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On the chairman assignment problem

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Abstract

Given m states, which form a union, every year a chairman has to be selected in such a way that at any time the accumulated number of chairmen from each state is proportional to its weight. In this paper an algorithm for a chairman assignment is given which, depending on the weights, guarantees a small discrepancy.

1. Introduction

Suppose a set of m ($\geqslant 2$) states $S = \{S_1, S_2, \ldots, S_m\}$ forms a union and a union chairman has to be selected every year. Each state S_i has a positive weight λ_i with $\sum_{i=1}^{m} \lambda_i = 1$. We denote the state designating the chairman in the jth year by ω_j. Hence, $\omega = \{\omega_j\}_{j=1}^{\infty}$ is a sequence in S. Let $A(N, i, \omega)$ denote the number of chairmen representing S_i in the first N years. Put

$$D_N(\omega) := \sup_i |N\lambda_i - A(N, i, \omega)|.$$

The assignment problem is to choose ω in such a way that the global discrepancy

$$D(\omega) := \sup_N D_N(\omega)$$

is minimal.

The problem was posed by Niederreiter [5], where it arose from a method for explicitly constructing uniformly distributed sequences in a compact space. Connections with a problem in combinatorial geometry were pointed out in [6]. The results of [5] were successively improved in [4, 7] and finally it was proved in [3] that for all sets of weights there is a sequence ω with $D(\omega) \leqslant 1 - 1/2(m-1)$. (In [7] it was shown, that for $\varepsilon > 0$ there are weights such that $D(\omega) \geqslant 1 - 1/2(m-1) - \varepsilon$ for every sequence ω.) However, the method of [3], using Hall's theorem on distinct representatives, does not provide an effective algorithm to construct a sequence with small global discrepancy when the weights are given. Such an algorithm was given in [8], where also the notion 'chairman assignment problem' was coined.

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In Section 2 a different algorithm to construct sequences ω is presented which, while yielding the same bound as that one in [8] in general, leads to sequences with $D(\omega)$ smaller than $1 - 1/2(m-1)$ for special weights.

A survey of questions related to those in this paper can be found in [9]. An algorithm for solving the related apportionment problem in the House of Representatives of the USA is described in [1].

2. An algorithm for constructing good sequences

Theorem 1. *Let $S = \{S_1, S_2, \ldots, S_m\}, m \geqslant 2$, be a finite set and let $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_m\}$ be positive weights for S with $\sum_{i=1}^{m} \lambda_i = 1$ and $\lambda_{i+1} \geqslant \lambda_i$ for $1 \leqslant i \leqslant m-1$.*

If there is an integer l, $1 \leqslant l \leqslant m/2$ such that for all j, $1 \leqslant j < l$ $\sum_{i=1}^{l-j} \lambda_i \geqslant 1 - 2(m-l+j)(m-l-1)/(2(m-1)(m-l)-m)$, then the following algorithm yields a sequence ω with

$$D(\omega) \leqslant 1 - \frac{1}{2(m-l)}.$$

To describe the algorithm we first need the following notations and conventions: Set $d_\Lambda := 1/2(m-l)$. For integers k, t, N with $k > 0$ and t, $N \geqslant 0$ and a sequence ω in S define

$$B(k,t,N) := \{S_i \in S \mid N\lambda_i - A(N,i,\omega) > k - d_\Lambda - t\lambda_i\}$$

and

$$u(N) := \min\{t \geqslant 0 \mid \exists\, S_i \in B(1,t,N) \text{ with } (N+1)\lambda_i - A(N,i,\omega) \geqslant d_\Lambda\}.$$

This minimum exists, otherwise

$$(N+1)\lambda_i - A(N,i,\omega) < d_\Lambda \quad \forall S_i \in S,$$

and so

$$1 = \sum_{i=1}^{m} ((N+1)\lambda_i - A(N,i,\omega)) < m \cdot d_\Lambda$$

which is clearly impossible with $l \leqslant m/2$.

For $N = 0$ we set $A(0,i,\omega) = 0$.

Algorithm. Suppose that $\omega_1, \omega_2, \ldots, \omega_N$ have already been defined. Choose S_{i_0} so that it is an element of $B(1,u(N),N)$ for which $(N+1)\lambda_i - A(N,i,\omega)$ is maximal and set $\omega_{N+1} = S_{i_0}$.

Proof of Theorem 1. We shall prove that the above algorithm creates a sequence which fulfills the requirements of the theorem. Let us write $A(N,i)$ instead of

$A(N, i, \omega)$. First we show by induction that

$$N\lambda_i - A(N,i) \geqslant -1 + d_\Lambda \quad \forall i, N \geqslant 0. \tag{1}$$

For $N = 0$ this is obvious.

Now if $\omega_{N+1} \neq S_i$ then $A(N,i) = A(N+1,i)$ and so

$$(N+1)\lambda_i - A(N+1,i) = N\lambda_i - A(N,i) + \lambda_i \geqslant -1 + d_\Lambda + \lambda_i \geqslant -1 + d_\Lambda.$$

If $\omega_{N+1} = S_i$ then $A(N+1,i) = A(N,i) + 1$ and $S_i \in B(1, u(N), N)$ with $(N+1)\lambda_i - A(N,i) \geqslant (N+1)\lambda_j - A(N,j)$ for all j with $S_j \in B(1, u(N), N)$. Since in $B(1, u(N), N)$ there is an S_j with

$$(N+1)\lambda_j - A(N,j) \geqslant d_\Lambda,$$

we have by construction of ω

$$(N+1)\lambda_i - A(N,i) \geqslant d_\Lambda,$$

and so

$$(N+1)\lambda_i - A(N+1,i) = (N+1)\lambda_i - A(N,i) - 1 \geqslant -1 + d_\Lambda.$$

To prove that also

$$N\lambda_i - A(N,i) \leqslant 1 - d_\Lambda \quad \forall i, N \geqslant 0 \tag{2}$$

we shall show that

$$\sum_{k=1}^{\infty} |B(k,t,N)| = \sum_i \max(k > 0 \mid N\lambda_i - A(N,i) > k - d_\Lambda - t\lambda_i) \leqslant t \quad \forall N, t \geqslant 0 \tag{3}$$

and so especially $|B(1,0,N)| \leqslant 0$ for all N from which (2) follows.

First suppose that $N = 0$. Then with $k_j = \max(k > 0 \mid N\lambda_j - A(N,j) > k - d_\Lambda - t\lambda_j)$

$$N\lambda_j - A(N,j) = 0 > k_j - d_\Lambda - t\lambda_j.$$

So we have

$$0 > \sum_{j=1}^{m} k_j - m \cdot d_\Lambda - t \sum_{j=1}^{m} \lambda_j,$$

and using that $m \cdot d_\Lambda \leqslant 1$ and $\sum_{j=1}^{m} \lambda_j = 1$ we obtain that

$$\sum_{j=1}^{m} k_j < t + 1,$$

and therefore $\sum_{k=1}^{\infty} |B(k,t,0)| \leqslant t$.

Now suppose that (3) has been proved for $N > 0$. An easy computation shows that

$$\max(k > 0 \mid (N+1)\lambda_j - A(N+1, j) > k - d_A - (t-1)\lambda_j)$$
$$= \max(k > 0 \mid N\lambda_j - A(N, j) > k - d_A - t\lambda_j) - 1 \text{ iff } S_j = \omega_{N+1} \text{ and }$$
$$S_j \in B(1, t, N)$$

and else

$$\max(k > 0 \mid (N+1)\lambda_j - A(N+1, j) > k - d_A - (t-1)\lambda_j)$$
$$= \max(k > 0 \mid N\lambda_j - A(N, j) > k - d_A - t\lambda_j).$$

If $t \geqslant u(N)$ then $\omega_{N+1} \in B(1, t, N)$ and so

$$\sum_{k=1}^{\infty} |B(k, t-1, N+1)| = \sum_{k=1}^{\infty} |B(k, t, N)| - 1 \leqslant t - 1.$$

If $t < u(N)$ then

$$\sum_{k=1}^{\infty} |B(k, t-1, N+1)| = \sum_{k=1}^{\infty} |B(k, t, N)|$$

and we have to show that the above sum is less than t.

So suppose $t < u(N)$. Defining $k_i := \max(k > 0 \mid N\lambda_i - A(N, i) > k - d_A - t\lambda_i)$ it follows from the construction of ω that for all $S_i \in B(1, t, N)$

$$(N+1)\lambda_i - A(N, i) < d_A$$

and

$$k_i - d_A - (t-1)\lambda_i < (N+1)\lambda_i - A(N, i)$$

and therefore

$$(t-1)\lambda_i > k_i - 2d_A.$$

If we can prove that $|B(1, u(N) - 1, N)| \leqslant m - l$ then it follows that

$$t - 1 \geqslant (t-1)\sum \lambda_j > \sum k_j - 2(m-l)d_A = \sum k_j - 1,$$

where the sums run over all j with $S_j \in B(1, t, N)$, and from the definition of $B(k, t, N)$ we get

$$\sum_{k=1}^{\infty} |B(k, t, N)| < t$$

and (3) is proved.

So suppose that with $v > m - l$, $S_1, \ldots, S_v \in B(1, u(N) - 1, N)$ and $S_{v+1}, \ldots, S_m \notin B(1, u(N) - 1, N)$. Then one has for $j > v$,

$$1 - d_A - (u(N) - 1)\lambda_j \geqslant N\lambda_j - A(N, j)$$

and for $j \leqslant v$,

$$d_\Lambda - \lambda_j > N\lambda_j - A(N, j). \tag{4}$$

Using that $\sum_{j=1}^{m} \lambda_i = 1$ and that $\sum_{j=1}^{m} (N\lambda_j - A(N,j)) = 0$ and adding the above inequalities one obtains that

$$(m - v - 1) - (m - 2v)d_\Lambda - (u(N) - 2) \sum_{j=v+1}^{m} \lambda_j > 0. \tag{5}$$

Also for $j \leqslant v$

$$N\lambda_j - A(N, j) > 1 - d_\Lambda - (u(N) - 1)\lambda_j, \tag{6}$$

and by (4) and (6)

$$d_\Lambda - \lambda_j > 1 - d_\Lambda - (u(N) - 1)\lambda_j,$$

and so

$$(u(N) - 2)\lambda_j > 1 - 2d_\Lambda \tag{7}$$

(from this it follows that $\lambda_j > 0$). Summing up (7) and transforming the resulting inequality one obtains

$$u(N) - 2 > \frac{v - 2vd_\Lambda}{\sum_{j=1}^{v} \lambda_j} = \frac{v - 2vd_\Lambda}{1 - \sum_{j=v+1}^{m} \lambda_j}.$$

Replacing $u(N) - 2$ in inequality (5), multiplying with $1 - \sum_{j=v+1}^{m} \lambda_j$ and using the definition of d_Λ yields after some transformations

$$\sum_{j=v+1}^{m} \lambda_j < 1 - \frac{2v(m - l - 1)}{2(m - 1)(m - l) - m}$$

contradicting the assumptions of the theorem. □

For some sets of weights Λ one can with the help of the above algorithm construct a sequence ω for which $D(\omega) \leqslant 1 - 1/m$. The following theorem shows that this cannot be improved without knowing more about the diophantine properties of Λ.

For the definition of linear independence and the result on uniform distribution used in the theorem see [2, Ch. 1, Section 6]. From the theorem also follows that for almost every Λ there is no ω with $D(\omega) < 1 - 1/m$.

Theorem 2. *Let $S = \{S_1, S_2, \ldots, S_m\}$, $m \geqslant 2$, be a finite set and let $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_m\}$ be positive weights for S. If 1, $\lambda_1, \lambda_2, \ldots, \lambda_{m-1}$ are linearly independent over the rationals, then for all sequences ω*

$$D(\omega) \geqslant 1 - \frac{1}{m}.$$

Proof. Suppose that ω is a sequence with

$$D_N(\omega) < 1 - \frac{1}{m} - \delta \quad \forall N \text{ and some } \delta > 0.$$

Since $1, \lambda_1, \lambda_2, \ldots, \lambda_{m-1}$ are linearly independent over the rationals the sequence $(N\lambda_1, N\lambda_2, \ldots, N\lambda_{m-1})$ is uniformly distributed mod 1. So for $\varepsilon = \delta/(m-1)$ there is an N having

$$1 - \frac{1}{m} + \varepsilon > \{N\lambda_i\} > 1 - \frac{1}{m} \quad \text{for } i \in \{1, \ldots, m-1\}.$$

Therefore, $A(N,i) = [N\lambda_i] + 1$ and

$$N\lambda_i - A(N,i) = N\lambda_i - [N\lambda_i] - 1 = \{N\lambda_i\} - 1$$

and so

$$-\frac{1}{m} + \varepsilon > N\lambda_i - A(N,i) > -\frac{1}{m}.$$

Summing up and using that $\sum_{i=0}^{m} (N\lambda_i - A(N,i)) = 0$ one obtains that

$$1 - \frac{1}{m} - (m-1)\varepsilon < N\lambda_m - A(N,m)$$

a contradiction. □

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