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Approximate analytical solutions of systems of PDEs by homotopy analysis method

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Abstract

In this paper, the homotopy analysis method (HAM) is applied to obtain series solutions to linear and nonlinear systems of first- and second-order partial differential equations (PDEs). The HAM solutions contain an auxiliary parameter which provides a convenient way of controlling the convergence region of series solutions. It is shown in particular that the solutions obtained by the variational iteration method (VIM) are only special cases of the HAM solutions.

Keywords: Homotopy analysis method; Systems of PDEs; Nonlinear systems; Variational iteration method; Adomian decomposition method

1. Introduction

The homotopy analysis method (HAM) was first proposed by Liao in his Ph.D. thesis [1]. A systematic and clear exposition on HAM is given in [2]. In recent years, this method has been successfully employed to solve many types of nonlinear, homogeneous or nonhomogeneous, equations and systems of equations as well as problems in science and engineering [3–23]. Very recently, Ahmad Bataineh et al. [24,25] presented two modifications of HAM to solve linear and nonlinear ODEs. The HAM contains a certain auxiliary parameter $\hbar$ which provides us with a simple way to adjust and control the convergence region and rate of convergence of the series solution. Moreover, by means of the so-called $\hbar$-curve, it is easy to determine the valid regions of $\hbar$ to gain a convergent series solution. Thus, through HAM, explicit analytic solutions of nonlinear problems are possible.

Systems of partial differential equations (PDEs) arise in many scientific models such as the propagation of shallow water waves and the Brusselator model of the chemical reaction-diffusion model. Very recently, Batiha et al. [26] improved Wazwaz's [27] results on the application of the variational iteration method (VIM) to solve some linear and nonlinear systems of PDEs. In [28], Saha Ray implemented the modified Adomian decomposition method (ADM) for solving the coupled sine-Gordon equation.

In this paper, we present an alternative approach based on HAM to approximate the solutions of linear and nonlinear systems of first- and second-order PDEs. Comparisons with the solutions obtained by VIM [27] and ADM [28] shall be made. It is shown in particular that the VIM [27] solutions are just special cases of the HAM solutions.

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2. Basic ideas of HAM

We consider the following differential equations,

$$N_i[z_i(x,t)] = 0, \quad i = 1, 2, \ldots, n,$$

where N_i are nonlinear operators that represent the whole equations, x and t denote the independent variables and $z_i(x,t)$ are unknowns function respectively. By means of generalizing the traditional homotopy method, Liao [2] constructed the so-called *zero-order deformation equations*

$$(1-q)L[\phi_i(x,t;q) - z_{i,0}(x,t)] = q\,\hbar_i\, N_i[\phi_i(x,t;q)], \tag{1}$$

where $q \in [0,1]$ is an embedding parameter, $\hbar_i$ are nonzero auxiliary functions, L is an auxiliary linear operator, $z_{i,0}(x,t)$ are initial guesses of $z_i(x,t)$ and $\phi_i(x,t;q)$ are unknown functions. It is important to note that, one has great freedom to choose auxiliary objects such as $\hbar_i$ and L in HAM. Obviously, when $q = 0$ and $q = 1$, both

$$\phi_i(x,t;0) = z_{i,0}(x,t) \quad \text{and} \quad \phi_i(x,t;1) = z_i(x,t),$$

hold. Thus as q increases from 0 to 1, the solutions $\phi_i(x,t;q)$ varies from the initial guesses $z_{i,0}(x,t)$ to the solutions $z_i(x,t)$. Expanding $\phi_i(x,t;q)$ in Taylor series with respect to q, one has

$$\phi_i(x,t;q) = z_{i,0}(x,t) + \sum_{m=1}^{+\infty} z_{i,m}(x,t)q^m, \tag{2}$$

where

$$z_{i,m} = \frac{1}{m!}\frac{\partial^m \phi_i(x,t;q)}{\partial q^m}\bigg|_{q=0}. \tag{3}$$

If the auxiliary linear operator, the initial guesses, the auxiliary parameters $\hbar_i$, and the auxiliary functions are properly chosen, then the series equation (2) converges at $q = 1$ and

$$\phi_i(x,t;1) = z_{i,0}(x,t) + \sum_{m=1}^{+\infty} z_{i,m}(x,t),$$

which must be one of solutions of the original nonlinear equations, as proved by Liao [2]. As $\hbar_i = -1$, Eq. (1) becomes

$$(1-q)L[\phi_i(x,t;q) - z_{i,0}(x,t)] + qN_i[\phi_i(x,t;q)] = 0, \tag{4}$$

which are mostly used in the homotopy-perturbation method [29].

According to (3), the governing equations can be deduced from the *zero-order deformation equations* (1). Define the vectors

$$\vec{z}_{i,n} = \{z_{i,0}(x,t), z_{i,1}(x,t), \ldots, z_{i,n}(x,t)\}.$$

Differentiating (1) m times with respect to the embedding parameter q and then setting $q = 0$ and finally dividing them by $m!$, we have the so-called *mth-order deformation equations*

$$L[z_{i,m}(x,t) - \chi_m z_{i,m-1}(x,t)] = \hbar_i\, R_{i,m}(\vec{z}_{i,m-1}), \tag{5}$$

where

$$R_{i,m}(\vec{z}_{i,m-1}) = \frac{1}{(m-1)!}\frac{\partial^{m-1} N_i\,[\phi_i(x,t;q)]}{\partial q^{m-1}}\bigg|_{q=0}, \tag{6}$$

and

$$\chi_m = \begin{cases} 0, & m \le 1, \\ 1, & m > 1. \end{cases}$$

It should be emphasized that $z_{i,m}(x,t)$ $(m \geq 1)$ is governed by the linear equation (5) with the linear boundary conditions that come from the original problem, which can be easily solved by symbolic computation softwares such as Maple and Mathematica.

3. Applications

We will apply the HAM to linear and nonlinear systems of PDEs to illustrate the strength of the method and to establish exact and/or approximate solutions for these problems. Comparisons with the VIM [27] and ADM [28] shall be made.

3.1. Homogeneous linear system

First we present the analytical solutions for the linear homogeneous system of PDEs:

$$u_t - v_x + (u + v) = 0, \tag{7}$$

$$v_t - u_x + (u + v) = 0, \tag{8}$$

subject to the initial conditions

$$u(x,0) = \sinh x, \qquad v(x,0) = \cosh x. \tag{9}$$

To solve system (7)–(9) by means of homotopy analysis method HAM, we choose the initial approximations

$$u_0(x,t) = \sinh x, \qquad v_0(x,t) = \cosh x,$$

and the linear operator

$$L[\phi_i(x,t;q)] = \frac{\partial \phi_i(x,t;q)}{\partial t}, \quad i = 1, 2, \tag{10}$$

with the property

$$L[c_i] = 0, \tag{11}$$

where c_i $(i = 1, 2)$ are integral constants. Furthermore, systems (7) and (8) suggest that we define a system of nonlinear operators as

$$N_1[\phi_i(x,t;q)] = \frac{\partial \phi_1(x,t;q)}{\partial t} - \frac{\partial \phi_2(x,t;q)}{\partial x} + \phi_1(x,t;q) + \phi_2(x,t;q),$$

$$N_2[\phi_i(x,t;q)] = \frac{\partial \phi_2(x,t;q)}{\partial t} - \frac{\partial \phi_1(x,t;q)}{\partial x} + \phi_1(x,t;q) + \phi_2(x,t;q).$$

Using the above definition, we construct the *zeroth-order deformation equations*

$$(1-q)L\left[\phi_i(x,t;q) - z_{i,0}(x,t)\right] = q\,\hbar_i\, N_i\left[\phi_i(x,t;q)\right], \quad i = 1, 2. \tag{12}$$

Obviously, when $q = 0$ and $q = 1$,

$$\phi_1(x,t;0) = z_{1,0}(x,t) = u_0(x,t), \qquad \phi_1(x,t;1) = u(x,t),$$

$$\phi_2(x,t;0) = z_{2,0}(x,t) = v_0(x,t), \qquad \phi_2(x,t;1) = v(x,t).$$

Therefore, as the embedding parameter q increases from 0 to 1, $\phi_i(x,t;q)$ varies from the initial guess $z_{i,0}(x,t)$ to the solution $z_i(x,t)$ for $i = 1, 2$. Expanding $\phi_i(x,t;q)$ in Taylor series with respect to q one has

$$\phi_i(x,t;q) = z_{i,0}(x,t) + \sum_{m=1}^{+\infty} z_{i,m}(x,t) q^m,$$

where

$$z_{i,m}(x,t) = \left.\frac{1}{m!}\frac{\partial^m \phi_i(x,t;q)}{\partial q^m}\right|_{q=0}.$$

If the auxiliary linear operator, the initial guesses and the auxiliary parameters $\hbar_i$ are properly chosen, the above series is convergent at $q = 1$, then one has

$$u(x,t) = z_{1,0}(x,t) + \sum_{m=1}^{+\infty} z_{1,m}(x,t),$$
$$u(x,t) = z_{2,0}(x,t) + \sum_{m=1}^{+\infty} z_{2,m}(x,t),$$

which must be one of the solutions of the original nonlinear equations, as proved by Liao [2]. Now we define the vector

$$\vec{z}_{i,n} = \{z_{i,0}(x,t),\ z_{i,1}(x,t), \ldots, z_{i,n}(x,t)\}.$$

So the *mth-order deformation equations* is

$$L\left[z_{i,m}(x,t) - \chi_m z_{i,m-1}(x,t)\right] = \hbar_i\, R_{i,m}\left(\vec{z}_{i,m-1}\right), \tag{13}$$

with the initial conditions

$$z_{i,m}(x,0) = 0, \tag{14}$$

where

$$R_{1,m}(\vec{z}_{i,m-1}) = (z_{1,m-1})_t - (z_{2,m-1})_x + z_{1,m-1} + z_{2,m-1},$$
$$R_{2,m}(\vec{z}_{i,m-1}) = (z_{2,m-1})_t - (z_{1,m-1})_x + z_{1,m-1} + z_{2,m-1}.$$

Now, the solution of the mth-order deformation equation (13) for $m \geq 1$ becomes

$$z_{i,m}(x,t) = \chi_m z_{i,m-1}(x,t) + \hbar_i \int_0^t R_{i,m}(\vec{z}_{i,m-1}) \mathrm{d}\tau + c_i, \tag{15}$$

where the integration constants c_i $(i = 1, 2)$ are determined by the initial conditions (14). We now successively obtain

$$z_{1,1}(x,t) = \hbar t \cosh x, \tag{16}$$

$$z_{1,2}(x,t) = \frac{\hbar t}{2}[2(1+\hbar)\cosh x + \hbar t \sinh x], \tag{17}$$

$$z_{1,3}(x,t) = \frac{\hbar t}{6}[(6 + 12\hbar + 6\hbar^2 + \hbar^2 t^2)\cosh x + 6\hbar t(1+\hbar)\sinh x], \tag{18}$$

$$z_{2,1}(x,t) = \hbar t \sinh x, \tag{19}$$

$$z_{2,2}(x,t) = \frac{\hbar t}{2}[2(1+\hbar)\sinh x + \hbar t \cosh x], \tag{20}$$

$$z_{2,3}(x,t) = \frac{\hbar t}{6}[(6 + 12\hbar + 6\hbar^2 + \hbar^2 t^2)\sinh x + 6\hbar t(1+\hbar)\cosh x], \tag{21}$$

etc. Then the series solutions expression by HAM can be written in the form

$$u(x,t) = z_{1,0}(x,t) + z_{1,1}(x,t) + z_{1,2}(x,t) + z_{1,3}(x,t) + \cdots, \tag{22}$$
$$v(x,t) = z_{2,0}(x,t) + z_{2,1}(x,t) + z_{2,2}(x,t) + z_{2,3}(x,t) + \cdots, \tag{23}$$

or specifically when $\hbar = -1$,

$$u(x,t) = \sinh x\left(1 + \frac{t^2}{2!} + \frac{t^4}{4!} + \cdots\right) - \cosh x\left(t + \frac{t^3}{3!} + \frac{t^5}{5!} + \cdots\right),$$
$$v(x,t) = \cosh x\left(1 + \frac{t^2}{2!} + \frac{t^4}{4!} + \cdots\right) - \sinh x\left(t + \frac{t^3}{3!} + \frac{t^5}{5!} + \cdots\right),$$

which are exactly the same as the solutions obtained by VIM [27] converging to the closed-form solutions,

$$u(x,t)=\sinh(x-t),\qquad v(x,t)=\cosh(x-t). \tag{24}$$

3.2. Nonhomogeneous linear system

Now consider the following nonhomogeneous linear system:

$$u_t - v_x - (u-v) = -2, \tag{25}$$
$$v_t - u_x - (u-v) = -2, \tag{26}$$

subject to the initial conditions

$$u(x,0)=1+\mathrm{e}^x,\qquad v(x,0)=-1+\mathrm{e}^x. \tag{27}$$

To solve system (25)–(27) by means of HAM, we choose the initial approximations

$$u_0(x,t)=1+\mathrm{e}^x,\qquad v_0(x,t)=-1+\mathrm{e}^x,$$

and the linear operator as in (10) with the property (11), the *zero-order deformation equations* (12) and the *mth-order deformation equations* (13) with the initial conditions (14), where

$$R_{1,m}(\vec{z}_{i,m-1}) = (z_{1,m-1})_t - (z_{2,m-1})_x - z_{1,m-1} + z_{2,m-1} + 2 - 2\chi_m,$$
$$R_{1,m}(\vec{z}_{i,m-1}) = (z_{2,m-1})_t + (z_{1,m-1})_x - z_{1,m-1} + z_{2,m-1} + 2 - 2\chi_m.$$

Now, the solution of the mth-order deformation for $m \geq 1$ are the same as (15), where the integration constants c_i $(i=1,2)$ are determined by the same initial conditions (14). We now successively obtain

$$z_{1,1}(x,t) = -\hbar t\mathrm{e}^x, \tag{28}$$
$$z_{1,2}(x,t) = \frac{\hbar t\mathrm{e}^x}{2}[-2+\hbar(t-2)], \tag{29}$$
$$z_{1,3}(x,t) = -\frac{\hbar t\mathrm{e}^x}{6}[6-6\hbar(t-2)+\hbar^2(t^2-6t+6)], \tag{30}$$
$$z_{2,1}(x,t) = \hbar t\mathrm{e}^x, \tag{31}$$
$$z_{2,2}(x,t) = \frac{\hbar t\mathrm{e}^x}{2}[2+\hbar(t+2)], \tag{32}$$
$$z_{2,3}(x,t) = \frac{\hbar t\mathrm{e}^x}{6}[6+6\hbar(t+2)+\hbar^2(t^2+6t+6)], \tag{33}$$

etc. In the special case $\hbar=-1$, we recover the VIM solutions [27],

$$u(x,t) = 1+\mathrm{e}^x\left(1+\frac{t^2}{2!}+\frac{t^3}{3!}+\cdots\right),$$
$$v(x,t) = 1+\mathrm{e}^x\left(1-t+\frac{t^2}{2!}-\frac{t^3}{3!}+\cdots\right),$$

which converge to the closed-form solutions

$$u(x,t)=1+\mathrm{e}^{x+t},\qquad v(x,t)=-1+\mathrm{e}^{x-t}.$$

3.3. Nonhomogeneous nonlinear system

Consider the following nonhomogeneous nonlinear system:

$$u_t + vu_x + u - 1 = 0, \tag{34}$$
$$v_t - uv_x - v - 1 = 0, \tag{35}$$

subject to the initial conditions

$$u(x,0) = e^{x}, \qquad v(x,0) = e^{-x}. \tag{36}$$

To solve system (34)–(36) by means of HAM, we choose the initial approximations

$$u_0(x,t) = e^{x}, \qquad v_0(x,t) = e^{-x},$$

and the linear operator as in (10) with the property (11), the *zero-order deformation equations* (12) and the *mth-order deformation equations* (13) with the initial conditions (14). We now successively obtain

$$z_{1,1}(x,t) = \hbar t e^{x}, \tag{37}$$

$$z_{1,2}(x,t) = \frac{\hbar t e^{x}}{2}[2 + \hbar(t+2)], \tag{38}$$

$$z_{1,3}(x,t) = \frac{\hbar t e^{x}}{6}[6 + 6\hbar(t+2) + \hbar^2(t^2+6t+6)], \tag{39}$$

$$z_{2,1}(x,t) = -\hbar t e^{-x}, \tag{40}$$

$$z_{2,2}(x,t) = \frac{\hbar t e^{-x}}{2}[-2 + \hbar(t-2)], \tag{41}$$

$$z_{2,3}(x,t) = -\frac{\hbar t e^{-x}}{6}[6 - 6\hbar(t-2) + \hbar^2(t^2-6t+6)], \tag{42}$$

etc. In the special case $\hbar = -1$, we recover the VIM solutions [27],

$$u(x,t) = e^{x}\left(1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \cdots\right),$$

$$v(x,t) = e^{-x}\left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots\right),$$

which are convergent to the closed-form solutions,

$$u(x,t) = e^{x-t}, \qquad v(x,t) = e^{-x+t}.$$

3.4. Homogeneous nonlinear system

We consider to examine the homogeneous nonlinear system [30]:

$$u_t + u_x v_x + u_y v_y + u = 0, \tag{43}$$

$$v_t + v_x w_x - v_y w_y - v = 0, \tag{44}$$

$$w_t + w_x u_x + w_y u_y - w = 0, \tag{45}$$

subject to the initial conditions

$$u(x,y,0) = e^{x+y}, \qquad v(x,y,0) = e^{x-y}, \qquad w(x,y,0) = e^{-x+y}. \tag{46}$$

According to HAM, the initial approximations are selected by using the given initial conditions

$$u_0(x,y,t) = e^{x+y}, \qquad v_0(x,y,t) = e^{x-y}, \qquad w_0(x,y,t) = e^{-x+y},$$

and the linear operator

$$L[\phi_i(x,t;q)] = \frac{\partial \phi_i(x,t;q)}{\partial t}, \quad i = 1,2,3,$$

with the property

$$L[c_i] = 0,$$

where c_i $(i = 1, 2, 3)$ are integral constants. Following similar procedure as in the previous subsections, the solution of the mth-order deformation equation for $m \geq 1$ becomes

$$z_{i,m}(x,t) = \chi_m z_{i,m-1}(x,t) + \hbar_i \int_0^t R_{i,m}(\vec{z}_{i,m-1})\mathrm{d}\tau + c_i, \tag{47}$$

where the integration constants c_i $(i = 1, 2, 3)$ are determined by the same initial conditions (14). We now successively obtain

$$z_{1,1}(x,y,t) = \hbar t \mathrm{e}^{x+y}, \tag{48}$$

$$z_{1,2}(x,y,t) = \frac{\hbar t \mathrm{e}^{x+y}}{2}[2 + \hbar(t+2)], \tag{49}$$

$$z_{1,3}(x,y,t) = \frac{\hbar t \mathrm{e}^{x+y}}{6}[6 + 6\hbar(t+2) + \hbar^2(t^2+6t+6)], \tag{50}$$

$$z_{2,1}(x,y,t) = -\hbar t \mathrm{e}^{x-y}, \tag{51}$$

$$z_{2,2}(x,y,t) = \frac{\hbar t \mathrm{e}^{x-y}}{2}[-2 + \hbar(t-2)], \tag{52}$$

$$z_{2,3}(x,y,t) = -\frac{\hbar t \mathrm{e}^{x-y}}{6}[6 - 6\hbar(t-2) + \hbar^2(t^2-6t+6)], \tag{53}$$

$$z_{3,1}(x,y,t) = -\hbar t \mathrm{e}^{-x+y}, \tag{54}$$

$$z_{3,2}(x,y,t) = \frac{\hbar t \mathrm{e}^{-x+y}}{2}[-2 + \hbar(t-2)], \tag{55}$$

$$z_{3,3}(x,y,t) = -\frac{\hbar t \mathrm{e}^{-x+y}}{6}[6 - 6\hbar(t-2) + \hbar^2(t^2-6t+6)], \tag{56}$$

etc. Then the series solutions expression by HAM can be written in the form

$$u(x,t) = z_{1,0}(x,t) + z_{1,1}(x,t) + z_{1,2}(x,t) + z_{1,3}(x,t) + \cdots \tag{57}$$

$$v(x,t) = z_{2,0}(x,t) + z_{2,1}(x,t) + z_{2,2}(x,t) + z_{2,3}(x,t) + \cdots \tag{58}$$

$$w(x,t) = z_{3,0}(x,t) + z_{3,1}(x,t) + z_{3,2}(x,t) + z_{3,3}(x,t) + \cdots. \tag{59}$$

So in the special case $\hbar = -1$, we obtain

$$u(x,t) = \mathrm{e}^{x+y}\left(1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \cdots\right),$$

$$v(x,t) = \mathrm{e}^{x-y}\left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots\right),$$

$$w(x,t) = \mathrm{e}^{-x+y}\left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots\right),$$

which are the VIM solutions [27] converging to the closed-form solutions,

$$u(x,t) = \mathrm{e}^{x+y-t}, \qquad v(x,t) = \mathrm{e}^{x-y+t}, \qquad v(x,t) = \mathrm{e}^{-x+y+t}.$$

3.5. Coupled sine-Gordon equations

We finally consider a system of nonlinear second-order PDEs given by the coupled sine-Gordon equations [28]:

$$u_{tt} - u_{xx} = -\delta^2 \sin(u - w), \tag{60}$$

$$w_{tt} - c^2 w_{xx} = \sin(u - w), \tag{61}$$

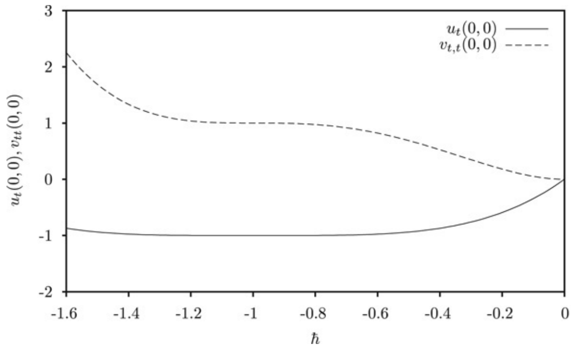

Fig. 1. The $\hbar$-curve of $u_t(0,0)$ and $v_{tt}(0,0)$ given by (22) and (23) based on the fifth-order HAM approximations.

subject to the initial conditions

$$u(x,0) = A\cos(kx), \qquad u_t(x,0) = 0, \tag{62}$$

$$w(x,0) = 0, \qquad w_t(x,0) = 0, \tag{63}$$

where c, δ, A and k are constants.

To solve (60)–(63) by means of HAM, we choose the initial approximations

$$u_0(x,t) = A\cos(kx), \qquad w_0(x,t) = 0,$$

and the linear operators for $i = 1, 2$

$$L[\phi_i(x,t;q)] = \frac{\partial^2\phi_i(x,t;q)}{\partial t^2},$$

with the property

$$L[c_i + c_i t] = 0,$$

where c_i $(i = 1, 2)$ are integral constants. Following similar procedure as in the previous examples, we obtain:

$$z_{1,1}(x,t) = \frac{\hbar}{2}[Ak^2\cos kx + \delta^2\sin(A\cos kx)]t^2,$$

$$z_{2,1}(x,t) = -\frac{\hbar}{2}\sin(A\cos kx)t^2.$$

So, the 2-term HAM series solutions in the special case $\hbar = -1$ are

$$u(x,t) = A\cos kx - \frac{1}{2}[Ak^2\cos kx + \delta^2\sin(A\cos kx)]t^2, \tag{64}$$

$$w(x,t) = \frac{1}{2}\sin(A\cos kx)t^2. \tag{65}$$

The validity of the method is based on such an assumption that the series (2) converges at $q = 1$. It is the auxiliary parameter $\hbar$ which ensures that this assumption can be satisfied. In general, by means of the so-called $\hbar$-curve, it is straightforward to choose a proper value of $\hbar$ which ensures that the solution series is convergent. The $\hbar$-curves for the five examples considered in this paper are presented in Figs. 1–3 which were obtained based on the fifth-order HAM approximations solutions. By HAM, it is easy to discover the valid region of $\hbar$, which corresponds to the line segments nearly parallel to the horizontal axis. In HAM it possible to obtain a large family of solutions. For the examples considered in this work, the special case $\hbar = -1$ yields the VIM solutions [27] and hence the exact solutions. Fig. 4 discussed the numerical comparison between the fifth-order HAM with different values of $\hbar$ and the exact solution of the problem in Section 3.1 in the interval $x \in [0, 1]$ at $t = 1$ the result shows that the proper value of $\hbar$ is -1. Figs. 5 and 6 show the comparisons between the 3-term of HAM and the 5-term of ADM solutions [28] for the case $\hbar = -1$, $c = \delta = A = 1$ and $k = 1.6$. The results presented in Figs. 5 and 6 clearly show

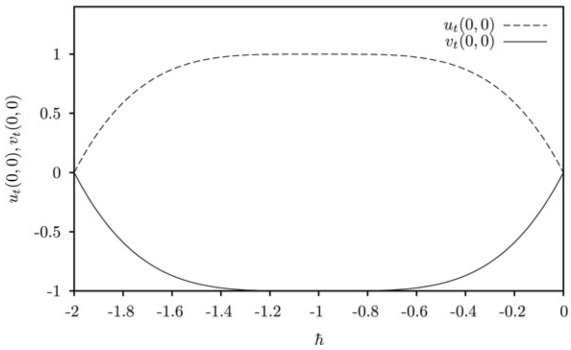

Fig. 2. The $\hbar$-curve of $u_t(0, 0)$ and $v_t(0, 0)$ given by (28)–(33) and (37)–(42) based on the fifth-order HAM approximations.

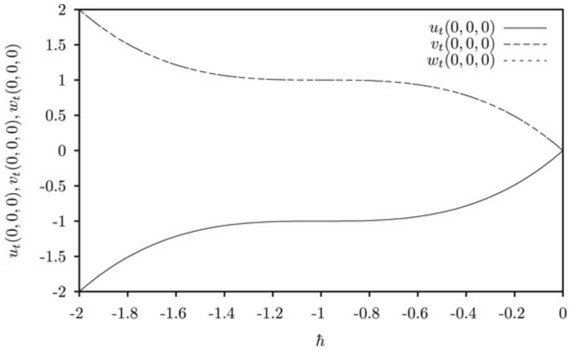

Fig. 3. The $\hbar$-curve of $u_t(0, 0, 0)$, $v_t(0, 0, 0)$ and $w_t(0, 0, 0)$ given by (57)–(59) based on the fifth-order HAM approximations.

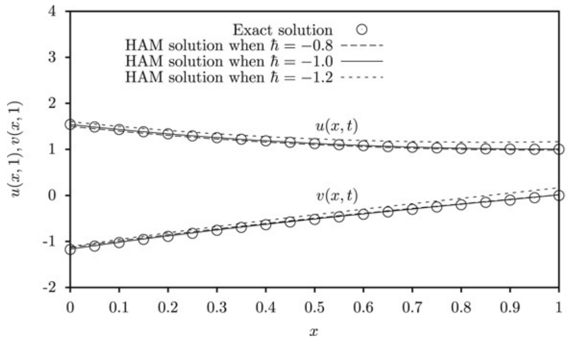

Fig. 4. The fifth-order HAM solutions (22) and (23) for different values of $\hbar$ vs the exact solution (24).

the good accuracy of HAM. In addition, HAM avoids the need for calculating the Adomian polynomials which can be complicated.

4. Conclusions

In this paper, it was shown how HAM can be applied to systems of PDEs. By HAM we obtained a family of solutions whose special cases are the solutions obtained by VIM and ADM. The advantage of HAM is the auxiliary parameter which provides a convenient way of controlling the convergence region of series solutions. This is not possible in other analytic methods like VIM and ADM. It is shown that the homotopy analysis method is a promising tool for other more complicated linear or nonlinear, homogeneous or nonhomogeneous, systems of PDEs.

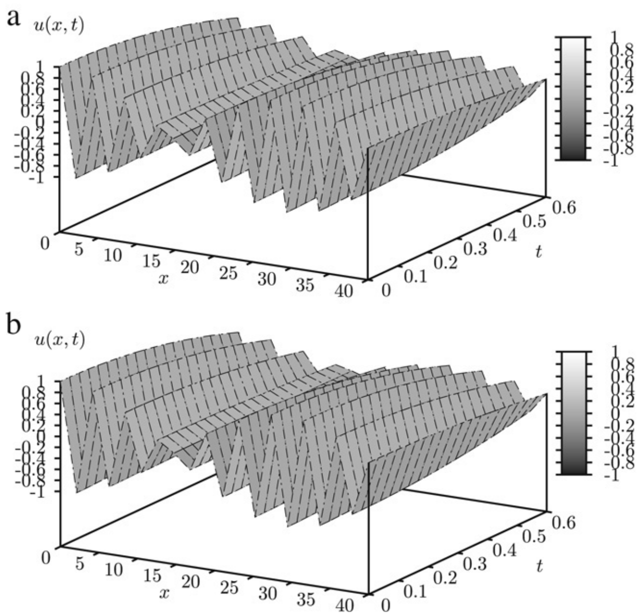

Fig. 5. (a) The numerical results for $u(x, t)$ by means of 3-term HAM solution. (b) The numerical results for $u(x, t)$ by means of 5-term ADM solution [28].

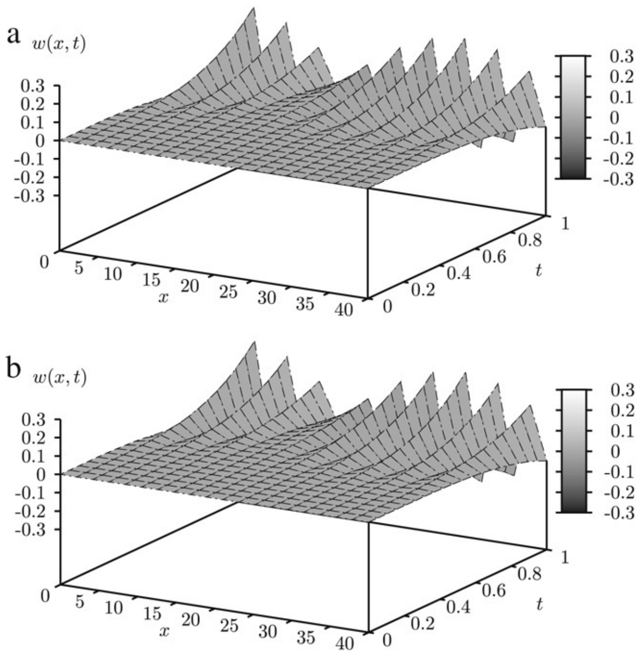

Fig. 6. (a) The numerical results for $w(x, t)$ by means of 3-term HAM solution. (b) The numerical results for $w(x, t)$ by means of 5-term ADM solution [28].

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