Existence and asymptotic behavior of ground states for quasilinear singular equations involving Hardy–Sobolev exponents

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Received 28 April 2005
Available online 12 October 2005
Submitted by P. Smith

Abstract

We study the existence and decaying rate of solutions for the quasilinear problem

$$
\begin{align*}
-\Delta_p u &= \rho(x)f(u) + \frac{\lambda}{|x|^\theta} g(u) \quad \text{in } \mathbb{R}^N, \\
u &> 0 \quad \text{in } \mathbb{R}^N, \quad u(x) \xrightarrow{|x| \to \infty} 0,
\end{align*}
$$

where $\Delta_p$ stands for the $p$-Laplacian operator, $1 < p < N$, $\rho: \mathbb{R}^N \to [0, \infty)$ is continuous and not identically zero, $\lambda \geq 0$ is a parameter, $|x|$ is the Euclidean norm of $x$, $0 \leq \theta \leq p$, $f, g: [0, \infty) \to [0, \infty)$ are continuous and nondecreasing, $f$ has sublinear growth and the Hardy–Sobolev exponent $p_{\theta}^* := p (N - \theta)/(N - p)$ bounds the growth of $g$. We deal with variational methods and the lower and upper solutions technique.

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Keywords: Quasilinear singular equations; Ground states; Variational methods; Lower–upper solutions

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Supported by IM-AGIMB, PADCT-620029/2004-8, CNPq/PQ, PRONEX/UnB, Brazil.

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1. Introduction

We study the existence and the decaying rate of solutions for the problem

\[
\begin{aligned}
-\Delta_p u &= \rho(x) f(u) + \lambda |x|^{\theta} g(u) \quad \text{in } \mathbb{R}^N, \\
u > 0 \quad \text{in } \mathbb{R}^N, \quad u(x) \xrightarrow{|x| \to \infty} 0,
\end{aligned}
\]

(1.1)

where \(\Delta_p\) is the \(p\)-Laplacian, \(1 < p < N\), \(\rho: \mathbb{R}^N \to [0, \infty)\) is continuous and not identically zero, \(\lambda \geq 0\) is a parameter, \(|x|\) is the Euclidean norm of \(x\), \(0 \leq \theta < p\), \(f, g: [0, \infty) \to [0, \infty)\) are continuous and nondecreasing, \(f\) has sublinear growth and the Hardy–Sobolev exponent \(p^*_\theta := p \frac{N - \theta}{N - p}\) bounds the growth of \(g\).

Let \(\Omega \subset \mathbb{R}^N\) be a smooth domain. Given \(s \in [1, \infty)\) set \(L^s_\theta(\Omega) := L^s(\Omega, 1/|x|^{\theta} \, dx)\) and \(L^s_\theta := L^s_\theta(\mathbb{R}^N)\). The usual notations \(L^s(\Omega)\) and \(L^s\) are used in the case \(\theta = 0\), while the corresponding norms are denoted by \(|\cdot|_s, \Omega\), \(|\cdot|_s\). Now, \(D_0^{1,p} \Theta(\Omega)\) denotes the closure of \(C_\infty^0(\Omega)\) under the norm \(\|\phi\|_p = \int_\Omega |\nabla \phi|^p \, dx, \phi \in C_\infty^0(\Omega)\), \(D_0^{1,p} := D_0^{1,p} \Theta(\mathbb{R}^N)\). \(W_1^1, p_{\text{rad}}(\Omega)\) is the closed subspace of radially symmetric functions of the usual Sobolev space \(W_1^1, p(\Omega)\) when \(\Omega\) is a ball and \(W_1^1, p_{\text{rad}} := W_1^1, p_{\text{rad}}(\mathbb{R}^N)\). To close this set of notations, \(B_r(0)\) is the ball of radius \(r\) centered at the origin of \(\mathbb{R}^N\), \(\omega_N\) is the volume of the unit ball, \(\int := \int_{\mathbb{R}^N}\), \(u^+, u^-\) are, respectively, the positive and negative parts of a measurable function \(u\), and \(C_1, C_2, \ldots\) will denote positive constants.

By Caffarelli, Kohn and Nirenberg [6], the embedding \(D_0^{1,p} \Theta(\Omega) \hookrightarrow L^{p^*_\theta}(\Omega)\) has as best constant:

\[
S_\theta := \inf \left\{ \frac{\int_\Omega |\nabla u|^p \, dx}{(\int_\Omega |u|^{p^*_\theta} \, dx)^{p/p^*_\theta}} \mid u \in D_0^{1,p} \Theta(\Omega), \ u \neq 0 \right\}.
\]

Notice that \(p^*_\theta = Np/(N - p) := p^*_\infty\) and \(S_0\) is the best constant in the Sobolev inequality. According to Ghoussoub and Yuan [14], if \(0 \leq \theta < p\) and \(\Omega = \mathbb{R}^N\), \(S_\theta\) is attained at

\[
w_\varepsilon(x) = \left( \varepsilon (N - \theta) \left( \frac{N - p}{p - 1} \right)^{p-1} \left( \varepsilon + |x|^{p - \theta} \right)^{p-N} \right)^{\frac{1}{p - \theta}},
\]

where \(\varepsilon > 0\) and \(\{w_\varepsilon\}\) are the only positive radial solutions of

\[-\Delta_p u = \frac{1}{|x|^{\theta}} u^{p^*_\theta - 1} \quad \text{in } \mathbb{R}^N.
\]

As a consequence,

\[
|w_\varepsilon|_{\theta, p^*_\theta} = \left( \int \frac{|w_\varepsilon|^{p^*_\theta}}{|x|^{\theta}} \, dx \right)^{\frac{1}{p^*_\theta}} = \left( \int \frac{w_\varepsilon^{p^*_\theta}}{|x|^{\theta}} \, dx \right)^{\frac{1}{p^*_\theta}} = S_\theta^{\frac{N - \theta}{p - \theta}},
\]

(1.2)
To date, there is a broad literature on the present class of singular problems. Regarding smooth bounded domains \( \Omega \subset \mathbb{R}^N \), Brézis and Cabré [3] showed that if \( \rho \in L^1(\Omega, \delta_0(x) \, dx) \), where \( \delta_0(x) := \text{dist}(x, \partial \Omega) \), then the problem
\[
-\Delta u = \rho(x) + \frac{u^2}{|x|^2} \quad \text{in} \ \Omega, \quad u = 0 \quad \text{on} \ \partial \Omega,
\]
has no distribution solution \( u \in L^1_{\text{loc}}(\Omega) \). On the other hand, Montefusco [20] showed that if \( \rho \) satisfies both \( \rho(x) \rightarrow 0 \) as \( |x| \rightarrow \infty \), \( \rho \geq \epsilon_0 \) in some open subset of \( \mathbb{R}^N \) for some \( \epsilon_0 > 0 \) and \( 0 < \lambda < S_p \), then the equation
\[
-\Delta_p u = \rho(x) + \lambda \frac{|u|^{p-2}u}{|x|^p} \quad \text{in} \ \mathbb{R}^N
\]
ads a solution \( u \in D^{1,p} \). Back to the case of bounded domains, Ghoussoub and Yuan [14] proved that the problem
\[
-\Delta_p u = |u|^{q-2}u + \lambda \frac{|u|^{p^*_0-2}u}{|x|^{p^*_0}} \quad \text{in} \ \Omega, \quad u = 0 \quad \text{on} \ \partial \Omega
\]
ads at least a positive weak solution when \( 0 \leq \theta < p, \ p^2 \leq N, \ p < q < p^*, \lambda > 0 \) and further has infinitely many weak solutions, one of which is positive, when \( \theta = p, \ 1 \leq p < N, \ p < q < p^* \) and \( \lambda \in (0, S_p) \). On the other hand, Chen and Li [9] showed that if the set \( \{x \in \mathbb{R}^N \mid \rho(x) > 0\} \) has positive Lebesgue measure, then the equation
\[
-\Delta_p u = \rho(x)|u|^{q-2}u + \lambda \frac{|u|^{p^*_0-2}u}{|x|^{p^*_0}} \quad \text{in} \ \mathbb{R}^N,
\]
ads an infinite sequence \( \{u_m\} \) of weak solutions with energy \( I \) satisfying \( I(u_m) \neq 0 \), provided \( 0 \leq \theta < p, \ 1 < q < p, \lambda \) is small and has an infinite sequence \( \{u_m\} \) of weak solutions with unbounded energy \( I(u_m) \) if \( \theta = p, \ p < q < p^* \) and \( \lambda \in (0, S_p) \).

The reader is further referred to Smets and Tesei [23], Dupaigne and Nedev [12], Dávila and Dupaigne [10] and their references.

We point out that [9,14,20] employ direct variational methods. In this paper this does not seem to be possible due the presence of the more general nonlinearities \( f \) and \( g \). We instead use lower and upper solutions whose construction employ variational methods.

The following conditions will be required in some of our results:

(i) \( f(t) \leq t^q \), where \( 0 \leq q < p - 1 \),

(ii) \( \frac{f(t)}{t^{p-1}} \) is decreasing,

(iii) \( \frac{f(t)}{t^{p-1}} \xrightarrow{t \to \infty} \infty \),

\[
(1.3)
\]

\[
g(t) \leq t^{\hat{p}_0-1},
\]

\[
(1.4)
\]

setting \( \hat{\rho}(r) := \max_{|x| = r} \rho(x) \), assume that

\[
\hat{\rho} \in L^{\hat{\mu}}, \quad \text{where} \ \hat{\mu} := \frac{p^*}{p^* - q - 1}.
\]

Additionally, \( \hat{\rho} \) will be required to satisfy

\[
\beta \left[ \frac{(p-q-1)\alpha}{(\hat{p}_0^*-p)} - \alpha \left( \frac{(p-q-1)\alpha}{(\hat{p}_0^*-p)} \right)^{\frac{\alpha+1-p}{\alpha}} < \frac{1}{p} \right],
\]

\[
(1.6)
\]
where
\[ \alpha := \frac{|\hat{\rho}|\mu}{(q + 1)s_0^{(q+1)/p}}, \quad \beta := \frac{1}{p^*_\theta S_\theta^p/p}. \]

Notice that \( \hat{\rho} \) is radially symmetric that is \( \hat{\rho}(x) = \hat{\rho}(|x|), x \in \mathbb{R}^N. \)

Our main result is

**Theorem 1.1.** Assume (1.3)–(1.6). If in addition, one of the conditions

(i) \( 0 \leq \theta < p \) and \( 0 \leq \lambda \leq 1, \)

(ii) \( \theta = p \) and \( 0 \leq \lambda < S_\theta \)

holds, then there is \( u \in D^{1,p} \) with \( u > 0 \) satisfying

\[
\int |\nabla u|^{p-2} \nabla u \nabla \phi \, dx = \int \left( \rho(x)f(u) + \frac{\lambda}{|x|^{p\theta}}g(u) \right) \phi \, dx, \quad \phi \in D^{1,p}. \tag{1.7}
\]

Moreover,

\[
\left[ u(x)^{\theta+1} \int _{B_{|x|}(0)} \rho(y) \, dy + \frac{\lambda \omega|N}{N - \theta} |x|^{N-\theta} u(x)^{p\theta} \right] \text{ is bounded in } \mathbb{R}^N \setminus \{0\}. \tag{1.8}
\]

Two key auxiliary results are established below. The first one extends to singular problems a result by Cañada et al. [7].

**Theorem 1.2.** Let \( \Omega \subset \mathbb{R}^N \) be a smooth bounded domain. Assume (1.3)–(1.5) and \( \lambda \geq 0. \) If \( \upsilon, \omega \in L^{p\theta}_\theta(\Omega) \cap W^{1,p}(\Omega) \) satisfy \( \omega = 0 \) and \( \upsilon \geq 0 \) on \( \partial \Omega, \) \( 0 \leq \omega \leq \upsilon \) in \( \Omega, \)

\[
\int \Omega |\nabla \omega|^{p-2} \nabla \omega \nabla \phi \, dx \leq \int \Omega \left( \rho(x)f(\omega) + \frac{\lambda}{|x|^{p\theta}}g(\omega) \right) \phi \, dx, \quad \phi \in D^{1,p}. \tag{1.9}
\]

and

\[
\int \Omega |\nabla \upsilon|^{p-2} \nabla \upsilon \nabla \phi \, dx \geq \int \Omega \left( \rho(x)f(\upsilon) + \frac{\lambda}{|x|^{p\theta}}g(\upsilon) \right) \phi \, dx \tag{1.10}
\]

for \( \phi \in D^{1,p}_0(\Omega) \) with \( \phi \geq 0, \) then there is \( u \in D^{1,p}_0(\Omega) \) such that both \( \omega \leq u \leq \upsilon \) and

\[
\int \Omega |\nabla u|^{p-2} \nabla u \nabla \phi \, dx = \int \Omega \left( \rho(x)f(u) + \frac{\lambda}{|x|^{p\theta}}g(u) \right) \phi \, dx, \quad \phi \in D^{1,p}_0(\Omega). \tag{1.11}
\]

The second one aims constructing an upper solution of (1.1). It, in fact, gives existence of a solution in the closed subspace \( D^{1,p}_\text{rad} \) of radially symmetric functions of \( D^{1,p} \) for the problem

\[
\begin{cases}
-\Delta_p \upsilon = \hat{\rho}(x)\upsilon^q + \frac{\lambda}{|x|^{p\theta}}u^{p\theta - 1} & \text{in } \mathbb{R}^N, \\
\upsilon > 0 & \text{in } \mathbb{R}^N, \quad \upsilon(x) \xrightarrow{|x| \to \infty} 0.
\end{cases} \tag{1.12}
\]
Theorem 1.3. Assume (1.5)–(1.6). If in addition, one of the conditions

(i) \( 0 \leq \theta < p \) and \( \lambda = 1 \),

(ii) \( \theta = p \) and \( 0 \leq \lambda < S_p \)

holds, then there is \( \nu \in D^{1,p}_{\text{rad}} \cap C^2(\mathbb{R}^N\setminus\{0\}) \) satisfying

\[
\int |\nabla \nu|^{p-2} \nabla \nu \nabla \phi \, dx = \int \left( \hat{\rho} \nu^q + \frac{\lambda}{|x|^\theta} \nu^{p^*_\theta - 1} \right) \phi \, dx, \quad \phi \in D^{1,p}_{\text{rad}}, \tag{1.13}
\]

\( \nu > 0 \) in \( \mathbb{R}^N \), \( \nabla \nu(x) \cdot x < 0 \) in \( \mathbb{R}^N \setminus\{0\} \),

and

\[
\nu(x)^{q+1} \int_{B_{|x|}(0)} \hat{\rho}(y) \, dy + \frac{\lambda \omega_N}{N - \theta} |x|^{N - \theta} \nu(x)^{p^*_\theta - 1} \leq \|\nu\|^p, \quad x \in \mathbb{R}^N \setminus\{0\}. \tag{1.15}
\]

The main result of this paper as well as its proof were greatly inspired by Brézis and Nirenberg [5] and Ambrosetti et al. [2]. The proof of Theorem 1.1 in fact consists in three steps. In a first one, we construct an upper solution \( \nu \) of (1.1) with the aid of Theorem 1.3. In a second step we construct a sequence of functions, say \( \{u_k\} \subset D^{1,p}_{\text{rad}}(B_k) \), with \( 0 < u_k < \nu \) in \( B_k \), where \( B_k := B_k(0) \), satisfying (1.11). In the last step we pass to the limit in \( k \) getting to a solution of (1.1).

2. Proof of Theorem 1.2

Denoting by \( D^{-1,p'}(\Omega) \) the dual space of \( D^{1,p}(\Omega) \), where \( 1/p + 1/p' = 1 \), consider \( T : D^{-1,p'}(\Omega) \to D^{1,p}(\Omega) \) defined by \( T(\psi) := \psi \), where \( u \in D^{1,p}(\Omega) \) is the unique solution of the equation \(-\Delta_p u = \psi \) in \( \Omega \). As is well known, \( T \) is continuous and monotone nondecreasing. Next, consider the set

\[
\mathcal{C} := \{ u \in D^{1,p}_0(\Omega) \mid \omega \leq u \leq \nu \}
\]

endowed with the pointwise convergence topology and the mapping \( S \) defined by

\[
\langle Su, \phi \rangle := \int_\Omega \xi(x,u) \phi \, dx, \quad u \in \mathcal{C}, \quad \phi \in D^{1,p}_0(\Omega),
\]

where

\[
\xi(x,u) := \rho(x) f(u) + \frac{\lambda}{|x|^\theta} g(u). \tag{2.1}
\]

We claim that \( S(\mathcal{C}) \subset D^{-1,p'}(\Omega) \) and \( S : \mathcal{C} \to D^{-1,p'}(\Omega) \) is continuous and nondecreasing. Indeed, take \( u \in \mathcal{C}, \phi \in D^{1,p}_0(\Omega) \) and notice that by (1.3)(i) and (1.4),

\[
\left| \langle Su, \phi \rangle \right| \leq \int_\Omega \rho(x) f(u) |\phi| \, dx + \lambda \int_\Omega \frac{u^{p^*_\theta - 1}}{|x|^{\theta p^*_\theta - 1}} \frac{|\phi|}{|x|^{p^*_\theta}} \, dx.
\]

Remarking that \( f(u) \leq 1 \) when \( q = 0 \), we have by first applying Hölder’s inequality in the integrals above and subsequently applying the Sobolev and Hardy–Sobolev inequalities that
\[ \left| \langle Su, \phi \rangle \right| \leq \int_\Omega \rho(x) f(u) |\phi| \, dx + \lambda \left( \int_\Omega \frac{|u|^{p_0^*}}{|x|^{\sigma}} \right)^{\frac{1}{p_0^*}} \left( \int_\Omega \frac{|\phi|^{p_0^*}}{|x|^{\sigma}} \right)^{\frac{1}{p_0^*}} \leq C_1 \|u\|^q \|\phi\| + C_2 \|u\|^{p_0^* - 1} \|\phi\|, \]

showing that \( S(\mathcal{C}) \subset D^{-1,p}'(\Omega) \) and in addition \( S \) maps \( D_0^{1,p}(\Omega) \)-bounded subsets of \( \mathcal{C} \) into bounded subsets of \( D^{-1,p'}(\Omega) \).

To show that \( S \) is continuous let \( \{u_n\} \) be a sequence in \( \mathcal{C} \) such that \( u_n \to u \) a.e. in \( \Omega \) for some \( u \in \mathcal{C} \). If \( \phi \in D_0^{1,p}(\Omega) \), we have

\[ \left| \langle Su_n - Su, \phi \rangle \right| \leq \int_\Omega \rho(x) \left| f(u_n) - f(u) \right| |\phi| \, dx + \lambda \int_\Omega \frac{1}{|x|^{\sigma}} \left( |g(u_n) - g(u)| + \left| \phi \right| \right) \, dx. \]  

(2.2)

By (1.3)(i), (1.4), the definition \( \mathcal{C} \), the Hölder and Hardy–Sobolev inequalities and the Lebesgue theorem,

\[ \left| \langle Su_n - Su, \phi \rangle \right| \leq o_n(1) \|\phi\|, \quad \text{where } o_n(1) \xrightarrow{\mathcal{D}} 0. \]  

(2.3)

Hence \( \|Su_n - Su\|_{D^{-1,p'}(\Omega)} \to 0 \), showing that \( S \) is continuous.

To show that \( S \) is nondecreasing notice that if \( u_1 \leq u_2 \) and \( \phi \in D_0^{1,p}(\Omega) \) is nonnegative,

\[ \langle Su_1, \phi \rangle = \int_\Omega \xi(x, u_1) \phi \, dx \leq \int_\Omega \xi(x, u_2) \phi \, dx \leq \langle Su_2, \phi \rangle. \]

Next let \( F : \mathcal{C} \to D_0^{1,p}(\Omega) \) given by \( F(u) := T(Su) \). Notice that \( u \in \mathcal{C} \) satisfies (1.11) if and only if \( u = F(u) \) that is \( u = T(Su) \). In order to find a fixed point \( u \) of \( F \) consider the sequence \( \omega_{n+1} := F(\omega_n) \), where \( \omega_1 := F(\omega) \).

We claim that

\[ \omega \leq \omega_1 \leq \cdots \leq \omega_n \leq \cdots \leq \nu. \]

Indeed, taking \( \phi \in D_0^{1,p}(\Omega) \) with \( \phi \geq 0 \), we notice that

\[ \int_\Omega |\nabla \omega|^{p-2} \nabla \omega \nabla \phi \, dx \leq \int_\Omega |\nabla \phi|^{p-2} \nabla \phi \, dx. \]

So the comparison principle for \( \Delta_p \) gives \( \omega \leq \omega_1 \). Iterating this argument, we have \( \omega_1 \leq \omega_2 \leq \cdots \leq \omega_n \leq \cdots \). By a similar argument \( \omega_n \leq \nu \) and the claim follows. Next noticing that \( \omega_n \to u \) pointwisely, one has of course that \( \omega \leq u \leq \nu \). We claim that \( u \in \mathcal{C} \) and \( u = F(u) \). Indeed, estimating as we did in (2.2) to get to (2.3), we find a constant \( C_{\omega, \nu} > 0 \) such that

\[ \|S\omega_n - S\omega_m\|_{D^{-1,p'}(\Omega)} \leq C_{\omega, \nu} |\omega_n - \omega_m|_{L_0^{p_0}(\Omega)}. \]

On the other hand, reminding that \( \{\omega_n\} \subset \mathcal{C} \) we find by applying Lebesgue’s theorem that \( |\omega_n - \omega_m|_{L_0^{p_0}(\Omega)} \to 0 \). Therefore, \( S\omega_n \) is a Cauchy sequence in \( D^{-1,p'} \) and thus \( S\omega_n \to \tilde{u} \) for some \( \tilde{u} \in D^{-1,p'}(\Omega) \). By the continuity of \( T \), \( T(S\omega_n) \to T(\tilde{u}) \) in \( D_0^{1,p}(\Omega) \). Since \( \omega_{n+1} = F(\omega_n) = T(S\omega_n) \), we get

\[ \omega_{n+1} \xrightarrow{D_0^{1,p}(\Omega)} T(\tilde{u}). \]
Therefore, \( \omega_{n+1} \rightarrow T(\bar{u}) \) pointwisely showing that \( u = T(\bar{u}) \) and so \( u \in \mathcal{C} \). To end the verification of the claim notice that
\[
u = T(\bar{u}) = \lim \omega_{n+1} = \lim F(\omega_n) = F(T(\bar{u})) = F(u).
\]
This proves Theorem 1.2.

3. Proof of Theorem 1.3

The energy functional associated with (1.12), namely
\[
I(\nu) = \frac{1}{p} \int |\nabla \nu|^p \, dx - \frac{1}{q+1} \int \hat{\rho}(x) \nu^{q+1}_+ \, dx - \frac{\lambda}{p^*_\theta} \int \frac{\nu_{p_\theta}^\theta}{|x|^\theta} \, dx, \quad \nu \in D^{1,p}_{\text{rad}},
\]
belongs to \( C^1(D^{1,p}_{\text{rad}}, \mathbb{R}) \) and
\[
\langle I'(\nu), \phi \rangle = \int |\nabla \nu|^{p-2} \nabla \nu \nabla \phi \, dx - \int \hat{\rho} \nu^q_+ \phi \, dx - \lambda \int \frac{\nu_{p_\theta}^{p_\theta-1}}{|x|^\theta} \phi \, dx, \quad \phi \in D^{1,p}_{\text{rad}}.
\]
In the case \( 0 \leq \theta < p \), we shall apply the technique by Brézis and Nirenberg [5] and, as a matter of fact, arguments in Alves and Goncalves [1]. This will be accomplished through the use of the Hardy–Sobolev inequality. In this regard, given \( e \in D^{1,p}_{\text{rad}} \) let
\[
c := \inf_{p \in \mathcal{P}} \max_{0 \leq t \leq 1} I(p(t)),
\]
where
\[
\mathcal{P} := \{ p \in C([0, 1], D^{1,p}_{\text{rad}}) \mid p(0) = 0, \ p(1) = e \}.
\]
In the case \( \theta = p \) minimization arguments will be explored.

In both cases we will make use of the following result on the concentration–compactness principle (limit case) of Lions [18,19], see also Montefusco [20] and Smets [22].

**Lemma 3.1.** Assume \( 0 \leq \theta \leq p \) and let \( \nu_n \in D^{1,p}_{\text{rad}} \) be a sequence with \( \nu_n^{D^{1,p}_{\text{rad}}} \rightarrow \nu \). Then
\[
\nu_n^p_{\text{rad}} \rightharpoonup \nu_+, \quad |\nabla \nu_n|^p \, dx \rightharpoonup \mu, \quad \nu_{p_\theta}^\theta \rightharpoonup \nu_{\text{rad}} \rightarrow v
\]
for some Radon measures \( \mu \) and \( v \). In addition, there are an at most denumerable set \( J_\theta \), a subset \( \{ x_j \}_{j \in J_\theta} \subseteq \mathbb{R}^N \) and positive numbers \( \mu_j, v_j \) such that
\[
v = \frac{\nu_{p_\theta}^\theta}{|x|^\theta} \, dx + \sum_{j \in J_\theta} \delta_{x_j} v_j, \quad \mu_{\text{rad}} \rightharpoonup \mu, \quad \nu_{\text{rad}} \rightharpoonup v
\]
for some \( \delta_{x_j} \) means the Dirac mass at \( x_j \). If \( 0 < \theta \leq p \) then \( x_j = 0 \) and \( J_\theta \) is a singleton.
The lemma below establishes the Mountain Pass Geometry in the case $0 \leq \theta < p$, gives an estimate to the critical level $c$ and establishes that $I$ is coercive when $\theta = p$.

**Lemma 3.2.** Assume (1.5)–(1.6). If $0 \leq \theta < p$ and $\lambda = 1$, then there are $\eta, \hat{\nu} > 0$ and $e \in D_{\text{rad}}^{1,p}$ such that

$$I(\upsilon) \geq \eta, \quad \|\upsilon\| = \hat{\nu},$$

$$\|e\| > \hat{\nu}, \quad I(e) \leq 0,$$  \hspace{1cm} (3.1)  \hspace{1cm} (3.2)

and furthermore,

$$0 < c < \frac{p - \theta}{p(N - \theta)} S^{(N-\theta)/(p-\theta)}_{\theta}.$$  \hspace{1cm} (3.3)

If, on the other hand, $\theta = p$ and $0 \leq \lambda < S_p$, then $I$ is coercive and $I(\phi_p) < 0$ for some $\phi_p \in D_{\text{rad}}^{1,p}$.

**Proof.** If $0 \leq \theta < p$ let $\upsilon \in D_{\text{rad}}^{1,p}$. We have, firstly, by (1.5) and Hölder’s inequality,

$$\int \hat{\rho} \upsilon_{+}^{q+1} dx \leq \left( \int \hat{\rho} \right)^{1/\hat{\mu}} \left( \int |\upsilon|^{p^*} \right)^{(q+1)/p^*}$$

and secondly, by the Hardy–Sobolev inequality,

$$\left( \int |\upsilon|^{p^*} \right)^{(q+1)/p^*} \leq S_0^{-(q+1)/p} \|\upsilon\|^{q+1} \quad \text{and} \quad \int \frac{|\upsilon_+|^{p^*_\theta}}{|x|^{\theta}} dx \leq S_{\theta}^{-p^*_\theta/p} \|\upsilon\|^{p^*_\theta}.$$  \hspace{1cm} (3.4)  \hspace{1cm} (3.5)

Now, applying (3.4) and (3.5) we find

$$I(\upsilon) \geq \frac{1}{p} \|\upsilon\|^p - \frac{|\hat{\rho}|_{\hat{\mu}}}{(q+1) S_0^{(q+1)/p}} \|\upsilon\|^{q+1} - \frac{1}{p^*_\theta S_{\theta}^{p^*_\theta/p}} \|\upsilon\|^{p^*_\theta}$$

$$= \|\upsilon\|^p \left( \frac{1}{p} - \alpha \|\upsilon\|^{(q+1-p)} - \beta \|\upsilon\|^{(p^*_\theta - p)} \right).$$

Using (1.6), (3.1) follows. Next (3.2) and (3.3) will be shown. In order to show (3.2), take $w_\epsilon$ as in (1.2). Picking $t := t_0 > 0$ large enough in the expression below,

$$I(t w_\epsilon) = \frac{t^p}{p} \int |\nabla w_\epsilon|^p dx - \frac{t^{q+1}}{q+1} \int \hat{\rho} w_{\epsilon}^{q+1} dx - \frac{t^{p^*_\theta}}{p^*_\theta} \int \frac{w_\epsilon^{p^*_\theta}}{|x|^{\theta}} dx, \quad t \geq 0,$$

and setting $e := t_0 w_\epsilon$, we get $I(e) \leq 0$. Next we infer by (1.2) and adapting arguments in Alves and Goncalves [1] that

$$\max_{t \geq 0} I(t w_\epsilon) < \left( \frac{1}{p} - \frac{1}{p^*_\theta} \right) S_{\theta}^{(N-\theta)/(p-\theta)}.$$ 

Reminding the definition of $c$, we get (3.3). Now, if $\theta = p$, observing that

$$|\upsilon_+|^p \leq |\upsilon|^p \quad \text{and} \quad \int \frac{\upsilon_+^p}{|x|^p} dx \leq S_p^{-1} \int |\nabla \upsilon|^p dx,$$

we get to the following estimate:
\[ I(\nu) \geq \frac{1}{p} \|\nu\|^p - \frac{C_3}{q + 1} |\hat{\rho}| \|\nu\|^{q+1} - \frac{\lambda}{pS_p} \|\nu\|^p \]

\[ = \frac{1}{p} \left( 1 - \frac{\lambda}{S_p} \right) \|\nu\|^p - \frac{C_3}{q + 1} |\hat{\rho}| \|\nu\|^{q+1}, \]

which shows that \( I \) is coercive. Now, choosing \( \phi \in D_{\text{rad}}^{1,p} \) with \( \phi > 0 \) and taking \( t > 0 \),

\[ I(t\phi) = \frac{t^p}{p} \left( \int |\nabla \phi|^p \, dx - \lambda \int \frac{\phi^p}{|x|^p} \, dx \right) - \frac{t^{q+1}}{q+1} \int \hat{\rho} \phi^{q+1} \, dx. \]

Since \( \lambda \in [0, S_p) \), the second term in the inequality above is positive and so \( I(t_0\phi) < 0 \) for some \( t_0 > 0 \). Setting \( \phi_p := t_0\phi \) ends the proof. \( \square \)

Lemma 3.3. Assume (1.5). If \( \theta = p \) and \( \lambda \in [0, S_p) \), then \( I : D_{\text{rad}}^{1,p} \to \mathbb{R} \) is weakly sequentially lower semicontinuous.

Proof. Let \( \nu_n \to \nu \) so that \( \nu_{n+} \to \nu_+ \). We point out that Lemma 3.1 applies to the sequence \( \nu_{n+} \). Using the fact that \( \hat{\rho} \in L^{\hat{\mu}} = (L^{p/(q+1)})' \), we find

\[ \lim_{n} I(\nu_n) = \lim_{n} \left( -I(\nu_n) \right) \]

\[ \geq -\frac{1}{p} \lim_{n} \left( -\int |\nabla \nu_n|^p \, dx \right) - \frac{1}{q+1} \lim_{n} \int \hat{\rho} \nu_{n+}^{q+1} \, dx - \frac{\lambda}{p} \lim_{n} \int \frac{\nu_{n+}^p}{|x|^p} \, dx \]

\[ = \frac{1}{p} \lim_{n} \left( \int |\nabla \nu_n|^p \, dx - \frac{1}{q+1} \int \hat{\rho} \nu_+^{q+1} \, dx - \frac{\lambda}{p} \lim_{n} \int \frac{\nu_{n+}^p}{|x|^p} \, dx \right. \]

\[ \left. + \frac{\lambda}{p} \lim_{n} \int \frac{\nu_{n+}^p}{|x|^p} \, dx \right) \]

\[ \geq I(\nu) + \frac{1}{p} \left( 1 - \frac{\lambda}{S_p} \right) \mu_o \]

\[ \geq I(\nu). \]

It follows by Evans [13, Theorem 3], Lemma 3.1, properties of Radon measures and the fact that \( J_p \) is a singleton, say \( J_p = \{o\} \) that

\[ \lim_{n} \int |\nabla \nu_n|^p \, dx \geq \int |\nabla \nu|^p \, dx + \mu_o \quad \text{and} \quad \lim_{n} \int \frac{\nu_{n+}^p}{|x|^p} \, dx \leq \int \frac{\nu_+^p}{|x|^p} \, dx + v_o. \]

Taking these into (3.6), using Lemma 3.1 and the assumption \( \lambda \in [0, S_p) \), we get

\[ \lim_{n} I(\nu_n) \geq \frac{1}{p} \int |\nabla \nu|^p \, dx - \frac{1}{q+1} \int \hat{\rho} \nu_+^{q+1} \, dx - \frac{\lambda}{p} \int \frac{\nu_+^p}{|x|^p} \, dx + \frac{1}{p} (\mu_o - \lambda v_o) \]

\[ \geq I(\nu) + \frac{1}{p} \left( 1 - \frac{\lambda}{S_p} \right) \mu_o \]

\[ \geq I(\nu). \]

Thus \( I \) is weakly sequentially lower semicontinuous, proving Lemma 3.3. \( \square \)

Remark 1. (i) \( 0 \leq \theta < p \). Using Lemma 3.2 and the Mountain Pass Theorem there is a sequence \( \nu_n \in D_{\text{rad}}^{1,p} \) satisfying

\[ I(\nu_n) \to c \geq \eta \quad \text{and} \quad I'(\nu_n) \to 0. \]

(ii) \( \theta = p \). Using Lemmas 3.2 and 3.3 there is \( \nu \in D_{\text{rad}}^{1,p} \) such that \( I(\nu) = \min_{D_{\text{rad}}^{1,p}} I \) and \( I(\nu) < 0 \). Thus \( \nu \neq 0 \) and verifies \( I'(\nu) = 0 \) because \( I \) is a \( C^1 \)-functional.
Lemma 3.4. Assume $0 \leq \theta < p$. Then there is $v \in D^{1,p}_{\text{rad}}$ such that $v \geq 0$,

$$v_n \overset{D^{1,p}_{\text{rad}}}{\rightharpoonup} v \quad \text{and} \quad v_n \overset{\text{a.e.}}{\to} v.$$

**Proof.** Noting that $p < p^*_\theta$, we have

$$I(v_n) - \frac{1}{p^*_\theta} I'(v_n, v_n) = \left( \frac{1}{p} - \frac{1}{p^*_\theta} \right) \|v_n\|^p + \left( \frac{1}{p^*_\theta} - \frac{1}{q+1} \right) \int \hat{\rho} v_n^{q+1}
\geq \left( \frac{1}{p} - \frac{1}{p^*_\theta} \right) \|v_n\|^p + \left( \frac{1}{p^*_\theta} - \frac{1}{q+1} \right) S_{\theta}^{-(q+1)/p} \|v_n\|^{q+1},$$

Taking $n$ large enough gives

$$\left( \frac{1}{p} - \frac{1}{p^*_\theta} \right) \|v_n\|^p + \left( \frac{1}{p^*_\theta} - \frac{1}{q+1} \right) S_{\theta}^{-(q+1)/p} \|v_n\|^{q+1} \leq \|v_n\| + C_4,$$

showing that $v_n$ is bounded in $D^{1,p}_{\text{rad}}$. As a consequence there is some $v \in D^{1,p}_{\text{rad}}$ such that $v_n \overset{D^{1,p}_{\text{rad}}}{\rightharpoonup} v$. Now, since $\|v_{n-}\| \to 0$. Now since $1 < p < N$, by the Rellich–Kondrachov theorem,

$$D^{1,p}_{\text{rad}}(B) \overset{\text{compactly}}{\hookrightarrow} L^q(B)$$

for each ball $B \subset \mathbb{R}^N$ and for some $q > 1$. By a diagonal argument one infers (eventually passing to a further subsequence) that $v_n \overset{\text{a.e.}}{\to} v$. As a consequence $v \geq 0$. □

**Lemma 3.5.** $\nabla v_n \overset{\text{a.e.}}{\to} \nabla v$.

The proof of Lemma 3.5 is quite technical and will be left to the end of this section.

**Remark 2.** Due to the fact that $\|v_{n-}\| \to 0$ and Lemma 3.5, we will assume on the proof of Theorem 1.3 below that $v_n \geq 0$.

We establish below Lemma 3.6. It slightly improves a remark by Brézis and Lieb [4] and will be used, for instance, in the proof of Theorem 1.3. Its proof is similar to that of [17, Lemma 4.8] and will be omitted.

**Lemma 3.6.** Let $\Omega$ be an open subset of $\mathbb{R}^N$ and let $u_n$ be a bounded sequence in $L^s_{\theta}(\Omega)$ where $1 < s < \infty$ and $0 \leq \theta \leq p$. If in addition, $u_n \overset{\text{a.e.}}{\to} u$ for some $u \in L^s_{\theta}(\Omega)$, then $u_n \overset{L^s_{\theta}}{\rightharpoonup} u$.

**Proof of Theorem 1.3** *(Completed)*. Regarding case (i) we have, $v_n \overset{D^{1,p}_{\text{rad}}}{\rightharpoonup} v$ and hence $|\nabla v_n|^{p-2} \nabla v_n$ is bounded in $(L^{p/(p-1)} \right)^N$. Assuming first that $0 < q < p - 1$ and using the Sobolev and Hardy–Sobolev inequalities it follows that $v_n^q$ and $v_n^{p-1}$ are bounded in $L^{p/q}$ and $L^{p^*_\theta/(p^*_\theta-1)}$, respectively. By Lemma 3.4, $v_n \overset{\text{a.e.}}{\to} v$ and by Lemma 3.5, $\nabla v_n \overset{\text{a.e.}}{\to} \nabla v$. By Lemma 3.6,

$$|\nabla v_n|^{p-2} \nabla v_n \rightharpoonup |\nabla v|^{p-2} \nabla v \quad \text{in} \quad (L^{p/(p-1)})^N,$$

$$v_n^{p-1} \rightharpoonup u^{p-1} \quad \text{in} \quad L^{p^*_\theta/(p^*_\theta-1)} \quad \text{and} \quad v_n^q \rightharpoonup u^q \quad \text{in} \quad L^{p^*_\theta/q}.$$
which means that for \( \phi \in D_{\text{rad}}^{1,p} \),
\[
\int |\nabla \upsilon_n|^{p-2} \nabla \upsilon_n \nabla \phi \, dx \to \int |\nabla \upsilon|^{p-2} \nabla \upsilon \nabla \phi \, dx,
\]
\[
\int \frac{\upsilon_n^{p^*_\theta-1}}{|x|^{\theta}} \phi \, dx \to \int \frac{\upsilon^{p^*_\theta-1}}{|x|^{\theta}} \phi \, dx \quad \text{and} \quad \int \hat{\rho} \upsilon_n^q \phi \, dx \to \int \hat{\rho} \upsilon^q \phi \, dx.
\]
If \( q = 0 \), the last convergence holds true, of course. Thus \( \langle I'(\upsilon_n), \phi \rangle \to \langle I'(\upsilon), \phi \rangle \). As for the case (ii), \( \upsilon \) is also a solution because it is a minimizer of \( I \). As a consequence, in both cases,
\[
\int |\nabla \upsilon|^{p-2} \nabla \upsilon \nabla \phi \, dx = \int \left[ \hat{\rho} \upsilon^q + \upsilon^{p^*_\theta-1} \right] \phi \, dx, \quad \phi \in D_{\text{rad}}^{1,p},
\]
that is, \( \upsilon \) is a weak solution of the equation in (1.12).

We claim that \( \upsilon \neq 0 \). In case (i), assume by the way of contradiction, that \( \upsilon = 0 \). By the boundedness of \( \upsilon_n \),
\[
\int |\nabla \upsilon_n|^p \, dx \to \ell \geq 0.
\]
But from the fact that \( I'(\upsilon_n) \to 0 \),
\[
\int |\nabla \upsilon_n|^p = \int \left[ \hat{\rho} \upsilon^q + \frac{\upsilon^{p^*_\theta}}{|x|^{\theta}} \right] \, dx + o_n(1).
\]
Arguing as we have done before and using (3.8),
\[
\int \hat{\rho} \upsilon_n^{q+1} \to 0 \quad \text{and} \quad \int \frac{\upsilon_n^{p^*_\theta}}{|x|^{\theta}} \, dx \to \ell.
\]
Now using the definition of \( \ell \) and the fact that \( I(\upsilon_n) \to c \),
\[
\left( \frac{1}{p} - \frac{1}{p^*_\theta} \right) \ell = c,
\]
which gives \( \ell > 0 \) since \( \theta \in [0, p) \). On the other hand, passing to the limit in
\[
\int |\nabla \upsilon_n|^p \geq S_\theta \left( \int \frac{\upsilon_n^{p^*_\theta}}{|x|^{\theta}} \, dx \right)^{p/p^*_\theta}
\]
leads to \( \ell \geq S_\theta \epsilon^{p/p^*_\theta} \) which in turn gives
\[
\ell \geq S_\theta^{(N-\theta)/(p-\theta)}.
\]
Thus, by (3.9)–(3.10),
\[
c = \frac{p-\theta}{p(N-\theta)} \ell \geq \frac{p-\theta}{p(N-\theta)} S_\theta^{(N-\theta)/(p-\theta)},
\]
a contradiction. So \( \upsilon \neq 0 \). Concerning case (ii), by Remark 1, \( I(\upsilon) < 0 \) and so \( \upsilon \neq 0 \) as well. Moreover, by construction, \( \upsilon \geq 0 \) in both cases.

For each \( \epsilon, r > 0 \), consider the function
\[
\upsilon_{r,\epsilon}(t) := \begin{cases} 
1 & \text{if } 0 \leq t \leq r, \\
\text{linear} & \text{if } r \leq t \leq r + \epsilon, \\
0 & \text{if } t \geq r + \epsilon.
\end{cases}
\]
Set \( \phi(x) := \nu_{r, \epsilon}(|x|) \) so that \( \phi \in D^{1,p}_{\text{rad}} \). Replacing \( \phi \) in (3.7), we get

\[
-\frac{1}{\epsilon} \int_r^{r+\epsilon} t^{N-1}|v'|^{p-2}v' \, dt = \int_0^r t^{N-1}(\hat{\rho} v^q + \lambda t^{-\theta} v_0^{p_0-1}) \, dt \\
+ \int_r^{r+\epsilon} t^{N-1}(\hat{\rho} v^q + \lambda t^{-\theta} v_0^{p_0-1}) \nu_{r, \epsilon} \, dt.
\]

Making \( \epsilon \to 0 \) gives

\[
-r^{N-1}|v'(r)|^{p-2}v'(r) = \int_0^r (t^{N-1} \hat{\rho} v^q + \lambda t^{N-\theta-1} v_0^{p_0-1}) \, dt,
\]

which leads to \( v' \leq 0 \). But this and the facts that \( v \neq 0 \) and \( v \geq 0 \) already shown give \( v' < 0 \) in \((0, \infty)\) and \( v > 0 \) in \((0, \infty)\). From (3.11),

\[
(-v'(r))^{p-1} = r^{1-N} \int_0^r (t^{N-1} \hat{\rho} v^q + \lambda t^{N-\theta-1} v_0^{p_0-1}) \, dt
\]

so that \( v \in C^2((0, \infty)) \). At this point, we have

\[
v \in D^{1,p}_{\text{rad}} \cap C^2(\mathbb{R}^N \setminus \{0\}) \quad \text{and both} \quad v > 0 \quad \text{and} \quad \nabla v(x) \cdot x < 0 \quad \text{in} \quad \mathbb{R}^N \setminus \{0\},
\]

which lead to (1.14). Setting \( \phi = v \) in (3.7) and using radial symmetry, we have

\[
\|v\|^p = \omega_N \int_0^\infty t^{N-1}(\hat{\rho}(t) v(t)^{q+1} + \lambda t^{-\theta} v(t)^{p_0}) \, dt \\
\geq \omega_N v(r)^{q+1} \int_0^r t^{N-1} \hat{\rho}(t) \, dt + \frac{\lambda \omega_N}{(N-\theta)} r^{N-\theta} v(r)^{p_0} \\
= v(x)^{q+1} \int_{B_{|x|}(0)} \hat{\rho}(y) \, dy + \frac{\lambda \omega_N}{(N-\theta)} |x|^{N-\theta} v(x)^{p_0}.
\]

Theorem 1.3 is proved. □

It remains to show Lemma 3.5.

**Proof of Lemma 3.5.** To begin with we claim that either \( J_\theta \) is a singleton or \( J_\theta = \emptyset \). (We write \( J_\theta = \{o\} \) in the first case.) Indeed, if \( 0 < \theta \leq p \) it follows by Lemma 3.1 that both \( J_\theta = \{o\} \) and \( x_0 = 0 \). If on the other hand \( \theta = 0 \), by Lemma 3.1,

\[
\int v_{n+}^p \phi \, dx \to \int v^p \phi \, dx + \sum_{j \in J_0} v_j \langle \delta_{x_j}, \phi \rangle, \quad \phi \in C^\infty_{0,\text{rad}}.
\]
If for some \( j \in J_0, x_j \neq 0 \), choose a ball \( B := B(x_j) \) which does not contain 0, \( \phi \in C_{0, \text{rad}}^\infty \), with \( \phi \geq 0 \) in \( B \), \( \phi > 0 \) in a smaller ball say \( B' \) centered at \( x_j \) and \( \phi = 0 \) on \( B \setminus B' \). Now observing that
\[
|\nabla (v_n + \phi)|^p \leq 2^p (|\phi \nabla v_n|^p + |v_n + \phi|^p) \quad \text{a.e. in } \mathbb{R}^N,
\]
and using the embedding \( D_{\text{rad}}^{1,p} \hookrightarrow L^p(B) \), we have the following inequalities:
\[
\int_B |v_n + \nabla \phi|^p \, dx \leq C_5 \int_B |v_n|^p \, dx, \quad \int_B |\phi \nabla v_n|^p \, dx \leq C_6 \|v_n\|^p.
\]
These inequalities show that the sequence \( \tilde{v}_n := v_n + \phi \) is bounded in \( W^{1,p}(B) \). In fact,
\[
\tilde{v}_n \rightharpoonup \tilde{v} \quad \text{a.e. in } \mathbb{R}^N \text{ with } \tilde{v} = v\phi.
\]
Using the fact that there is a positive constant \( C(N, p) \) such that for all \( u \in W^{1,p}(\text{rad}) \),
\[
|u(x)| \leq C(N, p)|x|^{-\frac{(N-1)}{p}} (|\nabla u|_p + |u|_p), \quad |x| \geq \epsilon,
\]
where \( \epsilon > 0 \), we get
\[
|(v_n + \phi)^p(x) - (\phi^p)^p(x)| \leq C_7(|\tilde{v}_n|_{W^{1,p}_{\text{rad}}} + |\tilde{v}|_{W^{1,p}_{\text{rad}}})|x|^{-\frac{(N-1)}{p}}
\]
\[
\leq C_8|x|^{-\frac{(N-1)}{p}} := g(x).
\]
Notice that \( g \in L^1(B(x_j)) \) because \( \{0\} \cap B(x_j) = \emptyset \). By Lebesgue’s theorem,
\[
\int_B v_n^{p^*} \, dx \to \int_B v^{p^*} \, dx.
\]
It follows as a consequence of this and (3.15) that \( \sum_{j \in J_\delta} \psi_j \langle \delta_{x_j}, \phi^{p^*} \rangle = 0 \), impossible. Therefore, \( x_j = 0 \) for \( j \in J_\delta \). Next we consider two cases:

**Case 1.** \( J_\delta = \{o\} \). Given \( \delta > 0 \) consider the set \( A_\delta := B_{1/2\delta} \setminus B_{\delta} \) and take \( \rho \in (0, \delta) \). We claim that
\[
\int_{A_\rho} (|\nabla v_n|^{p-2} \nabla v_n - |\nabla u|^{p-2} \nabla u) (\nabla v_n - \nabla u) \, dx \to 0. \tag{3.16}
\]
Indeed, let \( \phi \in C_0^\infty \) such that \( \phi = 1 \) in \( B_{1/2} \), \( \phi = 0 \) in \( B_1^c \) and \( 0 \leq \phi \leq 1 \). Set
\[
\psi_\epsilon(x) := \phi(\epsilon x) - \phi(x/\epsilon), \quad x \in \mathbb{R}^N,
\]
where \( \epsilon \in (0, \rho) \) and notice that
\[
\psi_\epsilon(x) = \begin{cases} 0 & \text{if } x \in B_{\epsilon/2}, \\ 1 & \text{if } x \in A_\epsilon. \end{cases}
\]
By Simon [21], there is a constant \( C_p > 0 \) such that
\[
|\langle x \rangle^{p-2} x - |y|^{p-2} y, x - y \rangle \geq \begin{cases} C_p |x - y|^p & \text{if } p \geq 2, \\ C_p \frac{|x-y|^2}{(1+|x|+|y|)^{2-p}} & \text{if } 0 < p < 1, \end{cases} \tag{3.17}
\]
for all \(x, y \in \mathbb{R}^N\). Applying (3.17), we get
\[
(|\nabla u_n|^{p-2}\nabla u_n - |\nabla v|^{p-2}\nabla v)(\nabla u_n - \nabla v) \geq 0 \quad \text{a.e. in } \mathbb{R}^N.
\]

Taking into account that \(A_\rho \subset A_\epsilon\), we have
\[
\int_{A_\rho} (|\nabla u_n|^{p-2}\nabla u_n - |\nabla v|^{p-2}\nabla v)(\nabla u_n - \nabla v) d\mathcal{L} \leq \int_{A_\epsilon} (|\nabla u_n|^{p-2}\nabla u_n - |\nabla v|^{p-2}\nabla v)(\nabla u_n - \nabla v) \psi_\epsilon d\mathcal{L} := \Upsilon_{n, \epsilon}.
\]

Employing arguments similar to those in Jianfu and Xiping [16], one shows that
\[
\lim_{\epsilon \to 0} \lim_{n} \Upsilon_{n, \epsilon} = 0.
\]

By (3.17) and (3.18), \(\nabla u_n \overset{a.e.}{\rightharpoonup} \nabla u\).

Case 2. \(J_\theta = \emptyset\). In this case \(\psi_\epsilon(x) := \phi(\epsilon x)\) and \(A_\rho = B_1/2\rho\). Arguments similar to the ones above apply, showing (3.16) in this case as well. It follows as above that \(\nabla u_n \overset{a.e.}{\rightharpoonup} \nabla u\).

4. Proof of Theorem 1.1

The lower and upper solutions technique will be applied.

Construction of an upper solution of (1.1). Let \(u\) as in Theorem 1.3. Since \(v(x) = v(r)\) and \(v \in C^2((0, \infty))\), it follows by (3.11) that
\[
-(r^{N-1}|u'(r)|^{p-2}u(r))' = r^{N-1}\left(\hat{\rho}uq + \frac{\lambda u^{p-1}}{\rho\theta} \right), \quad r > 0,
\]
which shows in both cases (i) and (ii) that
\[
-\Delta_{\rho} u = \hat{\rho}uq + \frac{\lambda u^{p-1}}{|x|^{\theta}} \quad \text{in } \mathbb{R}^N \setminus \{0\}
\]
in the classical sense. Multiplying (4.1) by \(\phi \in C_0^\infty(\mathbb{R}^N \setminus \{0\})\) and integrating gives
\[
\int |\nabla u|^{p-2}\nabla u \nabla \phi d\mathcal{L} = \int \left(\hat{\rho}(x)u^q + \frac{\lambda}{|x|^{\theta}} u^{p-1}\right) \phi d\mathcal{L}.
\]

Pick \(\epsilon > 0\) and a \(C^\infty\)-function \(\eta\) with \(0 \leq \eta \leq 1\), \(\eta(x) = 0\) if \(|x| \leq 1\) and \(\eta(x) = 1\) if \(|x| \geq 2\).

Consider the function \(\psi_\epsilon(x) = \eta(|x|/\epsilon)\). If \(\phi \in C_0^\infty\) then \(\psi_\epsilon \phi \in C_0^\infty(\mathbb{R}^N \setminus \{0\})\). Replacing \(\phi\) in (4.2) with this function gives
\[
\int |\nabla u|^{p-2}\nabla u \nabla \psi_\epsilon + \int |\nabla u|^{p-2}\nabla u \nabla \psi_\epsilon \phi = \int \left(\hat{\rho}u^q \psi_\epsilon + \frac{\lambda}{|x|^{\theta}} u^{p-1}\phi \psi_\epsilon \right).
\]

We claim that
\[
\int |\nabla u|^{p-2}\nabla u \nabla \psi_\epsilon d\mathcal{L} \to \int |\nabla u|^{p-2}\nabla u \nabla \phi d\mathcal{L},
\]
\[
\int |\nabla u|^{p-2}\nabla u \psi_\epsilon \phi d\mathcal{L} \to 0,
\]
\begin{align}
\int \hat{\rho} u^q \psi_\epsilon \, dx & \to \int \hat{\rho} u^q \, dx, \quad (4.6) \\
\int \frac{u \sigma_0^{p-1}}{|x|^p} \psi_\epsilon \, dx & \to \int \frac{u \sigma_0^{p-1}}{|x|^p} \, dx, \quad (4.7)
\end{align}
as \epsilon \to 0. Indeed, (4.4), (4.6) and (4.7) follow as a straightforward application of Lebesgue’s theorem. Regarding (4.5), we have by applying Hölder’s inequality,
\begin{align}
\left| \int |\nabla u|^{p-2} \nabla u \nabla \psi_\epsilon \phi \, dx \right| & \leq C_9 \left( \int |\nabla u|^p \, dx \right)^{\frac{p-1}{p}} \left( \int |\nabla \psi_\epsilon|^p \, dx \right)^{\frac{1}{p}} \\
& \leq C_{10} \left( \int |\nabla \psi_\epsilon|^p \, dx \right)^{\frac{1}{p}}.
\end{align}
Setting \( z := x/\epsilon \) and \( \eta(z) := \psi_\epsilon(x(z)) \), we get \( \frac{\partial \eta}{\partial z_j} = \epsilon \frac{\partial \psi_\epsilon}{\partial x_j} \) and \( |\nabla \eta|^p = \epsilon^p |\nabla \psi_\epsilon|^p \). Using Hölder’s inequality,
\begin{align}
\left| \int |\nabla u|^{p-2} \nabla u \nabla \psi_\epsilon \phi \, dx \right| & \leq C_{12} \left( \int |\nabla \eta|^p \, dz \right)^{\frac{1}{p}} \epsilon^{\frac{N-p}{p}} \to 0.
\end{align}
Making \( \epsilon \to 0 \) in (4.3) and using (4.4)–(4.7), we get to
\begin{align}
\int |\nabla u|^{p-2} \nabla u \nabla \phi \, dx = \int \left( \hat{\rho} u^q + \frac{\lambda}{|x|^\theta} u \sigma_0^{p-1} \right) \phi \, dx, \quad \phi \in D^{1,p},
\end{align}
where we remind that \( \lambda = 1 \) if \( \theta \in [0, p) \) and \( \lambda \in [0, S_p) \) if \( \theta = p \). So by (1.3)(i), (1.4), (1.5) and (i), (ii) in Theorem 1.1, we get
\begin{align}
\int |\nabla u|^{p-2} \nabla u \nabla \phi \, dx \geq \int \left( \rho(x) f(v) + \frac{\lambda}{|x|^\theta} g(v) \right) \phi \, dx, \quad \phi \in D^{1,p}, \quad \phi \geq 0. \quad (4.8)
\end{align}

**Remark 3.** As a consequence of (4.8) \( u \) satisfies (1.10) with \( \Omega = B_k \) and \( \lambda \) in accordance with either (i) or (ii) in Theorem 1.1.

**Construction of a family of lower solutions of (1.1).** In what follows we will refer several times to (1.8), (1.9), (1.11) and unless otherwise stated we mean \( \Omega = B_k \) and \( \lambda = 0 \) in those expressions. At this point of the proof we adapt some arguments in Carrião, Goncalves and Miyagaki [8].

Using (1.3)–(1.5), we get by Díaz and Saa [11, Theorems 1, 2] an only solution of (1.11) here labeled \( \omega_k \) with \( \omega_k \in D^{1,p}_0(B_k), \omega_k \geq 0, \omega_k \neq 0. \) Moreover, by Guedda and Veron [15, Corollary 1.1], \( \omega_k \in C^1(\bar{B}_k) \). Next, applying the maximum principle by Vázquez [24, Theorem 5], one infers that \( \omega_k > 0 \) in \( B_k \).

We contend that \( \omega_k \leq u \) in \( B_k \). Indeed, by Theorem 1.3, \( u \) is positive and continuous. So there is some \( \tau_k \in (0, 1) \) such that \( \tau_k \max_{\bar{B}_k} \omega_k < \min_{\bar{B}_k} u \). Thus \( \tau_k \omega_k < u \) in \( \bar{B}_k \). Now, given \( \phi \in D^{1,p}_0(B_k) \) with \( \phi \geq 0 \),
\[
\begin{align*}
\int_{B_k} |\nabla (\tau_k \omega_k)|^{p-2} \nabla (\tau_k \omega_k) \nabla \phi \, dx &- \int_{B_k} \rho(x) f (\tau_k \omega_k) \phi \, dx \\
= \tau_k^{p-1} \int_{B_k} |\nabla \omega_k|^{p-2} \nabla \omega_k \nabla \phi \, dx - \int_{B_k} \rho(x) \frac{f (\tau_k \omega_k)}{(\tau_k \omega_k)^{p-1}} (\tau_k \omega_k)^{p-1} \phi \, dx \\
\leq \tau_k^{p-1} \left( \int_{B_k} |\nabla \omega_k|^{p-2} \nabla \omega_k \nabla \phi \, dx - \int_{B_k} \rho(x) f (\omega_k) \phi \, dx \right) = 0,
\end{align*}
\]

showing that \( \tau_k \omega_k \) satisfies (1.9). On the other hand, by Remark 3, \( \upsilon \) satisfies (1.10). Applying Theorem 1.2 with \( \lambda = 0 \), there is \( \hat{\omega}_k \in D^{1,p}_0 (B_k) \) satisfying (1.11). By uniqueness, \( \hat{\omega}_k = \omega_k \) and so \( \omega_k \leq \upsilon \) in \( B_k \).

We claim that \( \omega_k \leq \omega_{k+1} \) in \( B_k \). Indeed, since for each \( k \), \( \omega_k \) is continuous on \( \bar{B}_k \) and positive in \( B_k \) there is \( \delta_k \in (0, 1) \) such that \( \delta_k \omega_k < \omega_{k+1} \) in \( B_k \).

But as above, \( \delta_k \omega_k \) satisfies (1.9) and \( \omega_{k+1} \) satisfies (1.10). So there is by Theorem 1.2 some \( \tilde{\omega}_k \in D^{1,p}_0 (B_k) \) satisfying (1.11). By the Díaz and Saa theorem referred to above, \( \tilde{\omega}_k = \omega_k \) which further shows that \( \omega_k \leq \omega_{k+1} \) in \( B_k \).

Making \( \omega_k = 0 \) outside \( B_k \), we have
\[
0 \leq \omega_1 \leq \omega_2 \leq \cdots \leq \omega_k \leq \cdots \leq \upsilon.
\]

Since \( \omega_k \) and \( \upsilon \) respectively satisfy (1.9) and (1.10), we get by Theorem 1.2 some \( u_k \) in \( D^{1,p}_0 (B_k) \) with \( \omega_k \leq u_k \leq \upsilon \) satisfying (1.11). We make \( u_k = 0 \) outside \( B_k \).

We claim that \( \{u_k\} \) is \( D^{1,p} \)-bounded. Indeed, set \( \phi = u_k \) in (1.11) and remark that
\[
\zeta(x, u_k) u_k \leq \zeta(x, \upsilon) \upsilon,
\]
where \( \zeta(x, \upsilon) \) was defined in (2.1). Noticing that \( \zeta(x, \upsilon) \upsilon \in L^1 \) it follows by (3.7) that \( \{u_k\} \) is bounded in \( D^{1,p} \) and hence \( u_k D^{1,p} \to u \). It will be shown that \( u \) is a solution of (1.1).

Consider the functional \( Z_k : D^{1,p} \to \mathbb{R} \) defined by
\[
\langle Z_k, \phi \rangle := \int |\nabla u_k|^{p-2} \nabla u_k \nabla \phi \, dx, \quad \phi \in D^{1,p},
\]
and notice that the sequence \( \{Z_k\} \) is bounded in \( (D^{1,p})' \). By compactness, \( Z_k \rightharpoonup \chi \) for some \( \chi \in (D^{1,p})' \), that is
\[
\langle Z_k, \phi \rangle \to \langle \chi, \phi \rangle, \quad \phi \in D^{1,p}.
\]

We claim that \( \chi = -\Delta_p u. \) Given \( \psi \in D^{1,p} \), by the monotonicity of \(-\Delta_p \),
\[
0 \leq \langle -\Delta_p u_k - (-\Delta_p \psi), u_k - \psi \rangle = \langle -\Delta_p u_k, u_k \rangle - \langle -\Delta_p \psi, u_k \rangle - \langle -\Delta_p \psi, u_k - \psi \rangle.
\]

Hence,
\[
0 \leq \int \zeta(x,u_k)u_k \, dx - \langle -\Delta_p u_k, \psi \rangle - \langle -\Delta_p \psi, u_k - \psi \rangle.
\]

Passing to the limit in \( k \),
\[
0 \leq \int \zeta(x,u)u \, dx - \langle \chi, \psi \rangle - \langle -\Delta_p \psi, u - \psi \rangle.
\]
Using the fact that $u_k$ satisfies (1.11) and passing to the limit in $k$, we get
\[
\langle \chi, u \rangle = \int \zeta(x, u) u \, dx.
\]
Setting $\psi := u - tw$, where $t > 0$ and $w \in D^{1,p}$,
\[
0 \leq \int \zeta(x, u) u \, dx - \langle \chi, u - tw \rangle - \langle -\Delta_p(u - tw), tw \rangle
\]
which gives
\[
0 \leq \langle \chi, w \rangle - \langle -\Delta_p(u - tw), w \rangle, \quad t > 0.
\]
Making $t \to 0$, we get
\[
0 \leq \langle \chi + \Delta_p u, w \rangle,
\]
which gives $\chi = -\Delta_p u$. As a consequence, $u$ satisfies (1.7). The facts that $u_k \overset{a.e.}{\to} u$, $u_k \geq \omega_k$ and (4.9) easily lead to $u > 0$. From $0 \leq u \leq \nu$ and (1.15), (1.8) follows. Theorem 1.1 is proved.

Acknowledgment

The authors are grateful to an anonymous referee for invaluable comments and suggestions which contributed to improve this paper.

References


