JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 3, 537–546 (1961)

On the Lifting Property (I)*

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Submitted by John G. Kemeny

1. Let $(E, \mathscr{E}, \mu)$ be a complete measure space of total mass one and let $\mathscr{N}$ be the set of all $A \in \mathscr{E}$ such that $\mu(A) = 0$.[1] We shall write $A \equiv B$ whenever $\mu(A \triangle B) = 0$; obviously $A \equiv B$ is an equivalence relation in $\mathscr{E}$. The tribes (= σ-algebras) $\mathscr{T} \subset \mathscr{E}$ considered below, will be always supposed to contain $\mathscr{N}$.

Let now $\mathscr{T} \subset \mathscr{E}$ be a tribe and let ρ be a mapping of $\mathscr{T}$ into $\mathscr{T}$. Properties of ρ, such as those listed here, will be considered in what follows:

(I) $\rho(A) \equiv A$;

(II) $A \equiv B$ implies $\rho(A) = \rho(B)$;

(III) $\rho(\phi) = \phi, \quad \rho(E) = E$;

(IV) $\rho(A \cap B) = \rho(A) \cap \rho(B)$;

(V) $\rho(A \cup B) = \rho(A) \cup \rho(B)$.

Using the results of [1], Maharam showed in [2] that:

(M) Given a tribe $\mathscr{T} \subset \mathscr{E}$ there exists a mapping ρ of $\mathscr{T}$ into $\mathscr{T}$ satisfying (I)–(V).

As a matter of fact, in [2] Maharam proved in detail only the existence of a mapping γ of $\mathscr{T}$ into $\mathscr{T}$ having the properties (I)–(IV). Once this is achieved, the existence of ρ satisfying (I)–(V) follows, as it was remarked in [2], by repeating an argument due to von Neumann [3, pp. 111–112]. The validity of property (M) had been previously established by von Neumann (see [3]; see also [4, Chap. VI] and [5]) for the case when $E = [0, 1]$ and $(E, \mathscr{T}, \mu)$ is the usual Lebesgue space.

* This paper was sponsored by the Office of Ordnance Research under Grant No. DA-ORD-12.

[1] For the notations and terminology concerning integration theory see [4] and [9].

We shall give here (Theorem 1) a different proof of property (M). In particular, we shall work directly on the abstract measure space $(E, \mathscr{E}, \mu)$ avoiding thus the use of any isomorphism theorem "with infinite products of unit intervals or 2-point spaces." Also, we shall not use the results of [1]; the classical martingale theorem (see [6]) will be enough for our purpose. Propositions 3 and 4 of this paper could be avoided, and instead we could essentially repeat the (above-mentioned) argument of von Neumann to deduce the existence of a mapping ρ of $\mathscr{T}$ into $\mathscr{T}$ satisfying (I)–(V), once we know the existence of a mapping γ of $\mathscr{T}$ into $\mathscr{T}$ satisfying (I)–(IV). However, we included here the detailed proofs of these propositions since they give an entirely different approach than that devised in [3]. Moreover, they provide supplementary information about the property (M) and make the paper self-contained.

The last part of the paper contains various applications of the above results.

2. The results given in this and the next section will be used in the proof of Theorem 1.

Let $\mathscr{F} \subset \mathscr{E}$ be a tribe, $H \in \mathscr{E}$ and $\mathscr{T}$ the smallest tribe containing $\mathscr{F}$ and H; every set $X \in \mathscr{T}$ can be written under the form

$$X = (A \cap H) \cup (B \cap \complement H)$$

with $A \in \mathscr{F}$, $B \in \mathscr{F}$. We shall prove now the:

Proposition 1. *Suppose that there is a mapping λ of $\mathscr{F}$ into $\mathscr{F}$ having the properties (I)–(V). Then there is a mapping ρ of $\mathscr{T}$ into $\mathscr{T}$ verifying (I)–(V) and coinciding with λ on $\mathscr{F}$.*

We shall divide the proof into five parts.

(a) For each $X \in \mathscr{E}$ let $\mathscr{F}(X)$ be the set of all $A \in \mathscr{F}$ such that $A \cap X \equiv \phi$ and $X_\infty = \bigcup_{A \in \mathscr{F}(X)} \lambda(A)$. It is easy to see that $X_\infty \in \mathscr{F}(X)$. In fact, let $(A_n)_{n \in N}$ be a sequence (which we may suppose increasing) of elements of $\mathscr{F}(X)$ such that

$$\sup_{A \in \mathscr{F}(X)} \mu(A) = \sup_{n \in N} \mu(A_n) = \mu(B);$$

here $B = \bigcup_{n \in N} A_n$. It is obvious that $B \in \mathscr{F}(X)$. Now if $A \in \mathscr{F}(X)$ then $A \cup B \in \mathscr{F}(X)$ and hence $\mu(A \cup B) = \mu(B)$; thus $\lambda(A) \subset \lambda(B)$. We deduce[2] that $X_\infty = \lambda(B) \in \mathscr{F}(X)$ (see also [2, p. 993]).

We remark that

$$X_\infty \cap (\complement X)_\infty = \phi \tag{1}$$

[2] Note that $\lambda(X_\infty) = X_\infty$.

In fact,

$$X_\infty \cap (\complement X)_\infty = (X_\infty \cap (\complement X)_\infty \cap X) \cup (X_\infty \cap (\complement X)_\infty \cap \complement X) \equiv \phi$$

and hence

$$X_\infty \cap (\complement X)_\infty = \lambda(X_\infty) \cap \lambda((\complement X)_\infty) = \lambda(\phi) = \phi.$$

(b) For each set $X \in \mathscr{E}$ write $\mathcal{O}(X) = X \cap X_\infty$. Define now:

$$\rho(H) = (H \cup \mathcal{O}(\complement H)) \cap \complement\mathcal{O}(H) = (H \cap \complement\mathcal{O}(H)) \cup \mathcal{O}(\complement H), \tag{2}$$

$$\rho(\complement H) = (\complement H \cup \mathcal{O}(H)) \cap \complement\mathcal{O}(\complement H) = (\complement H \cap \complement\mathcal{O}(\complement H)) \cup \mathcal{O}(H). \tag{3}$$

From (2) and (3) we deduce that $\rho(\complement H) = \complement\rho(H)$.

(c) For every $C \in \mathscr{F}(H)$ we have $\lambda(C) \cap \rho(H) = \phi$. Since $\lambda(C) \subset H_\infty$ we deduce

$$\lambda(C) \cap H \cap \complement O(H) = \lambda(C) \cap H \cap (\complement H \cup \complement H_\infty) = \phi;$$

on the other hand, by (1),

$$\lambda(C) \cap \mathcal{O}(\complement H) = \lambda(C) \cap \complement H \cap (\complement H)_\infty = \phi.$$

Hence (c) is proved.

(d) For every $C \in \mathscr{F}(\complement H)$ we have $\lambda(C) \cap \rho(\complement H) = \phi$. To prove (d) it is sufficient to replace H by $\complement H$ in (c).

(e) For an arbitrary $X = (A \cap H) \cup (B \cap \complement H) \in \mathscr{T}(A, B \in \mathscr{F})$ define

$$\rho(X) = (\lambda(A) \cap \rho(H)) \cup (\lambda(B) \cap \rho(\complement H)).$$

If $(A_1 \cap H) \cup (B_1 \cap \complement H) \equiv (A_2 \cap H) \cup (B_2 \cap \complement H)$ then $A_1 \Delta A_2 \in \mathscr{F}(H)$ and $B_1 \Delta B_2 \in \mathscr{F}(\complement H)$. From (c) and (d) it follows that ρ is well defined on $\mathscr{T}$ and that ρ coincides with λ on $\mathscr{F}$.

An immediate computation shows that ρ has the properties (I)–(V). Hence the proposition is completely proved.

3. For each tribe $\mathscr{T} \subset \mathscr{E}$ denote with $M^\infty(\mathscr{T})$ the Banach algebra of all bounded real-valued $\mathscr{T}$-measurable functions, defined on E, endowed with the norm $f \to \|f\|_\infty = \sup_{z \in E} |f(z)|$. We shall denote by $\mathscr{S}(\mathscr{T})$ the vector subspace of $M^\infty(\mathscr{T})$ consisting of all functions $f = \Sigma_{j=i}^q c_j \varphi_{A_j}$, where $c_1, \ldots, c_q$ are real numbers and $A_1, \ldots, A_q \in \mathscr{T}$ (φ_A is the characteristic function of the set A). Let $\mathscr{N}^\infty$ be the ideal of all $f \in M^\infty(\mathscr{T})$ which are equal to zero almost everywhere ($\mathscr{N}^\infty$ does not depend on $\mathscr{T}$). We shall write $f \equiv g$ whenever $f - g \in \mathscr{N}^\infty$; it is obvious that $f \equiv g$ is an equivalence relation on $M^\infty(\mathscr{T})$ and that $A \equiv B$ ($A, B \in \mathscr{T}$) if and only if $\varphi_A \equiv \varphi_B$. We shall denote by $f \to N_\infty(f)$ the essential supremum semi-norm on $M^\infty(\mathscr{T})$.

Let now $T\colon f \to T_f$ be a mapping of $M^\infty(\mathscr{T})$ into $M^\infty(\mathscr{T})$. Properties of T, such as those listed below, will be considered in what follows:

(I') $T_f \equiv f$;
(II') $f \equiv g$ implies $T_f = T_g$;
(III') $T_1 = 1$;
(IV') $f \geqslant 0$ implies $T_f \geqslant 0$;
(V') $T_{\alpha f + \beta g} = \alpha T_f + \beta T_g$;
(VI') $T_{fg} = T_f T_g$.

Let us remark that if $T\colon f \to T_f$ is a mapping of $M^\infty(\mathscr{T})$ into $M^\infty(\mathscr{T})$ satisfying (I')–(V'), then

$$||T_f||_\infty = N_\infty(f) \text{ for each } f \in M^\infty(\mathscr{T}). \tag{4}$$

We shall denote below by $\mathscr{C}(\mathscr{T})$ the set of all mappings $T\colon f \to T_f$ of $M^\infty(\mathscr{T})$ into $M^\infty(\mathscr{T})$ satisfying (I')–(V').

Remark. If $T \in \mathscr{C}(\mathscr{T})$ and $f, g \in M^\infty(\mathscr{T})$, then $T_{fg}(x) = T_f(x)T_g(x)$ for each $x \in E$ such that $\{x\} \notin \mathscr{T}$.

In fact, let $D = \{y | T_{fg}(y) \neq T_f(y)T_g(y)\}$. It follows from (I') that $D \in \mathscr{N}$. Hence if $x \in E$ is such that $\{x\} \notin \mathscr{T}$, then $x \notin D$ and thus $T_{fg}(x) = T_f(x)T_g(x)$.

Proposition 2. *The tribe $\mathscr{T}$ has the property* (M) *if and only if there is a mapping $T\colon f \to T_f$ of $M^\infty(\mathscr{T})$ into $M^\infty(\mathscr{T})$ having the properties (I')–(VI').*

Suppose that $\mathscr{T}$ has the property (M) and let ρ be a mapping of $\mathscr{T}$ into $\mathscr{T}$ satisfying (I)–(V). For $f = \Sigma_{j=1}^q c_j \varphi_{A_j} \in \mathscr{S}(\mathscr{T})$, define $T_f = \Sigma_{j=1}^q c_j \varphi_{\rho(A_j)}$. It is easy to see that T_f is well defined and that $||T_f||_\infty = N_\infty(f)$ for each $f \in \mathscr{S}(\mathscr{T})$; the properties (I')–(VI') are also satisfied by $f \to T_f$ on $\mathscr{S}(\mathscr{T})$. Since $\mathscr{S}(\mathscr{T})$ is dense in $M^\infty(\mathscr{T})$ (for the topology defined by N_∞), $T\colon f \to T_f$ can be extended by continuity to $M^\infty(\mathscr{T})$ and the extension continues to satisfy the conditions (I')–(VI'). Conversely, if there is a mapping $T\colon f \to T_f$ of $M^\infty(\mathscr{T})$ into $M^\infty(\mathscr{T})$ verifying the conditions (I')–(VI') and if for $A \in \mathscr{T}$ we set $\varphi_{\rho(A)} = T_{\varphi_A}$ (it is obvious that T_{φ_A} is a characteristic function), we obtain a mapping ρ of $\mathscr{T}$ into $\mathscr{T}$ having the properties (I)–(V).

Remark. The mappings ρ and T considered in the above proof verify the equations $\varphi_{\rho(A)} = T_{\varphi_A}$ for all $A \in \mathscr{T}$.

We shall denote below by R^I the locally convex space $\Pi_{(f,x)\in I} R^{(f,x)}$ where $I = M^\infty(\mathscr{T}) \times E$ and $R^{(f,x)} = R$ (= the real line with the usual topology) for all $(f, x) \in I$. Every $T \in \mathscr{C}(\mathscr{T})$ can be identified with the element $(T_f(x))_{(f,x)\in I}$ of R^I; hence $\mathscr{C}(\mathscr{T})$ can be identified with a convex part of R^I.

We denote by $L^\infty(\mathcal{T})$ the quotient Banach algebra $M^\infty(\mathcal{T})/\mathcal{N}^\infty$ and by $f \to \tilde{f}$ the canonical mapping of $M^\infty(\mathcal{T})$ onto $L^\infty(\mathcal{T})$. The norm on $L^\infty(\mathcal{T})$ is given by $\tilde{f} \to N_\infty(\tilde{f}) = N_\infty(f)$. Let now $\mathcal{P}$ be the set of all $x' \in (L^\infty(\mathcal{T}))'$ verifying the relations: i) $x'(\tilde{1}) = 1$, and ii) $x'(\tilde{f}) \geqslant 0$ if $\tilde{f} \geqslant 0$. We recall that the set of extremal points of $\mathcal{P}$ coincides with the set of all characters of $L^\infty(\mathcal{T})$ (see, for instance, [7, p. 443]).

Proposition 3. *Let θ', θ'' be two mappings of $\mathcal{T}$ into $\mathcal{T}$ and let $\mathcal{D}$ be the set of all $T \in \mathcal{C}(\mathcal{T})$ such that $\varphi_{\theta'(A)} \leqslant T_{\varphi_A} \leqslant \varphi_{\theta''(A)}$ for each $A \in \mathcal{T}$. Then $\mathcal{D} \subset \mathcal{C}(\mathcal{T})$ is convex and $T \in \mathcal{D}$ is extremal in $\mathcal{D}$ if and only if T is extremal in $\mathcal{C}(\mathcal{T})$.*

If $T \in \mathcal{D}$ is extremal in $\mathcal{C}(\mathcal{T})$, then obviously T is extremal in $\mathcal{D}$. Conversely, suppose that $T \in \mathcal{D}$ is extremal in $\mathcal{D}$ but is not extremal in $\mathcal{C}(\mathcal{T})$. There exist then $T^{(1)}, T^{(2)} \in \mathcal{C}(\mathcal{T})$, $T^{(1)} \neq T^{(2)}$, and $0 < t < 1$ such that $T = tT^{(1)} + (1-t)T^{(2)}$. Let now $A \in \mathcal{T}$; we have $T_{\varphi_A}(x) = tT^{(1)}_{\varphi_A}(x) + (1-t)T^{(2)}_{\varphi_A}(x)$ for all $x \in E$. If $x \in \theta'(A)$ then $T_{\varphi_A}(x) = 1$ and hence $T^{(j)}_{\varphi_A}(x) = 1$ for $j = 1, 2$; if $x \notin \theta'(A)$, then $\varphi_{\theta'(A)}(x) = 0$. It follows that for all $x \in E$, $T^{(j)}_{\varphi_A}(x) \geqslant \varphi_{\theta'(A)}(x)$, $(j = 1, 2)$. *If* $x \notin \theta''(A)$ then $T_{\varphi_A}(x) = 0$ and hence $T^{(j)}_{\varphi_A}(x) = 0$ for $j = 1, 2$; if $x \in \theta''(A)$ then $\varphi_{\theta''(A)}(x) = 1$. It follows that for all $x \in E$, $T^{(j)}_{\varphi_A}(x) \leqslant \varphi_{\theta''(A)}(x)$, $(j = 1, 2)$. Hence $T^{(j)} \in \mathcal{D}$ for $j = 1, 2$, and this leads to a contradiction. This completes the proof of the proposition.

Proposition 4. *Suppose that $\mathcal{C}(\mathcal{T})$ is nonvoid. Then there is a mapping T^*: $f \to T_f^*$ of $M^\infty(\mathcal{T})$ into $M^\infty(\mathcal{T})$ satisfying (I′)–(VI′). Moreover, if $T \in \mathcal{C}(\mathcal{T})$ is given, we can choose T^* so that $T^*_{\varphi_A} = T_{\varphi_A}$ for $A \in \mathcal{T}$, whenever T_{φ_A} is a characteristic function.*

Let $T \in \mathcal{C}(\mathcal{T})$ be given. We shall divide the proof into five parts:

(a) For each $A \in \mathcal{T}$ define $\theta'(A) = \{x | T_{\varphi_A}(x) = 1\}$ and $\theta''(A) = \{x | T_{\varphi_A}(x) \neq 0\}$; we have obviously $\theta'(A) \equiv \theta''(A) \equiv A$ and $\varphi_{\theta'(A)} \leqslant T_{\varphi_A} \leqslant \varphi_{\theta''(A)}$.

(b) Denote by $\mathcal{D}$ the set of all mappings $S \in \mathcal{C}(\mathcal{T})$ such that $\varphi_{\theta'(A)} \leqslant S_{\varphi_A} \leqslant \varphi_{\theta''(A)}$ for all $A \in \mathcal{T}$.

We shall identify $\mathcal{D}$ with a part of R^I; $\mathcal{D} \subset R^I$ is obviously *convex* and by hypothesis *nonvoid.* For $S \in \mathcal{D}$ and $(f, x) \in I$, we have by (4), $|S_f(x)| \leqslant N_\infty(f)$; hence $\mathcal{D}$ is a *bounded* part of R^I.

(c) We shall show now that $\mathcal{D}$ is a *closed* part of R^I. Denote by B^∞ the Banach algebra of all bounded real-valued functions on E, with the norm $f \to \|f\|_\infty = \sup_{z \in E} |f(z)|$. Let now $(T^{(j)})_{j \in J}$ be a filtering family of elements of $\mathscr{D}$ converging to an element of R^I. For each $f \in M^\infty(\mathscr{T})$ and $x \in E$ we have

$$\lim_{j \in J} T_f^{(j)}(x) = T_f^\infty(x). \tag{5}$$

It is obvious that T^∞, as a mapping of $M^\infty(\mathscr{T})$ into B^∞ satisfies the conditions (II')–(V'); in particular, it follows that $\|T_f^\infty\|_\infty \leqslant N_\infty(f)$ for each $f \in M^\infty(\mathscr{T})$. From (b) and (5) we deduce:

$$\varphi_{\theta'(A)} \leqslant T^\infty_{\varphi A} \leqslant \varphi_{\theta''(A)} \qquad \text{for all} \qquad A \in \mathscr{T}. \tag{6}$$

But (6) implies that $T^\infty_{\varphi_A} \in M^\infty(\mathscr{T})$ and that $T^\infty_{\varphi_A} \equiv \varphi_A$ for each $A \in \mathscr{T}$. Hence $T_f^\infty \in M^\infty(\mathscr{T})$ and $T_f^\infty \equiv f$ for each $f \in \mathscr{S}(\mathscr{T})$. Since $\mathscr{S}(\mathscr{T})$ is dense in $M^\infty(\mathscr{T})$ and $f \to T_f^\infty$ is a continuous mapping of $M^\infty(\mathscr{T})$ into B^∞, it follows that $T_f^\infty \equiv f$ for all $f \in M^\infty(\mathscr{T})$. Thus $T^\infty \in \mathscr{D}$ and $\mathscr{D}$ is closed.

(d) Since $\mathscr{D}$ is a *convex compact* part of R^I, there exists an extremal point $T^* \in \mathscr{D}$ (see [8, Chap. II, p. 84] and [7, p. 440]). We shall show that T^* satisfies also (VI'). By Proposition 3, T^* is extremal in $\mathscr{C}(\mathscr{T})$. Now every extremal point of $\mathscr{C}(\mathscr{T})$ satisfies (VI').[3] For the sake of completeness, we shall give here a direct proof of this assertion:

(e) Let $T^* \in \mathscr{C}(\mathscr{T})$ be an extremal point of $\mathscr{C}(\mathscr{T})$ and let $x_0 \in E$. If $\{x_0\} \notin \mathscr{T}$ we have, by the remark preceding Proposition 2, $T_{fg}^*(x_0) = T_f^*(x_0) T_g^*(x_0)$ for all $f, g \in M^\infty(\mathscr{T})$. If $\{x_0\} \in \mathscr{T}$ and $\mu(\{x_0\}) > 0$ then obviously $T_{fg}^*(x_0) = T_f^*(x_0) T_g^*(x_0)$ for all $f, g \in M^\infty(\mathscr{T})$. Finally, if $\{x_0\} \in \mathscr{T}$ and $\mu(\{x_0\}) = 0$, consider the mapping $\chi_{x_0}\colon \dot{f} \to T_f^*(x_0)$ of $L^\infty(\mathscr{T})$ into R; it is clear that $\chi_{x_0} \in \mathscr{P}$. We shall show that χ_{x_0} is extremal in $\mathscr{P}$. In fact, otherwise there would exist $\chi_1, \chi_2 \in \mathscr{P}$ $\chi_1 \neq \chi_2$ and $0 < t < 1$ such that $\chi_{x_0} = t\chi_1 + (1-t)\chi_2$. Define now $T^{(j)}$ by $T_f^{(j)}(y) = T_f^*(y)$ for $y \neq x_0$ and $T_f^{(j)}(x_0) = \chi_j(\dot{f})$ $(j = 1, 2)$; it is clear that $T^{(1)}, T^{(2)} \in \mathscr{C}(\mathscr{T})$, $T^{(1)} \neq T^{(2)}$, and that $T^* = tT^{(1)} + (1-t)T^{(2)}$. But this contradicts the fact that T^* is extremal in $\mathscr{C}(\mathscr{T})$; hence χ_{x_0} must be a character of $L^\infty(\mathscr{T})$ and thus $T_{fg}^*(x_0) = T_f^*(x_0) T_g^*(x_0)$ for all $f, g \in M^\infty(\mathscr{T})$. Therefore T^* satisfies (VI').

Since the last assertion of Proposition 4 is obvious, the proposition is completely proved.

4. We shall show now that $(E, \mathscr{E}, \mu)$ has always the property (**M**).

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[3] In fact, a more general result is valid: Let Z_1 and Z_2 be two compact spaces, $C(Z_1)$ and $C(Z_2)$ the corresponding Banach algebras of continuous real-valued functions on Z_1 and Z_2, respectively, and $\mathscr{C}$ the convex set of all linear positive mappings T of $C(Z_1)$ into $C(Z_2)$ mapping 1 into 1. Then $T \in \mathscr{C}$ is extremal in $\mathscr{C}$ if and only if it is multiplicative.

THEOREM 1. *Let $(E, \mathscr{E}, \mu)$ be a complete measure space of total mass one. Then there is a mapping $T: f \to T_f$ of $M^\infty(\mathscr{E})$ into $M^\infty(\mathscr{E})$ satisfying the conditions (I′)–(VI′).*

Let H be the set of all pairs $(\mathscr{T}, T^{\mathscr{T}})$ where $\mathscr{T} \subset \mathscr{E}$ is a tribe and $T^{\mathscr{T}}$ a mapping of $M^\infty(\mathscr{T})$ into $M^\infty(\mathscr{T})$ having the properties (I′)–(VI′); it is clear that H is nonvoid.[4] We shall order the set H as follows: $(\mathscr{F}, T^{\mathscr{F}}) \leqslant (\mathscr{G}, T^{\mathscr{G}})$ if $\mathscr{F} \subset \mathscr{G}$ and if the restriction of $T^{\mathscr{G}}$ to $M^\infty(\mathscr{F})$ coincides with $T^{\mathscr{F}}$. It is obvious that to prove the theorem, it is enough to show that every totally ordered part of H has an upper bound in H. In fact, by Zorn's theorem, there is then a maximal element $(\mathscr{L}, T^{\mathscr{L}})$ in H; by Proposition 1 (and Proposition 2), it follows that $\mathscr{L} = \mathscr{E}$ and hence that there is a mapping $T = T^{\mathscr{L}}$ of $M^\infty(\mathscr{E})$ into $M^\infty(\mathscr{E})$ satisfying (I′)–(VI′).

To prove that every totally ordered part of H has an upper bound in H we shall reason as follows: Let $\Phi = (\mathscr{T}_j, T^{(j)})_{j \in J}$ be a totally ordered family of elements of H (we denote here $T^{\mathscr{T}_j} = T^{(j)}$ for each $j \in J$) and let $\mathscr{T}_\infty$ be the tribe spanned by $\bigcup_{j \in J} \mathscr{T}_j$. We have to distinguish two cases:

(a) There is no countable cofinal part in J.

Let $f \in M^\infty(\mathscr{T}_\infty)$; there is then a countable part $I \subset J$ such that $f \in M^\infty(\mathscr{T}_I)$ where $\mathscr{T}_I$ is the tribe spanned by $\bigcup_{j \in I} \mathscr{T}_j$. If $h \in J$ is an element superior to all $j \in I$, then $f \in M^\infty(\mathscr{T}_h)$. Define $T_f^\infty = T_f^{(h)}$; it is clear that T_f^∞ is well defined and that T^∞: $f \to T_f^\infty$ is a mapping of $M^\infty(\mathscr{T}_\infty)$ into $M^\infty(\mathscr{T}_\infty)$ satisfying (I′)–(VI′) and the equation $T_f^\infty = T_f^{(j)}$ for all $j \in J$ and $f \in M^\infty(\mathscr{T}_j)$. This shows that $(\mathscr{T}_\infty, T^\infty)$ is an upper bound of the family Φ.

(b) There is a countable cofinal part K in J.

We may suppose that K is the set of elements of an increasing sequence $(j(n))_{n \in N}$ (the case when J is finite is obvious). Remark that in this case $\mathscr{T}_\infty$ is the tribe spanned by $\bigcup_{n \in N} \mathscr{T}_{j(n)}$. For each $f \in M^\infty(\mathscr{T}_\infty)$ and $n \in N$, denote by f_n (an arbitrary bounded determination of) the conditional expectation of f with respect to $\mathscr{T}_{j(n)}$; we have $\|T_{f_n}^{(j(n))}\|_\infty = N_\infty(f_n) \leqslant N_\infty(f)$. Let now $\mathscr{U}$ be an *ultrafilter*[5] on N finer than the Fréchet filter. For each $f \in M^\infty(\mathscr{T}_\infty)$ and $x \in E$ define:

$$T_f'(x) = \lim_{\mathscr{U}} T_{f_n}{}^{(j(n))}(x).$$

[4] It is sufficient to take for $\mathscr{T}$ the tribe consisting of all the sets $A \in \mathscr{N}$ and $\complement A$ with $A \in \mathscr{N}$, to define $\rho(B) = \phi$ if $B \in \mathscr{N}$ and $\rho(B) = E$ if $\complement B \in \mathscr{N}$ and to use Proposition 2.

[5] The use of the ultrafilter $\mathscr{U}$ was suggested in [5].

By the martingale theorem (see [6]) the sequence $(T_{f_n}^{(j(n))}(x))_{n \in N}$ converges to $f(x)$ for almost every $x \in E$, and hence $T_f' \in M^\infty(\mathscr{T}_\infty)$ and $T_f' \equiv f$. It is easy to see that T': $f \to T_f'$ is a mapping of $M^\infty(\mathscr{T}_\infty)$ into $M^\infty(\mathscr{T}_\infty)$ having the properties (I')–(V'). By Proposition 4, there is a mapping T^∞: $f \to T_f^\infty$ of $M^\infty(\mathscr{T}_\infty)$ into $M^\infty(\mathscr{T}_\infty)$ satisfying (I')–(VI') and the equation $T_{\varphi_A}^\infty = T_{\varphi_A}'$ for $A \in \mathscr{T}_\infty$, whenever T_{φ_A}' is a characteristic function. If $A \in \mathscr{T}_j$ for some $j \in J$, then for all n sufficiently large $A \in \mathscr{T}_{j(n)}$ and hence $T_{\varphi_A}' = T_{\varphi_A}^{(j(n))} = T_{\varphi_A}^{(j)}$ is a characteristic function. It follows that $T_{\varphi_A}^\infty = T_{\varphi_A}' = T_{\varphi_A}^{(j)}$ for every $j \in J$, $A \in \mathscr{T}_j$, and therefore $T_f^\infty = T_f^{(j)}$ for all $f \in M^\infty(\mathscr{T}_j)$, $j \in J$. This shows that $(\mathscr{T}_\infty, T^\infty)$ is an upper bound of the family Φ.

Hence the theorem is completely proved.

COROLLARY 1. *Let $(E, \mathscr{E}, \mu)$ be a complete totally σ-finite measure space. Then there is a mapping T: $f \to T_f$ of $M^\infty(\mathscr{E})$ into $M^\infty(\mathscr{E})$ satisfying the conditions (I')–(VI').*

The algebra $M^\infty(\mathscr{E})$ is defined here as before.

COROLLARY 2. *Let Z be a locally compact space and μ a positive Radon measure on Z. Then there is a mapping T: $f \to T_f$ of $M^\infty(Z)$ into $M^\infty(Z)$ satisfying the conditions (I')–(VI').*

We denote here by $M^\infty(Z)$ the (Banach) algebra of all real-valued bounded μ-measurable functions [4, Chap. IV] defined on Z and by $\mathscr{N}^\infty$ the ideal of all locally μ-negligible functions [4, Chap. IV] belonging to $M^\infty(Z)$; N_∞ is the essential supremum seminorm on $M^\infty(Z)$ [4, Chap. IV]. Corollary 2 is a consequence of Theorem 1 above and of Proposition 4 in [4, Chap. V, pp. 6–7].

REMARK. In connection with the corollary on p. 993 in [2], we wish to remark that the Haar measure on a compact group is completion regular (see [9, pp. 288–289]). This gives immediately the above mentioned corollary.

5. Using Corollary 2 above, we deduce that Theorem 1 and the Corollaries 1, 2, and 3 in [4, Chap. VI, § 2, No 5] remain valid *without* the assumption that the locally convex space F contains a *countable dense set*.[6] In the case when F is a normed space, the norm on $L_{F'_s}^\infty$ should be defined by ($f \to \dot{f}$ is the canonical mapping of $\mathscr{L}_{F'_s}^\infty$ onto $L_{F'_s}^\infty$):

$$N_\infty(\dot{f}) = \inf \{N_\infty(|g|) \mid g \in \mathscr{L}_{F'_s}^\infty, \dot{g} = \dot{f}\}. \tag{7}$$

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[6] The assertion of corollary 3 in [4, Chap. VI, § 2, No 5] that there is a determination of f such that $f(t) \in H'$ for each $t \in T$, can be proved using "the existence of a lifting" given by Theorem 1 of this paper.

We wish to mention explicitly that the above results show in particular that the Dunford-Pettis theorem (see [10] and [7, pp. 503–504]) holds without any countability hypothesis (this was shown in [5], at least for the case of Banach spaces, assuming that the result of Corollary 2 of this paper was valid).

The Proposition 10 in [4, Chap. VI, § 2, No 6], giving the dual of the space $L_F{}^1$, remains also valid without assuming that F contains a countable dense set [the norm on $L^\infty_{F's}$ is again defined by formula (7)].

Let now Z be a locally compact space, μ a positive Radon measure on Z and F a separated locally convex space; let P be the set of all continuous seminorms on F. For each mapping f of Z into F let

$$\bar{N}_{1,p}(f) = \overline{\int_Z}^{*} p(f(z))\, d\mu(z)$$

Denote by $\bar{\mathscr{F}}_F{}^1$ the vector space of all mappings f of Z into F such that $\bar{N}_{1,p}(f)$ is finite for each $p \in P$ ($\bar{N}_{1,p}$ is obviously a seminorm on $\bar{\mathscr{F}}_F{}^1$). Consider on $\bar{\mathscr{F}}_F{}^1$ the locally convex topology defined by the family of seminorms $(\bar{N}_{1,p})_{p \in P}$. Let $\mathscr{K}_F \subset \bar{\mathscr{F}}_F{}^1$ be the vector space of all continuous mappings of Z into F having compact support. Denote by $\bar{\mathscr{L}}_F{}^1$ the closure of $\mathscr{K}_F$ in $\bar{\mathscr{F}}_F{}^1$, by $L_F{}^1$ the associated separated space and by $f \to \tilde{f}$ the canonical mapping of $\bar{\mathscr{L}}_F{}^1$ onto $L_F{}^1$. It can be shown that:

(i) If $f \in \bar{\mathscr{L}}_F{}^1$, $g' \in \mathscr{L}^\infty_{F's}$ and $g'(Z)$ is an equicontinuous part of F', then $z \to \langle f(z), g'(z) \rangle$ is essentially integrable:

(ii) If $\tilde{f}_1 = \tilde{f}_2 \in L_F{}^1$, $\dot{g}' = \dot{g}'' \in L^\infty_{F's}$ and $g'(Z)$, $g''(Z)$ are equicontinuous parts of F', then $\langle f_1, g' \rangle = \langle f_2, g'' \rangle$ locally almost everywhere.

It follows that

$$\theta(\dot{g}) : \tilde{f} \to \int_Z < f, g' > d\mu$$

(here g' is an element belonging to the class $\dot{g} \in L^\infty_{F's}$ such that $g'(Z)$ is an equicontinuous part of F') is a well defined (continuous linear) mapping of $L_F{}^1$ into R. Moreover, we have the following

THEOREM 2. *Let F be a separated locally convex space. Then θ: $\dot{g} \to \theta(\dot{g})$ is a one-to-one linear mapping of $L^\infty_{F's}$ onto the dual of $L_F{}^1$.*

We shall not give here the proof of Theorem 2 (it can be obtained using essentially the same method as in the proof of Proposition 10 in [4, Chap. VI, § 2, No 5]).

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