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Eigenvalues of fourth order Sturm–Liouville problems using Fliess series

B. Chanane *

Department of Mathematical Sciences, K.F.U.P.M., Dhahran 31261, Saudi Arabia

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Abstract

We shall extend our previous results (Chanane, 1998) on the computation of eigenvalues of second order Sturm–Liouville problems to fourth order ones. The approach is based on iterated integrals and Fliess series. © 1998 Elsevier Science B.V. All rights reserved.

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1. Introduction

This paper is a sequel of our recent paper [3] on the computation of eigenvalues of Sturm–Liouville problems and their error bounds. We shall be concerned here in the approximation of eigenvalues of fourth order Sturm–Liouville problems using Fliess series [6, 7, 11].

Suppose we are given the fourth order equation

$$Ly := y^{(4)} - (s(x)y^{(1)})^{(1)} + q(x)y = \lambda y, \quad 0 \leqslant x \leqslant a \tag{1.1}$$

together with the self-adjoint boundary condition

$$\sum_{j=1}^{4} M_{ij} y^{(j-1)}(0) = 0, \ \sum_{j=1}^{4} N_{ij} y^{(j-1)}(a) = 0, \quad i = 1,2, \tag{1.2}$$

where q, s, $s' \in L^1(0,a)$. It is well known [5, 13] that such a problem has an infinite sequence of eigenvalues $\lambda_0 \leqslant \lambda_1 \leqslant \lambda_2 \leqslant \lambda_3 \leqslant \cdots$, $\lambda_n \to +\infty$ as $n \to \infty$ and each eigenvalue has multiplicity at

* E-mail: chanane@kfupm.edu.sa.

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most 2. For more details see [9] and the references therein. On the numerical side, the only code available to deal with fourth order Sturm–Liouville problems is SLEUTH (Sturm–Liouville eigenvalues using theta matrices) [10]. This situation contrats with the availability of many softwares dealing with the second order case, SLEIGN [2], SLEIGN2 [1], SLEDGE [8],... .

For definiteness we shall consider the case when the boundary condition reduces to

$$y(0,\lambda) = y^{(2)}(0,\lambda) = y(a,\lambda) = y^{(2)}(a,\lambda) = 0. \tag{1.3}$$

Other boundary conditions can be treated in the same manner.

With obvious notations we shall transform (1.1), (1.3) into

$$\begin{aligned} &Ly := y^{(4)} + \sum_{i=1}^{3} q_i(x) y^{(i-1)} = \lambda y, \quad 0 \leqslant x \leqslant a \\ &y(0,\lambda) = y^{(2)}(0,\lambda) = y(a,\lambda) = y^{(2)}(a,\lambda) = 0 \end{aligned} \tag{1.4}$$

then the eigenvalues of eq. (1.4) are the zeros of the boundary function B defined by

$$B(\lambda) := y(a,\lambda), \tag{1.5}$$

where $y(x,\lambda)$ is the solution of the initial value problem

$$\begin{aligned} &Ly := y^{(4)} + \sum_{i=1}^{3} q_i(x) y^{(i-1)} = \lambda y, \quad 0 \leqslant x \leqslant a \\ &y(0,\lambda) = y^{(2)}(0,\lambda) = 0, \quad y^{(1)}(0,\lambda) = 1, \quad y^{(3)}(0,\lambda) = \alpha \end{aligned} \tag{1.6}$$

the α is chosen so that $y^{(2)}(a,\lambda) = 0$. Thus, we are normalizing the eigenfunctions using $y^{(1)}(0,\lambda) = 1$.

Following [3] we shall use the concepts of iterated integrals [4] and Fliess series, well-known in control theory, to derive in Section 2 formal series representations for $y(x,\lambda)$ and the boundary function $B(\lambda)$. In Section 3, we address the problem of convergence of these series and their approximations by finite sums, the computation of eigenvalues and their error bounds. In Section 4, we work out few examples to illustrate the method.

2. The boundary function

Let $y_1 = y$, $y_2 = \mathrm{d}y/\mathrm{d}x$, $y_3 = \mathrm{d}^2y/\mathrm{d}x^2$, $y_4 = \mathrm{d}^3y/\mathrm{d}x^3$ and introduce $Y = (y_1, y_2, y_3, y_4)'$, $Y^0 = (0,1,0,\alpha)'$. Thus, (1.6) becomes

$$\frac{\mathrm{d}Y}{\mathrm{d}x} = AY + \sum_{i=1}^{3} q_i(x) N_i Y, \qquad Y(0) = Y^0, \tag{2.1}$$

where

$$N_0 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \lambda & 0 & 0 & 0 \end{pmatrix}, \qquad N_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix},$$

$$N_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \qquad N_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix},$$

and $A = N_0$.

We shall see (2.1) as a bilinear system where $q_i(x)$, $i = 1,...,3$ are the inputs and as such we may use the Fliess series representation of Y in term of the $q_i(x)$. Hence,

$$Y = Y^0 + \sum_{k\geqslant 0} \sum_{i_0,...,i_k=0}^{3} N_{i_k}....N_{i_0} Y^0 \int_0^x \mathrm{d}\zeta_{i_k} ... \mathrm{d}\zeta_{i_0} \tag{2.2}$$

and the iterated integrals $\int_0^x \mathrm{d}\zeta_{i_k} ... \mathrm{d}\zeta_{i_0}$ are defined as follows:

$$\begin{cases} \zeta_0(x) = x, \\ \zeta_i(x) = \int_0^x q_i(s)\,\mathrm{d}s, \quad i = 1,...,3, \\ \int_0^x \mathrm{d}\zeta_{i_k} ... \mathrm{d}\zeta_{i_0} = \int_0^x \mathrm{d}\zeta_{i_k}(z) \int_0^z \mathrm{d}\zeta_{i_{k-1}} ... \mathrm{d}\zeta_{i_0}. \end{cases} \tag{2.3}$$

Lemma 2.1. *The solution* $y(x,\lambda)$ *of* (1.6) *and the boundary function* $B(\lambda)$ *associated with the eigenvalue problem* (1.4) *have the following formal expansions:*

$$y(x,\lambda) = C^0 + \sum_{k\geqslant 0} \sum_{i_0,...,i_k=0}^{3} C^k_{i_k ... i_0}(\lambda) \int_0^x \mathrm{d}\zeta_{i_k} ... \mathrm{d}\zeta_{i_0}, \tag{2.4}$$

$$B(\lambda) = D^0 + \sum_{k\geqslant 0} \sum_{i_0,...,i_k=0}^{3} D^k_{i_k ... i_0}(\lambda) \int_0^a \mathrm{d}\zeta_{i_k} ... \mathrm{d}\zeta_{i_0}, \tag{2.5}$$

where

$$\begin{aligned} &C^0 = CY^0, \qquad C = (1,0,0,0), \\ &C^k_{i_k ... i_0}(\lambda) = CN_{i_k}....N_{i_0}Y^0, \qquad i_0,...,i_k = 0,...,3 \end{aligned} \tag{2.6}$$

and

$$\begin{aligned} &D^0 = EY^0, \qquad E = (1,0,0,0), \\ &D^k_{i_k ... i_0}(\lambda) = EN_{i_k}....N_{i_0}Y^0, \qquad i_0,...,i_k = 0,1. \end{aligned} \tag{2.7}$$

Proof. That the series (2.4) is indeed solution to (2.1) is seen from the facts

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}x} \int_0^x \mathrm{d}\zeta_0\, \mathrm{d}\zeta_{i_{k-1}} ... \mathrm{d}\zeta_{i_0} = \int_0^x \mathrm{d}\zeta_{i_{k-1}} ... \mathrm{d}\zeta_{i_0}, \\ &\frac{\mathrm{d}}{\mathrm{d}x} \int_0^x \mathrm{d}\zeta_i\, \mathrm{d}\zeta_{i_{k-1}} ... \mathrm{d}\zeta_{i_0} = q_i(x) \int_0^x \mathrm{d}\zeta_{i_{k-1}} ... \mathrm{d}\zeta_{i_0}, \quad i = 1,...,3 \end{aligned} \tag{2.8}$$

and a differentiation of (2.2), whereas, $B(\lambda) = EY(a,\lambda)$.

Remark 2.2. In the above equations (2.4), (2.6) and (2.5), (2.7) $E = C$ and the D's coincide with the C's with corresponding indices just because the boundary function is given by (1.5) otherwise $E \neq C$. Indeed, if the boundary condition is $y(0) = y^{(2)}(0) = y'(a) = y^{(2)}(a) = 0$ for example, then $E = (0,1,0,0)$ while $C = (1,0,0,0)$.

Remark 2.3. Although the expansion (2.2) originated from the method of successive approximations (Picard-Lindeloef method [5]), the way the terms of the series are arranged giving rise to iterated integrals, yielded the name Fliess series.

3. Eigenvalues approximation

In this section we shall address the problem of convergence of the series in (2.4) and (2.5) and their approximations by finite sums, the problem of computing the eigenvalues based on these finite sum approximations and their error bounds.

We shall be using the following norms, $||A|| = \max_{i=1,\dots,n} \sum_{j=1}^{n} |a_{i,j}|$ and $||x|| = \max_{i=1,\dots,n} |x_i|$ as matrix and vector norms respectively. We claim the following results whose proofs are immediate.

Lemma 3.1. *The coefficients in the series* (2.4) *and* (2.5) *satisfy the growth condition*

$$||C^k_{i_k\dots i_0}(\lambda)||, \quad ||D^k_{i_k\dots i_0}(\lambda)|| \leqslant \max(1,|\lambda|)^{k+1}, \quad k \geqslant 0. \tag{3.1}$$

Lemma 3.2. *If* $\gamma(x) = \sup(\int_0^x |q_1(\zeta)|\,\mathrm{d}\zeta, \dots, \int_0^x |q_3(\zeta)|\,\mathrm{d}\zeta, x) < \infty$ *then* $|\int_0^x \mathrm{d}\zeta_{i_k} \dots \mathrm{d}\zeta_{i_0}| \leqslant \gamma(x)^{k+1}/(k+1)!$, *and the series* (2.4) *and* (2.5) *are absolutely and uniformly convergent on compact sets.*

Proof. The result follows immediately from $|\int_0^x \mathrm{d}\zeta_{i_k} \dots \mathrm{d}\zeta_{i_0}| \leqslant (\gamma(x))^{k+1}/(k+1)!$ which can be proved by induction on k.

Lemma 3.3. *Let* $B_N(\lambda)$ *be defined by*

$$B_N(\lambda) := D^0 + \sum_{k=0}^{N} \sum_{i_0,\dots,i_k=0}^{3} D^k_{i_k\dots i_0}(\lambda) \int_0^a \mathrm{d}\zeta_{i_k} \dots \mathrm{d}\zeta_{i_0} \tag{3.2}$$

then

$$|B(\lambda) - B_N(\lambda)| \leqslant E(|\lambda|, N) \tag{3.3}$$

where

$$E(|\lambda|, N) = e^{3\gamma(a)\max(1,|\lambda|)}(\Gamma(N+1) - \Gamma(N+1, 3\gamma(a)\max(1,|\lambda|)))/\Gamma(N+1) \tag{3.4}$$

and $\Gamma(x,z)$ *is the incomplete gamma function defined by* $\Gamma(x,z) = \int_z^{\infty} \mathrm{e}^{-t} t^{x-1}\,\mathrm{d}t, \Gamma(x)$ *being the gamma function.*

Theorem 3.4. *Let* $q_i \in L^1(0,a), i=1,\dots,3$ *and* σ *be the set of eigenvalues of* (1.4) $\forall \varepsilon > 0, \forall \bar{\lambda} \in \sigma, \exists N(\varepsilon,\bar{\lambda}) > 0, \exists \bar{\lambda}_N$ *such that* $|\bar{\lambda} - \bar{\lambda}_N| \leqslant \varepsilon$ *where* $B(\bar{\lambda}) = 0$ *and* $B_N(\bar{\lambda}_N) = 0$. *Furthermore, we have the following estimate:*

$$|\bar{\lambda} - \bar{\lambda}_N| \leqslant \begin{cases} \dfrac{E(|\bar{\lambda}|,N)}{\inf_{\bar{\lambda}\leqslant\zeta\leqslant\bar{\lambda}_N} |B_N'(\zeta)|} & \text{if the eigenvalue is simple,} \\ \sqrt{\dfrac{2E(|\bar{\lambda}|,N)}{\inf_{\bar{\lambda}\leqslant\zeta\leqslant\bar{\lambda}_N} |B_N''(\zeta)|}} & \text{if the eigenvalue is double.} \end{cases} \tag{3.5}$$

Proof. The above estimate follows immediately from the $B_N(\bar{\lambda}) - B_N(\bar{\lambda}_N) = B_N(\bar{\lambda}) - B(\bar{\lambda})$, the use of the mean value theorem and the fact that $\bar{\lambda}$ and $\bar{\lambda}_N$ are either simple zeros of B and B_N respectively for N large enough or double zeroes. Furthermore, the right-hand side of (3.5) can be made arbitrarily small by choosing N large enough.

Remark 3.5. As in the second order case the error estimate (3.5) makes a sense only if $2\max(\int_0^a |q_1(\zeta)|\,d\zeta,\dots,\int_0^a |q_3(\zeta)|\,d\zeta, a)\max(1,|\lambda|)$ is small. Otherwise N becomes prohibitively large. We shall see in the next section that the actual error is much smaller than the computed one.

4. Numerical examples

As pointed out earlier the only available software to deal with the fourth order Sturm–Liouville problem is SLEUTH developed by Greenberg and Marletta [9], whereas in the second order case many such software exist. We quote SLEIGN (Sturm–Liouville Eigenvalue) [2], SLEDGE (Sturm–Liouville Estimates Determined by Global Error (control)) [8], SL02F [12] and SLEIGN2 [1].

In this section we shall solve two eigenvalue problems using the method introduced.

Example 4.1.

$$Ly \triangleq y^{(4)} - 0.02x^2 y^{(2)} - 0.04xy^{(1)} + (0.0001x^4 - 0.02)y = \lambda y,$$
$$y(0) = y^{(2)}(0) = y(5) = y^{(2)}(5) = 0.$$

Using $N = 40$ we obtain the first eigenvalue at 0.215050864369729 within 3×10^{-3}. The second eigenvalue is found to be 2.75480993468240. The corresponding error obtained from (3.5) is meaningless. It is easy to check that these eigenvalues are the square of the eigenvalues of the following second order Sturm–Liouville problem

$$ly \triangleq -y^{(2)} + 0.01x^2 y = \lambda y, \qquad y(0) = y(5) = 0.$$

The eigenvalues for this second order problem have been computed using the technique introduced in [3] and are 0.46373576999161440254, 1.65976201145918833222 which agree with those obtained using SLEIGN2 (0.463735819 and 1.65976214 to within an error of 0.75×10^{-7} and 0.909×10^{-7}, respectively). The error on the first eigenvalue is $0.60328468492722423180 \times 10^{-9}$, while the error on the second is meaningless. However, the absolute differences between these eigenvalues and the square of the eigenvalues of the corresponding second order system are 0.14×10^{-13} and 0.63×10^{-12}, respectively. Note that since the eigenvalues of the second order problem are positive, the corresponding eigenvalues for the fourth order problem are simple as by product of Theorem 6.1 in [10]. These values agree with the ones obtained using SLEUTH (0.2150508678096625 and 2.754809926369330 to within an error of 7.878973×10^{-11} and 3.19882435×10^{-10} respectively).

Example 4.2. We shall take the same differential equation as in the previous example, with different boundary conditions

$$Ly \triangleq y^{(4)} - 0.02x^2y^{(2)} - 0.04xy^{(1)} + (0.0001x^4 - 0.02)y = \lambda y,$$
$$y(0) = y^{(1)}(0) = y(5) = y^{(1)}(5) = 0.$$

Using $N = 40$ we obtain the first eigenvalue at 0.866902502399502 within 6.9×10^{-3}. The second eigenvalue is found to be 6.35768644815908 while the error obtained from (3.5) is meaningless. Note that here we took $Y_0 = (0,0,1,\alpha)'$ i.e., the eigenvalues are normalized using $y^{(2)}(0,\lambda) = 1$, and solved for α and λ the system

$$y(a,\lambda,\alpha) = 0, \qquad y^{(1)}(a,\lambda,\alpha) = 0,$$

where we have written explicitely a, λ, α in the arguments to show the dependence of y and $y^{(1)}$ on these quantities. These values agree with the ones obtained using SLEUTH (0.866902500921400 and 6.357686441644492 to within an error of 4.228077×10^{-11} and $7.861389796\times10^{-11}$, respectively).

5. Conclusion

In this paper we have been able to extend our previous results on the computation of eigenvalues of second order Sturm–Liouville problems [3] to fourth-order problems. Using the ideas of iterated integrals and Fliess series we derived series expansions for the solution of the initial value problem and the boundary function associated with the regular fourth order Sturm–Liouville problem. We computed the eigenvalues λ and provided error bounds for them. These bounds are meaningful only if $2\max(\int_0^a |q_1(\zeta)|\,d\zeta,\ldots,\int_0^a |q_3(\zeta)|\,d\zeta, a)\max(1,|\lambda|)$ is small. However, the actual error is much smaller than the computed one. We have worked out few examples to illustrate the theory and seen that our results are in agreement with those obtained using SLEUTH.

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