Finite solvable groups
whose character graphs are trees

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Abstract
The aim of this paper is to classify all the finite solvable groups $G$ whose character graphs $\Gamma(G)$ have no triangles: they are exactly the finite groups with at most two non-linear irreducible characters, and the symmetric group $S_4$.

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1. Introduction and preliminaries

1.1. Throughout, all groups are finite, and all characters are over an algebraically closed field of characteristic zero. Notation is standard and taken from [I]. In particular, denote by $G'$ the commutator subgroup of group $G$, Irr($G$) the set of irreducible characters of $G$, NL($G$) the set of non-linear irreducible characters of $G$, and c.d.($G$) the set of degrees of the irreducible characters of $G$.

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The character graph $\Gamma(G)$ is introduced by Manz, Staszewski, and Willems in [MSW]. By definition the set of the vertices of $\Gamma(G)$ is $NL(G)$, and two vertices $\chi$ and $\psi$ are connected by an edge provided that $\chi(1)$ and $\psi(1)$ have a common prime divisor. So, $G$ is abelian if and only if $\Gamma(G)$ has no vertices. It is proved in [MSW] that the number of the connected components of $\Gamma(G)$ is at most 3; and if $G$ is solvable then it is at most 2. Our aim is to prove

**Theorem 1.1.** Let $G$ be a non-abelian solvable group. Then $\Gamma(G)$ has no triangles if and only if either $|NL(G)| \leq 2$, or, $G$ is isomorphic to the symmetric group $S_4$.

It turns out that for a solvable group, the property that $\Gamma(G)$ has no triangles is equivalent to that $\Gamma(G)$ has no cycles, i.e. $\Gamma(G)$ is a tree. Note that Theorem 1.1 is not true for non-solvable groups. For example, the alternative group $A_5$ has 4 non-linear irreducible characters of degrees 4, 5, 3, 3, respectively.

The similar work has been done in [FZ] for the conjugacy graph $\Delta(G)$, introduced by Bertram, Herzog, and Mann in [BHM]. For more information on various related graphs attached to a group we refer to a survey article [L2] and [CHM,L1,LZ,MSW,MW,MWW,ZJ].

1.2. The groups with $|NL(G)| = 1$ have been classified by Seitz [S] as follows.

**Lemma 1.2.** [S] A group $G$ has exactly one non-linear irreducible character if and only if $G$ is isomorphic to one of the following:

(i) An extra-special 2-group (i.e. $|G| = 2^k$ with $k$ odd, and the center $Z(G) = G'$ is of order 2).

In this case, $c.d.(G) = \{1, \sqrt{|G'|} \}$.

(ii) A Frobenius group with the Frobenius kernel $N \cong \mathbb{Z}_p$, and a cyclic Frobenius complement $H$ of order $p^n - 1$, where $p^n \neq 2$. In this case, $c.d.(G) = \{1, p^n - 1\}$.

The groups with $|NL(G)| = 2$ have been classified by Pálfy [P] (see also G.X. Zhang [ZG]). We only need and record the following weaker description of this classification. This can be also directly proved by only using Lemma 12.3 in [I] and Lemma 1.2 above.

**Lemma 1.3.** ([P], also [ZG]) Let $G$ be a group with exactly two non-linear irreducible characters. Then $G$ is isomorphic to one of the following:

(i) An extra-special 3-group. In this case $c.d.(G) = \{1, \sqrt{|G'|} \}$.

(ii) A 2-group with $c.d.(G) = \{1, \sqrt{|G'|} \}$.

(iii) A Frobenius group with the Frobenius kernel $\mathbb{Z}_3^2$ and a Frobenius complement $Q_8$ (the quaternion group of order 8). In this case $c.d.(G) = \{1, 2, 8\}$.

(iv) A Frobenius group with the Frobenius kernel $K \cong \mathbb{Z}_p$, and an abelian Frobenius complement $H$ of order $p^n - 1$, where $p$ is an odd prime. In this case $c.d.(G) = \{1, p^n - 1\}$.

(v) There is a normal subgroup $L$ of order 2 such that $G/L$ is a Frobenius group with the Frobenius kernel $K \cong \mathbb{Z}_p$, and a cyclic Frobenius complement $H$ of order $p^n - 1 \neq 1$, where $p$ is a prime. In this case, $c.d.(G) = \{1, p^n - 1\}$.

In particular, $\Gamma(G)$ is connected in all the cases.
Lemma 1.4. [BCH] Let \( G \) be a non-abelian group. Then all the degrees of non-linear irreducible characters of \( G \) are pairwise distinct if and only if \( G \) is isomorphic to one of the following:

(i) The groups in Lemma 1.2.
(ii) The group in Lemma 1.3(iii).

In particular, \( |NL(G)| \leq 2 \) and \( \Gamma(G) \) is connected in all the cases.

Corollary 1.5. Let \( G \) be a non-abelian solvable group such that \( \Gamma(G) \) is not connected. Then there exist two distinct non-linear irreducible characters having the same degrees.

Corollary 1.6. Let \( G \) be a non-abelian group. Then \( \Gamma(G) \) has no edges if and only if \( NL(G) = 1 \).

Proof. If \( \Gamma(G) \) has no edges and \( NL(G) \geq 2 \), then by Lemma 1.4 we have \( c.d.(G) = \{1, 2, 8\} \), and hence \( \Gamma(G) \) has an edge. A contradiction.

2. Proof of Theorem 1.1

The proof given here uses Lemma 1.4, and hence the classification of finite simple groups (see [BCH]). We emphasize that there is a proof independent of the classification, but it is much longer.

2.1. We first give some facts which are used for “going-down” induction.

Lemma 2.1. Let \( G \) be a non-abelian solvable group. Then either all the non-linear irreducible characters of \( G \) have the same degree, or, there exists an abelian normal subgroup \( N \neq \{1\} \) of \( G \) such that \( G/N \) is non-abelian.

Proof. Since \( G \) is solvable and non-abelian, it follows that there exists an abelian normal subgroup \( N_1 \neq \{1\} \). If \( G/N_1 \) is non-abelian, then we are done. So assume that \( G/N_1 \) is abelian. Then \( G' \leq N_1 \), and \( G' \) is also abelian.

If \( G' \) is not a minimal normal subgroup of \( G \), then we have an abelian normal subgroup \( N \) of \( G \) with \( \{1\} \neq N \leq G' \). Then we are done since \( G/N \) is non-abelian.

If \( G' \) is the unique minimal normal subgroup of \( G \), then by Lemma 12.3 in [I], all the non-linear irreducible characters of \( G \) have the same degree.

The remainder case is that there is \( N \triangleleft G \) such that both \( G' \) and \( N \) are minimal normal subgroup of \( G \) with \( G' \neq N \). Then \( G/N \) is non-abelian. Note that a minimal normal subgroup of a solvable group is abelian. This completes the proof.

Lemma 2.2. Let \( G \) be an non-abelian group, and \( N \neq \{1\} \) be a normal subgroup of \( G \). Then \( |NL(G)| > |NL(G/N)| \).

Proof. Otherwise, we have the contradiction

\[
|G/N| > |G/N| - |L(G/N)| = |G| - |L(G)| = |G| \frac{|G'|-1}{|G'|} \geq \frac{|G|}{2},
\]

where \( L(G) := \text{Irr}(G/G') \) is the set of the linear characters of \( G \).

\( \square \)
Lemma 2.3. Let $p$ be a prime, $i$, $j$ two positive integers. Then

(i) If $(p^{n} - 1) | (p^{2i} + p^{2j})$ and $p^{n} - 1 \neq 1$, then $p = 3$ and $n = 1$.
(ii) If $p$ is odd and $\frac{p^{n} - 1}{2} \neq 1$, then $\frac{p^{n} - 1}{2} | p^{2i}$.
(iii) If $p$ is odd, $\frac{p^{n} - 1}{2} \neq 1$, and $\frac{p^{n} - 1}{2} | (p^{2i} + p^{2j})$, then $p = 5$ and $n = 1$.

Proof. Assume $j \geq i$. Then

$$
(p^{n} - 1, p^{2i} + p^{2j}) = (p^{n} - 1, 1 + p^{2j - 2i}) = (p^{n} - 1, p^{n} + p^{2j - 2i})
$$

$$
= \left\{ 
\begin{array}{ll}
(p^{n} - 1, 1 + p^{2j - 2i - n}), & \text{if } 2j - 2i > n; \\
(p^{n} - 1, 1 + p^{n - 2j + 2i}), & \text{if } n \geq 2j - 2i.
\end{array}
\right.
$$

If $2j - 2i - n \geq n$, then by repeating this process, we finally get $(p^{n} - 1, p^{2i} + p^{2j}) = (p^{n} - 1, p^{m} + 1)$ for some $0 \leq m < n$.

Now $(p^{n} - 1) | (p^{2i} + p^{2j})$ implies that $(p^{n} - 1) | (p^{m} + 1)$. By calculation together with $p^{n} - 1 \neq 1$ we get the assertion. (ii) is a special case of (i).

For (iii), $\frac{p^{n} - 1}{2} | (p^{2i} + p^{2j})$ implies that $\frac{p^{n} - 1}{2} | (p^{m} + 1)$. By calculation together with $\frac{p^{n} - 1}{2} \neq 1$ we get the assertion. □

2.2. For the readability of the proof of Lemma 2.6, we first treat two special cases.

Lemma 2.4. Let $G$ be a non-abelian solvable group such that $\Gamma(G)$ is connected. Then $\Gamma(G)$ has no triangles if and only if $|NL(G)| = 1$ or 2.

Proof. If $\Gamma(G)$ has no triangles and $|NL(G)| \geq 3$, then by Lemma 1.4 there exist two non-linear irreducible characters of $G$ having the same degree. Now by the connectivity of $\Gamma(G)$ we see that it has a triangle. □

Lemma 2.5. Let $G$ be a non-abelian solvable group such that $\Gamma(G)$ is not connected. Assume that there exists an abelian normal subgroup $N \neq \{1\}$ with $|NL(G/N)| = 1$. If $\Gamma(G)$ has no triangles, then $G \cong S_{4}$.

Proof. Since $\Gamma(G)$ is not connected, it follows from Lemma 1.3 that $|NL(G)| \geq 3$. Note that $G/N$ cannot be a 2-group; otherwise, by Ito’s theorem (see e.g. Theorem 6.15 in [I]) 2 divides the degrees of any non-linear irreducible characters of $G$, and then $\Gamma(G)$ has a triangle. So, by Lemma 1.2 $G/N$ has to be isomorphic to a Frobenius group with the Frobenius kernel $K \cong Z_{p}^{n}$ and a cyclic Frobenius complement $H$ of order $p^{n} - 1$, and $p^{n} \neq 2$. By Ito’s theorem we have $\psi(1) | |G/N| = p^{n}(p^{n} - 1)$ for $\psi \in \text{Irr}(G)$. For the later convenience, we divide the proof into several steps.

Step 1. character of a linear non-principle character of the Frobenius kernel is an irreducible character of a Frobenius group. See e.g. Proposition 14.4 in [CR], the unique non-linear irreducible character $\phi$ of $G/N$ is of degree $|H| = p^{n} - 1$. 

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Step 2. We claim that there exists a \( \psi \in \text{Irr}(G) \) such that \( p \mid \psi(1) \); otherwise, any degree of non-linear irreducible character of \( G \) can divide \( p^n - 1 \). Since \( |\text{NL}(G)| \geq 3 \), it follows from Step 1 that \( \Gamma(G) \) is connected.

Step 3. Since \( G/N \) is a Frobenius group with the Frobenius kernel \( K \), it follows that there is a normal subgroup \( M \) of \( G \) such that \( M/N \cong K \). Thus \( |M/N| = p^n \), \( |G/M| = p^n - 1 \). Such an \( M \) cannot be abelian (otherwise we have \( \chi(1) \mid |G/M| = p^n - 1 \) for every \( \chi \in \text{Irr}(G) \), which contradicts the claim in Step 2).

For \( \psi \in \text{Irr}(G) \) with \( p \mid \psi(1) \), \( \psi|_M \) has no linear summands (otherwise, by Corollary 11.29 in [I] \( \psi(1) \) divides \( |G/M| = p^n - 1 \). While for \( \chi \in \text{Irr}(G) \) with \( p \nmid \chi(1) \), by Clifford’s theorem \( \chi|_M \) is a sum of linear characters, since \( \phi(1) \mid |M/N| = p^n \) for every \( \phi \in \text{Irr}(M) \).

Step 4. Consider the irreducible decomposition \( \chi_{\text{reg},G} = \Sigma_1 + \Sigma_2 \) of the regular character of \( G \), where \( \Sigma_1 \) is the sum of \( \text{deg}(\chi) \chi \)'s, with \( \chi \) running over all the irreducible characters of \( G \) whose degrees cannot be divided by \( p \), and \( \Sigma_2 \) is the sum of \( \text{deg}(\psi) \psi \)'s, with \( \psi \) running over all the irreducible characters of \( G \) whose degrees can be divided by \( p \). Also, we have

\[
\chi_{\text{reg},G}|_M = |G/M|\chi_{\text{reg},M} = |G/M| (\Sigma_1 + \Sigma_2) = (p^n - 1) (\Sigma_3 + \Sigma_4),
\]

where \( \Sigma_3 \) is the sum of all the linear characters of \( M \), and \( \Sigma_4 \) is the sum of \( \text{deg}(\phi) \phi \)'s, with \( \phi \) running over all the non-linear irreducible characters \( \phi \) of \( M \). By Step 3 we have

\[
\Sigma_2|_M = (p^n - 1) \Sigma_4 \quad \text{and} \quad (p^n - 1) \mid \Sigma_2(1). \tag{2.1}
\]

Step 5. Let \( \{\psi_1, \ldots, \psi_t\} \) be the set of the irreducible characters of \( G \) whose degrees can be divided by \( p \), and \( \text{NL}(G) = \{\varphi, \psi_1, \ldots, \psi_t, \ldots, \psi_m\} \), where \( \varphi(1) = p^n - 1 \), \( \psi_i(1) = p^{a_i} \alpha_i \), \( 1 \leq \alpha_i \leq n \), \( 1 \leq i \leq t \); and \( \psi_i(1) = \alpha_i \), \( t + 1 \leq i \leq m \); and \( \alpha_i \mid (p^n - 1) \) for \( 1 \leq i \leq m \). Since \( \Gamma(G) \) has no triangles, it follows that \( t \leq 2 \), and \( (\alpha_i, \alpha_j) = 1 \) for \( i \neq j \).

Note that \( t \neq 1 \): otherwise \( (p^n - 1) \mid p^{2a_1} \alpha_1^2 \) by (2.1) and hence \( (p^n - 1) \mid \alpha_1^2 \). Since \( p^n \neq 2 \), it follows that \( \alpha_1 \neq 1 \), and hence \( \Gamma(G) \) is connected.

Thus \( t = 2 \), and then by (2.1) we have \( (p^n - 1) \mid (p^{2a_1} \alpha_1^2 + p^{2a_2} \alpha_2^2) \). Since \( \alpha_1 \mid (p^n - 1) \), it follows that \( \alpha_1 \mid (p^{2a_1} \alpha_1^2 + p^{2a_2} \alpha_2^2) \), and hence divides \( p^{2a_2} \alpha_2^2 \). It follows from \( (\alpha_i, p) = 1 = (\alpha_1, \alpha_2) \) that \( \alpha_1 \) divides \( \alpha_2 \) and hence \( \alpha_1 = 1 \). Similarly we have \( \alpha_2 = 1 \). By Lemma 2.3(i) we have \( n = 1 \), \( p = 3 \), and \( \alpha_1 = \alpha_2 = 1 \).

Now \( G \) has two irreducible characters of degree 3, \( c.d.(G) = \{1, 2, 3\} \), \( |G/M| = 2 \), \( |M/N| = 3 \), \( c.d.(M) = \{1, 3\} \), and \( |\text{NL}(M)| = 1 \) (by (2.1)). By Lemma 1.2(ii) we know that \( M \) is isomorphic to the alternative group \( A_4 \), and then \( |G| = 24 \). Note that the number of the Sylow 3-subgroups is 4 (otherwise, the Sylow 3-subgroup is an abelian normal subgroup of \( G \), and then by Ito’s theorem all the degrees of the irreducible characters of \( G \) divides \( 24 \) = 8). By using \( G \)'s conjugate action on the set of the Sylow 3-subgroups of \( G \), one easily gets \( G \cong S_4 \). \( \square \)

2.3. The following is the main lemma in this section. It claims that \( S_4 \) is the unique solvable group with the property that the character graph is not connected and has no triangles.

Lemma 2.6. Let \( G \) be a non-abelian solvable group such that \( \Gamma(G) \) is not connected. Then \( \Gamma(G) \) has no triangles if and only if \( G \) is isomorphic to \( S_4 \).
Proof. We only need to prove the necessity. Assume that $G$ is a counter-example of minimal order, that is, $|G|$ is minimal with respect to the property that $G$ is non-abelian solvable with $\Gamma(G)$ not connected, and that $\Gamma(G)$ has no triangles and $G \not\cong S_4$. Since $\Gamma(G)$ is not connected, it follows that we can apply Lemma 2.1 to get an abelian normal subgroup $N \neq \{1\}$ of $G$ with $G/N$ non-abelian. Since $\Gamma(G/N)$ has no triangles, it follows from the choice of the minimality of $|G|$ that either $G/N \cong S_4$, or $\Gamma(G/N)$ is connected.

Case 1. $G/N \cong S_4$.

By lifting from $\text{Irr}(S_4)$ one has $\chi_1, \chi_2, \chi_3 \in \text{Irr}(G)$ with $\chi_1(1) = 2, \chi_2(1) = 3 = \chi_3(1)$. Then $|\text{NL}(G)| = 4$ and $\chi_4(1) = 2, 4, \text{ or } 8$ (in fact, by Lemma 2.2 $\text{NL}(G) > 3$; if $\text{NL}(G) \geq 5$, then $\Gamma(G)$ has a triangle, since $\chi(1)$ divides $|G/N| = 24$ for each $\chi \in \text{Irr}(G)$). By calculations such a group $G$ does not exist. For example, if $\chi_4(1) = 8$, then $|G| = |L(G)| + 2^2 + 3^2 + 3^2 + 8^2 = |L(G)| + 86$. That is, $(|G'| - 1)|L(G)| = 86 = 2 \times 43$, and all the cases contradict $24 | |G|$.

Case 2. $\Gamma(G/N)$ is connected.

Since $\Gamma(G/N)$ is connected and has no triangles, it follows from Lemma 2.4 that $|\text{NL}(G/N)| = 1$ or $2$. But by Lemma 2.5 $|\text{NL}(G/N)| \neq 1$. So $|\text{NL}(G/N)| = 2$. By Lemma 1.3 we discuss as follows.

Subcase 1. If $G/N$ is a 2-group or a 3-group, then $\Gamma(G)$ has a triangle by Ito’s theorem. A contradiction.

Subcase 2. Let $G/N$ be a group in Lemma 1.3(iii), with $\text{NL}(G/N) = \{\psi_1, \psi_2\}$, $\psi_1(1) = 8$, $\psi_2(1) = 2$. Then any degree of the irreducible characters of $G$ other than $\psi_1$ and $\psi_2$ divides 9. By Corollary 1.5 there exist two distinct non-linear irreducible characters of $G$ having the same degrees, and hence $|\text{NL}(G)| = 4$ (otherwise $\Gamma(G)$ has a triangle). So, the possibilities of the vector $(\chi(1))_{\chi \in \text{NL}(G)}$ are $(2, 8, 3, 3)$ and $(2, 8, 9, 9)$. By calculations such a group $G$ does not exist. For example, if it is $(2, 8, 9, 9)$, then $|G| = |L(G)| + 2^2 + 8^2 + 9^2 + 9^2 = |L(G)| + 230$. That is, $(|G'| - 1)|L(G)| = 230 = 2 \times 5 \times 23$, and all the cases contradict $72 | |G|$.

Subcase 3. Let $G/N$ be a group in Lemma 1.3(iv), i.e., $G/N$ is a Frobenius group with the Frobenius kernel $K \cong \mathbb{Z}_p^n$ and an abelian Frobenius complement $H$, and $|H| = |K|^{-1}$. Note that $|G/N| = p^n p^n - 1$, and the two irreducible characters of $G/N$ are of degree $p^n - 1$.

By Ito’s theorem we have either $p \mid \psi(1)$ or $\psi(1) \mid p^n - 1$, for each $\psi \in \text{NL}(G)$. Let $t$ be the number of $\psi$’s in $\text{NL}(G)$ with $p \mid \psi(1)$. Since $\Gamma(G)$ has no triangles, it follows that $t = 1$ or 2, and hence

$$\text{NL}(G) = \{\psi_1, \psi_2, \psi_3\} \text{ or } \text{NL}(G) = \{\psi_1, \psi_2, \psi_3, \psi_4\},$$

with

$$\psi_1(1) = \psi_2(1) = \frac{p^n - 1}{2}, \quad \psi_3(1) = p^i, \quad \psi_4(1) = p^j, \quad 1 \leq i, j \leq n.$$
With the same notation and the same argument as in Step 4 in the proof of Lemma 2.5 (in particular, $M$ is the normal subgroup of $G$ such that $M/N = K$, and hence $|M/N| = p^n$ and $|G/M| = \frac{p^n - 1}{2}$), we get

$$\Sigma_2|_M = \frac{p^n - 1}{2} \Sigma_4 \quad \text{and} \quad \frac{p^n - 1}{2} | \Sigma_2(1).$$

So we have

$$\frac{p^n - 1}{2} | p^{2i} \quad \text{or} \quad \frac{p^n - 1}{2} | (p^{2i} + p^{2j}).$$

Since $\frac{p^n - 1}{2} \neq 1$, it follows from Lemma 2.3(ii) and (iii) that the first situation is impossible, and in the second situation we have $p = 5$, $n = 1$, $i = j = 1$. So we get $NL(M) = 1$, $c.d.(M) = \{1, 5\}$, which contradicts Lemma 1.2(ii).

**Subcase 4.** Let $G/N$ be a group in Lemma 1.3(v), i.e., $G/N$ has a normal subgroup $L/N$ of order 2 such that $G/L$ is a Frobenius group with the Frobenius kernel $K \cong \mathbb{Z}_p^n$, and a cyclic Frobenius complement $H$ of order $p^n - 1$, where $p$ is a prime and $p^n \neq 2$; and in this case, $|G/N| = 2p^n(p^n - 1)$, and the two non-linear irreducible characters of $G/N$ are of degree $p^n - 1$. As in Subcase 3 we have a normal subgroup $M$ of $G$ such that $M/L \cong K$, and hence $|M/N| = 2p^n$ and $|G/M| = p^n - 1$. Now we divide it into two situations.

(i) The situation of $p = 2$.

By Ito’s theorem we have either $2 | \psi(1)$, or $\psi(1) \mid (2^n - 1)$, for each $\psi \in NL(G)$. Let $t$ be the number of $\psi$’s in $NL(G)$ with $2 | \psi(1)$. Since $\Gamma(G)$ has no triangles and is disconnected, it follows that $t = 1$, or $2$, and hence

$$NL(G) = \{\psi_1, \psi_2, \psi_3\} \quad \text{or} \quad NL(G) = \{\psi_1, \psi_2, \psi_3, \psi_4\}$$

with

$$\psi_1(1) = \psi_2(1) = 2^n - 1, \quad \psi_3(1) = 2^i, \quad \psi_4(1) = 2^j, \quad 1 \leq i, j \leq n + 1.$$ 

With the same notation and the same argument as in Step 4 in the proof of Lemma 2.5 (as in Subcase 3) we get

$$\Sigma_2|_M = (2^n - 1) \Sigma_4 \quad \text{and} \quad (2^n - 1) | \Sigma_2(1).$$

So we have

$$(2^n - 1) | 2^{2i} \quad \text{or} \quad (2^n - 1) | (2^{2i} + 2^{2j}).$$

By Lemma 2.3 we get a contradiction.
(ii) The situation of $p \neq 2$.

Again by Ito’s theorem we get that either $p \mid \psi(1)$ or $\psi(1) \mid 2(p^n - 1)$, for each $\psi \in NL(G)$. Again let $t$ be the number of $\psi$’s in $NL(G)$ with $p \mid \psi(1)$. Then $t = 1$ or 2. Since $\Gamma(G)$ has no triangles, it follows that

$$NL(G) = \{\psi_1, \psi_2, \psi_3\} \quad \text{or} \quad NL(G) = \{\psi_1, \psi_2, \psi_3, \psi_4\},$$

with

$$\psi_1(1) = \psi_2(1) = p^n - 1, \quad \psi_3(1) = p^i, \quad \psi_4(1) = p^j, \quad 1 \leq i, j \leq n.$$

Hence we get (as we do before)

$$\Sigma_2 | M = (p^n - 1) \Sigma_4 \quad \text{and} \quad (p^n - 1) \mid \Sigma_2(1).$$

So we have

$$(p^n - 1) \mid p^{2i} \quad \text{or} \quad (p^n - 1) \mid (p^{2i} + p^{2j}).$$

By Lemma 2.3 the unique solution is $p = 3, n = 1$. Thus we have

$$NL(G) = \{\psi_1, \psi_2, \psi_3, \psi_4\},$$

with

$$\psi_1(1) = \psi_2(1) = 2, \quad \psi_3(1) = \psi_4(1) = 3.$$

Now we have $|G| = |L(G)| + 2^2 + 2^2 + 3^2 + 3^2 = |L(G)| + 26$, or $|L(G)|(|G'| - 1) = 26$, which contradicts that $|G|$ has factors 2 and 3.

This completes the proof. \(\square\)

2.4. Now Theorem 1.1 follows from Lemmas 2.4 and 2.6.

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