Salvaging the Thompson–Chandrasekhar Criterion: A Tribute to S. Chandrasekhar

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The sole surviving challenge in the linear theory of magnetothermoconvection, which emerges from an unsuccessful attempt of S. Chandrasekhar (Philos. Mag. 743 (1952)) and demands a mathematical proof of the existence of overstable motions when the boundaries are dynamically free and thermally and electrically perfectly conducting, is overcome herewith. As a consequence the linear theory, which prior to 1985 was mostly ridden with conjectures and controversies, is brought to a state of perfection where it is free from any such anomalies and a successful nonlinear investigation of magnetothermoconvection is a distinct possibility. © 1992 Academic Press, Inc.

INTRODUCTION

The classic and yet controversial calculations of S. Chandrasekhar in 1952 [1] on magnetothermoconvection wherein the electrically perfectly conducting and dynamically free boundaries are thermally perfectly conducting and overstability is valid has generated a spate of activities in the recent past. Banerjee et al. [2] have obtained the exact solution of this problem wherein the electrically perfectly conducting and dynamically free boundaries are thermally insulting and proved for this allied problem that the eigenvalues calculated by Chandrasekhar through his extremely simple solution of the governing equations that fails to satisfy any plausible set of boundary conditions on the magnetic field are the correct ones and thus all his conjectures except (i) (Banerjee et al. [2]) are valid.

Surprisingly there has not been any attempt, to the best of our knowledge, to construct the correct solution of the mathematical problem.
to which Chandrasekhar addressed himself. In the present paper we solve this problem of Chandrasekhar exactly and recover, as a consequence, the truth of the conjectures (ii)–(vii) that includes, in particular, the validity of the Thompson–Chandrasekhar criterion.

GOVERNING EQUATIONS AND BOUNDARY CONDITIONS

The governing equations and boundary conditions in their nondimensional forms for the magnetohydrodynamic thermal convection problem wherein the dynamically free boundaries are thermally and electrically perfectly conducting and a uniform vertical magnetic field opposite to gravity is impressed upon the system are given by (Banerjee et al. [2])

\[
(D^2 - a^2)(D^2 - a^2 - p/a)w = Ra^2\theta - QD(D^2 - a^2)h; \\
(D^2 - a^2 - p)\theta = w \\
(D^2 - a^2 - p\sigma_1/\sigma)h_z = -Dw \\
w = 0 = \theta = D^2w = h_z \quad \text{at} \quad z = -\frac{1}{2} \quad \text{and} \quad z = +\frac{1}{2}
\]

wherein the symbols have the same meanings as given in the above reference and the boundary conditions that are relevant to thermally perfectly conducting boundaries replace the corresponding ones that are relevant to thermally insulating boundaries.

Combining the above equations and boundary conditions in an appropriate manner we derive equations and the boundary conditions in terms of \(w\) alone,

\[
Lw = 0 \\
w = 0 = D^2w = L_1w = L_1'w \quad \text{at} \quad z = -\frac{1}{2} \quad \text{and} \quad z = +\frac{1}{2},
\]

where

\[
L = (D^2 - a^2 - p\sigma_1/\sigma)(D^2 - a^2 - p)(D^2 - a^2)(D^2 - a^2 - p/a) + Ra^2(D^2 - a^2 - p/a)D^2(D^2 - a^2 - p) \\
L_1 = D(D^2 - a^2)(D^2 - a^2 - p/a) - QD^3 - Qp\sigma_1/\sigma D \\
L' = D^3(D^2 - a^2)(D^2 - a^2 - p/\sigma) - QD^5 - Qp\sigma_1/\sigma D^3 \\
- Qp\sigma_1/\sigma(a^2 + p\sigma_1/\sigma)D + Ra^2D.
\]

We make an important observation here, namely that since Eqs. (1)–(3) are valid for all values of \(z\) in \([-\frac{1}{2}, +\frac{1}{2}]\), which includes the two end points
\[ z = -\frac{1}{2} \text{ and } z = +\frac{1}{2} \text{ also, it follows from Eq. (1) and boundary conditions (4) that the relevant solutions for } w \text{ and } h_z \text{ must satisfy the restriction} \]
\[
\left[ (D^2 - a^2)(D^2 - a^2 - p/\sigma)w \right]_z = \pm \frac{1}{2} = \left[ -QD(D^2 - a^2)h_z \right]_z = \pm \frac{1}{2} \]  
(10)

and therefore if \( h_z \) is expanded in terms of an appropriate complete set of functions such that the requisite boundary conditions on \( h_z \) are satisfied and \( w \) is evaluated in accord with the magnetic induction equation \( (3) \) then there must exist sufficient freedom in terms of adjustable constants in the expression for \( w \) and hence in the expression for \( h_z \) so that not only the requisite boundary conditions on \( w \) but also the restriction on \( w \) and \( h_z \) prescribed by Eq. (10) is satisfied. Herein lies the crucial difference between the mathematical analysis of Banerjee et al. [2] and the mathematical analysis that we now apply. In the analysis of Banerjee et al. [2] the relevant solutions for \( w \) and \( h_z \), that are constructed in the above manner and that satisfy the magnetic induction equation \( (3) \) and the requisite boundary conditions on \( w \) and \( h_z \), are not left with any freedom so that Eq. (1) can be satisfied on the boundaries which, in their framework, is necessary for it to be satisfied everywhere in the flow domain. In fact, in their case, \( w \) and \( h_z \) come out to be such that
\[
\left[ D(D^2 - a^2)(D^2 - a^2 - p/\sigma) \right]_{z=\pm \frac{1}{2}} = 0 = \left[ QD^2(D^2 - a^2)h_z \right]_{z=\pm \frac{1}{2}} \]  
(11)

and therefore it becomes necessary for them to assume that
\[
\left[ D\theta \right]_{z=\pm \frac{1}{2}} = 0 \]  
(1.2)

is valid so that Eq. (1) is not violated on the boundaries, namely at \( z = \pm \frac{1}{2} \). Herein lies the genesis of the consideration of the boundary conditions that are relevant to thermally insulating boundaries.

In the present paper we modify their analysis appropriately so that it is applicable to the case of thermally perfectly conducting boundaries on which \( \theta \) and not \( D\theta \) vanishes.

**Mathematical Analysis**

Following Banerjee et al. [2] we note that the proper solutions for \( w \) and \( \theta \) must be odd while that for \( h_z \) must be even. Therefore if \( d_1, d_2, \) and \( d_3 \) are constants the function \( h_z - d_1 \cos 3\pi z - d_2 \cos 5\pi z - d_3 \cos 7\pi z \) is even and since it is required to vanish at \( z = \pm \frac{1}{2} \), we can expand it in a Fourier cosine series in the form
\[
h_z = d_1 \cos 3\pi z - d_2 \cos 5\pi z - d_3 \cos 7\pi z \\
= \sum_{n=0}^{\infty} c_n \cos(2n + 1)\pi z. \]  
(13)
With \( h_z \) given by Eq. (13), Eq. (3) becomes

\[
-Dw = d_1(-3^2\pi^2 - a^2 - p\sigma_1/\sigma) \cos 3\pi z \\
   + d_2(-5^2\pi^2 - a^2 - p\sigma_1/\sigma) \cos 5\pi z \\
   + d_3(-7^2\pi^2 - a^2 - p\sigma_1/\sigma) \cos 7\pi z \\
   + \sum_{n=0}^{∞} c_n\left\{-(2n + 1)^2\pi^2 - a^2 - p\sigma_1/\sigma\right\} \cos(2n + 1)\pi z
\]

(14)

which upon integration yields

\[
w = \frac{d_1}{3\pi}(3^2\pi^2 + a^2 + p\sigma_1/\sigma) \sin 3\pi z \\
   + \frac{d_2}{5\pi}(5^2\pi^2 + a^2 + p\sigma_1/\sigma) \sin 5\pi z \\
   + \frac{d_3}{7\pi}(7^2\pi^2 + a^2 + p\sigma_1/\sigma) \sin 7\pi z \\
   + \sum_{n=0}^{∞} \frac{c_n}{(2n + 1)} \left\{(2n + 1)^2\pi^2 + a^2 + p\sigma_1/\sigma\right\} \sin(2n + 1)\pi z + d_4,
\]

(15)

where \( d_4 \) is a constant of integration.

The requirement that the above solution for \( w \) satisfy the boundary conditions specified by Eq. (4) and the above solutions for \( w \) and \( h_z \) satisfy the restriction prescribed by Eq. (10) lead to a unique determination of \( d_4 \), \( d_3 \), \( d_2 \), and \( d_1 \), which are given by

\[
d_4 = 0
\]

(16)

\[
d_3 = \left[ -\sum_{n=0}^{∞} \left\{\pi\gamma_1 c_n \alpha_n / \delta_1 \right\} + \sum_{n=0}^{∞} \left\{\pi\gamma_2 c_n \beta_n / \delta_2 \right\} \\
   - \sum_{n=0}^{∞} \left\{(-1)^n + 1 c_n \gamma_n / (2n + 1) \right\} \right] \left[ \frac{3\gamma_1 \delta_3}{2 \delta_1} - \frac{5\gamma_2 \delta_3}{2 \delta_2} + \gamma_3 \right]
\]

(17)

\[
d_2 = \sum_{n=0}^{∞} \left\{\pi c_n \beta_n / \delta_2 \right\} + \frac{5}{14} \frac{d_3 \delta_3}{\delta_2}
\]

(18)

\[
d_1 = \sum_{n=0}^{∞} \left\{\pi c_n \alpha_n / \delta_1 \right\} + \frac{3}{14} \frac{d_3 \delta_3}{\delta_1}
\]

(19)
where $\alpha_n$, $\beta_n$, $\gamma_n$, and $\delta_n$ are given by

$$
\alpha_n = \frac{(-1)^{n+1}}{16} \left[ (2n+1)^2 - 5^2 \right]
\times \frac{\left[ (2n+1)^2 \pi^2 + a^2 + \frac{p_0}{\sigma^2} \right]}{(2n+1)\pi}
$$  \hspace{1cm} (20)

$$
\beta_n = \frac{(-1)^{n+1}}{16} \left[ (2n+1)^2 - 3^2 \right]
\times \frac{\left[ (2n+1)^2 \pi^2 + a^2 + \frac{p_0}{\sigma^2} \right]}{(2n+1)\pi}
$$  \hspace{1cm} (21)

$$
\gamma_n = \left\{ (2n+1)^2 \pi^2 + a^2 \right\} \left[ \left\{ (2n+1)^2 \pi^2 + a^2 + \frac{p_0}{\sigma^2} \right\} + (2n+1)^2 \pi^2 Q \right] \frac{\gamma_1 \delta_3}{\psi \delta_2}
\times \left\{ \frac{3}{2} \sin 3\pi z + \frac{5}{2} \sin 5\pi z + \sin 7\pi z \right\}
$$  \hspace{1cm} (22)

$$
\delta_n = (2n+1)^2 \pi^2 + a^2 + \frac{p_0}{\sigma^2}.
$$  \hspace{1cm} (23)

Making use of Eqs. (15) to (19) we obtain the proper solution for $w$ as

$$
w = (\sin 3\pi z) \sum_{n=0}^{\infty} c_n \alpha_n + (\sin 5\pi z) \sum_{n=0}^{\infty} c_n \beta_n
\quad + \sum_{n=0}^{\infty} \left\{ c_n \gamma_n \sin(2n+1)\pi z \right\} + \left[ \frac{\gamma_1 \delta_3}{\psi \delta_2} \sum_{n=0}^{\infty} c_n \alpha_n
\quad + \frac{\gamma_2 \delta_3}{\psi \delta_2} \sum_{n=0}^{\infty} c_n \beta_n - \frac{\delta_3}{\psi} \sum_{n=0}^{\infty} \left\{ \frac{(-1)^n \gamma_1 \gamma_n}{(2n+1)\pi} \right\} \right]
\quad \times \left\{ \frac{3}{2} \sin 3\pi z + \frac{5}{2} \sin 5\pi z + \sin 7\pi z \right\}
$$  \hspace{1cm} (24)

where

$$
ev_n = \left\{ (2n+1)^2 \pi^2 + a^2 + \frac{p_0}{\sigma^2} \right\} \frac{1}{(2n+1)\pi}
$$  \hspace{1cm} (25)

$$
\psi = \frac{3 \gamma_1 \delta_3}{\delta_1} - \frac{5 \gamma_2 \delta_3}{\delta_2} + \gamma_3.
$$  \hspace{1cm} (26)

With $w$ given by Eq. (24), Eq. (5) becomes
\[
\sum_{n=0}^{\infty} \left\{ s_1 c_n x_n \sin 3\pi z + s_2 c_n \beta_n \sin 5\pi z \right. \\
+ s_n c_n \epsilon_n \sin(2n+1)\pi z \} \\
+ \left[ -\frac{\gamma_1 \delta_2}{\psi \delta_1} \sum_{n=0}^{\infty} c_n x_n + \frac{\gamma_2 \delta_3}{\psi \delta_2} \sum_{n=0}^{\infty} c_n \beta_n - \sum_{n=0}^{\infty} \frac{\delta_3}{\psi} \left( \frac{(-1)^{n+1} c_n \gamma_n}{(2n+1)\pi} \right) \right] \\
\times \left\{ \frac{3}{2} s_1 \sin 3\pi z + \frac{5}{2} s_2 \sin 5\pi z + s_3 \sin 7\pi z \right\} = 0, \quad (27)
\]

where

\[
s_n = \left\{ (2n+1)^2 \pi^2 + a^2 + \frac{p \sigma_1}{\sigma} \right\} \left\{ (2n+1)^2 \pi^2 + a^2 + p \right\} \\
\cdot \left\{ (2n+1)^2 \pi^2 + a^2 + \frac{p \sigma_1}{\sigma} \right\} \\
- Ra^2 \left\{ (2n+1)^2 \pi^2 + a^2 + \frac{p \sigma_1}{\sigma} \right\} \\
+ Q(2n+1)^2 \pi^2 \left\{ (2n+1)^2 \pi^2 + a^2 \right\} \\
\cdot \left\{ (2n+1)^2 \pi^2 + a^2 + p \right\}. \quad (28)
\]

Multiplying Eq. (27) by \( \sin(2m+1)\pi z \) (since the first derivative with respect to \( z \) of the left hand side of equation (27) vanishes at \( z = \pm \frac{1}{2} \)) and integrating the resulting equation over the range of \( z \), we obtain

\[
\sum_{n=0}^{\infty} c_n \left[ s_1 x_n \frac{\delta_{1m}}{2} + s_2 \beta_n \frac{\delta_{2m}}{2} + s_n \epsilon_n \frac{\delta_{mn}}{2} \\
+ \left\{ \frac{\gamma_1 \delta_3}{\psi \delta_1} x_n + \frac{\gamma_2 \delta_3}{\psi \delta_2} \beta_n - \frac{(-1)^{n+1} \gamma_n \delta_3}{(2n+1)\pi} \right\} \right. \\
\times \left\{ \frac{3}{2} s_1 \delta_{1m} + \frac{5}{2} s_2 \delta_{2m} + s_3 \delta_{3m} \right\} = 0, \quad (m = 0, 1, 2, \ldots). \quad (29)
\]

where \( \delta_{mn} \) is the Kronecker delta.

Equation (29) provides a set of linear and homogeneous equations for the determination of the constants \( c_n \) and the requirement that the determinant of this system of equations must vanish provides the characteristic equation for the determination of \( R \) and \( p_i \), where \( p_r = 0 \). We thus obtain

\[
\| s_1 x_n \delta_{1m} + s_2 \beta_n \delta_{2m} + s_n \epsilon_n \delta_{mn} + \{ K_1 x_n + K_2 \beta_n + K_3 \lambda_n \} \\
\times \{ \frac{3}{2} s_1 \delta_{1m} + \frac{5}{2} s_2 \delta_{2m} + s_3 \delta_{3m} \} \| = 0, \quad (30)
\]
where

\[ K_1 = \frac{\gamma_1 \delta_3}{\psi \delta_1}, \quad K_2 = \frac{\gamma_2 \delta_3}{\psi \delta_2}, \]
\[ K_3 = -\frac{\delta_3}{\psi}; \quad \text{and} \quad \lambda_n = \frac{(-1)^{n+1}}{(2n+1)\pi} \gamma_n. \]

The \( n \)th approximation to the characteristic values of \( R \) and \( p \), is obtained by setting the \( n \)th order determinant consisting of the first \( n \) rows and columns in the left hand side of Eq.(30) equal to zero. This corresponds to the retention of the first \( n \) terms only in the Fourier expansion of \( h_z - d_1 \cos 3\pi z - d_2 \cos 5\pi z - d_3 \cos 7\pi z \) as given by Eq. (13). The corresponding result is

\[
\begin{array}{cccc}
\varepsilon_0 s_0 & 0 & 0 & 0 \\
\alpha_0 s_1 + (K_1 \alpha_0) & \alpha_1 s_1 + \varepsilon_1 s_1 & \alpha_2 s_1 + (K_1 \alpha_2) & \cdots \\
+ K_2 \beta_0 + K_3 \lambda_0) & + (K_1 \alpha_1 + K_2 \beta_1) & + K_2 \beta_2 + K_3 \lambda_2) & \cdots \\
\cdot \left( \frac{3}{2} s_1 \right) & \cdot \left( \frac{3}{2} s_2 \right) & \cdot \left( \frac{3}{2} s_3 \right) & \cdots \\
\beta_0 s_2 + (K_1 \alpha_0) & \beta_1 s_2 + (K_1 \alpha_1) & \beta_2 s_2 + \varepsilon_2 s_2 & \beta_3 s_2 + (K_1 \alpha_3) \\
+ K_2 \beta_0 + K_3 \lambda_0) & + K_2 \beta_1 + K_3 \lambda_1) & + (K_1 \alpha_2 + K_2 \beta_2) & + K_2 \beta_3 + K_3 \lambda_3) \\
\cdot \left( \frac{3}{2} s_2 \right) & \cdot \left( \frac{3}{2} s_3 \right) & \cdot \left( \frac{3}{2} s_4 \right) & \cdots \\
(K_1 \alpha_0 + K_2 \beta_0 & (K_1 \alpha_1 + K_2 \beta_1 & (K_1 \alpha_2 + K_2 \beta_2) & \varepsilon_3 s_3 + (K_1 \alpha_3) \\
+ K_3 \lambda_0)(s_3) & + K_3 \lambda_1)(s_3) & + K_3 \lambda_2)(s_3) & + K_2 \beta_3 + K_3 \lambda_3)(s_3) \\
0 & 0 & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots \\
\varepsilon_4 s_4 & 0 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & \varepsilon_{n-1} s \\
\end{array} = 0
\]
from which it follows uniquely that the lowest characteristic value of $R$ and the associated value of $p$, are given by the equation (Banerjee et al. [2])

$$s_0 = 0. \quad (32)$$

Further since Eq. (32) is valid whatever the value of $n$, it follows that it is the unique solution that provides the lowest characteristic value of $R$ and the associated value of $p$, as given by the characteristic equation (30).

With $w$ given by Eq. (24), $\theta$ can be determined in accordance with Eq. (2) together with the relevant boundary conditions as specified by Eqs. (4).

We now complete the solution of the problem by demonstrating that $w, \theta, h_z$ determined in the manner shown above and satisfying Eqs. (2) and (3) along with the boundary conditions (4) also satisfy Eq. (1).

To prove this we consider Eq. (5) which can be written in an alternate form as

$$\left( D^2 - a^2 - \frac{pg_1}{\sigma} \right) (D^2 - a^2 - p) E = 0, \quad (33)$$

where

$$p = ip_1, \quad p_1 \neq 0 \quad \text{and}$$

$$E = (D^2 - a^2) \left( D^2 - a^2 - \frac{p}{\sigma} \right) w - Ra^2 \theta + QD(D^2 - a^2) h_z.$$  

With $w, \theta, h_z$ as determined above, we have

$$E = 0 = DE \quad \text{at} \quad z = \pm \frac{1}{2}.$$  

Multiplying Eq. (33) by $E^*$ (the complex conjugate of $E$) throughout and integrating the resulting equation over the range $z$, we get upon equating the imaginary part of this latter equation

$$p_i \left( \frac{\sigma + \sigma_1}{\sigma} \right) \int_{-\frac{1}{2}}^{\frac{1}{2}} (|DE|^2 + a^2 |E|^2) \, dz = 0. \quad (34)$$

But, since $p_i \neq 0$, it follows from Eq. (34) that

$$E = 0 \quad \forall z \in \left[ -\frac{1}{2}, \frac{1}{2} \right]$$  

which in turn implies that Eq. (1) is also satisfied.

After a lapse of 37 years since Chandrasekhar initiated the investigations on magnetothermoconvection with his seminal work in 1952, we are left upon completion of our own paper admiring the true genius of the man
who not only had the audacity to substitute an incorrect eigensolution for
the calculation of the correct eigenvalues but also had the conviction that
everything would work out right as we find it today, thus going beyond the
limits of variational theory of eigenvalues and indicating new directions in
the theory of differential equations along which investigations could be
fruitfully carried on.

REFERENCES

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