Positive Solutions of Second Order Nonlinear Differential Equations*

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Necessary and sufficient conditions are obtained for the existence of positive solutions of a nonlinear differential equation. Relations between this equation and an advanced type nonlinear differential equation are also discussed.

1. INTRODUCTION

This paper is concerned with a class of nonlinear differential equations of the form

\[ \left[ r(t)(y'(t))^{\sigma} \right]' + q(t)(y(t))^\sigma = 0, \quad t \geq t_0, \quad (1) \]

where \( \sigma \) is a quotient of positive odd integers, \( q: [t_0, \infty) \to [0, \infty) \) is a continuous function such that \( q(t) \neq 0 \), and \( r: [t_0, \infty) \to (0, \infty) \) is a continuous function such that

\[ \int_{t_0}^{\infty} \frac{ds}{r(s)^{1/\sigma}} = \infty. \quad (2) \]

By a solution of (1) is meant a function \( y \in C^1([t_0, \infty), t_0 \geq t_0 \), which has the property \( r(t)(y'(t))^{\sigma} \in C^1([t_0, \infty) \) and satisfies the equation for all \( t \geq t_0 \). The solutions vanishing in some neighborhood of infinity will be excluded from our consideration. A solution of (1) is said to be oscillatory

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if it has an infinite sequence of zeros clustering at \( t = \infty \); otherwise it is said to be nonoscillatory. Thus a nonoscillatory solution is eventually positive or negative.

When \( r(t) = 1 \), the oscillatory behavior of (1) has been studied by several authors; the reader is referred to the papers [2–4, 6, 10] for oscillation results. For the case \( r(t) \neq 1 \), a more general equation than (1) has been investigated by Li [8] and Wong and Agarwal [11–13]. In these papers, some sufficient conditions are obtained for oscillation of all solutions as well as for existence of positive monotone solutions. It is noted that the discrete analogs of (1) as well as related equations have been discussed in [1, 9, 14].

Recently, Kusano and Yoshida [6] considered (1) and obtained some results for nonoscillation of all solutions of (1) [6, Theorem 5]. Motivated by the results in [1, 9], here we are interested in the existence of a positive solution. Similar questions related to the equation of the form

\[
[r(t)(y'(t))^\sigma] + q(t)(y(t + \tau))^\sigma = 0, \quad t \geq t_0, \tag{3}
\]

are also considered in Section 3. These studies show that there are some subtle relations and differences between these two equations, as we will see below.

We base our investigation on a Riccati-type transformation. Monotone methods are then used to derive necessary and sufficient conditions for the existence of a positive solution of (1). Two useful implications are then derived as applications. In the final section, we will consider the relations of the two equations.

2. EXISTENCE CRITERIA

We begin by assuming that \( y(t) \) is a positive solution of (1). Then we see from (1) that

\[
[r(t)(y'(t))^\sigma]' = -q(t)(y(t))^\sigma \leq 0,
\]

for \( t \geq t_0 \). Furthermore, since \( q(t) \geq 0 \) and \( q(t) \neq 0 \), the nonincreasing function \( r(t)(y'(t))^\sigma \) is either eventually positive or negative. If the latter holds, then

\[
r(t)(y'(t))^\sigma \leq c < 0
\]
for $t$ greater than or equal to, say $T$. But then

$$y'(t) \leq \frac{c^{1/\sigma}}{r(t)^{1/\sigma}}, \quad t \geq T,$$

so that

$$y(t) - y(T) \leq c^{1/\sigma} \int_T^t \frac{ds}{r(s)^{1/\sigma}} \to -\infty,$$

which is a contradiction. We have thus shown that if $y(t)$ is a positive solution of (1), then $r(t)(y'(t))^\sigma$ is a positive nonincreasing function, and $y'(t)$ is a positive function.

Let the function $w(t)$ be defined by

$$w(t) = \frac{r(t)(y'(t))^\sigma}{y(t)^\sigma}, \quad t \geq t_0.$$  \hfill (4)

Then by means of what we have just shown, $w(t) > 0$ and $w'(t) \leq 0$ for $t \geq t_0$. Furthermore, since

$$\frac{y'(t)}{y(t)} = \left( \frac{w(t)}{r(t)} \right)^{1/\sigma},$$

we see from (1) that

$$w'(t) = -q(t) - w(t) \frac{\sigma y'(t)}{y(t)} = -q(t) - w(t) \sigma \left( \frac{w(t)}{r(t)} \right)^{1/\sigma},$$

or

$$w'(t) + w(t) \sigma \left( \frac{w(t)}{r(t)} \right)^{1/\sigma} + q(t) = 0, \quad t \geq t_0,$$  \hfill (5)

For the sake of convenience, we will write

$$F(x, y, z) = z \left( \frac{x}{y} \right)^{1/z}.$$  

Then we can also write (5) in the simpler form

$$w'(t) + w(t) F(w(t), r(t), \sigma) + q(t) = 0, \quad t \geq t_0.$$
Note that $F(x, y, z) > 0$ for $x, y, z > 0$. Furthermore,

$$F_x(x, y, z) = y^{-1/z}x^{1/z-1},$$

which is positive for $x, y, z > 0$; and

$$F_y(x, y, z) = -x^{1/z}y^{-1/z-1},$$

which is negative for $x, y, z > 0$. These properties of $F(w, r, \sigma)$ will be referred to later as the monotone nature of $F$.

**Theorem 1.** Equation (1) has a positive solution $y(t)$ for $t \geq t_0$ if, and only if, there is a positive and continuous function $u(t)$ for $t \geq t_0$ which satisfies the integral inequality

$$\int_t^\infty u(s)F(u(s), r(s), \sigma)ds + \int_t^\infty q(s)ds \leq u(t), \quad t \geq t_0. \quad (6)$$

**Proof.** If $y(t)$ is a positive solution of (1), then the function $w(t)$ defined by (4) is a positive solution of the inequality

$$\int_t^\infty w(s)F(w(s), r(s), \sigma)ds + \int_t^\infty q(s)ds \leq w(t), \quad t \geq t_0. \quad (7)$$

obtained by integrating (5) from $t$ to $\infty$. Conversely, let $u(t)$ be a positive and continuous function which satisfies (6). Let us define a mapping $T: C([t_0, \infty), (0, \infty)) \rightarrow C([t_0, \infty), (0, \infty))$ as

$$(Tv)(t) = \int_t^\infty v(s)F(v(s), r(s), \sigma)ds + \int_t^\infty q(s)ds, \quad t \geq t_0,$$

where $v(t) \in C([t_0, \infty), (0, \infty))$. Note that in view of (6), $(Tv)(t) \leq v(t)$ for $t \geq t_0$. Consider the successive approximating sequences $v_0 = v_0(t), v_1 = v_1(t), \ldots$, defined by

$$v_0(t) = 0, \quad (8)$$

and

$$v_{n+1}(t) = (Tv_n)(t), \quad t \geq t_0, n = 0, 1, \ldots. \quad (9)$$

By means of the monotone properties of $F(w, r, \sigma)$, it is not difficult to see that

$$v_0(t) \leq v_1(t) \leq v_2(t) \leq \cdots \leq v_n(t) \leq \cdots \leq u(t)$$
for $n \geq 0$ and $t \geq t_0$. Thus, by letting $v^*(t)$ be the positive function defined by

$$v^*(t) = \lim_{n \to \infty} v_n(t), \quad t \geq t_0,$$

we may then take limits on both sides of (9) and infer from Lebesgue's dominated convergence theorem that $v^* = Tv^*$.

In view of (4), the function $y(t)(t \geq t_0)$ defined by $y(t_0) = c_0 > 0$ and

$$y(t) = y(t_0) \exp \left( \int_{t_0}^t \left( \frac{v^*(s)}{r(s)} \right)^{1/\sigma} ds \right), \quad t \geq t_0$$

is a positive solution of (1). The proof is complete.

We remark that by slightly modifying the arguments used in the proof of the above theorem, we see that the following variant holds.

**Theorem 2.** Equation (1) has a positive solution if, and only if, the sequence $\{v_n\}$ defined by (8) and (9) is well defined and pointwise convergent.

The approximating sequence defined by (8) and (9) is not the only one that is available. Indeed, let us introduce another formal sequence of sequences $\{\phi_n\}$ defined as follows: First we define a mapping $S: C([t_0, \infty), (0, \infty)) \to C([t_0, \infty), (0, \infty))$ by

$$(Su)(t) = \int_{t_0}^t u(s)F(u(s), r(s), \sigma)ds, \quad t \geq t_0,$$

where $u \in C([t_0, \infty), (0, \infty))$. Then we define

$$\phi_0(t) = \int_{t_0}^t q(s)ds, \quad t \geq t_0,$$

$$\phi_1(t) = (Su)(t), \quad t \geq t_0,$$

and for $n = 1, 2, \ldots$,

$$\phi_{n+1}(t) = (Su)(t), \quad t \geq t_0.$$  

If the sequence $\{\phi_n\}$ is well defined, then by means of the monotone properties of $F(u, r, \sigma)$, $\phi_2(t) > 0$ for $t \geq t_0$. Furthermore,

$$\phi_2(t) = (Su)(t) = (S\phi_0)(t), \quad t \geq t_0,$$

and

$$\phi_3(t) = (S\phi_2)(t) \geq (S\phi_1)(t) = \phi_2(t), \quad t \geq t_0.$$
Inductively, we see that

\[ 0 < \phi_1(t) \leq \phi_2(t) \leq \cdots, \quad t \geq t_0. \]

Therefore, if we assume in addition that \((\phi_n)\) is pointwise convergent to \(\phi\), then by Lebesgue’s monotone convergence theorem, we see from (13) that

\[
\phi_0(t) + \phi(t) = \phi_0(t) + \int_t^\infty (\phi_0(s) + \phi(s))F(\phi_0(s) + \phi(s), r(s), \sigma)ds,
\]

\[ t \geq t_0. \]

In other words, we have found a positive function \(\phi_0 + \phi\) which satisfies (6). Conversely, if we assume that \(y(t)\) is a positive function which satisfies (6), i.e.,

\[ (Sy)(t) + \phi_0(t) \leq y(t), \quad t \geq t_0, \]

then

\[
\phi_0(t) \leq (Sy)(t) + \phi_0(t) \leq y(t), \quad t \geq t_0,
\]

\[
\phi_0(t) + \phi_1(t) = \phi_0(t) + (S\phi_0)(t)
\leq \phi_0(t) + (Sy)(t) \leq y(t), \quad t \geq t_0,
\]

and

\[
\phi_0 + \phi_{n+1}(t) = \phi_0(t) + (S(\phi_0 + \phi_n))(t)
\leq \phi_0(t) + (Sy)(t) \leq y(t), \quad t \geq t_0.
\]

In other words the sequence \(\{\phi_n\}\) is well defined. Therefore,

\[ 0 < \phi_0(t) \leq \phi_2(t) \leq \cdots \leq y(t), \quad t \geq t_0, \]

which implies that \(\{\phi_n\}\) is also pointwise convergent.

**Theorem 3.** Equation (1) has a positive solution if, and only if, the sequence \(\{\phi_n\}\) defined by (11), (12), and (13) is well defined and pointwise convergent.

We now deduce two important implications from these existence criteria. First of all, if \(y(t)\) is a positive function which satisfies (6), then by means of the monotone properties of the function \(F\), it also satisfies

\[
\int_t^\infty u(s)F(u(s), r(s), \sigma)ds + \int_t^\infty Q(s)ds \leq u(t), \quad t \geq t_0,
\]
where $Q: [t_0, \infty) \to [0, \infty)$ is a continuous function which satisfies
\[
\int_t^\infty Q(s) \, ds \leq \int_t^\infty q(s) \, ds, \quad t \geq t_0
\]  
and $R: [t_0, \infty) \to [0, \infty)$ is a continuous function which satisfies $0 < r(t) \leq R(t)$ for $t \geq t_0$. The following Hille–Wintner type comparison theorem is now clear from Theorem 1.

**Theorem 4.** Assume that $R(t)$ is a positive and continuous function which satisfies $0 < r(t) \leq R(t)$ for $t \geq t_0$, and
\[
\int_{t_0}^\infty \frac{ds}{R(s)^{1/\sigma}} = \infty,
\]
and that $Q(t)$ is a nonnegative and continuous function which satisfies (14). If Eq. (1) has a positive solution, then so is the equation
\[
\left[ R(t)\left( y'(t) \right)^{\sigma} \right] + Q(t)\left( y(t) \right)^{\sigma} = 0, \quad t \geq t_0.
\]

We remark that Theorem 4 is an extension of Theorem 2 of Kusano and Yoshida [6].

Next, let us assume that $\sigma = 1$ and
\[
\int_t^\infty \frac{1}{r(s)} \left( \int_t^\infty q(u) \, du \right)^2 \, ds \geq \frac{1 + \delta}{4} \int_t^\infty q(s) \, ds, \quad t \geq t_0,
\]  
where $\delta$ is an arbitrary positive number. In view of the function $\phi_0$ defined by (11), (15) is equivalent to saying that
\[
\int_t^\infty \frac{\left( \phi_0(s) \right)^2}{r(s)} \, ds \geq \frac{1 + \delta}{4} \phi_0(t), \quad t \geq t_0.
\]

Now the functions $\phi_1$ and $\phi_2$ defined above will satisfy
\[
\phi_1(t) \geq c_0 \phi_0(t), \quad t \geq t_0,
\]
where $c_0 = (1 + \delta)/4$, and
\[
\phi_2(t) \geq \int_t^\infty \left( \phi_0(s) + \phi_1(s) \right) F(\phi_0(s) + \phi_1(s), r(s), 1) \, ds
\]
\[
\geq \int_t^\infty (1 + c_0) \phi_0(s) F((1 + c_0) \phi_0(s), r(s), 1) \, ds
\]
\[
\geq (1 + c_0)^2 \int_t^\infty \phi_0(s) \frac{\phi_0(s)}{r(s)} \, ds
\]
\[
\geq (1 + c_0)^2 c_0 \phi_0(t) = c_1 \phi_0(t), \quad t \geq t_0,
\]
respectively. By induction we easily see that
\[ \phi_{n+1}(t) \geq c_n \phi_0(t), \quad t \geq t_0, \quad n = 1, 2, \ldots, \]
where
\[ c_n = (1 + c_{n-1})^2 c_0, \quad n = 1, 2, \ldots. \]
It is also easy to see that the sequence \( c_n \) is increasing. We assert further that it is unbounded. Otherwise, \( c_n \to c \) would imply \( c = (1 + c)^2 c_0 \), or
\[ c_0 c^2 + (2c_0 - 1)c + c_0 = 0. \]
However, the quadratic equation cannot have a real solution if \( c_0 > 1/4 \). Thus the assumption that \( c_n \to c \) is impossible. As a sequence, the sequence \( \{ \phi_n \} \) cannot be pointwise convergent. The following is now clear from Theorem 3.

**Corollary 1.** Assume that \( \sigma = 1 \) and that the function \( q(t) \) satisfies (15) for some positive number \( \delta \). Then Eq. (1) cannot have any positive solutions.

We remark that the condition (15) in Corollary 1 is sharp in the following sense.

**Corollary 2.** Assume that \( \sigma = 1 \), that
\[ \int_{t_0}^{\infty} q(s) \, ds < \infty, \]
and
\[ \int_t^{\infty} \left( \int_s^{\infty} q(u) \, du \right)^2 \, ds \leq \mu \int_t^{\infty} q(s) \, ds, \quad t \geq t_0, \]
where \( \mu \leq 1/4 \). Then (1) has a positive solution.

**Proof.** The crux of our proof lies in the observation that \( F(x, y, \sigma) = x/y \) when \( \sigma = 1 \). More specifically, consider the sequence \( \{ \phi_n(t) \} \) defined by (11), (12), and (13). Note that
\[ \phi_0(t) = \int_t^{\infty} q(s) \, ds < \infty, \quad t \geq t_0, \]
and
\[ \phi_2(t) = (S\phi_0)(t) = \int_t^{\infty} \phi_0(s) F(\phi_0(s), r(s), 1) \, ds \]
\[ = \int_t^{\infty} \left( \frac{\phi_0(s)}{r(s)} \right)^2 \, ds \leq c_0 \phi_0(t), \quad t \geq t_0, \]
where \( c_0 = \mu \). Next,
\[
\phi_2(t) = (S(\phi_0 + \phi_1))(t) = \int_t^\infty \left( \frac{\phi_0(s) + \phi_1(s)}{r(s)} \right)^2 ds
\]
\[
\leq \int_t^\infty \left( \frac{\phi_0(s) + c_0\phi_0(s)}{r(s)} \right)^2 ds
\]
\[
\leq \int_t^\infty \frac{(1 + c_0)^2(\phi_0(s))^2}{r(s)} ds \leq c_1\phi_0(t), \quad t \geq t_0,
\]
where \( c_1 = (1 + c_0)^2c_0 \). Inductively, we see that
\[
\phi_n(t) \leq c_n\phi_0(t), \quad t \geq t_0,
\]
where
\[
c_n = (1 + c_{n-1})c_0, \quad n = 1, 2, \ldots.
\]
It is also easy to see that the sequence \( \{c_n\} \) is nondecreasing and converges. We may see this as follows. Consider the fixed point problem \( x = g(x) \) where
\[
g(x) = \mu(1 + x)^2.
\]
As is customary, we find fixed points by means of the iteration scheme
\[
x_n = \mu(1 + x_{n-1})^2, \quad n = 1, 2, \ldots.
\]
Note that when \( \mu = 1/4 \), the graph of \( g \) is a parabola which has a unique minimum at \( x = -1 \) and touches the \( y = x \) line at \( (x, y) = (1, 1) \). Therefore, if we choose \( x_0 = \mu \), then we see that the approximating sequence \( \{x_n\} \) is strictly increasing and converges to \( x = 1 \). If \( c_0 < 1/4 \), then clearly \( c_n < x_n < 1 \) for all \( n \). This shows that \( \{c_n\} \) is bounded and hence converges.

We have thus shown that the sequence \( \{\phi_n\} \) is well defined and pointwise convergent. The proof is now complete in view of Theorem 3.

We remark that, on the basis of Theorem 3, it is easy to obtain Theorems 3 and 5 of Kusano and Yoshida [6].

3. AN ADVANCED TYPE EQUATION

In this section we will consider the following class of advanced type differential equations of the form
\[
[r(t)(y'(t))^\sigma] + q(t)(y(t + \tau))^\sigma = 0, \quad t \geq t_0,
\]
where \( \tau \geq 0 \). By means of the same Riccati transformation (4), we may proceed in a similar manner as in the last section and obtain the following extension of Theorem 1.

**Theorem 5.** Equation (16) has a positive solution \( y(t) \) if, and only if, there is a positive and continuous function \( u(t) \) which satisfies the inequality

\[
\int_{t_0}^{\infty} u(s) F(u(s), r(s), \sigma) \, ds + \int_{t_0}^{\infty} q(s) \left[ \exp \left( \int_{s}^{s + \tau} \left( \frac{u(v)}{r(v)} \right)^{1/\sigma} \, dv \right) \right]^{\sigma} \, ds \\
\leq u(t), \quad t \geq t_0.
\]

(17)

Since

\[
\int_{t_0}^{\infty} q(s) \, ds \leq \int_{t_0}^{\infty} q(s) \left[ \exp \left( \int_{s}^{s + \tau} \left( \frac{u(v)}{r(v)} \right)^{1/\sigma} \, dv \right) \right]^{\sigma} \, ds
\]

for \( u > 0 \), we immediately obtain from Theorems 1 and 5 the following corollary.

**Corollary 3.** If (16) has a positive solution, then so does (1).

A partial converse of the above corollary can be obtained as follows. Let \( y(t) \) be a positive solution of (1) such that

\[
1 \leq \left[ \exp \left( \int_{t_0}^{t + \tau} \left( \frac{u(s)}{r(s)} \right)^{1/\sigma} \, ds \right) \right]^{\sigma} \leq \Gamma, \quad t \geq t_0,
\]

(18)

where \( w(t) \) is defined by (4). Then

\[
\int_{t_0}^{\infty} u(s) F(u(s), r(s), \sigma) \, ds + \frac{1}{\Gamma} \int_{t_0}^{\infty} q(s) \left[ \exp \left( \int_{s}^{s + \tau} \left( \frac{u(v)}{r(v)} \right)^{1/\sigma} \, dv \right) \right]^{\sigma} \, ds \\
\leq u(t),
\]

for \( t \geq t_0 \), so that there exists a positive solution of

\[
(r(t)(y'(t))^\sigma + \frac{1}{\Gamma} q(t)(y(t))^\sigma = 0, \quad t \geq t_0.
\]

At this point, it is not clear what conditions are needed for a positive solution \( y(t) \) to exist such that the additional property (18) holds. However, we may twist the above arguments slightly and conclude that if \( y(t) \) is
a positive solution of
\[
(r(t)(y'(t))^\sigma)' + \beta q(t)(y(t))^\sigma = 0, \quad t \geq t_0, \tag{19}
\]
where \( \beta \geq 1 \) while the other parameters are the same as in (1), and if
\[
1 \leq \left[ \exp \left( \int_{t_0^t} \left( \frac{u(s)}{r(s)} \right)^{1/\sigma} \, ds \right) \right]^\sigma \leq \beta, \quad t \geq t_0, \tag{20}
\]
where \( w(t) \) is defined by (4), then (16) has a positive solution.

Now, we assert that if \( y(t) \) is a positive solution of (19), then
\[
\frac{r(t)(y'(t))^\sigma}{y(t)^\sigma} \to 0 \quad \text{as } t \to \infty.
\]
Indeed, from (19), we see that \( (r(t)(y'(t))^\sigma)' \leq 0 \), and \( r(t)(y'(t))^\sigma > 0 \) for \( t \geq t_0 \). Thus either \( r(t)(y'(t))^\sigma \) decreases to zero or to a positive constant \( c \). In the former case,
\[
\frac{r(t)(y'(t))^\sigma}{y(t)^\sigma} \leq \frac{r(t)(y'(t))^\sigma}{y(t_0)^\sigma} \to 0
\]
as desired. In the latter case, we have
\[
y'(t) \geq \left( \frac{c}{r(t)} \right)^{1/\sigma},
\]
which implies, in view of (2), that
\[
y(t) \geq y(t_0) + \int_{t_0^t} \frac{c^{1/\sigma}}{r(s)^{1/\sigma}} \, ds \to \infty.
\]
Thus,
\[
\frac{r(t)(y'(t))^\sigma}{y(t)^\sigma} \leq \frac{r(t_0)(y'(t_0))^\sigma}{y(t_0)^\sigma} \to 0
\]
also. As a consequence, if \( 1/r(t) \) is bounded, then given any number \( \beta > 1 \), the condition will automatically hold for all large \( t \). Therefore, we may now conclude that if \( \beta > 1 \) and \( 1/r(t) \) is bounded, and if (19) has a positive solution, then (16) will have an eventually positive solution.
We remark that it remains to analyze (1) in which the function \( r(t) \) satisfies the condition
\[
\int_{t_0}^{\infty} \frac{ds}{r(s)^{1/\alpha}} < \infty.
\]
Such an analysis will be the subject of our forthcoming paper.

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