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Realizations of AF-algebras as graph algebras, Exel–Laca algebras, and ultragraph algebras

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Abstract

We give various necessary and sufficient conditions for an AF-algebra to be isomorphic to a graph C^*-algebra, an Exel–Laca algebra, and an ultragraph C^*-algebra. We also explore consequences of these results. In particular, we show that all stable AF-algebras are both graph C^*-algebras and Exel–Laca algebras, and that all simple AF-algebras are either graph C^*-algebras or Exel–Laca algebras. In addition, we obtain a characterization of AF-algebras that are isomorphic to the C^*-algebra of a row-finite graph with no sinks.

Keywords: Graph C^*-algebras; Exel–Laca algebras; Ultragraph C^*-algebras; AF-algebras; Bratteli diagrams

Contents

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1. Introduction

In 1980 Cuntz and Krieger introduced a class of C^*-algebras constructed from finite matrices with entries in $\{0,1\}$ [4]. These C^*-algebras, now called Cuntz–Krieger algebras, are intimately related to the dynamics of topological Markov chains, and appear frequently in many diverse areas of C^*-algebra theory. Cuntz–Krieger algebras have been generalized in a number of ways, and two very natural generalizations are the *graph C^*-algebras* and the *Exel–Laca algebras*.

For graph C^*-algebras one views a $\{0,1\}$-matrix as an edge adjacency matrix of a graph, and considers the Cuntz–Krieger algebras as C^*-algebras of certain finite directed graphs. For a (not necessarily finite) directed graph E, one then defines the graph C^*-algebra $C^*(E)$ as the C^*-algebra generated by projections p_v associated to the vertices v of E and partial isometries s_e associated to the edges e of E that satisfy relations determined by the graph. Graph C^*-algebras were first studied using groupoid methods [17,18]. Due to technical constraints, the original theory was restricted to graphs that are *row-finite* and have *no sinks*; that is, the set of edges emitted by each vertex is finite and nonempty. In fact much of the early theory restricted to this case [2,17,18], and it was not until later [1,8,12] that the theory was extended to infinite graphs that are not row-finite. Interestingly, the non-row-finite setting is significantly more complicated than the row-finite case, with both new isomorphism classes of C^*-algebras and new kinds of C^*-algebraic phenomena exhibited.

Another approach to generalizing the Cuntz–Krieger algebras was taken by Exel and Laca, who defined what are now called the Exel–Laca algebras [10]. In this definition one allows a possibly infinite matrix with entries in $\{0,1\}$ and considers the C^*-algebra generated by a set of partial isometries indexed by the rows of the matrix and satisfying certain relations determined by the matrix. The construction of the Exel–Laca algebras contains the Cuntz–Krieger construction as a special case. Furthermore, for *row-finite matrices* (i.e., matrices in which each row contains a finite number of nonzero entries) with nonzero rows, the construction produces exactly the class of C^*-algebras of row-finite graphs with no sinks.

Despite the fact that the classes of graph C^*-algebras and Exel–Laca algebras agree in the row-finite case, they are quite different in the non-row-finite setting. In particular, there are C^*-algebras of non-row-finite graphs that are not isomorphic to any Exel–Laca algebra, and there are Exel–Laca algebras of non-row-finite matrices that are not isomorphic to the C^*-algebra of any graph [23]. In order to bring graph C^*-algebras and Exel–Laca algebras together under one theory, Tomforde introduced the notion of an ultragraph and described how to associate a C^*-algebra to such an object [22,23]. These ultragraph C^*-algebras contain all graph C^*-algebras

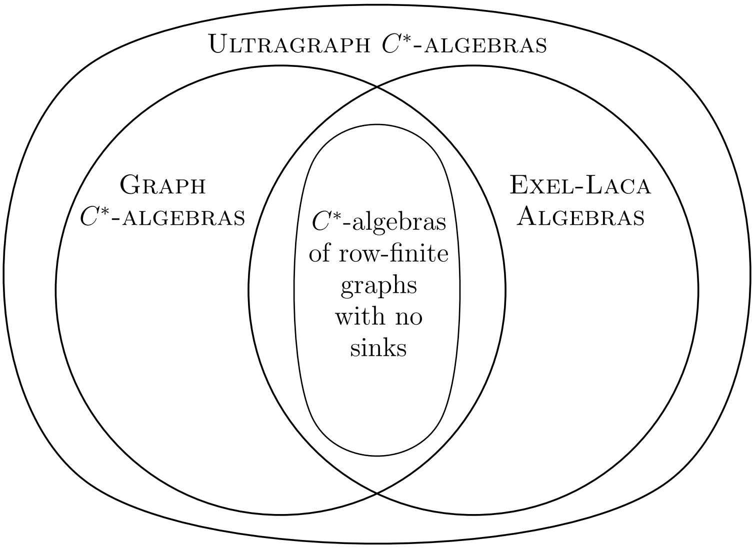


Fig. 1. The relationship among graph C^*-algebras, Exel–Laca algebras, and ultragraph C^*-algebras.

and all Exel–Laca algebras, as well as examples of C^*-algebras that are in neither of these two classes. The relationship among these classes is summarized in Fig. 1.

Given the relationship among these classes of C^*-algebras, it is natural to ask the following question.

Question. "How different are the C^*-algebras in the three classes of graph C^*-algebras, Exel–Laca algebras, and ultragraph C^*-algebras?"

There are various ways to approach this question, and one such approach was taken in [16], where it was shown that the classes of graph C^*-algebras, Exel–Laca algebras, and ultragraph C^*-algebras agree up to Morita equivalence. More specifically, given a C^*-algebra A in any of these three classes, one can always find a row-finite graph E with no sinks such that $C^*(E)$ is Morita equivalent to A.

Thus the three classes cannot be distinguished by Morita equivalence classes of C^*-algebras. The natural next question is to what extent they can be distinguished by isomorphism classes of C^*-algebras. A starting point for these investigations is to ask about AF-algebras.

While no Cuntz–Krieger algebra is an AF-algebra, the classes of graph C^*-algebras and Exel–Laca algebras each include many AF-algebras. In fact, one of the early results in the theory of graph C^*-algebras shows that if A is any AF-algebra, then there is a row-finite graph E with no sinks such that $C^*(E)$ is Morita equivalent to A [7]. From this fact and the result in [16] mentioned above, our three classes (graph C^*-algebras, Exel–Laca algebras, and ultragraph C^*-algebras) each contain all AF-algebras up to Morita equivalence.

The purpose of this paper is to examine the three classes of graph C^*-algebras, Exel–Laca algebras, and ultragraph C^*-algebras and determine which AF-algebras are contained, up to isomorphism, in each class. This turns out to be a difficult task, and we are unable to give a complete solution to the problem. Nonetheless, we are able to give a number of sufficient conditions and a number of necessary conditions for a given AF-algebra to belong to each of these three classes (see Sections 4.2 and 4.3). As special cases of our sufficient conditions, we obtain the following.

- If A is a stable AF-algebra, then A is isomorphic to the C^*-algebra of a row-finite graph with no sinks.
- If A is a simple AF-algebra, then A is isomorphic to either an Exel–Laca algebra or a graph C^*-algebra. In particular, if A is finite dimensional, then A is isomorphic to a graph C^*-algebra; and if A infinite dimensional, then A is isomorphic to an Exel–Laca algebra.
- If A is an AF-algebra with no nonzero finite-dimensional quotients, then A is isomorphic to an Exel–Laca algebra.

From our necessary conditions, we obtain the following.

- If an ultragraph C^*-algebra is a commutative AF-algebra then it is isomorphic to $c_0(X)$ for an at most countable discrete set X.
- No finite-dimensional C^*-algebra is isomorphic to an Exel–Laca algebra.
- No infinite-dimensional UHF algebra is isomorphic to a graph C^*-algebra.

Moreover, we are able to give a characterization of AF-algebras that are isomorphic to C^*-algebras of row-finite graphs with no sinks in Theorem 4.7.

Theorem. *Let A be an AF-algebra. Then the following are equivalent*:

(1) *A has no unital quotients.*
(2) *A is isomorphic to the C^*-algebra of a row-finite graph with no sinks.*

Our results allow us to make a fairly detailed analysis of the AF-algebras in each of our three classes, and in Fig. 2 at the end of this paper we draw a Venn diagram relating various classes of AF-algebras among the graph C^*-algebras, Exel–Laca algebras, and ultragraph C^*-algebras. Our results are powerful enough that we are able to give examples in each region of the Venn diagram, and also state definitively whether or not there are unital and nonunital examples in each region.

Finally, we remark that a particularly useful aspect of our sufficiency results is their constructive nature. When one first approaches the problem of identifying which AF-algebra are in our three classes, one may be tempted to use the K-theory classification of AF-algebras. There are, however, two problems with this approach: (1) Since any AF-algebra is Morita equivalent to the C^*-algebra of a row-finite graph with no sinks, we know that all ordered K_0-groups are attained by the AF-algebras in each of our three classes. Thus we need to identify which *scaled* ordered K_0-groups are attained by the AF-algebras in each class. Unfortunately, however, little is currently known about the scale for the K_0-groups of C^*-algebras in these three classes. (2) More importantly, even if we could decide exactly which scaled ordered K_0-groups are attained by, for example, graph AF-algebras, we would obtain at best an abstract characterization of which AF-algebras are graph C^*-algebras. Unless our understanding of the scaled ordered K_0-groups achieved by AF graph C^*-algebras extended to an algorithm for producing a graph whose C^*-algebra achieved a given scaled ordered K_0-group, we would be unable to take a given AF-algebra A and view it as a graph C^*-algebra. Most notably, we could not expect to "see" the canonical generators of $C^*(E)$ in A.

With an awareness of the limitations of an abstract characterization, we instead present constructive methods for realizing AF-algebras as C^*-algebras in our three classes. Given a certain type of AF-algebra A we show how to build an ultragraph $\mathcal{G}$ from a certain type of Bratteli diagram for A so that $C^*(\mathcal{G})$ is isomorphic to A (see Section 4.1). This ultragraph C^*-algebra is always an Exel–Laca algebra, and in special situations (see Section 4.2) it is also a graph C^*-algebra. Furthermore, one can extract from $\mathcal{G}$ a $\{0,1\}$-matrix for the Exel–Laca algebra or a directed graph for the graph C^*-algebra as appropriate.

This paper is organized as follows. In Section 2 we establish definitions and notation for graph C^*-algebras, Exel–Laca algebras, ultragraph C^*-algebras, and AF-algebras. In Section 3 we establish some technical lemmas regarding Bratteli diagrams and inclusions of finite-dimensional C^*-algebras. In Section 4 we state the main results of this paper. Specifically, in Section 4.1 we describe how to take a Bratteli diagram for an AF-algebra A with no nonzero finite-dimensional quotients and build an ultragraph $\mathcal{G}$. In Section 4.2 we prove that the associated ultragraph C^*-algebra $C^*(\mathcal{G})$ is isomorphic to A. We also show that $C^*(\mathcal{G})$ is always isomorphic to an Exel–Laca algebra, and describe conditions which imply $C^*(\mathcal{G})$ is also a graph C^*-algebra. These results give us a number of sufficient conditions for AF-algebras to be contained in our three classes of graph C^*-algebras, Exel–Laca algebras, and ultragraph C^*-algebras. We also present examples showing that none of our sufficient conditions are necessary. In Section 4.3 we give several necessary conditions for AF-algebras to be in each of our three classes. These conditions allow us to identify a number of obstructions to realizations of various AF-algebras in each class. We conclude in Section 5 by summarizing our containments. First, we characterize precisely which simple AF-algebras fall into each of our classes. Second, we summarize many of the relationships we have derived, including containments for the finite-dimensional and stable AF-algebras, and draw a Venn diagram to represent these containments. We are able to use our results from Section 4 to exhibit examples in each region of the Venn diagram, thereby showing these regions are nonempty. We are also able to describe precisely when unital and nonunital examples occur in these regions.

2. Preliminaries

In the following four subsections we establish definitions and notation for graph C^*-algebras, Exel–Laca algebras, ultragraph C^*-algebras, and AF-algebras. Since the literature for each of these classes of C^*-algebras is large and well developed, we present only the definitions and notation required in this paper. However, for each class we provide introductory references where more detailed information may be found.

2.1. Graph C^*-algebras

Introductory references include [2,20,25].

Definition 2.1. A *graph* $E = (E^0, E^1, r, s)$ consists of a countable set E^0 of vertices, a countable set E^1 of edges, and maps $r : E^1 \to E^0$ and $s : E^1 \to E^0$ identifying the range and source of each edge.

A *path* in a graph $E = (E^0, E^1, r, s)$ is a sequence of edges $\alpha := e_1 \dots e_n$ with $s(e_{i+1}) = r(e_i)$ for $1 \leqslant i \leqslant n-1$. We say that α has *length* n. We regard vertices as paths of length 0 and edges as paths of length 1, and we then extend our notation for the vertex set and the edge set by writing

E^n for the set of paths of length n for all $n \geqslant 0$. We write E^* for the set $\bigsqcup_{n=0}^{\infty} E^n$ of paths of finite length, and extend the maps r and s to E^* by setting $r(v) = s(v) = v$ for $v \in E^0$, and $r(\alpha_1 \dots \alpha_n) = r(\alpha_n)$ and $s(\alpha_1 \dots \alpha_n) = s(\alpha_1)$.

If α and β are elements of E^* such that $r(\alpha) = s(\beta)$, then $\alpha\beta$ is the path of length $|\alpha| + |\beta|$ obtained by concatenating the two. Given $\alpha, \beta \in E^*$, and a subset X of E^*, we let

$$\alpha X \beta := \{\gamma \in E^*\colon\ \gamma = \alpha\gamma'\beta \text{ for some } \gamma' \in X\}.$$

So when v and w are vertices, we have

$$\begin{aligned} vX &= \{\gamma \in X\colon\ s(\gamma) = v\}, \\ Xw &= \{\gamma \in X\colon\ r(\gamma) = w\}, \quad \text{and} \\ vXw &= \{\gamma \in X\colon\ s(\gamma) = v \text{ and } r(\gamma) = w\}. \end{aligned}$$

In particular, vE^1w denotes the set of edges from v to w and $|vE^1w|$ denotes the number of edges from v to w.

We say a vertex v is a *sink* if $vE^1 = \emptyset$ and an *infinite emitter* if vE^1 is infinite. A graph is called *row-finite* if it has no infinite emitters.

Definition 2.2 *(Graph C^*-algebras).* If $E = (E^0, E^1, r, s)$ is a graph, then the *graph C^*-algebra* $C^*(E)$ is the universal C^*-algebra generated by mutually orthogonal projections $\{p_v\colon\ v \in E^0\}$ and partial isometries $\{s_e\colon\ e \in E^1\}$ with mutually orthogonal ranges satisfying

(1) $s_e^* s_e = p_{r(e)}$ for all $e \in E^1$;
(2) $p_v = \sum_{e \in vE^1} s_e s_e^*$ for all $v \in E^0$ such that $0 < |vE^1| < \infty$;
(3) $s_e s_e^* \leqslant p_{s(e)}$ for all $e \in E^1$.

We write $v \geqslant w$ to mean that there is a path $\alpha \in E^*$ such that $s(\alpha) = v$ and $r(\alpha) = w$. A *cycle* in a graph E is a path $\alpha \in E^*$ of nonzero length with $r(\alpha) = s(\alpha)$. [17, Theorem 2.4] says that $C^*(E)$ is an AF-algebra if and only if E has no cycles.

2.2. Exel–Laca algebras

Introductory references include [10–12,21].

Definition 2.3 *(Exel–Laca algebras).* Let I be a finite or countably infinite set, and let $A = \{A(i,j)\}_{i,j \in I}$ be a $\{0,1\}$-matrix over I with no identically zero rows. The *Exel–Laca algebra* $\mathcal{O}_A$ is the universal C^*-algebra generated by partial isometries $\{s_i\colon\ i \in I\}$ with commuting initial projections and mutually orthogonal range projections satisfying $s_i^* s_i s_j s_j^* = A(i,j) s_j s_j^*$ and

$$\prod_{x \in X} s_x^* s_x \prod_{y \in Y} \bigl(1 - s_y^* s_y\bigr) = \sum_{j \in I} A(X, Y, j) s_j s_j^* \tag{2.1}$$

whenever X and Y are finite subsets of I such that $X \neq \emptyset$ and the function

$$j \in I \mapsto A(X, Y, j) := \prod_{x \in X} A(x, j) \prod_{y \in Y} \bigl(1 - A(y, j)\bigr)$$

is finitely supported. (We interpret the unit in (2.1) as the unit in the multiplier algebra of $\mathcal{O}_A$.)

We will see in Remark 2.10 that for a $\{0,1\}$-matrix A with no identically zero rows, the canonical ultragraph $\mathcal{G}_A$ of A satisfies $C^*(\mathcal{G}_A) \cong \mathcal{O}_A$. With this notation, [23, Theorem 4.1] implies that the Exel–Laca algebra $\mathcal{O}_A$ is an AF-algebra if and only if $\mathcal{G}_A$ has no cycle. The latter condition can be restated as: there does not exist a finite set $\{i_1, \dots, i_n\} \subseteq I$ with $A(i_k, i_{k+1}) = 1$ for all $1 \leqslant k \leqslant n-1$ and $A(i_n, i_1) = 1$.

It is well known that the class of graph C^*-algebras of row-finite graphs with no sinks and the class of Exel–Laca algebras of row-finite matrices coincide. However, we have been unable to find a reference, so we give a proof here.

Lemma 2.4. *The class of graph C^*-algebras of row-finite graphs with no sinks and the class of Exel–Laca algebras of row-finite matrices coincide. In particular,*

(1) *If $E = (E^0, E^1, r, s)$ is a row-finite graph with no sinks, and if we define a $\{0,1\}$-matrix A_E over E^1 by*

$$A_E(e,f) := \begin{cases} 1 & \text{if } r(e) = s(f), \\ 0 & \text{otherwise,} \end{cases}$$

then A_E is a row-finite matrix with no identically zero rows and $C^*(E) \cong \mathcal{O}_{A_E}$.

(2) *If A is a row-finite $\{0,1\}$-matrix over I with no identically zero rows, and if we define a graph E_A by setting $E_A^0 := I$ and drawing an edge from $v \in I$ to $w \in I$ if and only if $A(v,w) = 1$, then E_A is a row-finite graph with no sinks and $\mathcal{O}_A \cong C^*(E_A)$.*

Proof. For (1) let $E = (E^0, E^1, r, s)$ be a row-finite graph with no sinks, and define the matrix A_E as above. Since E is row-finite, A_E is also row-finite. Let $\{S_e\colon e \in E^1\}$ be a generating Exel–Laca A_E-family in $\mathcal{O}_{A_E}$. For $v \in E^0$ we define $P_v := \sum_{s(e)=v} S_e S_e^*$ in $\mathcal{O}_{A_E}$. (Note that this sum is always finite since A_E is row-finite.) We now show that $\{S_e, P_v\colon e \in E^1,\ v \in E^0\}$ is a Cuntz–Krieger E-family in $\mathcal{O}_{A_E}$. The S_e's have mutually orthogonal range projections by the Exel–Laca relations, and hence the P_v's are also mutually orthogonal projections. In addition, conditions (2) and (3) in the definition of graph C^*-algebras obviously hold from our definition of P_v. It remains to show condition (1) holds. If $e \in E^1$, let $X := \{e\}$ and $Y := \emptyset$. Then for $j \in E^1$, we have $A_E(X,Y,j) := 1$ if and only if $s(j) = r(e)$. Since E is row-finite, the function $j \mapsto A_E(X,Y,j)$ is finitely supported, and (2.1) gives $S_e^* S_e = \sum_{j \in E^1} A(X,Y,j) S_j S_j^* = \sum_{s(j)=r(e)} S_j S_j^* = P_{r(e)}$, so condition (1) holds. Thus $\{S_e, P_v\colon e \in E^1,\ v \in E^0\}$ is a Cuntz–Krieger E-family, and by the universal property of $C^*(E)$ we obtain a $*$-homomorphism $\phi : C^*(E) \to \mathcal{O}_{A_E}$ with $\phi(s_e) = S_e$ and $\phi(p_v) = P_v$ where $\{s_e, p_v\}$ is a generating Cuntz–Krieger E-family for $C^*(E)$. By checking on generators, one can see that ϕ is equivariant with respect to the gauge actions on $C^*(E)$ and $\mathcal{O}_{A_E}$, and thus the Gauge-Invariant Uniqueness Theorem [2, Theorem 2.1] implies that ϕ is injective. Since the image of ϕ contains the generators $\{S_e\colon e \in E^1\}$ of $\mathcal{O}_{A_E}$, ϕ is also surjective. Thus $C^*(E) \cong \mathcal{O}_{A_E}$.

For (2) let A be a row-finite $\{0,1\}$-matrix with no identically zero rows. Let $\mathcal{G}_A$ be the canonical ultragraph of A (see Remark 2.10). Then the source map of $\mathcal{G}_A$ is bijective and $C^*(\mathcal{G}_A) \cong \mathcal{O}_A$. Since A is a row-finite matrix, the range of each edge in $\mathcal{G}_A$ is a finite set. Thus $C^*(\mathcal{G}_A)$ is isomorphic to the C^*-algebra of the graph formed by replacing each edge in $\mathcal{G}_A$ with a set of edges from $s(e)$ to w for all $w \in r(e)$ [16, Remark 2.5]. But this is precisely the graph E_A described in the statement above. $\square$

2.3. Ultragraph C^*-algebras

Introductory references include [15,16,22,23]. For a set X, let $\mathcal{P}(X)$ denote the collection of all subsets of X.

Definition 2.5. (See [22, Definition 2.1].) An *ultragraph* $\mathcal{G} = (G^0, \mathcal{G}^1, r, s)$ consists of a countable set of vertices G^0, a countable set of ultraedges $\mathcal{G}^1$, and functions $s : \mathcal{G}^1 \to G^0$ and $r : \mathcal{G}^1 \to \mathcal{P}(G^0) \setminus \{\emptyset\}$.

Note that in the literature, ultraedges are typically just referred to as edges. However, since we will frequently be passing back and forth between graphs and ultragraphs in this paper, we feel that using the term ultraedge will serve as a helpful reminder that edges in ultragraphs behave differently than in graphs.

Definition 2.6. For a set X, a subset $\mathcal{C}$ of $\mathcal{P}(X)$ is called an *algebra* if

(i) $\emptyset \in \mathcal{C}$,
(ii) $A \cap B \in \mathcal{C}$ and $A \cup B \in \mathcal{C}$ for all $A, B \in \mathcal{C}$, and
(iii) $A \setminus B \in \mathcal{C}$ for all $A, B \in \mathcal{C}$.

Definition 2.7. For an ultragraph $\mathcal{G} = (G^0, \mathcal{G}^1, r, s)$, we let $\mathcal{G}^0$ denote the smallest algebra in $\mathcal{P}(G^0)$ containing the singleton sets and the sets $\{r(e)\colon e \in \mathcal{G}^1\}$.

Definition 2.8. A *representation* of an algebra $\mathcal{C}$ is a collection of projections $\{p_A\}_{A\in\mathcal{C}}$ in a C^*-algebra satisfying $p_\emptyset = 0$, $p_A p_B = p_{A\cap B}$, and $p_{A\cup B} = p_A + p_B - p_{A\cap B}$ for all $A, B \in \mathcal{C}$.

Observe that a representation of an algebra automatically satisfies $p_{A\setminus B} = p_A - p_A p_B$.

Definition 2.9. For an ultragraph $\mathcal{G} = (G^0, \mathcal{G}^1, r, s)$, the *ultragraph C^*-algebra $C^*(\mathcal{G})$* is the universal C^*-algebra generated by a representation $\{p_A\}_{A\in\mathcal{G}^0}$ of $\mathcal{G}^0$ and a collection of partial isometries $\{s_e\}_{e\in\mathcal{G}^1}$ with mutually orthogonal ranges that satisfy

(1) $s_e^* s_e = p_{r(e)}$ for all $e \in \mathcal{G}^1$,
(2) $s_e s_e^* \leqslant p_{s(e)}$ for all $e \in \mathcal{G}^1$,
(3) $p_v = \sum_{e\in v\mathcal{G}^1} s_e s_e^*$ whenever $0 < |v\mathcal{G}^1| < \infty$,

where we write p_v in place of $p_{\{v\}}$ for $v \in G^0$.

As with graphs, we call a vertex $v \in G^0$ a *sink* if $v\mathcal{G}^1 = \emptyset$ and an *infinite emitter* if $v\mathcal{G}^1$ is infinite. A *path* in an ultragraph $\mathcal{G}$ is a sequence of ultraedges $\alpha = e_1 e_2 \dots e_n$ with $s(e_{i+1}) \in r(e_i)$ for $1 \leqslant i \leqslant n-1$. A *cycle* is a path $\alpha = e_1 \dots e_n$ with $s(e_1) \in r(e_n)$. [23, Theorem 4.1] implies that $C^*(\mathcal{G})$ is an AF-algebra if and only if $\mathcal{G}$ has no cycles.

Remark 2.10. A graph may be regarded as an ultragraph in which the range of each ultraedge is a singleton set. The constructions of the two C^*-algebras then coincide: the graph C^*-algebra of a graph is the same as the ultragraph C^*-algebra of that graph when regarded as an ultragraph (see [22, §3] for more details).

For a $\{0,1\}$-matrix A over I with nonzero rows, the canonical associated ultragraph $\mathcal{G}_A = (G_A^0, \mathcal{G}_A^1, r, s)$ is defined by $G_A^0 = \mathcal{G}_A^1 = I$, $r(i) = \{j \in I\colon A(i,j) = 1\}$ and $s(i) = i$ for $i \in \mathcal{G}_A^1$ (see [22, Definition 2.5]). It follows from [22, Theorem 4.5] that $C^*(\mathcal{G}_A) \cong \mathcal{O}_A$. The ultragraph $\mathcal{G}_A$ has the property that s is bijective. Conversely an ultragraph $\mathcal{G} = (G^0, \mathcal{G}^1, r, s)$ with bijective s is isomorphic to $\mathcal{G}_A$ where A is the edge matrix of $\mathcal{G}$. Thus one can say that an Exel–Laca algebra is a C^*-algebra of a ultragraph with bijective source map.

From these observations, one can see that the class of ultragraph C^*-algebras contains both the class of graph C^*-algebras and the class of Exel–Laca algebras.

2.4. AF-algebras

Introductory references include [3,9,13] as well as [5, Chapter 6] and [19, §6.1, §6.2, and §7.2].

Definition 2.11. An AF-algebra is a C^*-algebra that is the direct limit of a sequence of finite-dimensional C^*-algebras. Equivalently, a C^*-algebra A is an AF-algebra if and only if $A = \overline{\bigcup_{n=1}^\infty A_n}$ for a sequence of finite-dimensional C^*-subalgebras $A_1 \subseteq A_2 \subseteq \cdots \subseteq A$.

To discuss AF-algebras, we need first to briefly discuss inclusions of finite-dimensional C^*-algebras. Fix finite-dimensional C^*-algebras $A = \bigoplus_{i=1}^m M_{a_i}(\mathbb{C})$ and $B = \bigoplus_{j=1}^n M_{b_j}(\mathbb{C})$. Let $M = (m_{i,j})_{i,j}$ be an $m \times n$ nonnegative integer matrix with no zero rows such that

$$\sum_{i=1}^m m_{i,j} a_j \leqslant b_j \quad \text{for all } j. \tag{2.2}$$

There exists an inclusion $\phi_M : A \hookrightarrow B$ with the following property. For an element $x = (x_i)_{i=1}^m \in A$, the image $\phi_M(x)$ of x has the form $(y_j)_{j=1}^n \in B$ where for each $j \leqslant n$, the matrix y_j is block-diagonal with $m_{i,j}$ copies of each x_i along the diagonal and 0's elsewhere. (Eq. (2.2) ensures that this is possible.) The map ϕ_M is not uniquely determined by this property, but its unitary equivalence class is.

Every inclusion ϕ of A into B is unitarily equivalent to ϕ_M for some matrix M. Specifically, $M = (m_{i,j})_{i.j}$ is the matrix such that $m_{i,j}$ is equal to the rank of $1_{B_j}\phi(p_i)$ where 1_{B_j} is the unit for the jth summand of B, and where p_i is any rank-1 projection in the ith summand of A. We refer to M as the *multiplicity matrix* of the inclusion ϕ.

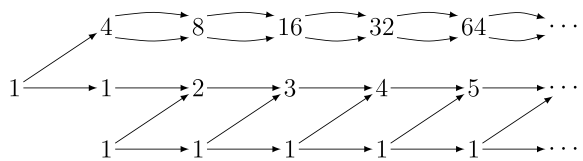
Definition 2.12. A *Bratteli diagram* (E,d) consists of a directed graph $E = (E^0, E^1, r, s)$ together with a collection $d = \{d_v\colon v \in E^0\}$ of positive integers satisfying the following conditions.

(1) E has no sinks;
(2) E^0 is partitioned as a disjoint union $E^0 = \bigsqcup_{n=1}^\infty V_n$ where each V_n is a finite set;
(3) for each $e \in E^1$ there exists $n \in \mathbb{N}$ such that $s(e) \in V_n$ and $r(e) \in V_{n+1}$; and
(4) for each vertex $v \in E^0$ we have $d_v \geqslant \sum_{e \in E^1 v} d_{s(e)}$ for all $v \in E^0$.

If (E,d) is a Bratteli diagram, then E is a row-finite graph with no sinks. We regard d as a labeling of the vertices by positive integers, so to draw a Bratteli diagram we sometimes just draw the directed graph, replacing each vertex v by its label d_v.

Remark 2.13. Those experienced with Bratteli diagrams will notice that our definition of a Bratteli diagram is slightly nonstandard. Specifically, a Bratteli diagram is traditionally specified as undirected graph in which each edge connects vertices in consecutive levels. Of course, an orientation of the edges is then implicitly chosen by the decomposition $E^0 = \bigsqcup V_n$, so it makes no difference if we instead draw a directed edge pointing from the vertex in level n to the vertex in level $n+1$.

Example 2.14. The following is an example of a Bratteli diagram.



Given a Bratteli diagram (E, d), we construct an AF-algebra A as follows. For each $v \in E^0$, let A_v be an isomorphic copy of $M_{d_v}(\mathbb{C})$, and for each $n \in \mathbb{N}$, let $A_n := \bigoplus_{v \in V_n} A_v$. For each n let $\phi_n : A_n \to A_{n+1}$ be the homomorphism whose multiplicity matrix is $(|vE^1w|)_{v \in V_n, w \in V_{n+1}}$. We then define A to be the direct limit $\varinjlim(A_n, \phi_n)$. Since the ϕ_n are determined up to unitary equivalence by (E, d), the isomorphism class of A is also uniquely determined by (E, d).

Example 2.15. In Example 2.14, we see that

$$A_1 = \mathbb{C} \qquad \phi_1(x) = \begin{pmatrix} x & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \oplus x \oplus 0$$

$$A_2 = M_4(\mathbb{C}) \oplus \mathbb{C} \oplus \mathbb{C} \qquad \phi_2(x, y, z) = \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} \oplus \begin{pmatrix} y & 0 \\ 0 & z \end{pmatrix} \oplus z$$

$$A_3 = M_8(\mathbb{C}) \oplus M_2(\mathbb{C}) \oplus \mathbb{C} \qquad \phi_3(x, y, z) = \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} \oplus \begin{pmatrix} y & 0 \\ 0 & z \end{pmatrix} \oplus z$$

$$\vdots \qquad \vdots$$

$$A_n = M_{2^n}(\mathbb{C}) \oplus M_{n-1}(\mathbb{C}) \oplus \mathbb{C} \qquad \phi_n(x, y, z) = \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} \oplus \begin{pmatrix} y & 0 \\ 0 & z \end{pmatrix} \oplus z$$

$$\vdots \qquad \vdots$$

The following *telescoping* operation on a Bratteli diagram preserves the associated AF-algebra. Given (E, d), we choose an increasing subsequence $\{n_m\}_{m=1}^{\infty}$ of $\mathbb{N}$. The set of the vertices of the new Bratteli diagram is $\bigcup_{m=1}^{\infty} V_{n_m}$, the set of the edges of the new Bratteli diagram is $\bigcup_{m=1}^{\infty}(V_{n_m} E^* V_{n_{m+1}})$, and the new function d is the restriction of the old d to $\bigcup_{m=1}^{\infty} V_{n_m}$. For example, if we have the portion of a Bratteli diagram shown below on the left and remove the middle column of vertices, we obtain the portion of the Bratteli diagram shown below on the right.

We say that two Bratteli diagrams (E, d) and (E', d') are equivalent if there is a finite sequence $(E_1, d_1), \dots, (E_n, d_n)$ such that $(E_1, d_1) = (E, d)$, $(E_n, d_n) = (E', d')$ and for each $1 \leqslant i \leqslant n-1$, one of (E_i, d_i) and (E_{i+1}, d_{i+1}) is a telescope of the other. Bratteli proved in [3] that two Bratteli diagrams give rise to isomorphic AF-algebras if and only if the diagrams are equivalent (see [3, §1.8 and Theorem 2.7] for details).

The class of AF-algebras is closed under forming ideals and quotients. On the other hand, the three classes of graph C^*-algebras, Exel–Laca algebras, and ultragraph C^*-algebras are not closed under forming ideals nor quotients. However we can show the following.

Lemma 2.16. *The class of graph AF-algebras is closed under forming ideals and quotients.*

Proof. If E is a graph and the graph C^*-algebra $C^*(E)$ is an AF-algebra, then E has no cycles by [17, Theorem 2.4]. Thus E vacuously satisfies Condition (K), and it follows that every ideal of $C^*(E)$ is gauge-invariant by [1, Corollary 3.8]. Thus every ideal of $C^*(E)$ as well as its quotient is a graph C^*-algebra by [6, Lemma 1.6] and [1, Theorem 3.6]. □

Remark 2.17. A quotient of an Exel–Laca AF-algebra need not be an Exel–Laca algebra. For example, if $\mathcal{K}^+$ is the minimal unitization of the compact operators $\mathcal{K}$ on a separable infinite-dimensional Hilbert space, then $M_2(\mathcal{K}^+)$ is an Exel–Laca AF-algebra that has a quotient, $M_2(\mathbb{C})$, that is not an Exel–Laca algebra — for details see Example 4.11 and Corollary 4.19. Whether ideals of Exel–Laca AF-algebras are necessarily Exel–Laca algebras is an open question. We also do not know whether ideals and quotients of ultragraph AF-algebras are necessarily ultragraph C^*-algebras. As we shall see later, this uncertainty causes problems in the analyses of Exel–Laca AF-algebras and ultragraph AF-algebras.

Lemma 2.18. *The three classes of graph AF-algebras, Exel–Laca AF-algebras, and ultragraph AF-algebras are closed under taking direct sums.*

Proof. Each of the four classes of AF-algebras, graph C^*-algebras, Exel–Laca algebras, and ultragraph C^*-algebras is closed under forming direct sums. The result follows. □

3. Some technical lemmas

In this section we establish some technical results for Bratteli diagrams and inclusions of finite-dimensional C^*-algebras. We will use these technical results to prove many of our realization results in Section 4.

Lemma 3.1. *Suppose A is an AF-algebra that has no quotients isomorphic to $\mathbb{C}$, and suppose that (E,d) is a Bratteli diagram for A. Let $H=\{v\in E^0\colon d_v=1\}$, and let F be the subgraph of E such that $F^0:=E^0\setminus H$ and $F^1:=\{e\in E^1\colon s(e)\notin H\}$ with $r,s:F^1\to F^0$ inherited from E. Let $d:F^0\to\mathbb{N}$ be the restriction of $d:E^0\to\mathbb{N}$. Then (F,d) is a Bratteli diagram for A.*

Proof. First note that if $e\in E^1$ with $r(e)\in H$, then $d_{r(e)}=1$ and hence $d_{s(e)}=1$ and $s(e)\in H$. Hence F is in fact a subgraph of E.

We claim that for any $n\in\mathbb{N}$ and $v\in V_n$, there exists $m\in\mathbb{N}$ such that whenever $w\in V_{n+m}$ and $v\geqslant w$, we have $d_w\geqslant 2$. We fix $n\in\mathbb{N}$ and $v\in V_n$, suppose that there is no such m, and seek a contradiction. Let $v_0:=v$. Inductively choose $e_i\in E^1$ such that $s(e_i)=v_{i-1}$ and such that for each $m\in\mathbb{N}$ there exists $w\in V_{n+i+m}$ with $r(e_i)\geqslant w$ and $d_w=1$, setting $v_i:=r(e_i)$. Then the infinite path $e_1e_2\ldots$ satisfies $d_{s(e_n)}=1$ for all n. Hence $\{x\in E^0\colon x\ngeqslant s(e_n)\text{ for any }n\}$ is a saturated hereditary subset and the quotient of A by the corresponding ideal is an AF-algebra with Bratteli diagram

$$1\longrightarrow 1\longrightarrow 1\longrightarrow 1\longrightarrow 1\longrightarrow\cdots$$

Hence this quotient is isomorphic to $\mathbb{C}$, which contradicts our hypothesis on A. This establishes the claim.

Let B be the AF-algebra associated to the Bratteli diagram F, and let $\iota_n:B_n\to A_n$ denote obvious inclusion of the nth approximating subalgebra of B determined by F into the nth approximating subalgebra of A determined by E. Let $\phi^E_{n,m}:A_n\to A_m$ be the connecting maps in the directed system associated to E, and let $\phi^E_{n,\infty}:A_n\to A$ be the inclusion of A_n into the direct limit algebra A. Likewise, let $\phi^F_{n,m}:B_n\to B_m$ be the connecting maps in the directed system associated to F, and let $\phi^F_{n,\infty}:B_n\to B$ be the inclusion of B_n into the direct limit algebra B.

We see that $\phi^E_{n,n+1}\circ\iota_n=\iota_{n+1}\circ\phi^F_{n,n+1}$ for all n, and thus by the universal property of the direct limit $B=\varinjlim(B_n,\phi^F_n)$, there is a $*$-homomorphism $\iota_\infty:B\to A$ with $\phi^E_{n,\infty}\circ\iota_n=\iota_\infty\circ\phi^F_{n,\infty}$. Since each ι_n is injective, it follows that ι_∞ is injective. We shall also show that ι_∞ is also surjective and hence an isomorphism. It suffices to show that for any $v\in V_n$ and for any a in the direct summand A_v of A_n corresponding to v, we have $\phi^E_{n,\infty}(a)\in\operatorname{im}\iota_\infty$. By the previous paragraph we may choose m so that whenever $w\in V_{n+m}$ and $v\geqslant w$, then $d_w\geqslant 2$. It follows that

$$\phi^E_{n,n+m}(a)\in\bigoplus_{\substack{w\in V_{n+m}\\ d_w\geqslant 2}}M_{d_w}(\mathbb{C})\subseteq\iota_{n+m}(B_{n+m}),$$

so that $\phi^E_{n,n+m}(a)=\iota_{n+m}(b)$ for some $b\in B_{n+m}$. Thus

$$\phi^E_{n,\infty}(a)=\phi^E_{n+m,\infty}\circ\phi^E_{n,n+m}(a)=\phi^E_{n+m,\infty}\circ\iota_{n+m}(b)=\iota_\infty\circ\phi^F_{n+m,\infty}(b)\in\operatorname{im}\iota_\infty$$

and ι_∞ is surjective. Hence ι_∞ is an isomorphism as required. □

Lemma 3.2. *Suppose A is an AF-algebra with no nonzero finite-dimensional quotients. Then any Bratteli diagram for A can be telescoped to obtain a second Bratteli diagram (E,d) for A such that for all $n\in\mathbb{N}$ and for each $v\in V_{n+1}$ either $d_v>\sum_{e\in E^1v}d_{s(e)}$ or there exists $w\in V_n$ with $|wE^1v|\geqslant 2$.*

Proof. Let (F,d) be a Bratteli diagram for A with F^0 partitioned into levels as $F^0=\bigsqcup_{n=1}^{\infty} W_n$. It suffices to show that for every m there exists $n\geqslant m$ such that for every $v\in W_n$ satisfying $d_v=\sum_{\alpha\in W_m F^* v} d_{s(\alpha)}$, there exists $w\in W_m$ with $|wF^*v|\geqslant 2$. We suppose not, and seek a contradiction. That is, we suppose that there exists m such that for every $n\geqslant m$ the set

$$X_n := \left\{x\in W_n\colon d_x=\sum_{\alpha\in W_m F^* x} d_{s(\alpha)} \text{ and } |wF^*x|\leqslant 1 \text{ for all } w\in W_m\right\}$$

is nonempty. By telescoping (F,d) to $\bigsqcup_{n=m}^{\infty} W_n$ we may assume $m=1$. We claim that if $n\leqslant p$, $x\in X_p$, and $v\in W_n$ with $v\geqslant x$, then $v\in X_n$. Indeed,

$$\begin{aligned} d_x &= \sum_{\alpha\in W_1F^*x} d_{s(\alpha)} \\ &= \sum_{\beta\in W_nF^*x}\left(\sum_{\gamma\in W_1F^*s(\beta)} d_{s(\gamma)}\right) \qquad (3.1)\\ &\leqslant \sum_{\beta\in W_nF^*x} d_{s(\beta)} \qquad (3.2)\\ &\leqslant d_x. \end{aligned}$$

Thus we have equality throughout, and the equality of (3.1) and (3.2) implies $d_{s(\beta)}=\sum_{\gamma\in W_1F^*s(\beta)} d_{s(\gamma)}$ for each $\beta\in W_nF^*x$. In particular, since $v\geqslant x$, we have that $d_v=\sum_{\gamma\in W_1F^*v} d_{s(\gamma)}$. Moreover for each $w\in W_1$,

$$1\geqslant |wF^*x|\geqslant |wF^*v||vF^*x|,$$

so $v\geqslant x$ implies that $|wF^*v|\leqslant 1$, and $v\in X_n$ as required.

We shall now construct an infinite path $\lambda=\lambda_1\lambda_2\ldots$ in F such that $s(\lambda_n)\in X_n$ for all n. If $x\in X_n$, then since d_x is nonzero and $d_x=\sum_{\alpha\in W_1F^*x} d_{s(\alpha)}$, there exists $w\in W_1$ such that $w\geqslant x$. Since W_1 is finite, there exists $w_1\in W_1$ such that for infinitely many n there exists $x\in X_n$ with $w_1\geqslant x$. Since w_1F^1 is finite, there exists $\lambda_1\in w_1F^1$ such that for infinitely many n, we have $r(\lambda_1)\geqslant x$ for some $x\in X_n$. We set $w_2:=r(\lambda_1)$ which is in X_2 by the claim above. Proceeding in this way, we produce an infinite path $\lambda=\lambda_1\lambda_2\ldots$ in F such that $s(\lambda_n)\in X_n$ for all n.

For each $w\in W_1$ such that $w\geqslant s(\lambda_n)$ for some n, we define $n_w:=\min\{n\colon w\geqslant s(\lambda_n)\}$. Let $N:=\max\{n_w\colon w\in W_1 \text{ and } w\geqslant s(\lambda_n) \text{ for some } n\}$. We claim that $F^1r(\lambda_n)=\{\lambda_n\}$ for all $n\geqslant N$. Fix $n\geqslant N$, and $e\in F^1r(\lambda_n)$. Since $r(\lambda_n)=s(\lambda_{n+1})\in X_{n+1}$, we have $s(e)\in X_n$. Hence $W_1F^*s(e)$ is nonempty, so we may fix $\beta\in W_1F^*s(e)$. Now βe is the unique path in $s(\beta)F^*r(\lambda_n)$ by definition of X_{n+1}. Let α be the unique path from $s(\beta)$ to $s(\lambda_{n_{s(\beta)}})$. Since $n_{s(\beta)}\leqslant N\leqslant n$, we have $\alpha\lambda_{n_{s(\beta)}}\lambda_{n_{s(\beta)}+1}\ldots\lambda_n$ in $s(\beta)F^*r(\lambda_n)$, and the uniqueness of this path then forces $\beta e=\alpha\lambda_{n_{s(\beta)}}\lambda_{n_{s(\beta)}+1}\ldots\lambda_n$, and in particular $e=\lambda_n$. Thus $F^1r(\lambda_n)=\{\lambda_n\}$ as required.

Since $F^1r(\lambda_n)=\{\lambda_n\}$, we have $W_1F^*r(\lambda_n)=W_1F^*\lambda_n=\{\beta\lambda_n\colon \beta\in W_1F^*s(\lambda_n)\}$. Hence that $r(\lambda_n)\in X_{n+1}$ and that $s(\lambda_n)\in X_n$ imply that

$$d_{r(\lambda_n)}=\sum_{\alpha\in W_1F^*r(\lambda_n)} d_{s(\alpha)}=\sum_{\beta\in W_1F^*s(\lambda_n)} d_{s(\beta)}=d_{s(\lambda_n)}$$

for all $n \geqslant N$. This implies $d_{s(\lambda_n)} = d_{s(\lambda_N)}$ for all $n \geqslant N$. Moreover, $\{y \in F^0\colon\ y \ngeqslant s(\lambda_n)$ for all $n\}$ is a saturated hereditary subset, and the quotient of A by the ideal corresponding to this set is an AF-algebra with a Bratteli diagram of the form

$$d_{s(\lambda_N)} \longrightarrow d_{s(\lambda_N)} \longrightarrow d_{s(\lambda_N)} \longrightarrow d_{s(\lambda_N)} \longrightarrow d_{s(\lambda_N)} \longrightarrow \cdots$$

Hence this quotient is isomorphic to $M_{d_{s(\lambda_N)}}(\mathbb{C})$, which contradicts the hypothesis that A has no finite-dimensional quotients. □

Lemma 3.3. *Let A be an AF-algebra. Then A has no nonzero finite-dimensional quotients if and only if there exists a Bratteli diagram (E, d) for A satisfying the following two properties*:

(1) $d_v \geqslant 2$ *for all* $v \in E^0$; *and*
(2) *for all* $n \in \mathbb{N}$ *and for each* $v \in V_{n+1}$ *either* $d_v > \sum_{e\in E^1 v} d_{s(e)}$ *or there exists* $w \in V_n$ *with* $|wE^1 v| \geqslant 2$.

Proof. If A has no nonzero finite-dimensional quotients, then by Lemma 3.1 there exists a Bratteli diagram for A satisfying (1). Lemma 3.2 shows that this Bratteli diagram may be telescoped to obtain a Bratteli diagram for A satisfying (2). The vertices of the telescoped diagram are a subset of those of the original diagram, and the values of d_v are the same for those vertices v common to both. In particular, telescoping preserves property (1), so the telescoped Bratteli diagram will then satisfy both (1) and (2).

Conversely, suppose that there exists a Bratteli diagram (E, d) for A satisfying (1) and (2). If I is a proper ideal of A, then I corresponds to a saturated hereditary subset H, and the complement $(E \setminus H, d)$ of H in (E, d) is a Bratteli diagram for A/I. Fix a vertex v in this complement. Since H is saturated hereditary, there exists an edge $e_1 \in E^1$ with $s(e_1) = v$ and $r(e_1)$ in the complement also. Inductively, we may produce an infinite path $e_1 e_2 \ldots$ in the complement. It follows from property (2) that $d_{s(e_i)} < d_{s(e_{i+1})}$ for all i, which implies that the function $d : (E \setminus H)^0 \to \mathbb{N}$ is unbounded. Hence A/I is infinite dimensional. □

Lemma 3.4. *Suppose A is an AF-algebra with no unital quotients. Then any Bratteli diagram for A can be telescoped to obtain a second Bratteli diagram (E, d) for A such that for all $v \in E^0$ we have $d_v > \sum_{e\in E^1 v} d_{s(e)}$.*

Proof. Let (F, d) be a Bratteli diagram for A with F^0 partitioned into levels as $F^0 = \bigsqcup_{n=1}^{\infty} W_n$. It suffices to show that for every m there exists $n \geqslant m$ such that for every $v \in W_n$ we have $d_v > \sum_{\alpha\in W_m F^* v} d_{s(\alpha)}$. Suppose not, and seek a contradiction. That is, we suppose that there exists m such that for every $n \geqslant m$ the set

$$Y_n := \left\{x \in W_n\colon\ d_x = \sum_{\alpha\in W_m F^* x} d_{s(\alpha)}\right\}$$

is nonempty. By telescoping (F, d) to $\bigsqcup_{n=m}^{\infty} W_n$ we may assume $m = 1$. If we let

$$T := \left\{w \in F^0\colon\ \text{for infinitely many } n \text{ there exists } x \in Y_n \text{ with } w \geqslant x\right\},$$

then the complement of T is a saturated hereditary subset, and the quotient of A by the ideal corresponding to this complement has a Bratteli diagram obtained by restricting to the vertices in T. Along similar lines to Lemma 3.2, one can show that if $n \leqslant p$, $x \in Y_p$, and $v \in W_n$ with $v \geqslant x$, then $v \in Y_n$. Hence each $v \in T \cap W_n$ is in Y_n. This implies that each $v \in T$ has the property that $d_v = \sum_{e \in F^1 v} d_{s(e)}$, and hence all the inclusions in the corresponding directed system are unital. Thus the quotient of A considered above is unital. This contradicts the hypothesis that A has no unital quotients. □

Lemma 3.5. *Let A be an AF-algebra. Then A has no unital quotients if and only if A has a Bratteli diagram (E, d) such that for all $v \in E^0$ we have both $d_v \geqslant 2$ and $d_v > \sum_{e \in E^1 v} d_{s(e)}$.*

Proof. If A has no unital quotients, then the existence of such a Bratteli diagram follows from Lemmas 3.3 and 3.4. Conversely, suppose that A has such a Bratteli diagram (E, d), and fix a nonzero quotient A/I of A. There is a subdiagram (F, d) of (E, d) which is a Bratteli diagram for A/I. In particular $d_v > \sum_{e \in F^1 v} d_{s(e)}$ for all $v \in F^0$. It follows that the inclusions in the direct limit decomposition of A corresponding to (F, d) are all nonunital. Hence A/I is nonunital. □

Lemma 3.6. *Let A be a C^*-algebra which is generated by finite-dimensional subalgebras B and C. Suppose that $B = \bigoplus_{v \in V} B^v$ where each $B^v \cong M_{b_v}(\mathbb{C})$ and that $C = \bigoplus_{w \in W} C^w$ where each $C^w \cong M_{c_w}(\mathbb{C})$. For each $v \in V$ suppose that q^v is a minimal projection in B^v such that $q^v \in C$ and $(1_{B^v} - q^v)C = \{0\}$. For each v, w, let $m_{v,w}$ denote the rank of $q^v 1_{C^w}$ in C^w, and let*

$$a_w := c_w + \sum_{v \in V} (b_v - 1) m_{v,w}.$$

Then $A = \bigoplus_{w \in W} A^w$ where each $A^w \cong M_{a_w}(\mathbb{C})$. Moreover, the inclusion $C^w \hookrightarrow A^w$ has multiplicity 1 for $w \in W$, and the inclusion $B \hookrightarrow A$ has multiplicity matrix $(m_{v,w})_{v \in V, w \in W}$. Finally, the unit 1_A of A is equal to $(1_B - \sum_{v \in V} q^v) + 1_C$.

Proof. The assumptions on the q^v imply that $(1_B - \sum_{v \in V} q^v) + 1_C$ is the unit of A. To obtain the desired decomposition of A, we construct a family of matrix units for A. We begin by fixing convenient systems of matrix units for the B^v and the C^w.

For $v \in V$, let $\{\beta^v_{r,s}\colon 0 \leqslant r, s \leqslant b_v - 1\}$ be a family of matrix units for B^v such that $\beta^v_{0,0} = q^v$. Similarly, for $w \in W$ let $\{\gamma^w_{k,l}\colon 0 \leqslant k, l \leqslant c_w - 1\}$ be a family of matrix units for C^w such that for each $v \in V$ there exists a subset $\kappa_{v,w} \subset \{0, 1, \dots, c_w - 1\}$ satisfying $q^v 1_{C^w} = \sum_{k \in \kappa_{v,w}} \gamma^w_{k,k}$. Note that the subsets $\{\kappa_{v,w}\}_{v \in V}$ of $\{0, 1, \dots, c_w - 1\}$ are mutually disjoint and satisfy $|\kappa_{v,w}| = m_{v,w}$.

We are now ready to define the desired matrix units for A; these matrix units will be indexed by the set

$$I_w := \bigl(\{0, 1, \dots, c_w - 1\} \times \{0\}\bigr) \sqcup \bigsqcup_{v \in V} \bigl(\kappa_{v,w} \times \{1, 2, \dots, b_v - 1\}\bigr)$$

for $w \in W$. We have $|I_w| = c_w + \sum_{v\in V} |\kappa_{v,w}|(b_v - 1) = a_w$. Define elements $\{\alpha^w_{(k,r),(l,s)}\colon w \in W,\ (k,r),(l,s) \in I_w\}$ by

$$\alpha^w_{(k,r),(l,s)} := \begin{cases} \gamma^w_{k,l} & \text{if } r = s = 0, \\ \gamma^w_{k,l}\beta^v_{0,s} & \text{if } r = 0,\, l \in \kappa_{v,w} \text{ and } s \geqslant 1, \\ \beta^{v'}_{r,0}\gamma^w_{k,l} & \text{if } k \in \kappa_{v',w},\, r \geqslant 1 \text{ and } s = 0, \\ \beta^{v'}_{r,0}\gamma^w_{k,l}\beta^v_{0,s} & \text{if } k \in \kappa_{v',w},\, r \geqslant 1,\, l \in \kappa_{v,w} \text{ and } s \geqslant 1. \end{cases}$$

We first claim that for each $w, w' \in W$, each $(k,r),(l,s) \in I_w$ and each $(k',r'),(l',s') \in I_{w'}$,

$$\alpha^w_{(k,r),(l,s)}\alpha^{w'}_{(k',r'),(l',s')} = \begin{cases} \alpha^w_{(k,r),(l',s')} & \text{if } w = w' \text{ and } (l,s) = (k',r'), \\ 0 & \text{otherwise.} \end{cases} \tag{3.3}$$

To verify (3.3), we consider four cases.

Case 1. $s = r' = 0$. Since $\gamma^w_{k,l}$ are matrix units and since the C^w are orthogonal, we have

$$\gamma^w_{k,l}\gamma^{w'}_{k',l'} = \begin{cases} \gamma^w_{k,l'} & \text{if } w = w' \text{ and } l = k', \\ 0 & \text{otherwise.} \end{cases}$$

This implies (3.3) in the case $s = r' = 0$.

Case 2. $s \geqslant 1$ and $r' = 0$. Then $\beta^v_{0,s}\gamma^{w'}_{k',l'} = \beta^v_{0,s}\beta^v_{s,s}\gamma^{w'}_{k',l'} = 0$ because $\beta^v_{s,s} \leqslant \sum_{s=1}^{b_v-1}\beta^v_{s,s} = 1_{B^v} - q^v$ which is orthogonal to C by assumption. This shows $\alpha^w_{(k,r),(l,s)}\alpha^{w'}_{(k',r'),(l',s')} = 0$.

Case 3. $s = 0$ and $r' \geqslant 1$. This case follows from Case 2 by taking adjoints.

Case 4. $s \geqslant 1$ and $r' \geqslant 1$. Then

$$\gamma^w_{k,l}\beta^v_{0,s}\beta^{v'}_{r',0}\gamma^{w'}_{k',l'} = \begin{cases} \gamma^w_{k,l}\beta^v_{0,0}\gamma^{w'}_{k',l'} & \text{if } v = v' \text{ and } s = r', \\ 0 & \text{otherwise.} \end{cases}$$

Since $\gamma^w_{k,l}\beta^v_{0,0} = \gamma^w_{k,l}$ we have

$$\gamma^w_{k,l}\beta^v_{0,0}\gamma^{w'}_{k',l'} = \gamma^w_{k,l}\gamma^{w'}_{k',l'} = \begin{cases} \gamma^w_{k,l'} & \text{if } w = w' \text{ and } l = k', \\ 0 & \text{otherwise.} \end{cases}$$

These show (3.3) in Case 4, completing the proof of the claim.

For each $w \in W$, let $A^w := \operatorname{span}\{\alpha^w_{(k,r),(l,s)}\colon (k,r),(l,s) \in I_w\} \subset A$. From (3.3), we see that A^w is isomorphic to $M_{a_w}(\mathbb{C})$ for each $w \in W$, and that $\{A^w\}_{w\in W}$ are orthogonal to each other. We next show that $A = \sum_{w\in W} A^w$. To see this, it suffices to show that all the matrix units $\beta^v_{r,s}$ and $\gamma^w_{k,l}$ for B and C belong to $\sum_{w\in W} A^w$. If $l \in \kappa_{v,w}$, then

$$\gamma^w_{k,l}\beta^v_{0,0} = \left(\gamma^w_{k,l}1_{C^w}\right)q^v = \gamma^w_{k,l}\left(\sum_{l'\in\kappa_{v,w}}\gamma^w_{l',l'}\right) = \gamma^w_{k,l}.$$

Similarly, we get $\beta^{v'}_{0,0}\gamma^w_{k,l} = \gamma^w_{k,l}$ if $k \in \kappa_{v',w}$. We may deduce from these two equalities that $\alpha^w_{(k,r),(l,s)} = \beta^{v'}_{r,0}\gamma^w_{k,l}\beta^v_{0,s}$ for all $k \in \kappa_{v',w}$, all $r \geqslant 0$, all $l \in \kappa_{v,w}$ and all $s \geqslant 0$. For each $v \in V$, we have

$$\beta^v_{0,0} = q^v = \sum_{w\in W} q^v 1_{C^w} = \sum_{w\in W}\sum_{k\in\kappa_{v,w}} \gamma^w_{k,k}.$$

It follows that

$$\beta^v_{r,s} = \beta^v_{r,0}\beta^v_{0,0}\beta^v_{0,s} = \sum_{w\in W}\sum_{k\in\kappa_{v,w}} \beta^v_{r,0}\gamma^w_{k,k}\beta^v_{0,s} = \sum_{w\in W}\sum_{k\in\kappa_{v,w}} \alpha^w_{(k,r),(k,s)} \in \sum_{w\in W} A^w$$

for all $v \in V$ and all $0 \leqslant r, s \leqslant b_v - 1$. We also have $\gamma^w_{k,l} = \alpha^w_{(k,0),(l,0)}$ for $w \in W$ and $0 \leqslant k, l \leqslant c_w - 1$. Thus we get $A = \sum_{w\in W} A^w$.

It is clear that the inclusion $C^w \hookrightarrow A^w$ has multiplicity 1 for $w \in W$. To see that the inclusion $B \hookrightarrow A$ has multiplicity matrix $(m_{v,w})_{v\in V, w\in W}$, it suffices to see that for each $v \in V$ and $w \in W$, the product of the minimal projection $q^v \in B^v$ and the unit 1_{A^w} of A^w has rank $m_{v,w}$ in $A^w \cong M_{a_w}(\mathbb{C})$. Since $q^v \in C$, we have

$$q^v 1_{A^w} = q^v 1_{C^w} = \sum_{k\in\kappa_{v,w}} \gamma^w_{k,k} = \sum_{k\in\kappa_{v,w}} \alpha^w_{(k,0),(k,0)}.$$

This shows that the rank of $q^v 1_{A^w} \in A^w$ is $|\kappa_{v,w}| = m_{v,w}$. □

4. Realizations of AF-algebras

4.1. A construction of an ultragraph from a certain type of Bratteli diagram

In this section we show how to construct ultragraphs from certain Bratteli diagrams and use these ultragraphs to realize particular classes of AF-algebras as ultragraph C^*-algebras, Exel–Laca algebras, and graph C^*-algebras.

Definition 4.1. Let A be an AF-algebra with no nonzero finite-dimensional quotients. By Lemma 3.3 there exists a Bratteli diagram (E, d) for A satisfying the following two properties:

(1) $d_v \geqslant 2$ for all $v \in E^0$; and
(2) for all $n \in \mathbb{N}$ and for each $v \in V_{n+1}$ either $d_v > \sum_{e\in E^1 v} d_{s(e)}$ or there exists $w \in V_n$ with $|wE^1 v| \geqslant 2$.

We define

$$\Delta_v := d_v - \sum_{e\in E^1 v} (d_{s(e)} - 1).$$

The symbol Δ has been chosen to connote "difference." Note that from the property (1), $\Delta_v = d_v$ if and only if v is a source. In addition, it follows from the properties of our Bratteli diagram that $\Delta_v \geqslant 2$ for all $v \in E^0$.

We claim that for each $v \in E^0$ there exists an injection $k_v : E^1 v \to \{0, 1, \ldots, \Delta_v - 1\}$ such that there exists $e \in E^1 v$ with $k_v(e) = 0$ if and only if $d_v = \sum_{e \in E^1 v} d_{s(e)}$, and in this case e is not the only element of $s(e)E^1 v$. To justify this claim, first observe that

$$\Delta_v = d_v - \sum_{e \in E^1 v} \bigl(d_{s(e)} - 1\bigr) = d_v - \sum_{e \in E^1 v} d_{s(e)} + \sum_{e \in E^1 v} 1 = \Bigl(d_v - \sum_{e \in E^1 v} d_{s(e)}\Bigr) + \bigl|E^1 v\bigr|.$$

Hence if $d_v > \sum_{e \in E^1 v} d_{s(e)}$ we may always choose an injection $k_v : E^1 v \to \{0, 1, \ldots, \Delta_v - 1\}$ so that its image does not contain 0. On the other hand if $d_v = \sum_{e \in E^1 v} d_{s(e)}$, then by hypothesis on the Bratteli diagram there exists $w \in E^0$ with $|wE^1 v| \geqslant 2$ so we may choose a bijection $k_v : E^1 v \to \{0, 1, \ldots, \Delta_v - 1\}$ such that $e \in E^1 v$ with $k_v(e) = 0$ satisfies $s(e) = w$. This establishes the claim.

We now define an ultragraph $\mathcal{G} = (G^0, \mathcal{G}^1, r_{\mathcal{G}}, s_{\mathcal{G}})$ by

$$G^0 := \bigl\{v_i\colon\ v \in E^0 \text{ and } 1 \leqslant i \leqslant \Delta_v - 1\bigr\} \quad \text{and} \quad \mathcal{G}^1 := \bigl\{e_{v_i}\colon\ v_i \in G^0\bigr\}$$

with

$$s_{\mathcal{G}}(e_{v_i}) := v_i \quad \text{for all } v_i \in G^0, \qquad r_{\mathcal{G}}(e_{v_i}) := \{v_{i-1}\} \quad \text{for } 2 \leqslant i \leqslant \Delta_v - 1$$

and

$$r_{\mathcal{G}}(e_{v_1}) := \bigl\{w_k\colon \text{ there exists a path } \lambda = \lambda_1\lambda_2\ldots\lambda_n \text{ such that } s(\lambda) = v,\ r(\lambda) = w,$$
$$k_{r(\lambda_i)}(\lambda_i) = 0 \text{ for } i = 1, 2, \ldots, n-1, \text{ and } k_w(\lambda_n) = k \geqslant 1\bigr\}.$$

To check that $\mathcal{G}$ is an ultragraph, we only need to see that $r_{\mathcal{G}}(e_{v_1}) \neq \emptyset$.

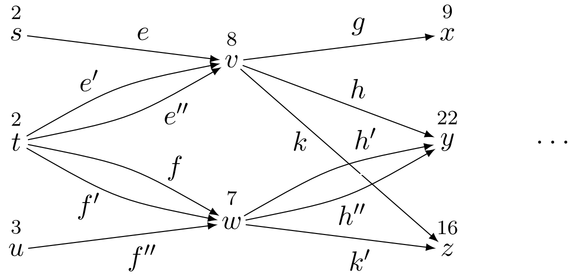
Lemma 4.2. *For all n and $v \in V_n$, the set $r_{\mathcal{G}}(e_{v_1})$ is nonempty and satisfies*

$$r_{\mathcal{G}}(e_{v_1}) = \bigl\{w_{k_w(e)}\colon\ w \in V_{n+1},\ e \in vE^1 w,\ k_w(e) \geqslant 1\bigr\} \cup \bigcup_{w \in V_{n+1},\ e \in vE^1 w,\ k_w(e) = 0} r_{\mathcal{G}}(e_{w_1}).$$

Proof. The latter equality follows from the definition of $r_{\mathcal{G}}(e_{v_1})$. For each $v \in V_n$, there exists $w \in V_{n+1}$ such that $vE^1 w \neq \emptyset$. By the assumption on k_w, there exists $e \in vE^1 w$ such that $k_w(e) \geqslant 1$. Thus $w_{k_w(e)} \in r_{\mathcal{G}}(e_{v_1})$. This shows that $r_{\mathcal{G}}(e_{v_1})$ is nonempty. □

Remark 4.3. By definition, $r_{\mathcal{G}}(e_{v_1}) \subset \bigcup_{k=n+1}^{\infty} V_k$ for $v \in V_n$. One can show that this property together with the equality in Lemma 4.2 uniquely determines $\{r_{\mathcal{G}}(e_{v_1})\}_{v \in E^0}$.

Example 4.4 *(An example of the ultragraph construction).* Consider a Bratteli diagram (E, d) satisfying conditions (1) and (2) of Lemma 3.3 and whose first three levels are as illustrated below. In the diagram, each vertex is labeled with its name, and above the label a appears the integer d_a.



The values of Δ for the vertices visible in the diagram are

$$\begin{array}{lll} \Delta_s = 2 & \Delta_v = 5 & \Delta_x = 2 \\ \Delta_t = 2 & \Delta_w = 3 & \Delta_y = 3 \\ \Delta_u = 3 & & \Delta_z = 3 \end{array}$$

So the corresponding section of the resulting ultragraph $\mathcal{G}$ will have vertices

$$G^0 = \{s_1, t_1, u_1, u_2, v_1, v_2, v_3, v_4, w_1, w_2, x_1, y_1, y_2, z_1, z_2, \ldots\},$$

and each of these vertices a_i will emit exactly one ultraedge e_{a_i}. For $i \neq 1$, we have $r_{\mathcal{G}}(e_{a_i}) = \{a_{i-1}\}$. To determine the ranges of the e_{a_1}, we must choose injections $k_a : E^1 a \to \{0, 1, \ldots, \Delta_a - 1\}$ for $a \in E^0$ with the properties described above; in particular, this necessitates that 0 is in the image of k_a only when $a = w$ or $a = y$, and also that $k_w(f'') \neq 0$ and $k_y(h) \neq 0$.

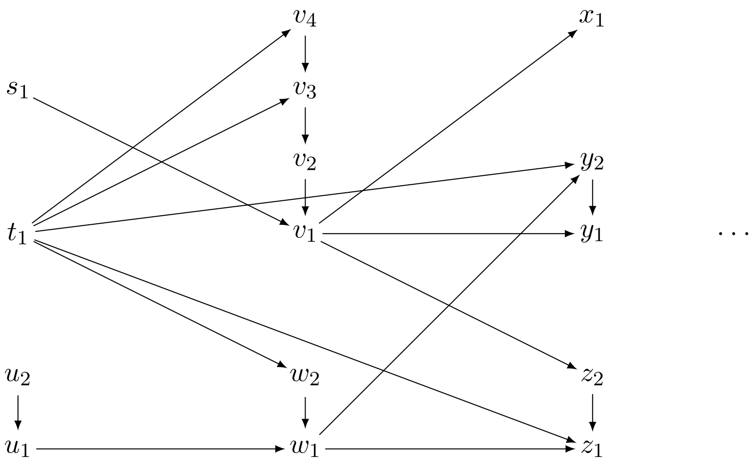
One possible set of choices of injections k_a is

$$\begin{array}{lllllll} k_v(e) = 1, & k_v(e') = 3, & k_v(e'') = 4, & k_x(g) = 1, & & \\ k_w(f) = 0, & k_w(f') = 2, & k_w(f'') = 1, & k_y(h) = 1, & k_y(h') = 0, & k_y(h'') = 2, \\ & & & k_z(k) = 2, & k_z(k') = 1. & \end{array}$$

We can calculate

$$r_{\mathcal{G}}(e_{s_1}) = \{v_1\}, \qquad r_{\mathcal{G}}(e_{t_1}) = \{v_3, v_4, w_2, y_2, z_1\} \cup r_{\mathcal{G}}(e_{y_1}), \qquad r_{\mathcal{G}}(e_{u_1}) = \{w_1\},$$
$$r(e_{v_1}) = \{x_1, y_1, z_2\}, \quad \text{and} \quad r(e_{w_1}) = \{z_1, y_2\} \cup r_{\mathcal{G}}(e_{y_1}).$$

We may now draw the fragment of the ultragraph $\mathcal{G}$ corresponding to the given fragment of the Bratteli diagram (E, d).



Note that by definition of the ultragraph $\mathcal{G}$, each vertex emits exactly one ultraedge, so in the picture any multiple arrows leaving the same vertex actually have the same label and constitute a single ultraedge of $\mathcal{G}$.

4.2. Sufficient conditions for realizations

Theorem 4.5. *Let A be an AF-algebra with a Bratteli diagram satisfying the conditions of Lemma* 3.3. *If $\mathcal{G}$ is an ultragraph constructed from this Bratteli diagram as in Definition* 4.1*, then $A \cong C^*(\mathcal{G})$. In addition, $C^*(\mathcal{G})$ is an Exel–Laca algebra.*

Proof. Let (E,d) be a Bratteli diagram for A with the vertices partitioned into levels as $E^0 = \bigsqcup_{n=1}^{\infty} V_n$ and satisfying the conditions of Lemma 3.3, and let $\mathcal{G}$ be an ultragraph constructed from (E,d) as in Definition 4.1. Our strategy is to find a direct limit decomposition of $C^*(\mathcal{G})$ so that at each level we may apply Lemma 3.6 to see that the inclusion of finite-dimensional algebras is the same as the corresponding inclusion in the direct limit decomposition of A determined by (E,d).

For each $v \in E^0$ let

$$C^v := C^*\big(\{s_{e_{v_i}}\colon\ 1 \leqslant i \leqslant \Delta_v - 1\}\big).$$

We have $s_{e_{v_i}} s_{e_{v_i}}^* = p_{v_i}$ for $1 \leqslant i \leqslant \Delta_v - 1$ and $s_{e_{v_i}}^* s_{e_{v_i}} = p_{v_{i-1}}$ for $2 \leqslant i \leqslant \Delta_v - 1$. We define a projection $q^v := p_{r_{\mathcal{G}}(e_{v_1})} = s_{e_{v_1}}^* s_{e_{v_1}} \in C^v$, which is orthogonal to p_{v_i} for $1 \leqslant i \leqslant \Delta_v - 1$. These computations show that there exist matrix units $\{\gamma_{k,l}^v\colon\ 0 \leqslant k,l \leqslant \Delta_v - 1\}$ in C^v such that $\gamma_{0,0}^v = q^v$, $\gamma_{i,i}^v = p_{v_i}$ and $\gamma_{i,i-1}^v = s_{e_{v_i}}$ for $1 \leqslant i \leqslant \Delta_v - 1$. Explicitly, $\gamma_{k,l}^v \in C^v$ is given by

$$\gamma_{k,l}^v := s_{e_{v_k}} s_{e_{v_{k-1}}} \cdots s_{e_{v_1}} q^v s_{e_{v_1}}^* s_{e_{v_2}}^* \cdots s_{e_{v_l}}^*$$

for $0 \leqslant k,l \leqslant \Delta_v - 1$. This shows that C^v is isomorphic to $M_{\Delta_v}(\mathbb{C})$ with minimal projection q^v and the unit $\sum_{i=1}^{\Delta_v - 1} p_{v_i} + q^v$. For each $n \in \mathbb{N}$

$$C_n := C^*\big(\{s_{e_{v_i}}\colon\ v \in V_n \text{ and } 1 \leqslant i \leqslant \Delta_v - 1\}\big)$$

is equal to $\bigoplus_{v\in V_n} C^v$. Moreover, for $n \in \mathbb{N}$, define

$$B_n := C^*\left(\bigcup_{j=1}^{n} C_j\right) = C^*\left(\left\{s_{e_{v_i}}\colon v \in \bigcup_{j=1}^{n} V_j \text{ and } 1 \leqslant i \leqslant \Delta_v - 1\right\}\right).$$

Claim. *For each $n \in \mathbb{N}$, the unit 1_{B_n} of B_n is given by $\sum_{v\in\bigcup_{j=1}^n V_j}\sum_{i=1}^{\Delta_v-1} p_{v_i} + \sum_{v\in V_n} q^v$, and there exists a decomposition $B_n = \bigoplus_{v\in V_n} B^v$ such that each $B^v \cong M_{d_v}(\mathbb{C})$ with minimal projection q^v; and for each $n \in \mathbb{N}$, the inclusion $B_n \hookrightarrow B_{n+1}$ has multiplicity matrix $(|vE^1w|)_{v\in V_n, w\in V_{n+1}}$.*

We proceed by induction on n. When $n = 1$, let $B^v := C^v$ for $v \in V_1$. Then $B_1 = C_1$ has the decomposition $B_1 = \bigoplus_{v\in V_1} B^v$. For each $v \in V_1$, we have $\Delta_v = d_v$ because v is a source. Hence $B^v = C^v$ is isomorphic to $M_{d_v}(\mathbb{C})$ with minimal projection q^v and the unit $\sum_{i=1}^{\Delta_v-1} p_{v_i} + q^v$. This shows the claim in the case $n = 1$. For the inductive step, assume that B_n has the desired decomposition. To apply Lemma 3.6 to the C^*-algebra B_{n+1} which is generated by B_n and C_{n+1}, we check that for each $v \in V_n$ the minimal projection $q^v \in B^v$ is in C_{n+1} and satisfies $(1_{B^v} - q^v)C_{n+1} = \{0\}$. We see that

$$\sum_{v\in V_n}\bigl(1_{B^v} - q^v\bigr) = 1_{B_n} - \sum_{v\in V_n} q^v = \sum_{v\in\bigcup_{j=1}^n V_j}\sum_{i=1}^{\Delta_v-1} p_{v_i}$$

which is orthogonal to C_{n+1}. This proves $(1_{B^v} - q^v)C_{n+1} = \{0\}$ for all $v \in V_n$. For each $v \in V_n$, Lemma 4.2 implies

$$\begin{aligned} q^v = p_{r_{\mathcal{G}}(e_{v_1})} &= \sum_{w\in V_{n+1}}\left(\sum_{\substack{e\in vE^1w\\ k_w(e)\geqslant 1}} p_{w_{k_w(e)}} + \sum_{\substack{e\in vE^1w\\ k_w(e)=0}} p_{r_G(e_{w_1})}\right)\\ &= \sum_{w\in V_{n+1}}\sum_{e\in vE^1w} \gamma^w_{k_w(e),k_w(e)}. \end{aligned} \tag{4.1}$$

Hence $q^v \in C_{n+1}$. Thus we can apply Lemma 3.6 to obtain the decomposition $B_{n+1} = \bigoplus_{w\in V_{n+1}} B^w$. Since the inclusion $C^w \hookrightarrow B^w$ has multiplicity 1 for $w \in W$, the projection q^w is minimal in B^w. From (4.1), $q^v 1_{C^w}$ has rank $|vE^1w|$ in C^w for $w \in V_{n+1}$. The definition of Δ_w implies that

$$d_w = \Delta_w + \sum_{w\in V_{n+1}} (d_v - 1)\bigl|vE^1w\bigr|.$$

Hence B^w is isomorphic to $M_{d_w}(\mathbb{C})$ for $w \in V_{n+1}$. The conclusion of Lemma 3.6 also shows that the inclusion $B_n \hookrightarrow B_{n+1}$ has multiplicity matrix $(|vE^1w|)_{v\in V_n, w\in V_{n+1}}$, and that the unit of B_{n+1} is equal to $\sum_{v\in\bigcup_{j=1}^{n+1} V_j}\sum_{i=1}^{\Delta_v-1} p_{v_i} + \sum_{w\in V_{n+1}} q^w$. This proves the claim.

We see that $\bigcup_{n=1}^\infty B^n$ contains $\{s_e\colon e \in \mathcal{G}^1\}$. Since each vertex v in $\mathcal{G}$ emits exactly one ultraedge e, $p_v = s_e s_e^*$ is contained in $\bigcup_{n=1}^\infty B^n$. Thus $\bigcup_{n=1}^\infty B^n$ contains all the generators of $C^*(\mathcal{G})$.

Hence $C^*(\mathcal{G}) = \overline{\bigcup_{n=1}^{\infty} B^n}$ is an AF-algebra, and the preceding paragraphs show that (E, d) is a Bratteli diagram for $C^*(\mathcal{G})$, giving $A \cong C^*(\mathcal{G})$. Since every vertex of $\mathcal{G}$ emits exactly one ultraedge, $C^*(\mathcal{G})$ is an Exel–Laca algebra (see Remark 2.10). □

Corollary 4.6. *If A is an AF-algebra with no nonzero finite-dimensional quotients, then A is isomorphic to an Exel–Laca algebra.*

Proof. Since A has no nonzero finite-dimensional quotients, Lemma 3.3 implies that A has a Bratteli diagram satisfying the conditions stated. It follows from Theorem 4.5 that A is isomorphic to an Exel–Laca algebra. □

The following result is important in that it is one of the few instances where we can give a complete characterization of AF-algebras in a certain graph C^*-algebra class. In particular, we give necessary and sufficient conditions for an AF-algebra to be the C^*-algebra of a row-finite graph with no sinks.

Theorem 4.7. *Let A be an AF-algebra. Then the following are equivalent*:

(1) *A has no (nonzero) unital quotients.*
(2) *A is isomorphic to the C^*-algebra of a row-finite graph with no sinks.*

Proof. We shall first prove that (1) implies (2). Suppose that A has no unital quotients. By Corollary 3.5 there is a Bratteli diagram (E, d) for A such that for all $v \in E^0$ we have both $d_v \geqslant 2$ and $d_v > \sum_{e \in E^1 v} d_{s(e)}$. Let $\mathcal{G}$ be an ultragraph constructed from (E, d) as in Definition 4.1. Theorem 4.5 implies that $A \cong C^*(\mathcal{G})$. Furthermore, since $d_v > \sum_{e \in E^1 v} d_{s(e)}$, we have $k_v(e) \geqslant 1$ for all $v \in E^0$ and $e \in E^1 v$. For $v \in E^0$, Lemma 4.2 implies $r_{\mathcal{G}}(e_{v_1}) = \{w_{k_w(e)}\colon\ w \in V_{n+1},\ e \in vE^1 w,\ k_w(e) \geqslant 1\}$. Thus, $r_{\mathcal{G}}(e)$ is finite for every $e \in \mathcal{G}^1$. Hence $C^*(\mathcal{G})$ is isomorphic to a graph C^*-algebra of a row-finite graph with no sinks (see [16, Remark 5.25]).

We next prove that (2) implies (1). Suppose that $A \cong C^*(E)$, where E is a row-finite graph with no sinks. Since $C^*(E)$ is an AF-algebra, it follows from [17, Theorem 2.4] that E has no cycles. Thus E satisfies Condition (K), and [2, Theorem 4.4] implies that every ideal of $C^*(E)$ is gauge invariant. Suppose I is a proper ideal of $C^*(E)$. Then $I = I_H$ for some saturated hereditary proper subset $H \subset E^0$, and $C^*(E)/I_H \cong C^*(E_H)$, where E_H is the nonempty subgraph of E with $E_H^0 := E^0 \setminus H$ and $E_H^1 := \{e \in E^1\colon\ r(e) \notin H\}$ (see [2, Theorem 4.1]). Since H is saturated hereditary, that E has no sinks implies that E_H has no sinks. Since E has no cycles, E_H also has no cycles. Because E_H is a nonempty graph with no cycles and no sinks, E_H^0 is infinite. Thus $C^*(E_H)$ is nonunital [17, Proposition 1.4]. □

Corollary 4.8. *Let A be a stable AF-algebra. Then there is a row-finite graph E with no sinks such that $A \cong C^*(E)$. In particular, A is isomorphic to a graph C^*-algebra, to an Exel–Laca algebra, and to an ultragraph C^*-algebra.*

Proof. Since any nonzero quotient of a stable C^*-algebra is stable, every quotient of A is stable, and in particular nonunital. The result then follows from Theorem 4.7. □

Lemma 4.9. *Let $\mathcal{G} = (G^0, \mathcal{G}^1, r, s)$ be an ultragraph. Let $\widetilde{\mathcal{G}} = (\widetilde{G}^0, \widetilde{\mathcal{G}}^1, \tilde{r}, \tilde{s})$ be the ultragraph defined by $\widetilde{G}^0 := G^0 \sqcup \{v_0\}$ and $\widetilde{\mathcal{G}}^1 := \mathcal{G}^1 \sqcup \{e_0\}$ with*

$$\tilde{s}|_{\mathcal{G}^1} = s, \qquad \tilde{s}(e_0) = v_0, \qquad \tilde{r}|_{\mathcal{G}^1} = r, \quad \text{and} \quad \tilde{r}(e_0) = G^0.$$

Then $C^*(\widetilde{\mathcal{G}}) \cong M_2(C^*(\mathcal{G})^+)$, where $C^*(\mathcal{G})^+$ is the minimal unitization of $C^*(\mathcal{G})$.

Proof. We first notice that the algebra $\widetilde{\mathcal{G}}^0$ is generated by the algebra $\mathcal{G}^0 \subseteq \mathcal{P}(\widetilde{G}^0)$ and the two elements $G_0, \{v_0\} \in \mathcal{P}(\widetilde{G}^0)$. The universal property of $C^*(\widetilde{\mathcal{G}})$ implies that there is a $*$-homomorphism $\phi : C^*(\widetilde{\mathcal{G}}) \to M_2(C^*(\mathcal{G})^+)$ satisfying

$$\phi(p_A) = \begin{pmatrix} p_A & 0 \\ 0 & 0 \end{pmatrix} \quad \text{for all } A \in \mathcal{G}^0 \quad \text{and} \quad \phi(s_e) = \begin{pmatrix} s_e & 0 \\ 0 & 0 \end{pmatrix} \quad \text{for all } e \in \mathcal{G}^1$$

and

$$\phi(p_{G^0}) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad \phi(p_{v_0}) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \text{and} \quad \phi(s_{e_0}) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

The Gauge-Invariant Uniqueness Theorem [22, Theorem 6.8] shows that ϕ is injective. Standard calculations show that the image under ϕ of the generating Cuntz–Krieger $\widetilde{\mathcal{G}}$-family in $C^*(\widetilde{\mathcal{G}})$ generates $M_2(C^*(\mathcal{G})^+)$. Hence ϕ is an isomorphism. □

Corollary 4.10. *Let A be a C^*-algebra, and let A^+ denote the minimal unitization of A. If A is isomorphic to an Exel–Laca algebra, then $M_2(A^+)$ is isomorphic to an Exel–Laca algebra.*

Proof. If A is isomorphic to an Exel–Laca algebra, then by Remark 2.10 $A \cong C^*(\mathcal{G})$ where $\mathcal{G}$ is an ultragraph with bijective source map. By Lemma 4.9 $C^*(\widetilde{\mathcal{G}}) \cong M_2(A^+)$, and since $\widetilde{\mathcal{G}}$ is an ultragraph with bijective source map, $C^*(\widetilde{\mathcal{G}})$ is an Exel–Laca algebra. □

The following example shows that the converse of Corollary 4.6 does not hold.

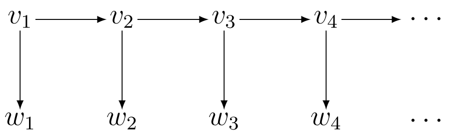
Example 4.11. Let A be a nonunital, simple AF-algebra (such as $\mathcal{K}$). By Corollary 4.22 A is isomorphic to an Exel–Laca algebra, and by Corollary 4.10 $M_2(A^+)$ is an Exel–Laca algebra. However, $M_2(A^+)$ has a quotient isomorphic to the finite-dimensional C^*-algebra $M_2(\mathbb{C})$. Thus the converse of Corollary 4.6 does not hold. (It is also worth mentioning that $M_2(\mathbb{C})$ is a quotient of an Exel–Laca algebra, but $M_2(\mathbb{C})$ is not itself an Exel–Laca algebra; cf. Corollary 4.19.)

The following elementary example shows that the C^*-algebra of a row-finite graph with sinks may admit unital quotients (cf. Theorem 4.7).

Example 4.12. The AF-algebra $M_2(\mathbb{C}) \oplus M_2(\mathbb{C})$ is isomorphic to the C^*-algebra of the graph $\bullet \longleftarrow \bullet \longrightarrow \bullet$ by [17, Corollary 2.3]. However, this C^*-algebra has $M_2(\mathbb{C})$ as a unital quotient. Thus graphs with sinks can have associated C^*-algebras that are AF-algebras with proper unital quotients.

The next example is more intriguing. Before considering this example, one is tempted to believe that if E is a row-finite graph, then $C^*(E)$ is isomorphic to a direct sum of a countable collection of algebras of compact operators on (finite or countably infinite dimensional) Hilbert spaces and the C^*-algebra of a row-finite graph with no sinks (see Proposition 4.14). This would give a characterization of AF-algebras associated to row-finite graphs along similar lines to Theorem 4.7. However, the example shows that this is not the case in general.

Example 4.13. Let E be the graph



Then for each $n \in \mathbb{N}$ the set $H_n := \{v_n, v_{n+1}, \ldots\} \cup \{w_n, w_{n+1}, \ldots\}$ is a saturated hereditary subset of E, and $C^*(E)/I_{H_n}$ is a finite-dimensional C^*-algebra. Thus $C^*(E)$ is an AF-algebra with infinitely many finite-dimensional quotients. This shows that, unlike what occurs for row-finite graphs with no sinks (cf. Theorem 4.7), the situation with sinks is much more complicated. It also shows that $C^*(E)$ does not have a Bratteli diagram of the types described in Lemma 3.4 or Lemma 3.5. Hence our construction of the ultragraph described in Section 4.1 cannot be applied.

By eliminating the bad behavior arising in the preceding example, we obtain a limited extension of Theorem 4.7 to graphs containing sinks.

Proposition 4.14. *Let A be an AF algebra. Then the following are equivalent*:

(1) *A is isomorphic to the C^*-algebra of a row-finite graph in which each vertex connects to at most finitely many sinks*; *and*
(2) *A has the form $(\bigoplus_{x\in X} M_{n_x}(\mathbb{C})) \oplus A'$ where X is an at most countably-infinite index set, each n_x is a positive integer, and A' is an AF algebra with no unital quotients.*

Proof. To see that (1) implies (2), we let E be a row-finite graph in which each vertex connects to at most finitely many sinks and such that $A \cong C^*(E)$. Since A is an AF-algebra, E has no cycles. Let $\mathrm{sinks}(E)$ denote the collection $\{v \in E^0\colon\ vE^1 = \emptyset\}$ of sinks in E. Let H be the smallest saturated hereditary subset of E^0 containing $\mathrm{sinks}(E)$. Since each vertex connects to at most finitely many sinks, H is equal to the set of $v \in E^0$ such that $vE^n = \emptyset$ for some n. Let F be the graph with vertices $F^0 := E^0 \setminus H$, edges $F^1 = \{e \in E^1\colon\ r(e) \notin H\}$ and range and source maps inherited from E. Note that the description of H above implies that F has no sinks; moreover F is row-finite because E is. We claim that

$$C^*(E) \cong \Bigg(\bigoplus_{v\in \mathrm{sinks}(E)} \mathcal{K}\big(\ell^2(E^*v)\big)\Bigg) \oplus C^*(F).$$

To prove this, we first define a Cuntz–Krieger E-family $\{q_v\colon\ v \in E^0\}$, $\{t_e\colon\ e \in E^1\}$ in $(\bigoplus_{v\in \mathrm{sinks}(E)} \mathcal{K}(\ell^2(E^*v))) \oplus C^*(F)$. We will denote the universal Cuntz–Krieger F-family by $\{p_v^F\colon\ v \in F^0\}$, $\{s_e^F\colon\ e \in F^1\}$, and we will denote the matrix units in each $\mathcal{K}(\ell^2(E^*v))$ by

$\{\Theta^v_{\alpha,\beta}\colon \alpha, \beta \in E^*v\}$. As a notational convenience, for $v \in E^0 \setminus F^0$, we write $p^F_v = 0$, and similarly for $e \in E^1 \setminus F^1$, we write $s^F_e = 0$. For $v \in E^0$, let

$$q_v := \left(\bigoplus_{w \in \mathrm{sinks}(E)} \sum_{\alpha \in vE^*w} \Theta^w_{\alpha,\alpha} \right) \oplus p^F_v$$

and for $e \in E^1$, let

$$t_e := \left(\bigoplus_{w \in \mathrm{sinks}(E)} \sum_{\alpha \in r(e)E^*w} \Theta^v_{e\alpha,\alpha} \right) \oplus s^F_e.$$

Routine calculations show that $\{q_v\colon v \in E^0\}$, $\{t_e\colon e \in E^1\}$ is a Cuntz–Krieger E-family. This family clearly generates $(\bigoplus_{v \in \mathrm{sinks}(E)} \mathcal{K}(\ell^2(E^*v))) \oplus C^*(F)$, and each q_v is nonzero because if $p^F_v = 0$ then v must connect to a sink w in which case q_v dominates some $\Theta^w_{\alpha,\alpha}$. An application of the Gauge-Invariant Uniqueness Theorem [2, Theorem 2.1] implies that there is an isomorphism

$$\pi_{q,t} : C^*(E) \to \left(\bigoplus_{v \in \mathrm{sinks}(E)} \mathcal{K}\big(\ell^2(E^*v)\big) \right) \oplus C^*(F)$$

such that $\pi_{q,t}(p_v) = q_v$ and $\pi_{q,t}(s_e) = t_e$.

To complete the proof of (1) implies (2), let $X \subset \mathrm{sinks}(E)$ denote the subset $\{v \in \mathrm{sinks}(E)\colon |E^*v| < \infty\}$, and for each $v \in X$ let $n_v := |E^*v|$. We have $\mathcal{K}(\ell^2(E^*v)) = M_{n_v}(\mathbb{C})$ for each $v \in X$. Recall that F is row-finite and has no sinks, so Theorem 4.7 implies that $C^*(F)$ has no unital quotient. For each $v \in \mathrm{sinks}(E) \setminus X$, the C^*-algebra $\mathcal{K}(\ell^2(E^*v))$ is simple and nonunital. Thus

$$A' := \left(\bigoplus_{v \in \mathrm{sinks}(E) \setminus X} \mathcal{K}\big(\ell^2(E^*v)\big) \right) \oplus C^*(F')$$

has no finite-dimensional quotients. We get

$$A \cong C^*(E) \cong \left(\bigoplus_{v \in \mathrm{sinks}(E)} \mathcal{K}\big(\ell^2(E^*v)\big) \right) \oplus C^*(F) \cong \left(\bigoplus_{v \in X} M_{n_v}(\mathbb{C}) \right) \oplus A'$$

as required.

To see that (2) implies (1), let $A = (\bigoplus_{x \in X} M_{n_x}(\mathbb{C})) \oplus A'$ as in (2). By Theorem 4.7, there is a row-finite graph E' with no sinks such that $C^*(E') \cong A'$. For each $x \in X$, let E_x be a copy of the graph

$$v_1 \longrightarrow v_2 \longrightarrow \cdots \longrightarrow v_{n_x}$$

A standard argument shows that $C^*(E_x) \cong M_{n_x}(\mathbb{C})$. Moreover $E := (\bigsqcup_{x \in X} E_x) \sqcup E'$ satisfies

$$C^*(E) \cong \left(\bigoplus_{x \in X} C^*(E_x) \right) \oplus C^*(E') \cong A$$

as required. □

For completeness, we conclude the section with the following well-known result.

Lemma 4.15. *A C^*-algebra A is finite dimensional if and only if it is isomorphic to the C^*-algebra of a finite directed graph with no cycles.*

Proof. If E is a finite directed graph with no cycles, then E^* is finite, and hence $C^*(E) = \overline{\operatorname{span}}\{s_\mu s_\nu^*\colon \mu, \nu \in E^*\}$ is finite dimensional.

On the other hand, if A is finite-dimensional, then there exist an integer $n \geqslant 1$ and nonnegative integers $d_1, \dots, d_n$ such that $A \cong \bigoplus_{i=1}^n M_{d_i}(\mathbb{C})$, and [17, Corollary 2.3] then implies that A is isomorphic to the C^*-algebra of a finite directed graph with no cycles. (Moreover, we remark that the last part of the proof of Proposition 4.14 actually shows that every finite-dimensional C^*-algebra is the C^*-algebra of a finite graph with no cycles.) □

4.3. Obstructions to realizations

Here we present a number of necessary conditions for an AF algebra to be an ultragraph C^*-algebra, an Exel–Laca algebra, or a graph C^*-algebra. Recall that an ultragraph C^*-algebra $C^*(\mathcal{G})$ is an AF-algebra if and only if $\mathcal{G}$ has no cycles by [23, Theorem 4.1].

Proposition 4.16. *Let $\mathcal{G}$ be an ultragraph and suppose that $C^*(\mathcal{G})$ is an AF-algebra. If $C^*(\mathcal{G})$ is commutative, then the ultragraph $\mathcal{G}$ has no ultraedges, and $C^*(\mathcal{G}) \cong c_0(G^0)$.*

Proof. It suffices to show that $\mathcal{G}$ has no ultraedges. Suppose that e is an ultraedge in $\mathcal{G}$, and let $v = s(e)$. Since $C^*(\mathcal{G})$ is commutative, we have $p_{r(e)} = s_e^* s_e = s_e s_e^* \leqslant p_{s(e)}$, and hence $r(e) = \{s(e)\}$. Thus e is a cycle. This contradicts the hypothesis that $C^*(\mathcal{G})$ is an AF-algebra. □

Proposition 4.17. *Let A be an AF-algebra that is also an Exel–Laca algebra. Then A does not have a quotient isomorphic to $\mathbb{C}$, and for each $n \in \mathbb{N}$ there is a C^*-subalgebra of A isomorphic to $M_n(\mathbb{C})$.*

Proof. There exists an ultragraph $\mathcal{G} = (G^0, \mathcal{G}^1, r, s)$ with bijective s such that $C^*(\mathcal{G}) \cong A$ (see Remark 2.10). The ultragraph $\mathcal{G}$ has no cycles. Let $\{p_v\}_{v \in G^0}$ and $\{s_e\}_{e \in \mathcal{G}^1}$ be the generator of $C^*(\mathcal{G})$ as in Definition 2.9.

Suppose, for the sake of contradiction, that there exists a nonzero $*$-homomorphism $\chi : C^*(\mathcal{G}) \to \mathbb{C}$. Since χ is nonzero, there exists $v \in G^0$ with $\chi(p_v) \neq 0$. Let $e \in \mathcal{G}^1$ be the unique ultraedge with $s(e) = v$. Since $\mathcal{G}$ has no cycles, we have $v \notin r(e)$. Hence p_v is orthogonal to $s_e^* s_e$. Thus

$$\left|\chi(s_e)\right|^2 \chi(p_v) = \overline{\chi(s_e)}\chi(s_e)\chi(p_v) = \chi\left(s_e^* s_e p_v\right) = 0,$$

and since $\chi(p_v) \neq 0$, it follows that $|\chi(s_e)|^2 = 0$ and $\chi(s_e) = 0$. But then $\chi(p_v) = \chi(s_e s_e^*) = \chi(s_e)\chi(s_e^*) = 0$, which is a contradiction. Hence $C^*(\mathcal{G})$ has no quotients isomorphic to $\mathbb{C}$.

Let $n \in \mathbb{N}$. We will construct a C^*-subalgebra of $C^*(\mathcal{G})$ isomorphic to $M_n(\mathbb{C})$. Choose $v_1 \in G^0$ and let $e_1 \in \mathcal{G}^1$ be the unique ultraedge with $s(e_1) = v_1$. Then choose a vertex $v_2 \in r(e_1)$. Since $\mathcal{G}$ has no cycles, we have $v_2 \neq v_1$. Continuing in this manner, we can find distinct vertices

$v_1, v_2, \ldots, v_n \in G^0$ such that $v_{k+1} \in r(e_k)$ for $k = 1, 2, \ldots, n-1$, where $e_k \in \mathcal{G}^1$ is the unique ultraedge with $s(e_k) = v_k$. For $1 \leqslant i, j \leqslant n$, we define

$$\Theta_{i,j} := s_{e_i} s_{e_{i+1}} \ldots s_{e_{n-1}} p_{v_n} s^*_{e_{n-1}} s^*_{e_{n-2}} \ldots s^*_{e_j}.$$

One can check that $\{\Theta_{i,j}\colon 1 \leqslant i, j \leqslant n\}$ is a family of matrix units, and thus the C^*-subalgebra of $C^*(\mathcal{G})$ generated by $\{\Theta_{i,j}\colon 1 \leqslant i, j \leqslant n\}$ is isomorphic to $M_n(\mathbb{C})$. □

Corollary 4.18. *If A is an AF-algebra that is also an Exel–Laca algebra, then A has a Bratteli diagram (E, d) such that $d_v \geqslant 2$ for all $v \in E^0$.*

Proof. Since A has no quotient isomorphic to $\mathbb{C}$, the result follows from Lemma 3.1. □

Corollary 4.19. *No finite-dimensional C^*-algebra is isomorphic to an Exel–Laca algebra.*

Definition 4.20. We recall that a C^*-algebra A is said to be *Type* I if whenever $\pi : A \to \mathcal{B}(\mathcal{H})$ is a nonzero irreducible representation, then $\mathcal{K}(\mathcal{H}) \subseteq \pi(A)$. In the literature, the terms *postliminary*, *GCR*, and *smooth* are all synonymous with Type I.

Proposition 4.21. *Let $C^*(E)$ be a graph C^*-algebra that is also an AF-algebra. Then every unital quotient of $C^*(E)$ is Type* I *and has finitely many ideals.*

Proof. By Lemma 2.16, it suffices to show that if a graph C^*-algebra $C^*(E)$ is a unital AF-algebra then $C^*(E)$ is Type I and has finitely many ideals. Note that $C^*(E)$ is a unital AF-algebra if and only if E has a finite number of vertices and no cycles.

We first show that $C^*(E)$ has finitely many ideals. Since E has no cycles, it satisfies Condition (K). Hence any ideal of $C^*(E)$ is of the form $I_{(H,S)}$ for a saturated hereditary subset H of E^0 and a subset $S \subseteq E^0$ of the set of breaking vertices for H [8, Theorem 3.5]. Since the set E^0 of vertices of E is finite, there are only a finite number of such pairs (H, S). Thus $C^*(E)$ has finitely many ideals.

To prove that $C^*(E)$ is of Type I, first observe that any graph with finitely many vertices and no cycles contains a sink v, and the ideal I_v generated by p_v is then a nontrivial gauge-invariant ideal which is Morita equivalent to $\mathbb{C}$ and hence of Type I (see [14, Proposition 2] and the subsequent remark in [14]).

We shall show by induction on the number of nonzero ideals of $C^*(E)$ that $C^*(E)$ is Type I. Our basis case is when has just one nontrivial ideal I. That is, $C^*(E)$ is simple, and then the Type I ideal I_v of the preceding paragraph is $C^*(E)$ itself, proving the result. Now suppose as an inductive hypothesis that the result holds whenever $C^*(E)$ has at most n distinct nonzero ideals, and suppose that $C^*(E)$ has $n + 1$ such. Let v be a sink in E and let I_v be the corresponding nonzero Type I ideal as in the preceding paragraph. If $C^*(E)/I_v$ is trivial, then $C^*(E) = I_v$ is of Type I, so we may assume that $C^*(E)/I_v$ is nonzero. Then Lemma 2.16 implies that $C^*(E)/I_v$ is a unital AF-algebra that is a graph C^*-algebra. Moreover, $C^*(E)/I_v$ has strictly fewer ideals than $C^*(E)$, so the inductive hypothesis implies that $C^*(E)/I_v$ is of Type I. Since an extension of a Type I C^*-algebra by a Type I C^*-algebra is Type I (see [19, Theorem 5.6.2]), it follows that $C^*(E)$ is of Type I. □

Theorem 4.22. *For a simple AF-algebra A we have the following.*

(1) *If A is finite dimensional then A is isomorphic to a graph C^*-algebra but not isomorphic to an Exel–Laca algebra.*
(2) *If A is infinite dimensional and unital then A is isomorphic to an Exel–Laca algebra but not isomorphic to a graph C^*-algebra.*
(3) *If A is infinite dimensional and nonunital then A is isomorphic to a C^*-algebra of a row-finite graph with no sinks* (*which is also isomorphic to the Exel–Laca algebra of a row-finite matrix by Lemma* 2.4).

In particular, each simple AF-algebra A is isomorphic to either an Exel–Laca algebra or a graph C^*-algebra.

Proof. The statement in (1) follows from Lemma 4.15 and Corollary 4.19.

For (2) we observe that if A is simple, infinite dimensional, and unital, then it follows from Corollary 4.6 that A is isomorphic to an Exel–Laca algebra. Since A is in particular unital, to see that A is not a graph C^*-algebra, it suffices by Proposition 4.21 to show that it is not of Type I. If we suppose for contradiction that A is of Type I, then as it is simple, we must have $A \cong \mathcal{K}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. Since A is unital, $\mathcal{H}$ and hence $\mathcal{K}(\mathcal{H})$ must be finite-dimensional, contradicting that A is infinite dimensional.

The statement in (3) follows from Theorem 4.7. The final assertion follows from (1), (2), and (3). □

Corollary 4.23. *If A is an infinite-dimensional UHF algebra, then A is not isomorphic to a graph C^*-algebra.*

5. A summary of known containments

In this section we use our results to describe how various classes of AF-algebras are contained in the classes of graph C^*-algebras, Exel–Laca algebras, and ultragraph algebras. We first examine the simple AF-algebras, where we have a complete description. Moreover, we see that the simple AF-algebras allow us to distinguish among the four classes of C^*-algebras of row-finite graphs with no sinks, graph C^*-algebras, Exel–Laca algebras, and ultragraph algebras. Second, we consider general AF-algebras, and while our description in this case is not complete, we are able to describe how the finite-dimensional and stable AF-algebras are contained in the classes of graph C^*-algebras, Exel–Laca algebras, and ultragraph algebras. Furthermore, we use our results to show that there are numerous other AF-algebras in the various intersections of these classes.

5.1. Simple AF-algebras

Consider the following partition of the simple AF-algebras.

$$\mathrm{AF}^{\text{simple}}_{\text{finite}} := \text{finite-dimensional simple AF-algebras},$$

$$\mathrm{AF}^{\text{simple}}_{\infty,\text{unital}} := \text{infinite-dimensional simple AF-algebras that are unital},$$

$$\mathrm{AF}^{\text{simple}}_{\infty,\text{nonunital}} := \text{infinite-dimensional simple AF-algebras that are nonunital}.$$

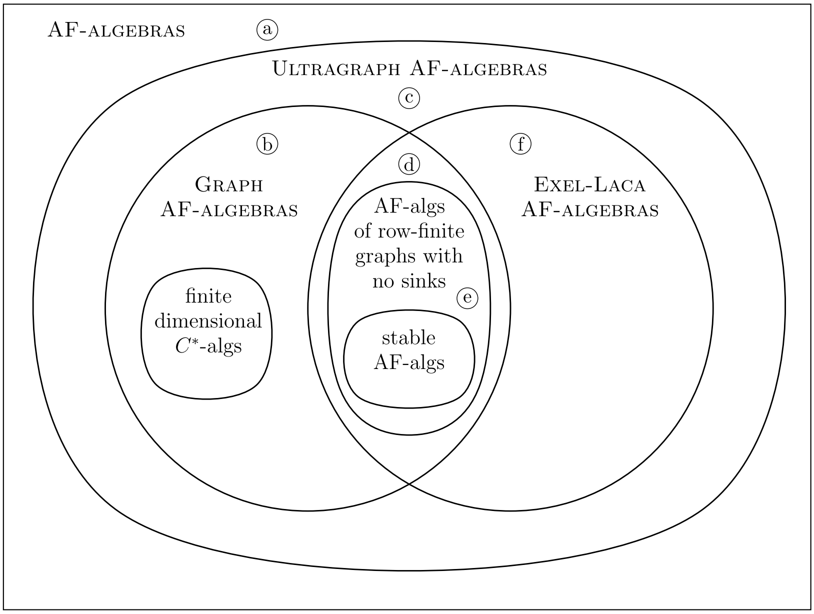


Fig. 2. A Venn diagram summarizing AF-algebra containments.

Theorems 4.22 and 4.7 imply that

$$\begin{aligned}
\mathrm{AF}^{\text{simple}}_{\infty,\text{nonunital}} &= \text{simple AF-algebras that are } C^*\text{-algebras of} \\
&\quad \text{row-finite graphs with no sinks}, \\
\mathrm{AF}^{\text{simple}}_{\text{finite}} \cup \mathrm{AF}^{\text{simple}}_{\infty,\text{nonunital}} &= \text{simple AF-algebras that are graph } C^*\text{-algebras}, \\
\mathrm{AF}^{\text{simple}}_{\infty,\text{unital}} \cup \mathrm{AF}^{\text{simple}}_{\infty,\text{nonunital}} &= \text{simple AF-algebras that are Exel–Laca algebras}
\end{aligned}$$

and

$$\mathrm{AF}^{\text{simple}}_{\text{finite}} \cup \mathrm{AF}^{\text{simple}}_{\infty,\text{unital}} \cup \mathrm{AF}^{\text{simple}}_{\infty,\text{nonunital}} = \text{simple AF-algebras that are ultragraph algebras.}$$

Hence these three classes of simple AF-algebras allow us to distinguish among the four classes of C^*-algebras of row-finite graphs with no sinks, graph C^*-algebras, Exel–Laca algebras, and ultragraph algebras. However, they do not allow us to distinguish between the classes of C^*-algebras of row-finite graphs with no sinks and the intersection of graph C^*-algebras and Exel–Laca algebras. Nor do they allow us to distinguish between the classes of ultragraph C^*-algebras and the union of graph C^*-algebras and Exel–Laca algebras. To distinguish these classes we will need nonsimple examples.

5.2. More general AF-algebras

For nonsimple AF-algebras, we cannot give such an explicit description. Nevertheless, in Fig. 2 we present a Venn diagram summarizing the relationships we have established for finite-

Table 1
Examples of C^*-algebras lying in each region of Fig. 2.

Region	Unital C^*-algebra	Nonunital C^*-algebra
(a)	c_c	$c_0 \oplus c_c$
(b)	$\mathcal{K}^+$	c_0
(c)	$M_{2^\infty} \oplus \mathbb{C}$	$M_{2^\infty} \oplus \mathbb{C} \oplus \mathcal{K}$
(d)	$M_2(\mathcal{K}^+)$	$M_2(\mathcal{K}^+) \oplus \mathcal{K}$
(e)	—	$C^*(F_2)$
(f)	M_{2^∞}	$M_{2^\infty} \oplus \mathcal{K}$

dimensional and stable AF-algebras, and also give various examples in the intersections of our classes of graph C^*-algebras, Exel–Laca algebras, and ultragraph C^*-algebras.

Table 1 presents, for each region of the Venn diagram of Fig. 2, both a unital and a nonunital example belonging to that region, with three exceptions: we give no examples of finite-dimensional or stable AF algebras, nor any example of a unital AF algebra which is the C^*-algebra of a row-finite graph with no sinks. Our reasons for these omissions are as follows: examples of finite-dimensional and stable AF algebras are obvious, and necessarily unital and nonunital respectively; and no unital example exists in region (e) by Theorem 4.7.

In Table 1, we use the following notation:

- M_{2^∞} denotes the UHF algebra of type 2^∞.
- $\mathcal{K}$ denotes the compact operators on a separable infinite-dimensional Hilbert space.
- $\mathcal{K}^+$ denotes the minimal unitization of the C^*-algebra $\mathcal{K}$.
- c_0 denotes the space $\{f : \mathbb{N} \to \mathbb{C} \mid \lim_{n\to\infty} f(n) = 0\}$.
- c_c denotes the space $\{f : \mathbb{N} \to \mathbb{C} \mid \lim_{n\to\infty} f(n) \in \mathbb{C}\}$.
- F_2 denotes the graph

$$v_1 \rightrightarrows v_2 \rightrightarrows v_3 \rightrightarrows v_4 \cdots$$

We now justify that the examples listed have the desired properties.

(a) • The unital AF-algebra c_c is not an ultragraph C^*-algebra since it is commutative and its spectrum is not discrete (see Proposition 4.16).
• The nonunital AF-algebra $c_0 \oplus c_c$ is not an ultragraph algebra for precisely the same reason that c_c is not.

(b) • The minimal unitization $\mathcal{K}^+$ of the compact operators is isomorphic to the C^*-algebra of the graph

$$v \xrightarrow{(\infty)} w$$

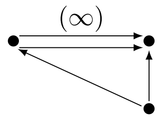
with two vertices v, w and infinitely many edges from v to w. Since, $\mathcal{K}^+$ has a quotient isomorphic to $\mathbb{C}$, it is not an Exel–Laca algebra by Proposition 4.17.
• The nonunital AF-algebra c_0 is the C^*-algebra of the graph with infinitely many vertices and no edges. It is not an Exel–Laca algebra by Proposition 4.17.

(c) • Since M_{2^∞} is an infinite-dimensional simple AF-algebra, Theorem 4.22 implies that M_{2^∞} is an Exel–Laca algebra and hence also an ultragraph algebra. In addition, $\mathbb{C}$ is a graph

C^*-algebra so also an ultragraph C^*-algebra. Since the class of ultragraph C^*-algebras is closed under direct sums, $M_{2^\infty} \oplus \mathbb{C}$ is a unital ultragraph C^*-algebra. It is not an Exel–Laca algebra because it has a quotient isomorphic to $\mathbb{C}$ (see Proposition 4.17), and it is not a graph C^*-algebra because it has a unital quotient M_{2^∞} that is not Type I (see Proposition 4.21).

- Since $\mathcal{K}$ and $M_{2^\infty} \oplus \mathbb{C}$ are both ultragraph C^*-algebras, the direct sum $M_{2^\infty} \oplus \mathbb{C} \oplus \mathcal{K}$ is a nonunital ultragraph C^*-algebra. It is neither a graph C^*-algebra nor an Exel–Laca algebra as above.

(d) • The unital AF-algebra $M_2(\mathcal{K}^+)$ is isomorphic to the C^*-algebra of the following graph



and it is also isomorphic to the Exel–Laca algebra of the matrix

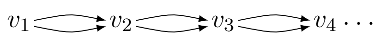
$$\begin{pmatrix} 0 & 1 & 1 & 1 & 1 & \cdots \\ 0 & 0 & 1 & 0 & 0 & \\ 0 & 0 & 0 & 1 & 0 & \\ 0 & 0 & 0 & 0 & 1 & \\ \vdots & & & & & \ddots \end{pmatrix}.$$

It is not isomorphic to the C^*-algebra of a row-finite graph with no sinks by Theorem 4.7.

- The nonunital AF-algebra $M_2(\mathcal{K}^+) \oplus \mathcal{K}$ is isomorphic to both a graph C^*-algebra and an Exel–Laca algebra because its two direct summands have this property. It is not the C^*-algebra of a row-finite graph with no sinks by Theorem 4.7 because it admits the unital quotient $M_2(\mathcal{K}^+)$.

(e) • There is no unital example in this region by Theorem 4.7.

- Let F_2 denote the graph



Then $C^*(F_2)$ is a graph C^*-algebra, and since F_2 is cofinal with no cycles and no sinks, $C^*(F_2)$ is simple by [17, Corollary 3.10]. In addition, $C^*(F_2)$ is nonunital because F_2 has infinitely many vertices. Since $C^*(F_2)$ is the C^*-algebra of a row-finite graph with no sinks, it is both a graph C^*-algebra and an Exel–Laca algebra (see Lemma 2.4). The function $g : F_2^0 \to \mathbb{R}^+$ defined by $g(v_i) = 2^{-i}$ is a graph trace with norm 1 (see [24, Definition 2.2]), and the existence of such a function implies that $C^*(F_2)$ is not stable by (a) $\Rightarrow$ (c) of [24, Theorem 3.2].

(f) • As in example (c), the unital AF-algebra M_{2^∞} is an Exel–Laca algebra but not a graph C^*-algebra.

- As in example (c), the nonunital AF-algebra $M_{2^\infty} \oplus \mathcal{K}$ is an Exel–Laca algebra but not a graph C^*-algebra.

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