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Nordhaus–Gaddum for treewidth

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ABSTRACT

We prove that, for every n-vertex graph G, the treewidth of G plus the treewidth of the complement of G is at least $n-2$. This bound is tight.

Nordhaus–Gaddum-type theorems establish bounds on $f(G)+f(\overline{G})$ for some graph parameter f, where $\overline{G}$ is the complement of a graph G. The literature has numerous examples; see [3,8,4,6,13,14,11] for a few. Our main result is the following Nordhaus–Gaddum-type theorem for treewidth,[1] which is a graph parameter of particular importance in structural and algorithmic graph theory. Let $\text{tw}(G)$ denote the treewidth of a graph G.

Theorem 1. *For every graph G with n vertices,*

$$\text{tw}(G)+\text{tw}(\overline{G}) \geq n-2.$$

The following lemma is the key to the proof of Theorem 1.

Lemma 2. *For every n-vertex graph G with no induced 4-cycle and no k-clique,*

$$\text{tw}(\overline{G}) \geq n-k.$$

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[1] While treewidth is normally defined in terms of tree decompositions (see [2]), it can also be defined as follows. A graph G is a *k-tree* if $G \cong K_{k+1}$ or $G-v$ is a k-tree for some vertex v whose neighbours induce a k-clique. Then the *treewidth* of a graph G is the minimum integer k such that G is a spanning subgraph of a k-tree. See [1,10] for surveys on treewidth.

Let G be a graph. Two subsets of vertices A and B in G *touch* if $A \cap B \neq \emptyset$, or some edge of G has one endpoint in A and the other endpoint in B. A *bramble* in G is a set of subsets of $V(G)$ that induce connected subgraphs and pairwise touch. A set S of vertices in G is a *hitting set* of a bramble $\mathcal{B}$ if S intersects every element of $\mathcal{B}$. The *order* of $\mathcal{B}$ is the minimum size of a hitting set. Seymour and Thomas [12] proved the Treewidth Duality Theorem, which says that a graph G has treewidth at least k if and only if G contains a bramble of order at least $k+1$.

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Proof. Let $\mathcal{B} := \{\{v, w\} : vw \in E(\overline{G})\}$. If $\{v, w\}$ and $\{x, y\}$ do not touch for some $vw, xy \in E(\overline{G})$, then the four endpoints are distinct and (v, x, w, y) is an induced 4-cycle in G, which is a contradiction. Thus $\mathcal{B}$ is a bramble in $\overline{G}$. Let S be a hitting set for $\mathcal{B}$. Thus no edge in $\overline{G}$ has both endpoints in $V(\overline{G}) \setminus S$. Hence $V(G) \setminus S$ is a clique in G. Therefore $n - |S| \leq k - 1$ and $|S| \geq n - k + 1$. That is, the order of $\mathcal{B}$ is at least $n - k + 1$. By the Treewidth Duality Theorem, $\text{tw}(\overline{G}) \geq n - k$, as desired. □

Proof of Theorem 1. Let $k := \text{tw}(G)$. Let H be a k-tree that contains G as a spanning subgraph. Thus H has no induced 4-cycle (it is chordal) and has no $(k+2)$-clique. By Lemma 2, and since $\overline{G} \supseteq \overline{H}$, we have $\text{tw}(\overline{G}) \geq \text{tw}(\overline{H}) \geq n - k - 2$. Therefore $\text{tw}(G) + \text{tw}(\overline{G}) \geq n - 2$. □

Lemma 2 immediately implies the following result of independent interest.

Theorem 3. *For every n-vertex graph G with girth at least 5,*

$$\text{tw}(\overline{G}) \geq n - 3.$$

We now show that Theorem 1 is tight.

Lemma 4. *Let G be a graph with treewidth k that contains a $(k+1)$-clique C such that each vertex in C has a neighbour outside of C. Then*

$$\text{tw}(G) + \text{tw}(\overline{G}) = n - 2.$$

Proof. We describe an $(n-k-2)$-tree H that contains $\overline{G}$. Let $A := V(G) \setminus C$ be the starting $(n-k-1)$-clique of H. For each vertex $x \in C$, add x to H adjacent to $A \setminus \{y\}$, where y is a neighbour of x outside of C. Observe that H is an $(n-k-2)$-tree and $\overline{G}$ is a spanning subgraph of H. Thus $\text{tw}(\overline{G}) \leq n - k - 2$ and $\text{tw}(G) + \text{tw}(\overline{G}) \leq n - 2$, with equality by Theorem 1. □

For k-trees, we have the following precise result. Let Q_n^k be the k-tree consisting of a k-clique C with $n - k$ vertices adjacent only to C.

Theorem 5. *For every n-vertex k-tree G,*

$$\text{tw}(G) + \text{tw}(\overline{G}) = \begin{cases} n - 1 & \text{if } G \cong Q_n^k \\ n - 2 & \text{otherwise.} \end{cases}$$

Proof. First, suppose that $G \cong Q_n^k$. Then $\overline{G}$ consists of K_{n-k} and k isolated vertices. Thus $\text{tw}(\overline{G}) = n - k - 1$, and $\text{tw}(G) + \text{tw}(\overline{G}) = n - 1$. Now assume that $G \ncong Q_n^k$. By the definition of a k-tree, $V(G)$ can be labelled $v_1, \ldots, v_n$ such that $\{v_1, \ldots, v_{k+1}\}$ is a clique, and, for $j \in \{k+2, \ldots, n\}$, the neighbourhood of v_j in $G[\{v_1, \ldots, v_{j-1}\}]$ is a k-clique C_j. If $C_{k+2}, \ldots, C_n$ are all equal, then $G \cong Q_n^k$. Thus $C_j \neq C_{k+2}$ for some minimum integer j. Observe that each vertex in C_j has a neighbour outside of C_j. The result follows from Lemma 4. □

In view of Theorem 1, it is natural to also consider how large $\text{tw}(G) + \text{tw}(\overline{G})$ can be. Every n-vertex graph G satisfies $\text{tw}(G) \leq n - 1$, implying that $\text{tw}(G) + \text{tw}(\overline{G}) \leq 2n - 2$. It turns out that this trivial upper bound is tight up to lower-order terms. Indeed, Perarnau and Serra [9] proved that, if $G \in \mathcal{G}(n, p)$ is a random n-vertex graph with edge probability $p = \omega(\frac{1}{n})$ in the sense of Erdős and Rényi, then asymptotically almost surely $\text{tw}(G) = n - o(n)$; see [5,7] for related results. Setting $p = \frac{1}{2}$, it follows that, asymptotically almost surely, $\text{tw}(G) = n - o(n)$ and $\text{tw}(\overline{G}) = n - o(n)$, and hence $\text{tw}(G) + \text{tw}(\overline{G}) = 2n - o(n)$. Theorems 1 and 5 can be reinterpreted as follows, where, for graphs G_1 and G_2, the union $G_1 \cup G_2$ is the graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$ (where G_1 and G_2 may intersect).

Proposition 6. *For all graphs G_1 and G_2, the union $G_1 \cup G_2$ contains no clique on $\text{tw}(G_1) + \text{tw}(G_2) + 3$ vertices. This result is sharp, since there exist graphs G_1 and G_2 such that $G_1 \cup G_2$ contains a clique on $\text{tw}(G_1) + \text{tw}(G_2) + 2$ vertices.*

Proof. For the first claim, we may assume that $V(G_1) = V(G_2)$ and $E(G_1) \cap E(G_2) = \emptyset$. Let S be a clique in $G_1 \cup G_2$. Thus $\overline{G_1[S]} = G_2[S]$. By Theorem 1, $\text{tw}(G_1) + \text{tw}(G_2) \geq \text{tw}(G_1[S]) + \text{tw}(G_2[S]) \geq |S| - 2$. Thus $|S| \leq \text{tw}(G_1) + \text{tw}(G_2) + 2$ as desired. The sharpness example follows from Theorem 5. □

Proposition 6 suggests studying $G_1 \cup G_2$ further. For example, what is the maximum of $\chi(G_1 \cup G_2)$ taken over all graphs G_1 and G_2 with $\text{tw}(G_1) \leq k$ and $\text{tw}(G_2) \leq k$? By Proposition 6, the answer is at least $2k + 2$. A minimum-degree greedy algorithm shows that $\chi(G_1 \cup G_2) \leq 4k$. This question is somewhat similar to Ringel's earth–moon problem, which asks for the maximum chromatic number of the union of two planar graphs.

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