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Finite solvable groups whose character graphs are trees ☆

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Abstract

The aim of this paper is to classify all the finite solvable groups G whose character graphs $\Gamma(G)$ have no triangles: they are exactly the finite groups with at most two non-linear irreducible characters, and the symmetric group S_4.
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1. Introduction and preliminaries

1.1. Throughout, all groups are finite, and all characters are over an algebraically closed field of characteristic zero. Notation is standard and taken from [I]. In particular, denote by G' the commutator subgroup of group G, $\mathrm{Irr}(G)$ the set of irreducible characters of G, $NL(G)$ the set of non-linear irreducible characters of G, and $c.d.(G)$ the set of degrees of the irreducible characters of G.

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The character graph $\Gamma(G)$ is introduced by Manz, Staszewski, and Willems in [MSW]. By definition the set of the vertices of $\Gamma(G)$ is $NL(G)$, and two vertices χ and ψ are connected by an edge provided that $\chi(1)$ and $\psi(1)$ have a common prime divisor. So, G is abelian if and only if $\Gamma(G)$ has no vertices. It is proved in [MSW] that the number of the connected components of $\Gamma(G)$ is at most 3; and if G is solvable then it is at most 2. Our aim is to prove

Theorem 1.1. *Let G be a non-abelian solvable group. Then $\Gamma(G)$ has no triangles if and only if either $|NL(G)| \leqslant 2$, or, G is isomorphic to the symmetric group S_4.*

It turns out that for a solvable group, the property that $\Gamma(G)$ has no triangles is equivalent to that $\Gamma(G)$ has no cycles, i.e. $\Gamma(G)$ is a tree. Note that Theorem 1.1 is not true for non-solvable groups. For example, the alternative group A_5 has 4 non-linear irreducible characters of degrees $4, 5, 3, 3$, respectively.

The similar work has been done in [FZ] for the conjugacy graph $\Delta(G)$, introduced by Bertram, Herzog, and Mann in [BHM]. For more information on various related graphs attached to a group we refer to a survey article [L2] and [CHM,L1,LZ,MQS,MSW,MW,MWW,ZJ].

1.2. The groups with $|NL(G)| = 1$ have been classified by Seitz [S] as follows.

Lemma 1.2. [S] *A group G has exactly one non-linear irreducible character if and only if G is isomorphic to one of the following*:

(i) *An extra-special 2-group (i.e. $|G| = 2^k$ with k odd, and the center $Z(G) = G'$ is of order 2). In this case, $c.d.(G) = \{1, \sqrt{\frac{|G|}{2}}\}$.*
(ii) *A Frobenius group with the Frobenius kernel $N \cong \mathbb{Z}_p^n$, and a cyclic Frobenius complement H of order $p^n - 1$, where $p^n \neq 2$. In this case, $c.d.(G) = \{1, p^n - 1\}$.*

The groups with $|NL(G)| = 2$ have been classified by Pálfy [P] (see also G.X. Zhang [ZG]). We only need and record the following weaker description of this classification. This can be also directly proved by only using Lemma 12.3 in [I] and Lemma 1.2 above.

Lemma 1.3. ([P], also [ZG]) *Let G be a group with exactly two non-linear irreducible characters. Then G is isomorphic to one of the following*:

(i) *An extra-special 3-group. In this case $c.d.(G) = \{1, \sqrt{\frac{|G'|}{3}}\}$.*
(ii) *A 2-group with $c.d.(G) = \{1, \frac{\sqrt{|G|}}{2}\}$.*
(iii) *A Frobenius group with the Frobenius kernel $\mathbb{Z}_3^2$ and a Frobenius complement Q_8 (the quaternion group of order 8). In this case $c.d.(G) = \{1, 2, 8\}$.*
(iv) *A Frobenius group with the Frobenius kernel $K \cong \mathbb{Z}_p^n$, and an abelian Frobenius complement H of order $\frac{p^n-1}{2} \neq 1$, where p is an odd prime. In this case $c.d.(G) = \{1, \frac{p^n-1}{2}\}$.*
(v) *There is a normal subgroup L of order 2 such that G/L is a Frobenius group with the Frobenius kernel $K \cong \mathbb{Z}_p^n$, and a cyclic Frobenius complement H of order $p^n - 1 \neq 1$, where p is a prime. In this case, $c.d.(G) = \{1, p^n - 1\}$.*

In particular, $\Gamma(G)$ is connected in all the cases.

Lemma 1.4. [BCH] *Let G be a non-abelian group. Then all the degrees of non-linear irreducible characters of G are pairwise distinct if and only if G is isomorphic to one of the following*:

(i) *The groups in Lemma* 1.2.
(ii) *The group in Lemma* 1.3(iii).

In particular, $|NL(G)| \leqslant 2$ and $\Gamma(G)$ is connected in all the cases.

Corollary 1.5. *Let G be a non-abelian solvable group such that $\Gamma(G)$ is not connected. Then there exist two distinct non-linear irreducible characters having the same degrees.*

Corollary 1.6. *Let G be a non-abelian group. Then $\Gamma(G)$ has no edges if and only if $NL(G) = 1$.*

Proof. If $\Gamma(G)$ has no edges and $NL(G) \geqslant 2$, then by Lemma 1.4 we have $c.d.(G) = \{1, 2, 8\}$, and hence $\Gamma(G)$ has an edge. A contradiction. □

2. Proof of Theorem 1.1

The proof given here uses Lemma 1.4, and hence the classification of finite simple groups (see [BCH]). We emphasize that there is a proof independent of the classification, but it is much longer.

2.1. We first give some facts which are used for "going-down" induction.

Lemma 2.1. *Let G be a non-abelian solvable group. Then either all the non-linear irreducible characters of G have the same degree, or, there exists an abelian normal subgroup $N \neq \{1\}$ of G such that G/N is non-abelian.*

Proof. Since G is solvable and non-abelian, it follows that there exists an abelian normal subgroup $N_1 \neq \{1\}$. If G/N_1 is non-abelian, then we are done. So assume that G/N_1 is abelian. Then $G' \leqslant N_1$, and G' is also abelian.

If G' is not a minimal normal subgroup of G, then we have an abelian normal subgroup N of G with $\{1\} \neq N \lneqq G'$. Then we are done since G/N is non-abelian.

If G' is the unique minimal normal subgroup of G, then by Lemma 12.3 in [I], all the non-linear irreducible characters of G have the same degree.

The remainder case is that there is $N \lhd G$ such that both G' and N are minimal normal subgroup of G with $G' \neq N$. Then G/N is non-abelian. Note that a minimal normal subgroup of a solvable group is abelian. This completes the proof. □

Lemma 2.2. *Let G be an non-abelian group, and $N \neq \{1\}$ be a normal subgroup of G. Then $|NL(G)| > |NL(G/N)|$.*

Proof. Otherwise, we have the contradiction

$$|G/N| > |G/N| - \left|L(G/N)\right| = |G| - \left|L(G)\right| = |G|\frac{|G'| - 1}{|G'|} \geqslant \frac{|G|}{2},$$

where $L(G) := \mathrm{Irr}(G/G')$ is the set of the linear characters of G. □

Lemma 2.3. *Let p be a prime, i, j two positive integers. Then*

(i) *If $(p^n-1) \mid (p^{2i}+p^{2j})$ and $p^n-1 \neq 1$, then $p=3$ and $n=1$.*
(ii) *If p is odd and $\frac{p^n-1}{2} \neq 1$, then $\frac{p^n-1}{2} \nmid p^{2i}$.*
(iii) *If p is odd, $\frac{p^n-1}{2} \neq 1$, and $\frac{p^n-1}{2} \mid (p^{2i}+p^{2j})$, then $p=5$ and $n=1$.*

Proof. Assume $j \geqslant i$. Then

$$\begin{aligned}\left(p^n-1, p^{2i}+p^{2j}\right) &= \left(p^n-1, 1+p^{2j-2i}\right) = \left(p^n-1, p^n+p^{2j-2i}\right) \\ &= \begin{cases} (p^n-1, 1+p^{2j-2i-n}), & \text{if } 2j-2i>n; \\ (p^n-1, 1+p^{n-2j+2i}), & \text{if } n \geqslant 2j-2i. \end{cases}\end{aligned}$$

If $2j-2i-n \geqslant n$, then by repeating this process, we finally get $(p^n-1, p^{2i}+p^{2j}) = (p^n-1, p^m+1)$ for some $0 \leqslant m < n$.

Now $(p^n-1) \mid (p^{2i}+p^{2j})$ implies that $(p^n-1) \mid (p^m+1)$. By calculation together with $p^n-1 \neq 1$ we get the assertion. (ii) is a special case of (i).

For (iii), $\frac{p^n-1}{2} \mid (p^{2i}+p^{2j})$ implies that $\frac{p^n-1}{2} \mid (p^m+1)$. By calculation together with $\frac{p^n-1}{2} \neq 1$ we get the assertion. □

2.2. For the readability of the proof of Lemma 2.6, we first treat two special cases.

Lemma 2.4. *Let G be a non-abelian solvable group such that $\Gamma(G)$ is connected. Then $\Gamma(G)$ has no triangles if and only if $|NL(G)| = 1$ or 2.*

Proof. If $\Gamma(G)$ has no triangles and $|NL(G)| \geqslant 3$, then by Lemma 1.4 there exist two non-linear irreducible characters of G having the same degree. Now by the connectivity of $\Gamma(G)$ we see that it has a triangle. □

Lemma 2.5. *Let G be a non-abelian solvable group such that $\Gamma(G)$ is not connected. Assume that there exists an abelian normal subgroup $N \neq \{1\}$ with $|NL(G/N)| = 1$. If $\Gamma(G)$ has no triangles, then $G \cong S_4$.*

Proof. Since $\Gamma(G)$ is not connected, it follows from Lemma 1.3 that $|NL(G)| \geqslant 3$. Note that G/N cannot be a 2-group: otherwise, by Ito's theorem (see e.g. Theorem 6.15 in [I]) 2 divides the degrees of any non-linear irreducible characters of G, and then $\Gamma(G)$ has a triangle. So, by Lemma 1.2 G/N has to be isomorphic to a Frobenius group with the Frobenius kernel $K \cong Z_p^n$ and a cyclic Frobenius complement H of order p^n-1, and $p^n \neq 2$. By Ito's theorem we have $\psi(1) \mid |G/N| = p^n(p^n-1)$ for $\psi \in \mathrm{Irr}(G)$. For the later convenience, we divide the proof into several steps.

Step 1. character of a linear non-principle character of the Frobenius kernel is an irreducible character of a Frobenius group. See e.g. Proposition 14.4 in [CR], the unique non-linear irreducible character φ of G/N is of degree $|H| = p^n-1$.

Step 2. We claim that there exists a $\psi \in \mathrm{Irr}(G)$ such that $p \mid \psi(1)$: otherwise, any degree of non-linear irreducible character of G can divide $p^n - 1$. Since $|NL(G)| \geqslant 3$, it follows from Step 1 that $\Gamma(G)$ is connected.

Step 3. Since G/N is a Frobenius group with the Frobenius kernel K, it follows that there is a normal subgroup M of G such that $M/N \cong K$. Thus $|M/N| = p^n$, $|G/M| = p^n - 1$. Such an M cannot be abelian (otherwise we have $\chi(1) \mid |G/M| = p^n - 1$ for every $\chi \in \mathrm{Irr}(G)$, which contradicts the claim in Step 2).

For $\psi \in \mathrm{Irr}(G)$ with $p \mid \psi(1)$, $\psi|_M$ has no linear summands (otherwise, by Corollary 11.29 in [I] $\psi(1)$ divides $|G/M| = p^n - 1$). While for $\chi \in \mathrm{Irr}(G)$ with $p \nmid \chi(1)$, by Clifford's theorem $\chi|_M$ is a sum of linear characters, since $\phi(1) \mid |M/N| = p^n$ for every $\phi \in \mathrm{Irr}(M)$.

Step 4. Consider the irreducible decomposition $\chi_{reg,G} = \Sigma_1 + \Sigma_2$ of the regular character of G, where Σ_1 is the sum of $\deg(\chi)\chi$'s, with χ running over all the irreducible characters of G whose degrees cannot be divided by p, and Σ_2 is the sum of $\deg(\psi)\psi$'s, with ψ running over all the irreducible characters of G whose degrees can be divided by p. Also, we have

$$\chi_{reg,G}|_M = |G/M|\chi_{reg,M} = |G/M|(\Sigma_3 + \Sigma_4) = \big(p^n - 1\big)(\Sigma_3 + \Sigma_4),$$

where Σ_3 is the sum of all the linear characters of M, and Σ_4 is the sum of $\deg(\phi)\phi$'s, with ϕ running over all the non-linear irreducible characters ϕ of M. By Step 3 we have

$$\Sigma_2|_M = \big(p^n - 1\big)\Sigma_4 \quad \text{and} \quad \big(p^n - 1\big) \,\big|\, \Sigma_2(1). \tag{2.1}$$

Step 5. Let $\{\psi_1, \ldots, \psi_t\}$ be the set of the irreducible characters of G whose degrees can be divided by p, and $NL(G) = \{\varphi, \psi_1, \ldots, \psi_t, \ldots, \psi_m\}$, where $\varphi(1) = p^n - 1$, $\psi_i(1) = p^{a_i}\alpha_i$, $1 \leqslant a_i \leqslant n$, $1 \leqslant i \leqslant t$; and $\psi_i(1) = \alpha_i$, $t + 1 \leqslant i \leqslant m$; and $\alpha_i \mid (p^n - 1)$ for $1 \leqslant i \leqslant m$. Since $\Gamma(G)$ has no triangles, it follows that $t \leqslant 2$, and $(\alpha_i, \alpha_j) = 1$ for $i \neq j$.

Note that $t \neq 1$: otherwise $(p^n - 1) \mid p^{2a_1}\alpha_1^2$ by (2.1) and hence $(p^n - 1) \mid \alpha_1^2$. Since $p^n \neq 2$, it follows that $\alpha_1 \neq 1$, and hence $\Gamma(G)$ is connected.

Thus $t = 2$, and then by (2.1) we have $(p^n - 1) \mid (p^{2a_1}\alpha_1^2 + p^{2a_2}\alpha_2^2)$. Since $\alpha_i \mid (p^n - 1)$, it follows that α_1 divides $(p^{2a_1}\alpha_1^2 + p^{2a_2}\alpha_2^2)$, and hence divides $p^{2a_2}\alpha_2^2$. It follows from $(\alpha_i, p) = 1 = (\alpha_1, \alpha_2)$ that α_1 divides α_2 and hence $\alpha_1 = 1$. Similarly we have $\alpha_2 = 1$. By Lemma 2.3(i) we have $n = 1$, $p = 3$, and $a_1 = a_2 = 1$.

Now G has two irreducible characters of degree 3, $c.d.(G) = \{1, 2, 3\}$, $|G/M| = 2$, $|M/N| = 3$, $c.d.(M) = \{1, 3\}$, and $|NL(M)| = 1$ (by (2.1)). By Lemma 1.2(ii) we know that M is isomorphic to the alternative group A_4, and then $|G| = 24$. Note that the number of the Sylow 3-subgroups is 4 (otherwise, the Sylow 3-subgroup is an abelian normal subgroup of G, and then by Ito's theorem all the degrees of the irreducible characters of G divides $\frac{24}{3} = 8$). By using G's conjugate action on the set of the Sylow 3-subgroups of G, one easily gets $G \cong S_4$. $\square$

2.3. The following is the main lemma in this section. It claims that S_4 is the unique solvable group with the property that the character graph is not connected and has no triangles.

Lemma 2.6. *Let G be a non-abelian solvable group such that $\Gamma(G)$ is not connected. Then $\Gamma(G)$ has no triangles if and only if G is isomorphic to S_4.*

Proof. We only need to prove the necessity. Assume that G is a counter-example of minimal order, that is, $|G|$ is minimal with respect to the property that G is non-abelian solvable with $\Gamma(G)$ not connected, and that $\Gamma(G)$ has no triangles and $G \ncong S_4$. Since $\Gamma(G)$ is not connected, it follows that we can apply Lemma 2.1 to get an abelian normal subgroup $N \neq \{1\}$ of G with G/N non-abelian. Since $\Gamma(G/N)$ has no triangles, it follows from the choice of the minimality of $|G|$ that either $G/N \cong S_4$, or $\Gamma(G/N)$ is connected.

Case 1. $G/N \cong S_4$.

By lifting from $\mathrm{Irr}(S_4)$ one has $\chi_1, \chi_2, \chi_3 \in \mathrm{Irr}(G)$ with $\chi_1(1) = 2$, $\chi_2(1) = 3 = \chi_3(1)$. Then $|NL(G)| = 4$ and $\chi_4(1) = 2, 4$, or 8 (in fact, by Lemma 2.2 $NL(G) > 3$; if $NL(G) \geqslant 5$, then $\Gamma(G)$ has a triangle, since $\chi(1)$ divides $|G/N| = 24$ for each $\chi \in \mathrm{Irr}(G)$). By calculations such a group G does not exist. For example, if $\chi_4(1) = 8$, then $|G| = |L(G)| + 2^2 + 3^2 + 3^2 + 8^2 = |L(G)| + 86$. That is, $(|G'| - 1)|L(G)| = 86 = 2 \times 43$, and all the cases contradict $24 \mid |G|$.

Case 2. $\Gamma(G/N)$ is connected.

Since $\Gamma(G/N)$ is connected and has no triangles, it follows from Lemma 2.4 that $|NL(G/N)| = 1$ or 2. But by Lemma 2.5 $|NL(G/N)| \neq 1$. So $|NL(G/N)| = 2$. By Lemma 1.3 we discuss as follows.

Subcase 1. If G/N is a 2-group or a 3-group, then $\Gamma(G)$ has a triangle by Ito's theorem. A contradiction.

Subcase 2. Let G/N be a group in Lemma 1.3(iii), with $NL(G/N) = \{\varphi_1, \varphi_2\}$, $\varphi_1(1) = 8$, $\varphi_2(1) = 2$. Then any degree of the irreducible characters of G other than φ_1 and φ_2 divides 9. By Corollary 1.5 there exist two distinct non-linear irreducible characters of G having the same degrees, and hence $|NL(G)| = 4$ (otherwise $\Gamma(G)$ has a triangle). So, the possibilities of the vector $(\chi(1))_{\chi \in NL(G)}$ are $(2, 8, 3, 3)$ and $(2, 8, 9, 9)$. By calculations such a group G does not exist. For example, if it is $(2, 8, 9, 9)$, then $|G| = |L(G)| + 2^2 + 8^2 + 9^2 + 9^2 = |L(G)| + 230$. That is, $(|G'| - 1)|L(G)| = 230 = 2 \times 5 \times 23$, and all the cases contradict $72 \mid |G|$.

Subcase 3. Let G/N be a group in Lemma 1.3(iv), i.e., G/N is a Frobenius group with the Frobenius kernel $K \cong \mathbb{Z}_p^n$ and an abelian Frobenius complement H, and $|H| = \frac{|K|-1}{2}$. Note that $|G/N| = p^n \frac{p^n-1}{2}$, and the two irreducible characters of G/N are of degree $\frac{p^n-1}{2}$.

By Ito's theorem we have either $p \mid \psi(1)$ or $\psi(1) \mid \frac{p^n-1}{2}$, for each $\psi \in NL(G)$. Let t be the number of ψ's in $NL(G)$ with $p \mid \psi(1)$. Since $\Gamma(G)$ has no triangles, it follows that $t = 1$ or 2, and hence

$$NL(G) = \{\psi_1, \psi_2, \psi_3\} \quad \text{or} \quad NL(G) = \{\psi_1, \psi_2, \psi_3, \psi_4\},$$

with

$$\psi_1(1) = \psi_2(1) = \frac{p^n - 1}{2}, \qquad \psi_3(1) = p^i, \qquad \psi_4(1) = p^j, \quad 1 \leqslant i, j \leqslant n.$$

With the same notation and the same argument as in Step 4 in the proof of Lemma 2.5 (in particular, M is the normal subgroup of G such that $M/N = K$, and hence $|M/N| = p^n$ and $|G/M| = \frac{p^n-1}{2}$), we get

$$\Sigma_2|_M = \frac{p^n-1}{2}\Sigma_4 \quad \text{and} \quad \frac{p^n-1}{2} \bigm| \Sigma_2(1).$$

So we have

$$\frac{p^n-1}{2} \bigm| p^{2i} \quad \text{or} \quad \frac{p^n-1}{2} \bigm| \left(p^{2i}+p^{2j}\right).$$

Since $\frac{p^n-1}{2} \neq 1$, it follows from Lemma 2.3(ii) and (iii) that the first situation is impossible, and in the second situation we have $p = 5, n = 1, i = j = 1$. So we get $NL(M) = 1, c.d.(M) = \{1, 5\}$, which contradicts Lemma 1.2(ii).

Subcase 4. Let G/N be a group in Lemma 1.3(v), i.e., G/N has a normal subgroup L/N of order 2 such that G/L is a Frobenius group with the Frobenius kernel $K \cong \mathbb{Z}_p^n$, and a cyclic Frobenius complement H of order $p^n - 1$, where p is a prime and $p^n \neq 2$; and in this case, $|G/N| = 2p^n(p^n - 1)$, and the two non-linear irreducible characters of G/N are of degree $p^n - 1$. As in Subcase 3 we have a normal subgroup M of G such that $M/L \cong K$, and hence $|M/N| = 2p^n$ and $|G/M| = p^n - 1$. Now we divide it into two situations.

(i) The situation of $p = 2$.

By Ito's theorem we have either $2 \mid \psi(1)$, or $\psi(1) \mid (2^n - 1)$, for each $\psi \in NL(G)$. Let t be the number of ψ's in $NL(G)$ with $2 \mid \psi(1)$. Since $\Gamma(G)$ has no triangles and is disconnected, it follows that $t = 1$, or 2, and hence

$$NL(G) = \{\psi_1, \psi_2, \psi_3\} \quad \text{or} \quad NL(G) = \{\psi_1, \psi_2, \psi_3, \psi_4\}$$

with

$$\psi_1(1) = \psi_2(1) = 2^n - 1, \qquad \psi_3(1) = 2^i, \qquad \psi_4(1) = 2^j, \quad 1 \leqslant i, j \leqslant n + 1.$$

With the same notation and the same argument as in Step 4 in the proof of Lemma 2.5 (as in Subcase 3) we get

$$\Sigma_2|_M = \left(2^n - 1\right)\Sigma_4 \quad \text{and} \quad \left(2^n - 1\right) \bigm| \Sigma_2(1).$$

So we have

$$\left(2^n - 1\right) \bigm| 2^{2i} \quad \text{or} \quad \left(2^n - 1\right) \bigm| \left(2^{2i} + 2^{2j}\right).$$

By Lemma 2.3 we get a contradiction.

(ii) The situation of $p \neq 2$.

Again by Ito's theorem we get that either $p \mid \psi(1)$ or $\psi(1) \mid 2(p^n - 1)$, for each $\psi \in NL(G)$. Again let t be the number of ψ's in $NL(G)$ with $p \mid \psi(1)$. Then $t = 1$ or 2. Since $\Gamma(G)$ has no triangles, it follows that

$$NL(G) = \{\psi_1, \psi_2, \psi_3\} \quad \text{or} \quad NL(G) = \{\psi_1, \psi_2, \psi_3, \psi_4\},$$

with

$$\psi_1(1) = \psi_2(1) = p^n - 1, \qquad \psi_3(1) = p^i, \qquad \psi_4(1) = p^j, \quad 1 \leqslant i, j \leqslant n.$$

Hence we get (as we do before)

$$\Sigma_2|_M = \left(p^n - 1\right)\Sigma_4 \quad \text{and} \quad \left(p^n - 1\right) \bigm| \Sigma_2(1).$$

So we have

$$\left(p^n - 1\right) \bigm| p^{2i} \quad \text{or} \quad \left(p^n - 1\right) \bigm| \left(p^{2i} + p^{2j}\right).$$

By Lemma 2.3 the unique solution is $p = 3$, $n = 1$. Thus we have

$$NL(G) = \{\psi_1, \psi_2, \psi_3, \psi_4\},$$

with

$$\psi_1(1) = \psi_2(1) = 2, \qquad \psi_3(1) = \psi_4(1) = 3.$$

Now we have $|G| = |L(G)| + 2^2 + 2^2 + 3^2 + 3^2 = |L(G)| + 26$, or $|L(G)|(|G'| - 1) = 26$, which contradicts that $|G|$ has factors 2 and 3.

This completes the proof. □

2.4. Now Theorem 1.1 follows from Lemmas 2.4 and 2.6.

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