implicative filter.

**Proof.** It is similar to the proof of Theorem 2.4. \( \square \)

3. Interval valued \( (e, \in \lor q) \)-fuzzy Boolean filters

Based on [20,21], we extend the concept of quasi-coincidence of fuzzy point with a fuzzy set to the concept of quasi-coincidence of a fuzzy interval value with an interval valued fuzzy set as follows.

An interval valued fuzzy set \( F \) of a \( BL \)-algebra \( L \) of the form

\[ \bar{\mu}_F(y) = \begin{cases} \{\text{rmin}(\bar{\mu}_F(x), \bar{\mu}_F(y))\} & \text{if } y = x, \\ [0, 0] & \text{if } y \neq x. \end{cases} \]

is said to be fuzzy interval value with support \( x \) and interval value \( \bar{t} \) and is denoted by \( U(x; \bar{t}) \). A fuzzy interval value \( U(x; \bar{t}) \) is said to belong to (resp. be quasi-coincident with) an interval valued fuzzy set \( F \), written as \( U(x; \bar{t}) \in F \) (resp. \( U(x; \bar{t})qF \)) if \( \bar{\mu}_F(x) \geq \text{rmin}[\bar{\mu}_F(x \rightarrow (z' \rightarrow y)), \bar{\mu}_F(y \rightarrow z)] \), for all \( x, y, z \in L \).

In what follows, let \( L \) be a \( BL \)-algebra unless otherwise specified. Also we emphasize \( \bar{\mu}_F(x) = [\mu^-_F(x), \mu^+_F(x)] \) must satisfy the following properties:

1. Any two elements of \( D[0, 1] \) are comparable;
2. \( [\mu^-_F(x), \mu^+_F(x)] \leq [0.5, 0.5] \) or \( [0.5, 0.5] < [\mu^-_F(x), \mu^+_F(x)] \), for all \( x \in L \).

**Definition 3.1** ([25]). An interval valued fuzzy set \( F \) of \( L \) is said to be an interval valued \( (e, \in \lor q) \)-fuzzy filter of \( L \) if for all \( t, r \in (0, 1) \) and \( x, y \in L \),

\[ (F1) \ U(x; \bar{t}) \in F \text{ and } U(y; \bar{t}) \in F \] \[ \text{imply } U(x \circ y; \text{rmin}(\bar{t}, \bar{t})) \in \lor qF, \]
\[ (F2) \ U(x; \bar{t}) \in F \text{ implies } U(y; \bar{t}) \in \lor qF \text{ with } x \leq y. \]

**Theorem 3.2.** The conditions (F1) and (F2) in Definition 3.1, are equivalent to the following conditions respectively:

\[ (F3) \ \bar{\mu}_F(x \circ y) \geq \text{rmin}[\bar{\mu}_F(x), \bar{\mu}_F(y), [0.5, 0.5]], \text{ for all } x, y \in L, \]
\[ (F4) \forall x, y \in L, x \leq y \Rightarrow \bar{\mu}_F(y) \geq \text{rmin}[\bar{\mu}_F(x), [0.5, 0.5]]. \]
Proof. (F1) $\rightarrow$ (F3): Suppose that $x, y \in L$, we consider the following cases:
(a) $\mu_{\min}(\overline{\mu}_L(x), \overline{\mu}_L(y)) \leq [0.5, 0.5]$.
(b) $\mu_{\min}(\overline{\mu}_L(x), \overline{\mu}_L(y)) > [0.5, 0.5]$.
Case (a): Assume that $\overline{\mu}_L(x \circ y) < \mu_{\min}(\overline{\mu}_L(x), \overline{\mu}_L(y))$, which implies $\overline{\mu}_L(x \circ y) < \mu_{\min}(\overline{\mu}_L(x), \overline{\mu}_L(y))$. Choose $t$ such that $\overline{\mu}_L(t \circ y) < \overline{\mu}_L(x \circ y)$. Then $U(x; t) \in F$ and $U(y; t) \in F$, but $U(x \circ y; t) \notin \nu F$, which contradicts (F1).
Case (b): Assume that $\overline{\mu}_L(x \circ y) < [0.5, 0.5]$, then $U(x; [0.5, 0.5]) \in F$ and $U(y; [0.5, 0.5]) \in F$, but $U(x \circ y; [0.5, 0.5]) \notin \nu F$, a contradiction. Hence (F3) holds.

(F2) $\rightarrow$ (F4): Suppose that $x, y \in L$, we consider the following two cases:
(a) $\overline{\mu}_L(x) < [0.5, 0.5]$.
(b) $\overline{\mu}_L(x) > [0.5, 0.5]$.
Case (a): Let $x \leq y$. Assume that $\overline{\mu}_L(x) = \overline{\tau} < [0.5, 0.5]$ and $\overline{\mu}_L(y) = \overline{\tau} < \overline{\mu}_L(x)$. Choose $s$ such that $\overline{\tau} < \overline{s} < \overline{\tau}$ and $\overline{s} \in [1, 1]$. Then $U(y; s) \in F$, but $U(x; s) \notin \nu F$, which contradicts (F2). So $\overline{\mu}_L(y) > \mu_{\min}(\overline{\mu}_L(x), [0.5, 0.5])$.
Case (b): Let $x \leq y$ and $\overline{\mu}_L(x) > [0.5, 0.5]$. If $\overline{\mu}_L(y) < \mu_{\min}(\overline{\mu}_L(x), [0.5, 0.5])$, then $U(y; [0.5, 0.5]) \in F$, but $U(x; [0.5, 0.5]) \notin \nu F$, which contradicts (F2). Hence (F4) holds.

(F3) $\Rightarrow$ (F1): Let $U(x; t) \in F$ and $U(y; t) \in F$, then $\overline{\mu}_L(x) \geq \overline{t}$ and $\overline{\mu}_L(y) \geq \overline{t}$. Now, we have
$\overline{\mu}_L(x \circ y) \geq \mu_{\min}(\overline{\mu}_L(x), \overline{\mu}_L(y), [0.5, 0.5]) \geq \mu_{\min}(\overline{t}, \overline{t}, [0.5, 0.5])$.
If $\overline{\mu}(\overline{t}, \overline{t}) > [0.5, 0.5]$, then $\overline{\mu}_L(x \circ y) \geq [0.5, 0.5]$, which implies $\overline{\mu}_L(x \circ y) + \mu_{\min}(\overline{t}, \overline{t}) > [1, 1]$. If $\overline{\mu}(\overline{t}, \overline{t}) \leq [0.5, 0.5]$, then $\overline{\mu}_L(x \circ y) \geq \mu_{\min}(\overline{t}, \overline{t})$. Therefore $U(x \circ y; \mu_{\min}(\overline{t}, \overline{t})) \notin \nu F$.

(F4) $\Rightarrow$ (F2): Let $x = y$. Suppose that $U(x; t) \in F$, then $\overline{\mu}_L(x) \geq \overline{t}$, and so $\overline{\mu}_L(y) \geq \mu_{\min}(\overline{\mu}_L(x), [0.5, 0.5]) \geq \mu_{\min}(\overline{\mu}_L(x), [0.5, 0.5])$, which implies $\overline{\mu}_L(y) \geq \overline{t}$ or $\overline{\mu}_L(y) \geq [0.5, 0.5]$ according as $\overline{t} \leq [0.5, 0.5]$ or $\overline{t} > [0.5, 0.5]$. Therefore $U(y; t) \notin \nu F$. □

Remark 3.3. (i) Every interval valued fuzzy filter of $L$ is an interval valued ($\varepsilon, \in \nu F$)-fuzzy filter; the converse may not be true.
(ii) If $F$ is an interval valued ($\varepsilon, \in \nu F$)-fuzzy filter of $L$ with $\overline{\mu}(1) < [0.5, 0.5]$, then $F$ is an interval valued fuzzy filter.

Theorem 3.4. An interval valued fuzzy set $F$ of $L$ is an interval valued ($\varepsilon, \in \nu F$)-fuzzy filter of $L$ if and only if it satisfies:
(F5) $\forall x \in L$, $\overline{\mu}_L(1) \geq \mu_{\min}(\overline{\mu}_L(x), [0.5, 0.5])$.
(F6) $\forall x, y \in L$, $\overline{\mu}_L(y \circ x) \geq \mu_{\min}(\overline{\mu}_L(x), \overline{\mu}_L(x \circ y), [0.5, 0.5])$.

Proof. Assume that $F$ satisfies the conditions (F5) and (F6). Let $x, y \in L$ be such that $x \leq y$. Then $x = y = 1$, and so $\overline{\mu}_L(1) \geq \mu_{\min}(\overline{\mu}_L(x), \overline{\mu}_L(1), [0.5, 0.5]) = \mu_{\min}(\overline{\mu}_L(x), [0.5, 0.5])$, which proves (F4).
Since $x = y$ and $\mu_{\min}(\overline{\mu}_L(x), \overline{\mu}_L(x \circ y), [0.5, 0.5])$, it follows from (F5) and (F6) that
$\overline{\mu}_L(x \circ y) \geq \mu_{\min}(\overline{\mu}_L(x), \overline{\mu}_L(x \circ y), [0.5, 0.5])$
$\geq \mu_{\min}(\overline{\mu}_L(y), \mu_{\min}(\overline{\mu}_L(x), \overline{\mu}_L(x \circ y)), [0.5, 0.5])$
$= \mu_{\min}(\overline{\mu}_L(y), \mu_{\min}(\overline{\mu}_L(x), [0.5, 0.5]), [0.5, 0.5])$
$= \mu_{\min}(\overline{\mu}_L(y), \overline{\mu}_L(x), [0.5, 0.5]),$
which proves (F3).
Hence $F$ is an interval valued ($\varepsilon, \in \nu F$)-fuzzy filter of $L$.

Conversely, assume that $F$ is an interval valued ($\varepsilon, \in \nu F$)-fuzzy filter of $L$. Since $x \leq 1$ for all $x \in L$, it follows (F4) that $\overline{\mu}_L(1) \geq \mu_{\min}(\overline{\mu}_L(x), [0.5, 0.5])$, for all $x \in L$. Let $x, y \in L$. Since $x \leq (x \to y)$, we have $x \circ (x \to y) \leq y$, by the Galois correspondence. Hence $\overline{\mu}_L(1) \geq \mu_{\min}(\overline{\mu}_L(x \circ (x \to y)), [0.5, 0.5]) \geq \mu_{\min}(\overline{\mu}_L(x), \overline{\mu}_L(x \circ y), [0.5, 0.5])$. This completes the proof. □

Definition 3.5. An interval valued ($\varepsilon, \in \nu F$)-fuzzy filter of $L$ is called an interval valued ($\varepsilon, \in \nu F$)-fuzzy Boolean filter of $L$ if it satisfies:
(F7) $\forall x \in L$, $\overline{\mu}_L(x \vee x) \geq [0.5, 0.5]$. Clearly, every interval valued fuzzy Boolean filter of $L$ is an interval valued ($\varepsilon, \in \nu F$)-fuzzy Boolean filter.

Example 3.6. Let $L = \{0, a, b, 1\}$ be a chain with Cayley table as follows:

<table>
<thead>
<tr>
<th>$\circ$</th>
<th>0</th>
<th>a</th>
<th>b</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>a</td>
<td>a</td>
<td>a</td>
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<tr>
<td>a</td>
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<td>a</td>
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</tr>
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<td>b</td>
<td>b</td>
<td>b</td>
<td>b</td>
<td>b</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

Define $\wedge$ and $\vee$ operations on $L$ as min and max, respectively. Then $(L, \wedge, \vee, \circ, \rightarrow, 0, 1)$ is a BL-algebra. Define an interval valued fuzzy set $F$ in $L$ by $\overline{\mu}_F(0) = [0.3, 0.4]$, $\overline{\mu}_F(1) = \overline{\mu}_F(b) = \overline{\mu}_F(a) = [0.7, 0.8]$. One can easily verify that $F$ is an interval valued ($\varepsilon, \in \nu F$)-fuzzy Boolean filter of $L$. 

Let $F$ be an interval valued fuzzy set. For every $t \in (0, 1)$, the set $U(F; \tilde{t}) = \{x \in L | \tilde{\mu}_F(x) \geq \tilde{t}\}$ is called the level subset of $F$.

By their level Boolean filters of BL-algebras, we characterize interval valued $(\varepsilon, \in \mathcal{V})$-fuzzy Boolean filters as follows:

**Lemma 3.7.** Let $F$ be an interval valued $(\varepsilon, \in \mathcal{V})$-fuzzy filter of $L$. Then for all $[0, 0] < \tilde{t} \leq [0.5, 0.5]$, $U(F; \tilde{t})$ is an empty set or a filter of $L$. Conversely, if $F$ is an interval valued fuzzy set of $L$ such that $U(F; \tilde{t}) (\neq \emptyset)$ is a filter of $L$ for all $[0, 0] < \tilde{t} \leq [0.5, 0.5]$, then $F$ is an interval valued $(\varepsilon, \in \mathcal{V})$-fuzzy filter of $L$.

**Proof.** Let $F$ be an interval valued $(\varepsilon, \in \mathcal{V})$-fuzzy filter of $L$ and $[0, 0] < \tilde{t} \leq [0.5, 0.5]$. Let $x, y \in U(F; \tilde{t})$, then $\tilde{\mu}_F(x) \geq \tilde{t}$ and $\tilde{\mu}_F(y) \geq \tilde{t}$. Now we have

\[
\tilde{\mu}_F(x \circ y) \geq \min(\tilde{\mu}_F(x), \tilde{\mu}_F(y), [0.5, 0.5]) \geq \min(\tilde{t}, [0.5, 0.5]) = \tilde{t}.
\]

which implies, $x \circ y \in U(F; \tilde{t})$. Let $x, y \in L$ be such that $x \leq y$. If $x \in U(F; \tilde{t})$, then by (F3), we have

\[
\tilde{\mu}_F(y) \geq \min(\tilde{\mu}_F(x), [0.5, 0.5]) \geq \min(\tilde{t}, [0.5, 0.5]) = \tilde{t},
\]

which implies $y \in U(F; \tilde{t})$. Hence $U(F; \tilde{t})$ is a filter of $L$.

Conversely, let $F$ be an interval valued fuzzy set of $L$ such that $U(F; \tilde{t}) (\neq \emptyset)$ is a filter of $L$ for all $[0, 0] < \tilde{t} \leq [0.5, 0.5]$. For every $x, y \in L$, we can write

\[
\tilde{\mu}_F(x) \geq \min(\tilde{\mu}_F(x), \tilde{\mu}_F(y), [0.5, 0.5]) = \tilde{t}_0,
\]

\[
\tilde{\mu}_F(y) \geq \min(\tilde{\mu}_F(x), \tilde{\mu}_F(y), [0.5, 0.5]) = \tilde{t}_0,
\]

then $x, y \in U(F; \tilde{t}_0)$, and so $x \circ y \in U(F; \tilde{t}_0)$. Hence

\[
\tilde{\mu}_F(x \circ y) \geq \min(\tilde{\mu}_F(x), \tilde{\mu}_F(y), [0.5, 0.5]).
\]

Also, let $x, y \in L$ be such that $x \leq y$. Putting $\tilde{\mu}_F(x) \geq \min(\tilde{\mu}_F(x), [0.5, 0.5]) = \tilde{t}_0$. Hence $x \in U(F; \tilde{t}_0)$, and so $y \in U(F; \tilde{t}_0)$. Thus

\[
\tilde{\mu}_F(y) \geq \tilde{t}_0 = \min(\tilde{\mu}_F(x), [0.5, 0.5]).
\]

Therefore, $F$ is an interval valued $(\varepsilon, \in \mathcal{V})$-fuzzy filter of $L$. □

**Theorem 3.8.** Let $F$ be an interval valued $(\varepsilon, \in \mathcal{V})$-fuzzy Boolean filter of $L$. Then for all $[0, 0] < \tilde{t} \leq [0.5, 0.5]$, $U(F; \tilde{t})$ is an empty set or a Boolean filter of $L$. Conversely, if $F$ is an interval valued fuzzy set of $L$ such that $U(F; \tilde{t}) (\neq \emptyset)$ is a Boolean filter of $L$ for all $[0, 0] < \tilde{t} \leq [0.5, 0.5]$, then $F$ is an interval valued $(\varepsilon, \in \mathcal{V})$-fuzzy Boolean filter of $L$.

**Proof.** Let $F$ be an interval valued $(\varepsilon, \in \mathcal{V})$-fuzzy Boolean filter of $L$ and $[0, 0] < \tilde{t} \leq [0.5, 0.5]$. By Lemma 3.7, we know that $U(F; \tilde{t})$ is a filter of $L$. Since $F$ is an interval valued $(\varepsilon, \in \mathcal{V})$-fuzzy Boolean filter of $L$, it follows that $\tilde{\mu}_F(x \circ x) \geq [0.5, 0.5] \geq \tilde{t}$, and so $x \circ x \in U(F; \tilde{t})$. Hence $U(F; \tilde{t})$ is a Boolean filter of $L$.

Conversely, let $F$ be an interval valued fuzzy set of $L$ such that $U(F; \tilde{t}) (\neq \emptyset)$ is a Boolean filter of $L$ for all $[0, 0] < \tilde{t} \leq [0.5, 0.5]$. Then by Lemma 3.7, we know that $F$ is an interval valued $(\varepsilon, \in \mathcal{V})$-fuzzy filter of $L$. Since $U(F; [0.5, 0.5])$ is a Boolean filter of $L$, then, for all $x \in L, x \circ x' \in U(F; [0.5, 0.5])$, i.e., $\tilde{\mu}_F(x \circ x') \geq [0.5, 0.5]$. Therefore, $F$ is an interval valued $(\varepsilon, \in \mathcal{V})$-fuzzy Boolean filter of $L$. □

Naturally, a corresponding result should be considered when $U(F; \tilde{t})$ is a Boolean filter of $L$ for all $[0.5, 0.5] < \tilde{t} \leq [1, 1]$. Now, we first give a Lemma as follows:

**Lemma 3.9.** Let $F$ be an interval valued fuzzy set of $L$. Then $U(F; \tilde{t})(\neq \emptyset)$ is a filter of $L$ for all $[0.5, 0.5] < \tilde{t} \leq [1, 1]$ if and only if for all $x, y \in L$:

\[
(F8) \ rmax(\tilde{\mu}_F(x \circ y), [0.5, 0.5]) \geq \min(\tilde{\mu}_F(x), \tilde{\mu}_F(y));
\]

\[
(F9) \ rmax(\tilde{\mu}_F(y), [0.5, 0.5]) \geq \tilde{\mu}_F(x) \text{ with } x \leq y.
\]

**Proof.** Assume that $U(F; \tilde{t})(\neq \emptyset)$ is a filter of $L$. Suppose that for some $x, y \in L$, $rmax(\tilde{\mu}_F(x \circ y), [0.5, 0.5]) < \min(\tilde{\mu}_F(x), \tilde{\mu}_F(y)) = \tilde{t}$, then $[0.5, 0.5] < \tilde{t} \leq [1, 1]$, $\tilde{\mu}_F(x \circ y) < \tilde{t}$, and $x, y \in U(F; \tilde{t})$. Since $x, y \in U(F; \tilde{t})$ and $U(F; \tilde{t})$ is a filter of $L$, so $x \circ y \in U(F; \tilde{t})$ or $\tilde{\mu}_F(x \circ y) \geq \tilde{t}$, which is a contradiction with $\tilde{\mu}_F(x \circ y) < \tilde{t}$. Hence (F8) holds.

If there exist $x, y \in L$ such that $rmax(\tilde{\mu}_F(y), [0.5, 0.5]) < \tilde{\mu}_F(x) = \tilde{t}$, then $[0.5, 0.5] < \tilde{t} \leq [1, 1]$, $\tilde{\mu}_F(y) < \tilde{t}$ and $x \in U(F; \tilde{t})$. Since $x \in U(F; \tilde{t})$, we have $y \in U(F; \tilde{t})$ or $\tilde{\mu}_F(y) \geq \tilde{t}$, which is a contradiction. Hence (F9) holds.

Conversely, suppose that the conditions (F8) and (F9) hold. To prove $U(F; \tilde{t})$ is a filter of $L$. Assume that $[0.5, 0.5] < \tilde{t} \leq [1, 1]$, $x, y \in U(F; \tilde{t})$. Then $[0.5, 0.5] < \tilde{t} \leq \min(\tilde{\mu}_F(x), \tilde{\mu}_F(y)) \leq rmax(\tilde{\mu}_F(x \circ y), [0.5, 0.5]) = \tilde{\mu}_F(x \circ y)$, which implies, $x \circ y \in U(F; \tilde{t})$. Let $x \leq y$ and $x \in U(F; \tilde{t})$, then $[0.5, 0.5] < \tilde{t} \leq \tilde{\mu}_F(x) \leq rmax(\tilde{\mu}_F(y), [0.5, 0.5]) = \tilde{\mu}_F(y)$, and so $y \in U(F; \tilde{t})$. Therefore $U(F; \tilde{t})$ is a filter of $L$. □
Theorem 3.10. Let $F$ be an interval valued fuzzy set of $L$. Then $U(F; \tilde{t}) (\neq \emptyset)$ is a Boolean filter of $L$ for all $[0.5, 0.5] < \tilde{t} \leq [1, 1]$ if and only if for all $x, y \in L$,

(F8) $\max[\mu_{F}(x \circ y), [0.5, 0.5]] \geq \min[\mu_{F}(x), \mu_{F}(y)]$;

(F9) $\max[\mu_{F}(y), [0.5, 0.5]] \geq \mu_{F}(x)$ with $x \leq y$;

(F10) $\mu_{F}(x \vee x') \geq \tilde{t}$.

Proof. Assume that $U(F; \tilde{t}) (\neq \emptyset)$ is a Boolean filter of $L$. It follows from Lemma 3.9 that (F8) and (F9) hold. Since $U(F; \tilde{t})$ is Boolean, then, for all $x \in L$, $x \vee x' \in U(F; \tilde{t})$, that is, $\mu_{F}(x \vee x') \geq \tilde{t}$. Thus (F10) holds.

Conversely, suppose that the conditions (F8), (F9) and (F10) hold. It follows from Lemma 3.9 that $U(F; \tilde{t})$ is a filter of $L$. By (F10), for all $x \in L$, $\mu_{F}(x \vee x') \geq \tilde{t}$, i.e., $x \vee x' \in U(F; \tilde{t})$. Thus $U(F; \tilde{t})$ is a Boolean filter of $L$. \(\square\)

Based on [16], we introduce the concept of interval valued $(\varepsilon, \in \vee \emptyset)$-fuzzy implicative filters in BL-algebras as follows:

Definition 3.11. An interval valued $(\varepsilon, \in \vee \emptyset)$-fuzzy filter of $L$ is called an interval valued $(\varepsilon, \in \vee \emptyset)$-fuzzy implicative filter of $L$ if it satisfies:

(F11) $\mu_{F}(x \rightarrow z) \geq \min[\mu_{F}(x \rightarrow (z' \rightarrow y)), \mu_{F}(y \rightarrow z), [0.5, 0.5]]$, for all $x, y, z \in L$.

Remark 3.12. (i) Every interval valued implicative filter of $L$ is an interval valued $(\varepsilon, \in \vee \emptyset)$-fuzzy implicative filter;

(ii) If $F$ is an interval valued $(\varepsilon, \in \vee \emptyset)$-fuzzy implicative filter of $L$ with $\mu_{F}(1) < [0.5, 0.5]$, then $F$ is an interval valued fuzzy implicative filter.

Theorem 3.13. Let $F$ be an interval valued $(\varepsilon, \in \vee \emptyset)$-fuzzy filter of $L$. The following are equivalent:

(i) $F$ is an interval valued $(\varepsilon, \in \vee \emptyset)$-fuzzy implicative filter;

(ii) $\mu_{F}(x \rightarrow z) \geq \min[\mu_{F}(x \rightarrow (z' \rightarrow z)), [0.5, 0.5]]$, for all $x, z \in L$;

(iii) $\mu_{F}(x \rightarrow z) \geq \min[\mu_{F}(y \rightarrow (x \rightarrow (z' \rightarrow z))), \mu_{F}(y), [0.5, 0.5]]$, for all $x, y, z \in L$.

Proof. (i)\(\Rightarrow\) (ii) Assume that $F$ is an interval valued $(\varepsilon, \in \vee \emptyset)$-fuzzy implicative filter of $L$. Putting $y = z$ in (F11), we have

$$\mu_{F}(x \rightarrow z) \geq \min[\mu_{F}(x \rightarrow (z' \rightarrow z)), \mu_{F}(z \rightarrow z), [0.5, 0.5]]$$

$$= \min[\mu_{F}(x \rightarrow (z' \rightarrow z)), \mu_{F}(1), [0.5, 0.5]]$$

$$= \min[\mu_{F}(x \rightarrow (z' \rightarrow z)), [0.5, 0.5]].$$

Thus, (ii) holds.

(ii)\(\Rightarrow\) (iii) For any $x, y, z \in L$, by Theorem 3.2 (F4), we have

$$\mu_{F}(x \rightarrow (z' \rightarrow z)) \geq \min[\mu_{F}(y \rightarrow (x \rightarrow (z' \rightarrow z))), \mu_{F}(y), [0.5, 0.5]].$$

Using (ii), we obtain

$$\mu_{F}(x \rightarrow z) \geq \min[\mu_{F}(x \rightarrow (z' \rightarrow z)), [0.5, 0.5]]$$

$$\geq \min[\mu_{F}(y \rightarrow (x \rightarrow (z' \rightarrow z))), \mu_{F}(y), [0.5, 0.5]],$$

which proves (iii).

(iii)\(\Rightarrow\) (i) Let $F$ be an interval valued $(\varepsilon, \in \vee \emptyset)$-fuzzy filter of $L$ satisfying the condition (iii). Then, for all $x, y, z \in L$, we have

$$\mu_{F}(x \circ z' \rightarrow z) \geq \min[\mu_{F}(x \circ z' \rightarrow y), \mu_{F}(y \rightarrow z), [0.5, 0.5]].$$

That is,

$$\mu_{F}(x \rightarrow (z' \rightarrow z)) \geq \min[\mu_{F}(x \rightarrow (z' \rightarrow y)), \mu_{F}(y \rightarrow z), [0.5, 0.5]].$$

Putting $y = 1$ in (iii), we get

$$\mu_{F}(x \rightarrow z) \geq \min[\mu_{F}(1 \rightarrow (x \rightarrow (z' \rightarrow z))), \mu_{F}(1), [0.5, 0.5]]$$

$$= \min[\mu_{F}(x \rightarrow (z' \rightarrow z)), [0.5, 0.5]]$$

$$\geq \min[\mu_{F}(x \rightarrow (z' \rightarrow y)), \mu_{F}(y \rightarrow z), [0.5, 0.5]].$$

This shows that (F11) holds. Thus, $F$ is an interval valued $(\varepsilon, \in \vee \emptyset)$-fuzzy implicative filter of $L$. \(\square\)

Next, we investigate the relation between interval valued $(\varepsilon, \in \vee \emptyset)$-fuzzy implicative filters and interval valued $(\varepsilon, \in \vee \emptyset)$-fuzzy Boolean filters.

Theorem 3.14. An interval valued fuzzy set of $L$ is an interval valued $(\varepsilon, \in \vee \emptyset)$-fuzzy implicative filter of $L$ if and only if it is an interval valued $(\varepsilon, \in \vee \emptyset)$-fuzzy Boolean filter of $L$.\(\square\)
Proof. Suppose that \( F \) is an interval valued \((\varepsilon, \in \lor q)\)-fuzzy Boolean filter of \( L \). Then, for all \( x, y, z \in L \),
\[
\mu_F(x \to z) \geq \min[\mu_F((z \lor z') \to (x \to z)), \mu_F(z \lor z'), [0.5, 0.5]]
\]
\[
\geq \min[\mu_F((z \lor z') \to (x \to z)), [0.5, 0.5]].
\]
Since
\[
(z \lor z') \to (x \to z) = (z \to (x \to z)) \land (z' \to (x \to z))
\]
\[
= z' \to (x \to z)
\]
\[
= x \to (z' \to z),
\]
it follows that
\[
\mu_F(x \to z) \geq \min[\mu_F(x \to (z' \to z)), [0.5, 0.5]].
\]
By Theorem 3.13(ii), \( F \) is an interval valued \((\varepsilon, \in \lor q)\)-fuzzy implicative filter of \( L \).
Conversely, let \( F \) be an interval valued \((\varepsilon, \in \lor q)\)-fuzzy implicative filter of \( L \). Then we consider the following two cases:
Case (i) \( \mu_F(1) < [0.5, 0.5] \).
By Remark 3.12(ii), \( F \) is an interval valued fuzzy implicative filter of \( L \). It follows from Theorem 2.8 that \( F \) is an interval valued fuzzy Boolean filter of \( L \). Thus, \( F \) is an interval valued \((\varepsilon, \in \lor q)\)-fuzzy Boolean filter of \( L \).
Case (ii) \( \mu_F(1) \geq [0.5, 0.5] \).
From Theorem 3.13(ii), we have
\[
\mu_F((x' \to x) \to x) \geq \min[\mu_F((x' \to x) \to (x' \to x)), [0.5, 0.5]]
\]
\[
= \min[\mu_F(1), [0.5, 0.5]]
\]
\[
= [0.5, 0.5]
\]
and
\[
\mu_F((x \to x') \to x') \geq \min[\mu_F((x \to x') \to (x' \to x)), [0.5, 0.5]]
\]
\[
= \min[\mu_F(1), [0.5, 0.5]]
\]
\[
= [0.5, 0.5].
\]
Thus,
\[
\mu_F(x \lor x') = \min[\mu_F((x \to x') \to (x' \to x)), \mu_F((x' \to x) \to x)]
\]
\[
\geq \min([0.5, 0.5], [0.5, 0.5])
\]
\[
= [0.5, 0.5].
\]
Therefore, \( F \) is an interval valued \((\varepsilon, \in \lor q)\)-fuzzy Boolean filter of \( L \). \( \Box \)

From the above discussion, interval valued \((\varepsilon, \in \lor q)\)-fuzzy implicative filters are equivalent to interval valued \((\varepsilon, \in \lor q)\)-fuzzy Boolean filters in \( \mathcal{BL} \)-algebras.

Now, we continue to study the characterizations of interval valued \((\varepsilon, \in \lor q)\)-fuzzy Boolean (implicative) filters.

Theorem 3.15. Let \( F \) be an interval valued \((\varepsilon, \in \lor q)\)-fuzzy filter of \( L \). Then the following are equivalent:
(i) \( F \) is an interval valued \((\varepsilon, \in \lor q)\)-fuzzy Boolean (implicative) filter;
(ii) \( \mu_F(x) \geq \min[\mu_F(x' \to x), [0.5, 0.5]], \) for all \( x \in L \);
(iii) \( \mu_F(x) \geq \min[\mu_F((x \to y) \to x), [0.5, 0.5]], \) for all \( x, y \in L \);
(iv) \( \mu_F(x) \geq \min[\mu_F((x \to y) \to (x \to y)), \mu_F(z)], \) for all \( x, y, z \in L \).

Proof. (i) \( \implies \) (ii) From Theorem 3.13(ii), we have
\[
\mu_F(x) = \mu_F(1 \to x) \geq \min[\mu_F(1 \to (x' \to x)), [0.5, 0.5]]
\]
\[
= \min[\mu_F(x' \to x), [0.5, 0.5]].
\]
which proves (ii).
(ii) \( \implies \) (iii) Since \( x' \leq x \to y \), we have \( (x \to y) \to x \leq x' \to x \). Since \( F \) is an interval valued \((\varepsilon, \in \lor q)\)-fuzzy filter of \( L \), then
\[
\mu_F(x' \to x) \geq \min[\mu_F((x \to y) \to x), [0.5, 0.5]].
\]
Hence (ii) holds.
Let $0 \leq 1 \leq 1 \leq 1$, we know that $\mu_i (x \to y) \geq \mu_i (z \to ((x \to y) \to x)), \mu_i (z), [0.5, 0.5]$. It follows from (ii) that

$$\mu_i (x) \geq \min(\mu_i ((x \to y) \to x), [0.5, 0.5])$$

$$\geq \min(\mu_i ((x \to z) \to ((x \to y) \to x)), \mu_i (z), [0.5, 0.5]).$$

Thus, (iv) holds.

(iv)$\Rightarrow$(i) Since $z \leq x \to z$, we have $(x \to z)' \leq z'$ and $z' \to (x \to z) \leq (x \to z)' \to (x \to z)$. Thus,

$$\mu_i ((x \to z)' \to (x \to z)) \geq \min(\mu_i (z' \to (x \to z)), [0.5, 0.5]).$$

It follows from (iv) that

$$\mu_i (x \to z) \geq \min(\mu_i (1 \to (((x \to z) \to 0) \to (x \to z))), \mu_i (1), [0.5, 0.5])$$

$$= \min(\mu_i ((x \to z)' \to (x \to z)), [0.5, 0.5])$$

$$\geq \min(\mu_i (z' \to (x \to z)), [0.5, 0.5]).$$

Thus, by Theorem 3.13(ii), $F$ is an interval valued $(\in, \in \circ \in)$-fuzzy Boolean (implicative) filter of $L$. □

4. Interval valued $(\in, \in \circ \in)$-fuzzy positive implicative filters

In this section, we introduce the concept of interval valued $(\in, \in \circ \in)$-fuzzy positive implicative filters in $BL$-algebras and investigate some of their properties.

Definition 4.1. An interval valued $(\in, \in \circ \in)$-fuzzy filter of $L$ is called an interval valued $(\in, \in \circ \in)$-fuzzy positive implicative filter of $L$ if it satisfies:

(F12) $\mu_i (x \to z) \geq \min(\mu_i (x \to (y \to z)), \mu_i (x \to y), [0.5, 0.5])$, for all $x, y, z \in L$.

The following example shows that interval valued $(\in, \in \circ \in)$-fuzzy positive implicative filters exist.

Example 4.2. Let $L = [0, a, b, 1]$ be a chain with Cayley table as follows:

<table>
<thead>
<tr>
<th>⊕</th>
<th>0</th>
<th>a</th>
<th>b</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>a</td>
<td>b</td>
<td>1</td>
</tr>
<tr>
<td>a</td>
<td>0</td>
<td>a</td>
<td>a</td>
<td>a</td>
</tr>
<tr>
<td>b</td>
<td>0</td>
<td>b</td>
<td>b</td>
<td>b</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

Define $\land$ and $\lor$ operations on $L$ as min and max, respectively. Then $(L, \land, \lor, \circ, \rightarrow, 0, 1)$ is a $BL$-algebra. Define an interval valued fuzzy set $F$ in $L$ by $\mu_i (1) = [0.7, 0.8], \mu_i (0) = \mu_i (a) = \mu_i (b) = [0.3, 0.4]$. One can easily verify that $F$ is an interval valued $(\in, \in \circ \in)$-fuzzy positive implicative filter of $L$.

By their level positive implicative filters of $BL$-algebras, we characterize interval valued $(\in, \in \circ \in)$-fuzzy positive implicative filters as follows:

Theorem 4.3. Let $F$ be an interval valued $(\in, \in \circ \in)$-fuzzy positive implicative filter of $L$. Then for all $[0, 0] < \bar{\mu} \leq [0.5, 0.5]$, $U(F; \bar{\mu})$ is an empty set or a positive implicative filter of $L$. Conversely, if $F$ is an interval valued fuzzy set of $L$ such that $U(F; \bar{\mu})(\neq \in)$ is a positive implicative filter of $L$ for all $[0, 0] < \bar{\mu} \leq [0.5, 0.5]$, then $F$ is an interval valued $(\in, \in \circ \in)$-fuzzy positive implicative filter of $L$.

Proof. Let $F$ be an interval valued $(\in, \in \circ \in)$-fuzzy positive implicative filter of $L$ and $[0, 0] < \bar{\mu} \leq [0.5, 0.5]$. By Lemma 3.7, we know that $U(F; \bar{\mu})$ is a filter of $L$. If $x \to (y \to z) \in U(F; \bar{\mu})$ and $x \to y \in U(F; \bar{\mu})$, then $\mu_i (x \to (y \to z)) \geq \bar{\mu}$ and $\mu_i (x \to y) \geq \bar{\mu}$. Using (F12), it follows that $\mu_i (x \to z) \geq \min(\mu_i (x \to (y \to z)), \mu_i (x \to y), [0.5, 0.5]) \geq \min(\bar{\mu}, [0.5, 0.5]) = \bar{\mu}$, that is, $x \to z \in U(F; \bar{\mu})$. Thus, $U(F; \bar{\mu})$ is a positive implicative filter of $L$.

Conversely, let $F$ be an interval valued fuzzy set of $L$ such that $U(F; \bar{\mu})(\neq \in)$ is a positive implicative filter of $L$ for all $[0, 0] < \bar{\mu} \leq [0.5, 0.5]$. Then by Lemma 3.7, we know that $F$ is an interval valued $(\in, \in \circ \in)$-fuzzy filter of $L$. For every $x, y, z \in L$, we can write

$$\mu_i (x \to (y \to z)) \geq \min(\mu_i (x \to (y \to z)), \mu_i (x \to y), [0.5, 0.5]) = \tilde{\mu},$$

$$\mu_i (x \to y) \geq \min(\mu_i (x \to (y \to z)), \mu_i (x \to y), [0.5, 0.5]) = \tilde{\mu},$$

then $x \to (y \to z) \in U(F; \tilde{\mu})$ and $x \to y \in U(F; \tilde{\mu})$. Since $U(F; \tilde{\mu})$ is a positive implicative filter of $L$, we have $x \to z \in U(F; \tilde{\mu})$. Thus

$$\mu_i (x \to z) \geq \tilde{\mu} = \min(\mu_i (x \to (y \to z)), \mu_i (x \to y), [0.5, 0.5]).$$

Therefore, $F$ is an interval valued $(\in, \in \circ \in)$-fuzzy positive implicative filter of $L$. □
Naturally, a corresponding result should be considered when $U(F; \tilde{r})$ is a positive implicative filter of $L$ for all $[0.5, 0.5] < \tilde{r} \leq [1, 1]$.

**Theorem 4.4.** Let $F$ be an interval valued fuzzy set of $L$. Then $U(F; \tilde{r})(\neq \emptyset)$ is a positive implicative filter of $L$ for all $[0.5, 0.5] < \tilde{r} \leq [1, 1]$ if and only if for all $x, y \in L$,

(F8) $\max(\widetilde{\mu}_F(x \lor y), [0.5, 0.5]) \geq \min(\widetilde{\mu}_F(x), \widetilde{\mu}_F(y))$;

(F9) $\max(\widetilde{\mu}_F(y), [0.5, 0.5]) \geq \widetilde{\mu}_F(x)$ with $x \leq y$;

(F13) $\max(\widetilde{\mu}_F(x \rightarrow z), [0.5, 0.5]) \geq \min(\widetilde{\mu}_F(x \rightarrow (y \rightarrow z)), \widetilde{\mu}_F(x \rightarrow y))$.

**Proof.** Assume that $U(F; \tilde{r})(\neq \emptyset)$ is a positive implicative filter of $L$. It follows from Lemma 3.9 that (F8) and (F9) hold. If there exist $x, y, z \in L$ such that $\max(\widetilde{\mu}_F(x \rightarrow z), [0.5, 0.5]) < \min(\widetilde{\mu}_F(x \rightarrow (y \rightarrow z)), \widetilde{\mu}_F(x \rightarrow y), [0.5, 0.5]) = \tilde{r}$, then $[0.5, 0.5] < \tilde{r} \leq [1, 1], \widetilde{\mu}_F(x \rightarrow z) < \tilde{r}$ and $x \rightarrow (y \rightarrow z), x \rightarrow y \in U(F; \tilde{r})$. Since $U(F; \tilde{r})$ is a positive implicative filter of $L$, then $x \rightarrow z \in U(F; \tilde{r})$, and so $\widetilde{\mu}_F(x \rightarrow z) \geq \tilde{r}$, which is a contradiction. Hence (F13) holds.

Conversely, suppose that the conditions (F8), (F9) and (F13) hold. It follows from Lemma 3.9 that $U(F; \tilde{r})$ is a filter of $L$. Assume that $[0.5, 0.5] < \tilde{r} \leq [1, 1], x \rightarrow (y \rightarrow z), x \rightarrow y \in U(F; \tilde{r})$, then

$$[0.5, 0.5] < \tilde{r} \leq \min(\widetilde{\mu}_F(x \rightarrow (y \rightarrow z)), \widetilde{\mu}_F(x \rightarrow y))$$

$$\leq \max(\widetilde{\mu}_F(x \rightarrow z), [0.5, 0.5]) = \widetilde{\mu}_F(x \rightarrow z),$$

which implies, $x \rightarrow z \in U(F; \tilde{r})$. Thus $U(F; \tilde{r})$ is a positive implicative filter of $L$.

In [30], Yuan et al. gave the definition of a fuzzy subgroup with thresholds which is a generalization of Rosenfeld’s fuzzy subgroup, and Bhakat and Das’s fuzzy subgroup. Based on [30], we can extend the concept of a fuzzy subgroup with thresholds to the concept of a fuzzy positive implicative filter with thresholds in the following way:

**Definition 4.5.** Let $\widetilde{\alpha}, \widetilde{\beta} \in D[0, 1]$ with $\widetilde{\alpha} < \widetilde{\beta}$. Then an interval valued fuzzy set $F$ of $L$ is an interval valued fuzzy positive implicative filter with thresholds $(\widetilde{\alpha}, \widetilde{\beta})$ of $L$ if for all $x, y \in L$,

(F14) $\max(\widetilde{\mu}_F(x \lor y), \widetilde{\alpha}) \geq \min(\widetilde{\mu}_F(x), \widetilde{\mu}_F(y), \widetilde{\beta})$,

(F15) $\max(\widetilde{\mu}_F(y), \widetilde{\alpha}) \geq \min(\widetilde{\mu}_F(x), \widetilde{\beta})$ with $x \leq y$,

(F16) $\max(\widetilde{\mu}_F(x \rightarrow z), \widetilde{\alpha}) \geq \min(\widetilde{\mu}_F(x \rightarrow (y \rightarrow z)), \widetilde{\mu}_F(x \rightarrow y), \widetilde{\beta})$.

Now, we characterize interval valued fuzzy positive implicative filters with thresholds by their level positive implicative filters.

**Theorem 4.6.** An interval valued fuzzy set $F$ of $L$ is an interval valued fuzzy positive implicative filter with thresholds $(\widetilde{\alpha}, \widetilde{\beta})$ of $L$ if and only if $U(F; \tilde{r})(\neq \emptyset)$ is a positive implicative filter of $L$ for all $\widetilde{\alpha} < \tilde{r} \leq \widetilde{\beta}$.

**Proof.** The proof is similar to the proof of Theorems 4.3 and 4.4.

Next, we discuss some properties of interval valued $(\varepsilon, \in \vee q)$-fuzzy positive implicative filters in BL-algebras.

**Theorem 4.7.** Let $F$ be an interval valued $(\varepsilon, \in \vee q)$-fuzzy filter of $L$. Then the following are equivalent:

(i) $F$ is an interval valued $(\varepsilon, \in \vee q)$-fuzzy positive implicative filter;

(ii) $\widetilde{\mu}_F(x \rightarrow y) \geq \min(\widetilde{\mu}_F(x \rightarrow (x \rightarrow y)), [0.5, 0.5]),$ for all $x, y \in L$;

(iii) $\widetilde{\mu}_F(x \rightarrow (y \rightarrow z)) \geq \min(\widetilde{\mu}_F(x \rightarrow (y \rightarrow z)), [0.5, 0.5]),$ for all $x, y, z \in L$;

(iv) $\widetilde{\mu}_F(x \land (y \rightarrow z)) \geq \min(\widetilde{\mu}_F((x \land y) \rightarrow z), [0.5, 0.5]),$ for all $x, y, z \in L$.

**Proof.** (i) $\Rightarrow$ (ii) Assume that $F$ is an interval valued $(\varepsilon, \in \vee q)$-fuzzy positive implicative filter of $L$. Then from (F12), we have

$$\widetilde{\mu}_F(x \rightarrow y) \geq \min(\widetilde{\mu}_F(x \rightarrow (x \rightarrow y)), \widetilde{\mu}_F(x \rightarrow x), [0.5, 0.5])$$

$$= \min(\widetilde{\mu}_F(x \rightarrow (x \rightarrow y)), \widetilde{\mu}_F(1), [0.5, 0.5])$$

$$= \min(\widetilde{\mu}_F(x \rightarrow (x \rightarrow y)), [0.5, 0.5]).$$

Thus (ii) holds.

(ii) $\Rightarrow$ (i) Since $F$ is an interval valued $(\varepsilon, \in \vee q)$-fuzzy filter of $L$. Then, for all $x, y, z \in L$,

$$\widetilde{\mu}_F(x \rightarrow (y \rightarrow z)) \geq \min(\widetilde{\mu}_F(x \rightarrow (y \rightarrow z)), \widetilde{\mu}_F(x \rightarrow y), [0.5, 0.5]).$$

By (ii), we have

$$\widetilde{\mu}_F(x \rightarrow z) \geq \min(\widetilde{\mu}_F(x \rightarrow (y \rightarrow z)), [0.5, 0.5])$$

$$\geq \min(\widetilde{\mu}_F(x \rightarrow (y \rightarrow z)), \widetilde{\mu}_F(x \rightarrow y), [0.5, 0.5]),$$

which proves (F12). Thus $F$ is an interval valued $(\varepsilon, \in \vee q)$-fuzzy positive implicative filter of $L$.

(i) $\Rightarrow$ (iii) Let $F$ be an interval valued $(\varepsilon, \in \vee q)$-fuzzy positive implicative filter of $L$. Then, for all $x, y, z \in L$,

$$\widetilde{\mu}_F((x \rightarrow (y \rightarrow z)) \geq \min(\widetilde{\mu}_F((x \rightarrow (y \rightarrow z)), \widetilde{\mu}_F(x \rightarrow y), [0.5, 0.5]).$$
Since $x \rightarrow ((x \rightarrow y) \rightarrow z) = (x \rightarrow y) \rightarrow (x \rightarrow z)$ and
\[x \rightarrow ((y \rightarrow z) \rightarrow ((x \rightarrow y) \rightarrow z)) = (y \rightarrow z) \rightarrow (x \rightarrow ((x \rightarrow y) \rightarrow z)) = (y \rightarrow z) \rightarrow ((x \rightarrow y) \rightarrow z) = 1,
\]
it follows that
\[
\tilde{\mu}_F((x \rightarrow y) \rightarrow (x \rightarrow z)) = \tilde{\mu}_F((x \rightarrow ((x \rightarrow y) \rightarrow z))) \\
\geq \min\{\tilde{\mu}_F(1), \tilde{\mu}_F(x \rightarrow (y \rightarrow z)), [0.5, 0.5]\} \\
= \min\{\tilde{\mu}_F(x \rightarrow (y \rightarrow z)), [0.5, 0.5]\}.
\]
Thus (iii) holds.

(iii)$\implies$(iv) Since $(x \land y) \rightarrow z = x \rightarrow (y \rightarrow z)$ and $(x \lor y) \rightarrow z = (x \lor (x \land y)) \rightarrow z = (x \rightarrow y) \rightarrow (x \rightarrow z)$, we have
\[
\tilde{\mu}_F((x \land y) \rightarrow z) = \tilde{\mu}_F((x \rightarrow y) \rightarrow (x \rightarrow z)) \\
\geq \min\{\tilde{\mu}_F(x \rightarrow (y \rightarrow z)), [0.5, 0.5]\} \\
= \min\{\tilde{\mu}_F(x \rightarrow (x \rightarrow z)), [0.5, 0.5]\}.
\]
Thus (iv) holds.

(iv)$\implies$(i) Since $F$ is an interval valued $(\varepsilon, \in \lor q)$-fuzzy filter of $L$, then, for all $x, y, z \in L$,
\[
\tilde{\mu}_F(x \rightarrow (x \rightarrow z)) \geq \min\{\tilde{\mu}_F(x \rightarrow (y \rightarrow z)), \tilde{\mu}_F(x \rightarrow y), [0.5, 0.5]\}.
\]
Since $x \rightarrow (x \rightarrow z) = x \land x \rightarrow z$, it follows from (iv) that
\[
\tilde{\mu}_F(x \rightarrow z) = \tilde{\mu}_F((x \land x) \rightarrow z) \\
\geq \min\{\tilde{\mu}_F((x \land x) \rightarrow z), [0.5, 0.5]\} \\
= \min\{\tilde{\mu}_F(x \rightarrow (x \rightarrow z)), [0.5, 0.5]\} \\
\geq \min\{\tilde{\mu}_F(x \rightarrow (y \rightarrow z)), \tilde{\mu}_F(x \rightarrow y), [0.5, 0.5]\}.
\]
which proves (F12). Thus $F$ is an interval valued $(\varepsilon, \in \lor q)$-fuzzy positive implicative filter of $L$. □

The relation between interval valued $(\varepsilon, \in \lor q)$-fuzzy Boolean (implicative) filters and interval valued $(\varepsilon, \in \lor q)$-fuzzy positive implicative filters is as follows:

**Theorem 4.8.** Each interval valued $(\varepsilon, \in \lor q)$-fuzzy Boolean (implicative) filter is an interval valued $(\varepsilon, \in \lor q)$-fuzzy positive implicative filter; the converse may not be true.

**Proof.** Let $F$ be an interval valued $(\varepsilon, \in \lor q)$-fuzzy Boolean (implicative) filter of $L$. Then, for all $x, y, z \in L$, we have
\[
\tilde{\mu}_F(x \rightarrow z) \geq \min\{\tilde{\mu}_F((x \lor x') \rightarrow (x \rightarrow z)), \tilde{\mu}_F(x \lor x'), [0.5, 0.5]\} \\
= \min\{\tilde{\mu}_F((x \lor x') \rightarrow (x \rightarrow z)), \tilde{\mu}_F(1), [0.5, 0.5]\} \\
= \min\{\tilde{\mu}_F((x \lor x') \rightarrow (x \rightarrow z)), [0.5, 0.5]\}.
\]
Since
\[
(x \lor x') \rightarrow (x \rightarrow z) = (x \rightarrow (x \rightarrow z)) \land (x' \rightarrow (x \rightarrow z)) = x \rightarrow (x \rightarrow z),
\]
we have
\[
\tilde{\mu}_F(x \rightarrow z) \geq \min\{\tilde{\mu}_F(x \rightarrow (x \rightarrow z)), [0.5, 0.5]\}.
\]
It follows from Theorem 4.7 that $F$ is an interval valued $(\varepsilon, \in \lor q)$-fuzzy positive implicative filter of $L$.

The last part is shown by Example 4.2. We know that $F$ is an interval valued $(\varepsilon, \in \lor q)$-fuzzy positive implicative filter of $L$, but $F$ is not an interval valued $(\varepsilon, \in \lor q)$-fuzzy Boolean (implicative) filter of $L$, because
\[
\tilde{\mu}_F(a \lor a') = \tilde{\mu}_F(a) = [0.3, 0.4] < [0.5, 0.5].
\]

### 5. Interval valued $(\varepsilon, \in \lor q)$-fuzzy fantastic filters

In this section, we introduce the concept of interval valued $(\varepsilon, \in \lor q)$-fuzzy fantastic filters in $BL$-algebras and investigate some of their properties.
Definition 5.1. An interval valued \((\varepsilon, \varepsilon \lor q)\)-fuzzy filter of \(L\) is called an interval valued \((\varepsilon, \varepsilon \lor q)\)-fuzzy fantastic filter of \(L\) if it satisfies:
\[
(F17) \quad \tilde{\mu}_{F}(((x \rightarrow y) \rightarrow y) \rightarrow x) \geq \min(\tilde{\mu}_{F}(z \rightarrow (y \rightarrow x)), \tilde{\mu}_{F}(z), [0.5, 0.5]), \quad \text{for all } x, y, z \in L.
\]

The following example shows that interval valued \((\varepsilon, \varepsilon \lor q)\)-fuzzy fantastic filters exist.

Example 5.2. Let \(L = \{0, a, b, 1\}\) be a chain with Cayley table as follows:

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>a</th>
<th>b</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>a</td>
<td>a</td>
</tr>
<tr>
<td>b</td>
<td>0</td>
<td>0</td>
<td>b</td>
<td>b</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>a</td>
<td>b</td>
</tr>
</tbody>
</table>

Define \(\land\) and \(\lor\) operations on \(L\) as \(\min\) and \(\max\), respectively. Then \((L, \land, \lor, \lor, \rightarrow, 0, 1)\) is a BL-algebra. Define an interval valued fuzzy set \(F\) in \(L\) by \(\tilde{\mu}_{F}(1) = [0.7, 0.8], \tilde{\mu}_{F}(0) = \tilde{\mu}_{F}(b) = \tilde{\mu}_{F}(a) = [0.3, 0.4]\). One can easily verify that \(F\) is an interval valued \((\varepsilon, \varepsilon \lor q)\)-fuzzy fantastic filter of \(L\).

By their level fantastic filters of BL-algebras, we characterize interval valued \((\varepsilon, \varepsilon \lor q)\)-fuzzy fantastic filters as follows:

Theorem 5.3. Let \(F\) be an interval valued \((\varepsilon, \varepsilon \lor q)\)-fuzzy fantastic filter of \(L\). Then for all \(0 \leq t \leq 0.5\), \(U(F; \tilde{t})\) is an empty set or a fantastic filter of \(L\). Conversely, if \(F\) is an interval valued fuzzy set of \(L\) such that \(U(F; t) \neq \emptyset\) is a fantastic filter of \(L\) for all \(0 \leq t \leq 0.5\), then \(F\) is an interval valued \((\varepsilon, \varepsilon \lor q)\)-fuzzy fantastic filter of \(L\).

Proof. The proof is similar to the proof of Theorem 4.3 and we omit the details. \(\Box\)

Theorem 5.4. Let \(F\) be an interval valued fuzzy set of \(L\). Then \(U(F; \tilde{t})(\neq \emptyset)\) is a fantastic filter of \(L\) for all \(0 \leq t \leq 1\) if and only if for all \(x, y, z \in L\),
\[
(F8) \quad \max(\tilde{\mu}_{F}(x \circ y), [0.5, 0.5]) \geq \min(\tilde{\mu}_{F}(x), \tilde{\mu}_{F}(y)),
\]
\[
(F9) \quad \max(\tilde{\mu}_{F}(y), [0.5, 0.5]) \geq \tilde{\mu}_{F}(x) \quad \text{with } x \leq y.
\]
\[
(F18) \quad \max(\tilde{\mu}_{F}(((x \rightarrow y) \rightarrow y) \rightarrow x), [0.5, 0.5]) \geq \min(\tilde{\mu}_{F}(z \rightarrow (y \rightarrow x)), \tilde{\mu}_{F}(z)).
\]

Proof. The proof is similar to the proof of Theorem 4.3 and we omit the details. \(\Box\)

Definition 5.5. Let \(\tilde{\alpha}, \tilde{\beta} \in D[0, 1]\), then interval valued fuzzy set \(F\) is called an interval valued fuzzy fantastic filter with thresholds \((\tilde{\alpha}, \tilde{\beta})\) if for all \(x, y \in L\),
\[
(F14) \quad \max(\tilde{\mu}_{F}(x \circ y), \tilde{\alpha}) \geq \min(\tilde{\mu}_{F}(x), \tilde{\mu}_{F}(y), \tilde{\beta}),
\]
\[
(F15) \quad \max(\tilde{\mu}_{F}(y), \tilde{\alpha}) \geq \min(\tilde{\mu}_{F}(x), \tilde{\beta}) \quad \text{with } x \leq y.
\]
\[
(F19) \quad \max(\tilde{\mu}_{F}(((x \rightarrow y) \rightarrow y) \rightarrow x), \tilde{\alpha}) \geq \min(\tilde{\mu}_{F}(z \rightarrow (y \rightarrow x)), \tilde{\mu}_{F}(z), \tilde{\beta}).
\]

Now, we characterize interval valued fuzzy fantastic filters with thresholds by their level fantastic filters.

Theorem 5.6. An interval valued fuzzy set \(F\) of \(L\) is an interval valued fuzzy fantastic filter with thresholds \((\tilde{\alpha}, \tilde{\beta})\) if and only if \(U(F; \tilde{t})(\neq \emptyset)\) is a fantastic filter of \(L\) for all \(\tilde{\alpha} < \tilde{t} \leq \tilde{\beta}\).

Proof. The proof is similar to the proof of Theorems 5.3 and 5.4. \(\Box\)

Next, we investigate some properties of interval valued \((\varepsilon, \varepsilon \lor q)\)-fuzzy fantastic filters in BL-algebras.

Theorem 5.7. Let \(F\) be an interval valued \((\varepsilon, \varepsilon \lor q)\)-fuzzy filter of \(L\). Then \(F\) is an interval valued \((\varepsilon, \varepsilon \lor q)\)-fuzzy fantastic filter of \(L\) if and only if it satisfies:
\[
(F20) \quad \tilde{\mu}_{F}(((x \rightarrow y) \rightarrow y) \rightarrow x) \geq \min(\tilde{\mu}_{F}(y \rightarrow x), [0.5, 0.5]), \quad \text{for all } x, y \in L.
\]

Proof. Let \(F\) be an interval valued \((\varepsilon, \varepsilon \lor q)\)-fuzzy fantastic filter of \(L\). Putting \(z = 1\) in (F17), we get
\[
\tilde{\mu}_{F}(((x \rightarrow y) \rightarrow y) \rightarrow x) \geq \min(\tilde{\mu}_{F}(1 \rightarrow (y \rightarrow x)), \tilde{\mu}_{F}(1), [0.5, 0.5])
\]
\[
= \min(\tilde{\mu}_{F}(1 \rightarrow (y \rightarrow x)), [0.5, 0.5]).
\]

Thus (F20) holds.

Conversely, for any \(x, y, z \in L\), since \(F\) is an interval valued \((\varepsilon, \varepsilon \lor q)\)-fuzzy filter of \(L\), we have
\[
\tilde{\mu}_{F}(y \rightarrow x) \geq \min(\tilde{\mu}_{F}(z \rightarrow (y \rightarrow x)), \tilde{\mu}_{F}(z), [0.5, 0.5]).
\]

It follows from (F20) that
\[
\tilde{\mu}_{F}(((x \rightarrow y) \rightarrow y) \rightarrow x) \geq \min(\tilde{\mu}_{F}(y \rightarrow x), [0.5, 0.5])
\]
\[
= \min(\tilde{\mu}_{F}(z \rightarrow (y \rightarrow x)), [0.5, 0.5]).
\]

Thus, \(F\) is an interval valued \((\varepsilon, \varepsilon \lor q)\)-fuzzy fantastic filter of \(L\). \(\Box\)
The relation between interval valued \((\varepsilon, \in \vee q)\)-fuzzy Boolean (implicative) filters and interval valued \((\varepsilon, \in \vee q)\)-fuzzy fancy filters is as follows:

**Theorem 5.8.** Each interval valued \((\varepsilon, \in \vee q)\)-fuzzy Boolean (implicative) filter is an interval valued \((\varepsilon, \in \vee q)\)-fuzzy fancy filter; the converse may not be true.

**Proof.** Let \(F\) be an interval valued \((\varepsilon, \in \vee q)\)-fuzzy Boolean (implicative) filter of \(L\). Since \(x \circ ((x \to y) \to y) \leq x\), we have \(x \leq ((x \to y) \to y) \to x\), and so

\[
(((x \to y) \to y) \to x) \to y \leq x \to y,
\]

which implies

\[
(((x \to y) \to y) \to x) \to y \to (((x \to y) \to y) \to x) \geq (x \to y) \to (((x \to y) \to y) \to x)
\]

\[= (x \to y) \to ((x \to y) \to x)
\]

\[\geq y \to x.
\]

Hence,

\[
\tilde{\mu}_F(((x \to y) \to y) \to x) \to y \to (((x \to y) \to y) \to x)) \geq \min[\tilde{\mu}_F(y \to x), [0.5, 0.5]].
\]

It follows from Theorem 3.15 that

\[
\tilde{\mu}_F(((x \to y) \to y) \to x) \geq \min[\tilde{\mu}_F(((x \to y) \to y) \to x) \to y \to (((x \to y) \to y) \to x)), [0.5, 0.5)]
\]

\[\geq \min[\tilde{\mu}_F(y \to x), [0.5, 0.5]].
\]

Thus, by Theorem 5.7, \(F\) is an interval valued \((\varepsilon, \in \vee q)\)-fuzzy fancy filter of \(L\).

The last part is shown by Example 5.2. We know that \(F\) is an interval valued \((\varepsilon, \in \vee q)\)-fuzzy fancy filter of \(L\), but \(F\) is not an interval valued \((\varepsilon, \in \vee q)\)-fuzzy Boolean (implicative) filter of \(L\), because

\[
\tilde{\mu}_F(a \vee a') = \tilde{\mu}_F(b) = [0.3, 0.4] < [0.5, 0.5].
\]

Finally, we discuss the relation among these generalized fuzzy filters in BL-algebras.

**Theorem 5.9.** An interval valued fuzzy set \(F\) of \(L\) is an interval valued \((\varepsilon, \in \vee q)\)-fuzzy Boolean (implicative) filter of \(L\) if and only if it is both an interval valued \((\varepsilon, \in \vee q)\)-fuzzy positive implicative filter and an interval valued \((\varepsilon, \in \vee q)\)-fuzzy fancy filter of \(L\).

**Proof.** Assume that \(F\) is an interval valued \((\varepsilon, \in \vee q)\)-fuzzy Boolean (implicative) filter of \(L\). Then, by Theorems 4.8 and 5.8, we know that \(F\) is both an interval valued \((\varepsilon, \in \vee q)\)-fuzzy positive implicative filter and an interval valued \((\varepsilon, \in \vee q)\)-fuzzy fancy filter of \(L\).

Conversely, if \(F\) is both an interval valued \((\varepsilon, \in \vee q)\)-fuzzy positive implicative filter and an interval valued \((\varepsilon, \in \vee q)\)-fuzzy fancy filter of \(L\). Since \((x \to y) \to x \leq (x \to y) \to ((x \to y) \to y)\), then

\[
\tilde{\mu}_F((x \to y) \to ((x \to y) \to y)) \geq \min[\tilde{\mu}_F((x \to y) \to x), [0.5, 0.5]].
\]

Since \(F\) is an interval valued \((\varepsilon, \in \vee q)\)-fuzzy positive implicative of \(L\), then, by Theorem 4.7(ii), we have

\[
\tilde{\mu}_F((x \to y) \to y) \geq \min[\tilde{\mu}_F((x \to y) \to ((x \to y) \to y)), [0.5, 0.5]].
\]

Thus, we have

\[
\tilde{\mu}_F((x \to y) \to y) \geq \min[\tilde{\mu}_F((x \to y) \to x), [0.5, 0.5]]. \tag{*}
\]

On the other hand, since \(F\) is an interval valued \((\varepsilon, \in \vee q)\)-fuzzy fancy filter of \(L\), then, by Theorem 5.7, we have

\[
\tilde{\mu}_F(((x \to y) \to y) \to x) \geq \min[\tilde{\mu}_F(y \to x), [0.5, 0.5]].
\]

Since \((x \to y) \to x \leq y \to x\), then

\[
\tilde{\mu}_F(y \to x) \geq \min[\tilde{\mu}_F((x \to y) \to x), [0.5, 0.5]].
\]

Thus, we have

\[
\tilde{\mu}_F(((x \to y) \to y) \to x) \geq \min[\tilde{\mu}_E((x \to y) \to x), [0.5, 0.5]]. \tag{**}
\]

Hence, using (\(\ast\)) and (\(\ast\ast\)), we get

\[
\tilde{\mu}_F(x) \geq \min[\tilde{\mu}_F((x \to y) \to y), \tilde{\mu}_F(((x \to y) \to y) \to x), [0.5, 0.5)]
\]

\[\geq \min[\tilde{\mu}_F((x \to y) \to x), [0.5, 0.5]].
\]

It follows from Theorem 3.15 that \(F\) is an interval valued \((\varepsilon, \in \vee q)\)-fuzzy Boolean (implicative) filter of \(L\). \(\square\)
A BL-algebra \( L \) is called an MV-algebra if \( x'' = x \), or equivalently, \((x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x\), for all \( x, y \in L \). Summarizing the above theorems, we have the following:

**Corollary 5.10.** Let \( F \) be an interval valued fuzzy set of an MV-algebra \( L \). Then the following are equivalent:

(i) \( F \) is an interval valued \((e, \in \lor q)\)-fuzzy filter;

(ii) \( F \) is an interval valued \((e, \in \lor q)\)-fuzzy fantastic filter.

**Corollary 5.11.** Let \( F \) be an interval valued fuzzy set of a BL-algebra \( L \). Then the following are equivalent:

(i) \( L \) is an MV-algebra;

(ii) \( F \) is an interval valued \((e, \in \lor q)\)-fuzzy Boolean (implicative) filter;

(iii) \( F \) is an interval valued \((e, \in \lor q)\)-fuzzy positive implicative filter.

6. Conclusions

In this paper, we introduced the notions of interval valued \((e, \in \lor q)\)-fuzzy Boolean (implicative, positive implicative and fantastic) filters in BL-algebras and investigated some related properties. It was proved that an interval valued \((e, \in \lor q)\)-fuzzy filter in BL-algebras is Boolean (implicative) if and only if it is both positive implicative and fantastic. The obtained results can be applied to other logical algebraic structures.

In our future work, we will consider the interval valued \((\alpha, \beta)\)-fuzzy algebra (implicative, positive implicative and fantastic) filters of BL-algebras, where \( \alpha, \beta \) is any one of \( e, q, \in \lor q \) or \( \in \land q \). In the notions of an interval valued \((\alpha, \beta)\)-fuzzy Boolean (implicative, positive implicative and fantastic) filter of BL-algebras, we can consider twelve different types of such structures resulting from three choices of \( \alpha \) and four choices of \( \beta \). But, in this report, we only discuss the \((e, \in \lor q)\)-type. We shall focus on other types and their relationships among them, and also consider these generalized rough fuzzy filters of BL-algebras and other logical algebras.

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References