Affine images of compact convex sets and maximal measures ✤

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Abstract

Let \( \varphi : X \rightarrow Y \) be an affine continuous mapping of a compact convex set \( X \) onto a compact convex set \( Y \). We show that the induced mapping \( \varphi_* \) need not map maximal measures on \( X \) to maximal measures on \( Y \) even in case \( \varphi \) maps extreme points of \( X \) to extreme points of \( Y \). This disproves Théorème 6 of [S. Teleman, Sur les mesures maximales, C. R. Acad. Sci. Paris Sér. I Math. 318 (6) (1994) 525–528]. We prove the statement of Théorème 6 under an additional assumption that \( \text{ext} \ Y \) is Lindelöf or \( Y \) is a simplex. We also show that under either of these two conditions injectivity of \( \varphi \) on \( \text{ext} \ X \) implies injectivity of \( \varphi_* \) on maximal measures. A couple of examples illustrate the results.

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Résumé

Soit \( \varphi : X \rightarrow Y \) une application affine et continue d’un compact convexe \( X \) sur un compact convexe \( Y \). Nous montrons que l’image d’une mesure maximale par l’application induite \( \varphi_* \) n’est pas nécessairement une mesure maximale, même pas, si les images des points extrémaux sont des points extrémaux. Ceci réfute Théorème 6 dans [S. Teleman, Sur les mesures maximales, C. R. Acad. Sci. Paris Sér. I Math. 318 (6) (1994) 525–528]. Nous prouvons l’énoncé de ce théorème sous l’hypothèse supplémentaire que \( \text{ext} \ Y \) est Lindelöf ou \( Y \) est un simplexe. En plus, nous démontrons que, en supposant l’une ou l’autre de ces deux propriétés, l’injectivité de \( \varphi \) sur \( \text{ext} \ X \) implique l’injectivité de \( \varphi \) pour les mesures maximales. Quelques exemples explicitent les résultats.

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1. Introduction

All topological spaces are considered to be Hausdorff. If $X$ is a compact convex subset of a real locally convex space, we write ext $X$ for the set of extreme points of $X$ and $\mathcal{M}^1_{\text{max}}(X)$ for the set of all maximal probability Radon measures on $X$ (see [1, Chapter I, §3], we also refer the reader to [6, Chapter 6], [10, Sections 1–3], [2, Chapter 1], [15] and [13, Chapter 7]). If $\varphi: X \to Y$ is a continuous mapping of a compact space $X$ to a compact set $Y$, it induces a continuous mapping $\varphi^*: \mathcal{M}^1_{\text{max}}(X) \to \mathcal{M}^1_{\text{max}}(Y)$ from the set of all probability Radon measures on $X$ to the set of all probability Radon measures on $Y$ by the formula $\varphi^*\mu = \mu \circ \varphi^{-1}$ (see [11, Theorem 418I]). The induced mapping $\varphi^*$ is surjective if $\varphi$ is surjective.

For any $\mu \in \mathcal{M}^1_{\text{max}}(X)$ we write $r(\mu)$ for the barycenter of $\mu$ (see [1, Chapter I, §2]). If $x \in X$, we write $M^*_x$ for the set of all measures $\mu \in \mathcal{M}^1_{\text{max}}(X)$ satisfying $r(\mu) = x$. We recall that a set $F \subset X$ is extremal if $x, y \in F$ whenever $x, y \in X$, $\alpha \in (0, 1)$ and $\alpha x + (1 - \alpha)y \in F$. It is a face if $F$ is a convex extremal set. We also mention the well-known fact that ext $F = F \cap \text{ext} X$ for any face $F$.

Let $\varphi: X \to Y$ be a continuous affine mapping of a compact convex set $X$ to a compact convex set $Y$. If $\varphi: X \to Y$ is surjective, it is easy to see that $\varphi(\text{ext} X) \supset \text{ext} Y$ and $\varphi^*(\mathcal{M}^1_{\text{max}}(X)) \supset \mathcal{M}^1_{\text{max}}(Y)$. In order to ensure the reverse inclusion $\varphi^*(\mathcal{M}^1_{\text{max}}(X)) \subset \mathcal{M}^1_{\text{max}}(Y)$, it is necessary to assume that $\varphi(\text{ext} X) \subset \text{ext} Y$. This observation prompts the following two questions.

**Question.** Let $\varphi: X \to Y$ be a continuous affine mapping of a compact convex set $X$ to a compact convex set $Y$.

1. If $\varphi(\text{ext} X) \subset \text{ext} Y$, does it imply that $\varphi^*(\mathcal{M}^1_{\text{max}}(X)) \subset \mathcal{M}^1_{\text{max}}(Y)$?
2. If $\varphi(\text{ext} X) \subset \text{ext} Y$ and $\varphi$ is injective on ext $X$, does it imply that $\varphi^*$ is injective on $\mathcal{M}^1_{\text{max}}(X)$?

If $Y$ is a simplex (see [1, Chapter II, §3]), both questions were answered affirmatively in [8, Corollaries 2 and 3]. For $X$ and $Y$ being simplices, the result can be found in [7, Lemma 6] and [12, Theorem 1]. It is claimed in [18, Théorème 6] without a proof that Question (1) has the affirmative answer without any restrictions. The author also suggests to study Question (2) in [18, Conjecture].

Unfortunately, the answer to Question (1) is in general negative as the following example shows (see also [3, Example 1]).

**Example 1.1.** There exists a continuous affine surjection $\varphi$ of a simplex $X$ onto a compact convex set $Y$ and a measure $\mu \in \mathcal{M}^1_{\text{max}}(X)$ such that

- $\varphi(\text{ext} X) = \text{ext} Y$ and $\varphi$ is injective on ext $X$,
- $\varphi^*\mu \notin \mathcal{M}^1_{\text{max}}(Y)$.

Nevertheless, we prove in Theorem 1.2 that the answer to both questions is positive if we assume that ext $Y$ is a Lindelöf space (see [9, Section 3.8]).
Theorem 1.2. Let \( \varphi : X \to Y \) be a continuous affine map of a compact convex set \( X \) to a compact convex set \( Y \) and let \( \text{ext}Y \) be a Lindelöf space.

(a) Then the following assertions are equivalent:
   (i) \( \varphi(\text{ext}X) \subset \text{ext}Y \),
   (ii) \( \varphi_\sharp(\mathcal{M}^1_{\text{max}}(X)) \subset \mathcal{M}^1_{\text{max}}(Y) \).

(b) Further, the following assertions are equivalent:
   (i) \( \varphi(\text{ext}X) \subset \text{ext}Y \) and \( \varphi \) is injective on \( \text{ext}X \),
   (ii) \( \varphi_\sharp(\mathcal{M}^1_{\text{max}}(X)) \subset \mathcal{M}^1_{\text{max}}(Y) \) and \( \varphi_\sharp \) is injective on \( \mathcal{M}^1_{\text{max}}(X) \).

We also provide in Theorem 1.3(a) a slightly different proof of [8, Corollary 2]. The case of injectivity is described in Theorem 1.3(b), where the proof is based upon the results of E.A. Reznichenko from [16]. We indicate in Remark 2.4 an alternative proof of this assertion that uses a notion of induced measures on the set of extreme points, which is a technique developed by S. Teleman and C.J.K. Batty in [19] and [4].

Theorem 1.3. Let \( \varphi : X \to Y \) be a continuous affine map of a compact convex set \( X \) to a simplex \( Y \).

(a) Then the following assertions are equivalent:
   (i) \( \varphi(\text{ext}X) \subset \text{ext}Y \),
   (ii) \( \varphi_\sharp(\mathcal{M}^1_{\text{max}}(X)) \subset \mathcal{M}^1_{\text{max}}(Y) \),
   (iii) \( \varphi(F) \) is a face for each closed face \( F \subset X \),
   (iv) \( \varphi(F) \) is a closed extremal set for each closed extremal \( F \subset X \).

(b) Further, the following assertions are equivalent:
   (i) \( \varphi(\text{ext}X) \subset \text{ext}Y \) and \( \varphi \) is injective on \( \text{ext}X \),
   (ii) \( \varphi_\sharp(\mathcal{M}^1_{\text{max}}(X)) \subset \mathcal{M}^1_{\text{max}}(Y) \) and \( \varphi_\sharp \) is injective on \( \mathcal{M}^1_{\text{max}}(X) \),
   (iii) \( \varphi \) is a homeomorphism onto \( \varphi(X) \).

The following example shows that Theorem 1.3(b) need not hold if we omit the condition imposed on \( Y \).

Example 1.4. There exists a continuous affine surjection \( \varphi \) of a metrizable simplex \( X \) onto a compact convex set \( Y \) such that
   - \( \varphi \) is injective on \( \text{ext}X \),
   - \( \varphi_\sharp(\mathcal{M}^1_{\text{max}}(X)) \subset \mathcal{M}^1_{\text{max}}(Y) \) and \( \varphi_\sharp \) is injective on \( \mathcal{M}^1_{\text{max}}(X) \),
   - \( \varphi \) is not injective on \( X \).

Our last example shows that even if \( \varphi_\sharp \) maps maximal measures to maximal measures and \( \varphi \) is injective on \( \text{ext}X \), the induced mapping need not be injective on \( \mathcal{M}^1_{\text{max}}(X) \).

Example 1.5. There exists a continuous affine surjection \( \varphi \) of a simplex \( X \) onto a compact convex set \( Y \) such that
   - \( \varphi \) is injective on \( \text{ext}X \),
   - \( \varphi_\sharp(\mathcal{M}^1_{\text{max}}(X)) \subset \mathcal{M}^1_{\text{max}}(Y) \),
   - \( \varphi_\sharp \) is not injective on \( \mathcal{M}^1_{\text{max}}(X) \).
2. Proofs of the positive results

If \( f : X \to \mathbb{R} \) is a function on a compact convex set \( X \), we recall the definition from [1, p. 4] of the upper envelope \( f^* \) of \( f \) defined as

\[
f^*(x) = \inf \{ h(x) : h \geq f, \ h \text{ continuous affine on } X \}, \quad x \in X.
\]

Before embarking on the proof of the main theorems, we need a couple of auxiliary results.

**Proposition 2.1.** Let \( f, g \), be upper semicontinuous real functions on \( X \) such that \( f \) is concave, \( g \) is convex and \( f \geq g \) on \( \text{ext} X \). Then \( f \geq g \) on \( X \).

**Proof.** Given \( f \) and \( g \) as in the premise, let \( x \) be a point of \( X \). We fix \( \varepsilon > 0 \) and use [1, Corollary I.1.3] to find a concave continuous function \( f' \) such that \( f' \geq f \) and \( f(x) \geq f'(x) - \varepsilon \).

Then \( f' - g \) is a lower semicontinuous concave function on \( X \) such that \( f' - g \geq 0 \) on \( X \). According to Bauer’s minimum principle [1, Theorem I.5.3], \( f' - g \geq 0 \) on \( X \). Thus

\[
g(x) \leq f'(x) \leq f(x) + \varepsilon.
\]

Since \( \varepsilon > 0 \) is arbitrary, we are done. \( \Box \)

**Proposition 2.2.** Let \( \text{ext} X \) be Lindelöf and \( \mu \in M^1(X) \). Then the following assertions are equivalent:

(i) \( \mu \in M^1_{\text{max}}(X) \),

(ii) \( \mu_*(X \setminus \text{ext} X) = 0 \) (here \( \mu_* \) stands for the inner measure induced by \( \mu \)).

**Proof.** Let \( \mu \in M^1_{\text{max}}(X) \) be given and \( F \subset X \setminus \text{ext} X \) be an arbitrary closed set. For any point \( x \in \text{ext} X \) we can find a cozero set \( U_x \) such that \( x \in U_x \subset X \setminus F \). (We recall that a subset of a normal space is cozero if and only if it is an open \( F_\sigma \) set, see [9, p. 42].) By the Lindelöf property of \( \text{ext} X \), there exists a cozero set \( U \) such that

\[
\text{ext} X \subset U \subset X \setminus F.
\]

According to [6, Theorem 27.11], \( \mu(U) = 1 \) and hence \( \mu(F) = 0 \). Thus \( \mu_*(X \setminus \text{ext} X) = 0 \) and (i) \( \Rightarrow \) (ii).

For the proof of (ii) \( \Rightarrow \) (i), let \( \mu \) satisfy (ii). For any continuous function \( f \) on \( X \), [1, p. 32] yields

\[
\text{ext} X \subset \{ x \in X : f^*(x) = f(x) \}.
\]

Hence \( \mu(\{ x \in X : f^*(x) = f(x) \}) = 1 \) and \( \mu(f^*) = \mu(f) \). By [1, Proposition I.4.5], \( \mu \in M^1_{\text{max}}(X) \). \( \Box \)

**Proof of Theorem 1.2.** For the proof of (a) we first notice that the implications (ii) \( \Rightarrow \) (i) and (ii') \( \Rightarrow \) (i') are obvious. We start the proof of the converse implications by showing (i) \( \Rightarrow \) (ii). To this end, let \( \mu \in M^1_{\text{max}}(X) \) be given. We fix an arbitrary closed set \( F \subset Y \setminus \text{ext} Y \). Since \( \text{ext} Y \) is Lindelöf, there exists a countable family of cozero sets \( \{ U_n : n \in \mathbb{N} \} \) in \( Y \) such that

\[
\text{ext} Y \subset \bigcup_{n=1}^{\infty} U_n \subset Y \setminus F.
\]
Then $G = \varphi^{-1}(\bigcup_{n=1}^{\infty} U_n)$ is an $F_\sigma$ set. By the assumptions, $\text{ext } X \subset G$ and hence $\mu(G) = 1$. Thus
\[
(\varphi\sharp\mu)\left(\bigcup_{n=1}^{\infty} U_n\right) = \mu(G) = 1,
\]
and hence $\mu(F) = 0$.
Thus $(\varphi\sharp\mu)_* (Y \setminus \text{ext } Y) = 0$, and $\varphi\sharp\mu \in \mathcal{M}_{\text{max}}^1(Y)$ by virtue of Proposition 2.2.

We proceed with the proof of (i) $\Rightarrow$ (ii). We start by proving
\[
\varphi(X \setminus \text{ext } X) \subset Y \setminus \text{ext } Y.
\]
Indeed, given $y \in \text{ext } Y \cap \varphi(X)$, the set $\varphi^{-1}(y)$ is a closed face. Since $\varphi^{-1}(y) = \text{co}(\text{ext } \varphi^{-1}(y)) = \text{co}(\varphi^{-1}(y) \cap \text{ext } X)$, the assumption yields that $\varphi^{-1}(y)$ is a singleton. Hence (1) follows.

Let $\mu \in \mathcal{M}_{\text{max}}^1(X)$ be given. For any set $F \subset X \setminus \text{ext } X$, inclusion (1) gives
\[
\varphi(F) \subset Y \setminus \text{ext } Y.
\]
This along with Proposition 2.2 and the first part of the proof yields
\[
(\varphi\sharp\mu)(\varphi(F)) = 0, \quad F \subset X \setminus \text{ext } X \text{ closed}.
\]
Hence
\[
\mu(F) \leq \mu(\varphi^{-1}(\varphi(F))) = (\varphi\sharp\mu)(\varphi(F)) = 0, \quad F \subset X \setminus \text{ext } X \text{ closed},
\]
and thus
\[
\mu(\varphi^{-1}(\varphi(F))) = \mu(F), \quad F \subset X \text{ closed}.
\]
If $\mu, \nu \in \mathcal{M}_{\text{max}}^1(X)$ are measures with $\varphi\sharp\mu = \varphi\sharp\nu$, then (2) yields
\[
\mu(F) = \mu(\varphi^{-1}(\varphi(F))) = (\varphi\sharp\mu)(\varphi(F)) = (\varphi\sharp\nu)(\varphi(F)) = \nu(\varphi^{-1}(\varphi(F))) = \nu(F)
\]
for any closed $F \subset X$. Hence $\mu = \nu$ and $\varphi\sharp$ is injective on $\mathcal{M}_{\text{max}}^1(X)$. \hfill \Box

**Remark 2.3.** It can be easily verified that the mapping $\varphi : X \to Y$ is a homeomorphism of $\text{ext } X$ onto $\varphi(\text{ext } X)$ if $\varphi(\text{ext } X) \subset \text{ext } Y$ and $\varphi$ is injective on $\text{ext } X$.

Indeed, since
\[
\varphi(\text{ext } X) \subset \text{ext } Y \quad \text{and} \quad \varphi(X \setminus \text{ext } X) \subset Y \setminus \text{ext } Y,
\]
it is not difficult to realize that $\varphi(F \cap \text{ext } X) = \varphi(F) \cap \text{ext } Y$ for any $F \subset X$. Hence $\varphi : \text{ext } X \to \text{ext } Y$ is a closed mapping, and thus a homeomorphism on $\text{ext } X$.

Hence we obtain that $\text{ext } X$ is a Lindelöf space if $\text{ext } Y$ is Lindelöf and $\varphi$ as above.

**Proof of Theorem 1.3.** For the proof of (a), we first verify (i) $\Rightarrow$ (ii). To this end, let $\mu$ be a maximal probability measure on $X$. To show that $\varphi\sharp\mu$ is maximal on $Y$, we use Mokobodzki's maximality test [1, Proposition I.4.5].

Let $g$ be a convex continuous function on $Y$. Since $Y$ is a simplex, $g^*$ is an affine function (see [1, Theorem II.3.7]). By the assumption and [1, Proposition I.4.1],
\[
g^* \circ \varphi = (g \circ \varphi)^* \quad \text{on } \text{ext } X.
\]
By Proposition 2.1, \( g^* \circ \varphi \leq (g \circ \varphi)^* \) on \( X \).

On the other hand, given \( x \in X \), there exists a measure \( \lambda \in \mathcal{M}_X \) such that \( \lambda(g \circ \varphi) = (g \circ \varphi)^*(x) \) (see [1, Proposition I.3.5]). Then \( \varphi_\# \lambda \in \mathcal{M}_{\varphi(x)} \) and
\[
(g \circ \varphi)^*(x) = \lambda(g \circ \varphi) = (\varphi_\# \lambda)(g) \leq g^*(\varphi(x)).
\]
Hence \( g^* \circ \varphi = (g \circ \varphi)^* \) on \( X \).

Thus we may use [16, Proposition 1.6] to get that

\[
\varphi(F) \text{ is supported by } \nu = \varphi_\# \mu \text{ as } \varphi \text{ is injective on } \text{ext} X.
\]

By Proposition 2.1, \( g^* \circ \varphi \leq (g \circ \varphi)^* \) on \( X \).

On the other hand, given \( x \in X \), there exists a measure \( \lambda \in \mathcal{M}_X \) such that \( \lambda(g \circ \varphi) = (g \circ \varphi)^*(x) \) (see [1, Proposition I.3.5]). Then \( \varphi_\# \lambda \in \mathcal{M}_{\varphi(x)} \) and
\[
(g \circ \varphi)^*(x) = \lambda(g \circ \varphi) = (\varphi_\# \lambda)(g) \leq g^*(\varphi(x)).
\]
Hence \( g^* \circ \varphi = (g \circ \varphi)^* \) on \( X \).

Thus we may use [16, Proposition 1.6] to get that

\[
\varphi(F) \text{ is supported by } \nu = \varphi_\# \mu \text{ as } \varphi \text{ is injective on } \text{ext} X.
\]

Let now an arbitrary \( \nu' \in \mathcal{M}^1(Y) \) satisfy \( r(\nu') \in \varphi(F) \). We find a maximal measure \( \mu \in \mathcal{M}^1(\text{ext} \lambda) \) such that \( r(\mu) = \nu \). Since \( F \) is a closed face, \( \mu \in \mathcal{M}^1(\text{ext} \lambda) \). Then \( \varphi_\# \mu \) is supported by \( \varphi(F) \) and by the assumption, \( \varphi_\# \mu \) is maximal. Since
\[
r(\varphi_\# \mu) = \varphi(r(\mu)) = r(\nu')
\]
and \( Y \) is a simplex, \( \varphi_\# \mu = \nu \). Hence \( \nu \in \mathcal{M}^1(\varphi(F)) \).

Let now an arbitrary \( \nu' \in \mathcal{M}^1(Y) \) satisfy \( r(\nu') \in \varphi(F) \). We find a maximal measure \( \nu \in \mathcal{M}^1(Y) \) such that \( \nu' \preceq \nu \) (here \( \preceq \) is the Choquet ordering, see [1, Chapter I, §3] and [1, Lemma I.4.7]). Since \( r(\nu) = r(\nu') \), \( \nu \) is supported by \( \varphi(F) \) according to the paragraph above. Since it is easy to see that \( \text{spt} \nu' \subset \text{clos} \text{spt} \nu \), the measure \( \nu' \) is supported by \( \varphi(F) \) as well. Thus \( \varphi(F) \) is a face as needed.

Since a closed set is extremal if and only if it is a union of closed faces (see [14, §4, Theorem 7]), we get (iii) \( \Rightarrow \) (iv). We proceed to the proof of (iv) \( \Rightarrow \) (i). But this is immediate, because a set \( \{x\} \) is extremal if and only if \( x \in \text{ext} X \). This concludes the proof of (a).

We start the proof of (b) by showing (i') \( \Rightarrow \) (iii'). We know from the part (a) that \( \varphi(X) \) is a face of \( Y \) and hence a simplex. Since \( \text{ext} \varphi(X) = \varphi(X) \cap \text{ext} X \), we may assume from now on that \( \varphi \) is a surjective mapping onto a simplex \( Y \).

Thus we may use [16, Proposition 1.6] to get that \( \varphi \) is a simplicial map, that is, the function
\[
\tilde{a}(y) = \inf\{a(x) : x \in \varphi^{-1}(y)\}, \quad y \in Y,
\]
is affine on \( Y \) for any continuous affine function \( a \) on \( X \) (see [16, Definition 1.3]). Since \( \varphi \) is injective on \( \text{ext} X \), [16, Theorem 1.5] yields that \( \varphi \) is a homeomorphism.

Since the remaining implications are obvious, the proof is finished. \( \square \)

Remark 2.4. We remark that Theorem 1.3(b) can be deduced from results of S. Teleman and C.J.K. Batty on maximal measures.

For the proof of (i') \( \Rightarrow \) (ii') we realize that \( F = \varphi^{-1}(\varphi(F)) \) for any closed face \( F \subset X \) and hence also for any closed extremal set \( F \subset X \). It is shown in [4, Section 6] or in [19, Theorem 5.2] and [20, Theorem 6] that
\[
\mu(B) = \sup\{\mu(F) : F \subset B \text{ is closed extremal}\}, \quad B \subset X \text{ Baire},
\]
for any measure \( \mu \in \mathcal{M}^1_{\text{max}}(X) \). From this fact we get that \( \varphi_\# \mu \) is injective on \( \mathcal{M}^1_{\text{max}}(X) \).

To verify (ii') \( \Rightarrow \) (iii'), it is enough to check injectivity of \( \varphi \) on \( X \). Let \( x_1, x_2 \in X \) satisfy \( y = \varphi(x_1) = \varphi(x_2) \). For \( i = 1, 2 \), we find a maximal measure \( \mu_i \in \mathcal{M}_{\chi_i} \). Then the measure
\[\varphi_x \mu_i \in \mathcal{M}_Y, i = 1, 2,\] and thus \(\varphi_x \mu_1 = \varphi_x \mu_2\) (we remind that \(Y\) is a simplex). By the assumption, \(\mu_1 = \mu_2\) and thus \(x_1 = x_2\).

Obviously, \((iii') \Rightarrow (i')\) which finishes this remark.

3. Construction of examples

All the constructions are based upon the notion of a function space \(\mathcal{H}\), which is a subspace of the space \(C(K)\) of all continuous functions on a compact space \(K\) such that \(\mathcal{H}\) contains constant functions and separates points of \(K\). Then the state space

\[X = \{\xi \in \mathcal{H}^* : \xi \geq 0, \; \xi(1) = 1\}\]

endowed with the weak* topology is a convex compact set that inherits many properties from \(\mathcal{H}\).

The mapping \(\phi : K \rightarrow X\), where \(\phi(x)\) is the evaluation mapping at a point \(x \in K\), is a homeomorphic embedding. (We refer the reader to \([15, \text{Chapter 6}], [6, \text{Chapter 6, §29}]\) and \([17]\) for a detailed information on the issue.)

Construction of Example 1.1. Let \(K_1 = [0, 1] \times \{-1, 0, 1\}\) with the “porcupine” topology (see \([5, \text{Section VII}]\) or \([1, \text{Proposition I.4.15}]\)) and let \(K_2\) be the quotient of \(K_1\) where all points of \([0, 1] \times \{0\}\) are identified with the point \((0, 0)\) (see \([9, \text{Section 2.4}]\)). We write \(q : K_1 \rightarrow K_2\) for the quotient mapping and take

\[\mathcal{H}_1 = \{f \in C(K_1) : 2f((t, 0)) = f((t, -1)) + f((t, 1)), \; t \in [0, 1]\}\quad \text{and} \quad \mathcal{H}_2 = \{f \in C(K_2) : 2f((0, 0)) = f((t, -1)) + f((t, 1)), \; t \in [0, 1]\}.

Let \(X, Y\) be the state space of \(\mathcal{H}_1, \mathcal{H}_2\), respectively, and \(\phi_1, \phi_2\) be the respective embeddings. Then \(\text{ext} X = \phi_1(K_1 \setminus ([0, 1] \times \{0\}))\) and \(\text{ext} Y = \phi_2(K_2 \setminus \{0, 0\})\). We denote by \(\varphi : X \rightarrow Y\) the restriction of the adjoint operator \(h \mapsto h \circ q, h \in \mathcal{H}_2\). Then \(X\) is a simplex and \(\varphi_{x_1} \lambda \in \mathcal{M}_1(X)\) is maximal for any continuous measure \(\lambda \in \mathcal{M}_1([0, 1] \times \{0\})\), even though \(\varphi_{x_1} \lambda\) is supported by a compact set disjoint with \(\text{ext} X\) (see \([1, \text{Chapter I, §4, p. 42}]\)). (We recall that \(\lambda\) is continuous if \(\lambda(\{x\}) = 0\) for each \(x \in X\).)

Then \(\varphi(\text{ext} X) = \text{ext} Y\) and \(\varphi\) is even injective on \(\text{ext} X\). On the other hand, if \(\lambda\) is any continuous probability measure on \(\phi_1([0, 1] \times \{0\})\), then \(\lambda\) is maximal on \(X\), yet the measure \(\varphi_{x_1} \lambda\) equals the Dirac measure at the point \(\phi_2((0, 0))\), and hence \(\varphi_{x_1} \lambda\) is not maximal. \(\square\)

Construction of Example 1.4. Let \(K_1 = \{x_1, x_2, x_3, y_1, y_2, y_3\}\) and \(K_2\) be the quotient of \(K_1\), if we identify \(y_2\) with \(x_2\). Again we denote by \(q : K_1 \rightarrow K_2\) the quotient mapping. Let

\[\mathcal{H}_1 = \{f \in C(K_1) : 2f(x_2) = f(x_1) + f(x_3), \; 2f(y_2) = f(y_1) + f(y_3)\}\quad \text{and} \quad \mathcal{H}_2 = \{f \in C(K_2) : f(x_1) + f(x_3) = 2f(x_2) = f(y_1) + f(y_3)\}.

We take \(X, Y, \phi_1, \phi_2\) and \(\varphi : X \rightarrow Y\) as above. Then \(X\) is a simplex, \(\text{ext} X = \phi_1(K_1 \setminus \{x_2, y_2\})\), \(\text{ext} Y = \phi_2(K_2 \setminus \{x_2\})\), \(\psi : \text{ext} X \rightarrow \text{ext} Y\) is a bijection, yet \(\varphi\) is not injective on \(X\). Obviously, \(\varphi_x\) maps injectively maximal measures to maximal measures. \(\square\)

Construction of Example 1.5. Let \(K_1 = [0, 1] \cup [2, 3] \times \{-1, 0, 1\}\) endowed again with the “porcupine” topology and let \(K_2\) be the quotient of \(K_1\) after identifying points \((t + 2, 0)\) with \((t, 0), t \in [0, 1]\). Let
\[ H_1 = \{ f \in C(K_1): \ 2 f((t + i, 0)) = f((t + i, -1)) + f((t + i, 1)), \ t \in [0, 1], \ i = 0, 2 \} , \]
\[ H_2 = \{ f \in C(K_2): \ 2 f((t, 0)) = f((t + i, -1)) + f((t + i, 1)), \ t \in [0, 1], \ i = 0, 2 \} , \]
and let \( X, Y, \phi_1, \phi_2 \) and \( \varphi \) be as above.

Then
\[ \text{ext } X = \phi_1(K_1 \setminus ([0, 1] \cup [2, 3] \times \{0\})), \quad \text{ext } Y = \phi_2(K_2 \setminus ([0, 1] \times \{0\})), \]
and \( \varphi \) maps injectively \( \text{ext } X \) onto \( \text{ext } Y \).

We claim that \( \varphi_2(\mathcal{M}^{1}_{\text{max}}(X)) \subset \mathcal{M}^{1}_{\text{max}}(Y) \). Indeed, a probability measure \( \lambda \) is maximal on \( X \) if and only if \( \lambda = (\phi_1)_{*} \mu \) for some measure \( \mu \in \mathcal{M}^{1}(K_1) \) that is continuous on \([0, 1] \cup [2, 3] \times \{0\} \). Similarly, any maximal probability measure on \( Y \) is of the form \( (\phi_2)_{*} \mu \) for some measure \( \mu \in \mathcal{M}^{1}(K_2) \) that is continuous on \([0, 1] \times \{0\} \). From these observations the claim follows.

Finally, if we take the Lebesgue measure \( \lambda_1 \) on \([0, 1] \times \{0\} \) and \( \lambda_2 \) on \([2, 3] \times \{0\} \), then
\[ \varphi_2((\phi_1)_{*} \lambda_1) = \varphi_2((\phi_1)_{*} \lambda_2). \]
Hence \( \varphi_2 \) is not injective on \( \mathcal{M}^{1}_{\text{max}}(X) \). \( \square \)

References