POLAR GEOMETRY. IV

BY

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(Communicated by Prof. H. FREUDENTHAL at the meeting of October 31, 1959)

IV. THE EMBEDDING OF AXIOMATIC POLAR GEOMETRY IN A PROJECTIVE SPACE

1. In this chapter **S** will be a system that satisfies the axioms I–X of the preceding chapter.

2. By $\mathfrak{P}^*$ we denote the set of all flat subsets X^* of **S** with the following properties:

1. If $u \in \mathbf{S}$ with $r(u) > 0$, then there exists a $v \leqslant u$, $r(v) \geqslant r(u) - 1$, such that $v \in X^*$.
2. X^* is a proper subset of **S**.

$\mathfrak{P}$ will be a replica of $\mathfrak{P}^*$; to each element $X^* \in \mathfrak{P}^*$ there corresponds an element $X \in \mathfrak{P}$.

For every point x of **S** we define an $X^* \in \mathfrak{P}^*$ as the set of all elements u of **S** such that $x \vee u$ exists; it can easily be verified that such a set has all the properties of the sets that belong to $\mathfrak{P}^*$. As $x = y$ is equivalent to $X^* = Y^*$ (where Y^* corresponds to y in the same way as X^* to x), we can identify any point $x \in \mathbf{S}$ to the element X of $\mathfrak{P}$ which corresponds to $X^* \in \mathfrak{P}^*$. Instead of saying that X^* is the set of elements that are joined to $x \in \mathbf{S}$ we shall often say $X \in \mathbf{S}$.

3. LEMMA. *If X^* and Y^* are arbitrary elements of $\mathfrak{P}^*$ and $X^* \subset Y^*$, then $X^* = Y^*$.*

Proof: Let us suppose that Y^* contains a point p that does not belong to X^*. If $u > p$, we can find a $v < u$ of rank $r(v) \geqslant r(u) - 1$ such that $v \in X^*$. $p \notin X^*$, hence $u = p + v$. Both p and $v \in Y^*$, hence $u \in Y^*$.

Hence $x \in Y^*$ if $x \vee p$ exists. Now we can reason in a way similar to the proof of III, 11 to show that all elements of **S** belong to Y^*. This is in contradiction with the property of Y^* to be a proper subset of **S**.

4. In $\mathfrak{P}$ we are going to introduce a notion of dependence. Note that if p, q and r are points in **S** and P, Q and R respectively are the corresponding elements of $\mathfrak{P}$, r is dependent on p and q if, and only if, $R^* \supset P^* \cap Q^*$.

Now we generalize this:

Definition. *If* P, Q *and* R *are arbitrary elements of* $\mathfrak{P}$, *then* R *is called dependent on* P *and* Q *if, and only if,* $R^* \supset P^* \cap Q^*$.

It follows from section 3 that if $P=Q$, R is dependent on P and Q if, and only if, $R=P$.

It is trivial that P and Q themselves depend on P and Q.

5. Proposition. *If* P, Q *and* $R \in \mathfrak{P}$, $Q \neq R$ *and* R *is dependent on* P *and* Q, *then* P *is dependent on* Q *and* R.

Proof: We know: $R^* \supset P^* \cap Q^*$ and we have to prove: $P^* \supset Q^* \cap R^*$. Suppose this statement not to be true. Then there exists a point $a \in \mathbf{S}$ such that $a \in Q^* \cap R^*$ but $a \notin P^*$. Then we shall prove $Q^* \subset R^*$, hence $Q^* = R^*$, which leads to a contradiction with the assumption $Q \neq R$.

First we consider an arbitrary line $l > a$, $l \in Q^*$. There must be a point $b < l$, $b \in P^*$. Since $a \notin P^*$, $a \neq b$. As $b \in P^* \cap Q^*$, $b \in R^*$. Hence $l \in R^*$.

Now let c be an arbitrary point $\in Q^*$. If $c \in P^*$, then $c \in P^* \cap Q^*$ and hence $c \in R^*$. Suppose $c \notin P^*$.

If $c \vee a$ exists, $c \in R^*$ as we have just proved.

Suppose $c \vee a$ does not exist. Take a line $l > a$, $l \in Q^*$. On l we can find a point b such that $b \vee c$ exists. We know that $b \in Q^* \cap R^*$; if moreover $b \notin P^*$, we can prove that $c \in R^*$ in exactly the same way as we did above with a instead of b.

Now suppose such a point b cannot be found, that is to say: if $b \in Q^*$ and $b \vee a$ and $b \vee c$ exist, then $b \in P^*$. We can easily find two points b_1 and b_2, both $\in Q^*$, such that $b_i \vee a$ and $b_i \vee c$ exist but $b_1 \vee b_2$ does not. By a similar reasoning as in III, 11, part 2*c*., we conclude that in this case too $c \in R^*$.

Hence $Q^* \subset R^*$ as we intended to prove.

6. Definition. *If* P *and* $Q \in \mathfrak{P}$, *the line* $P+Q$ *is defined as the set of all elements of* $\mathfrak{P}$ *that depend on* P *and* Q.

Proposition. *For every two elements of* $\mathfrak{P}$ *there is one and only one line containing those elements.*

Proof: Let R and T be two different points on $P+Q$; by applying proposition 5 we conclude $P+Q=R+T$.

In the sequel we shall often speak of *points* instead of elements of $\mathfrak{P}$.

7. Lemma. *Let* P, Q *and* R *be points of* $\mathfrak{P}$ *such that* $R^* \supset P^* \cap Q^*$ *and* $P \neq Q$. *Let* x *be a point* $\in \mathbf{S}$ *that is* $\in R^*$, *but* $x \notin P^* \cap Q^*$.

Then R^* *is the smallest flat subset of* $\mathbf{S}$ *that contains both* $P^* \cap Q^*$ *and* x.

Proof: We may suppose, for instance, $R \neq P$. Let θ be a flat subset of $\mathbf{S}$ such that $\theta \supset P^* \cap Q^*$, $x \in \theta$ and $\theta \subset R^*$.

Let y be an arbitrary point in R^*. If $x \vee y$ exists, it is $\in R^*$. There must exist a point $z < x \vee y$, $z \in P^*$. Then $z \in P^* \vee R^*$ and hence $z \in Q^*$ (proposition 5). Hence $z \in \theta$ and $x \in \theta$ and therefore $x \vee z \in \theta$. Hence $y \in \theta$.

If $x \vee y$ does not exist, a reasoning like that in III, 11, part 2*b*. and *c*., leads to $R^* \subset \theta$. Hence $\theta = R^*$.

8. PROPOSITION. *Let x and y be points of* **S** *identified with X and $Y \in \mathfrak{P}$. Suppose that $x \vee y$ exists. Then $X+Y$ is the set of all $Z \in \mathfrak{P}$ corresponding to the points $z < x \vee y$. (In this case we shall often write $X \vee Y$ instead of $X+Y$).*

Proof: If $z < x \vee y$, it is trivial that $Z \in X+Y$. Suppose conversely $Z \in X+Y$.

Choose a point $r \in Z^*$, $r \notin X^* \cap Y^*$. There is a point $t < x \vee y$ such that $t \vee r$ exists. Let $T \in \mathfrak{P}$ correspond to t. Then $T^* \supset X^* \cap Y^*$ and $r \in T^*$. It follows from the preceding lemma that $T^* \supset Z^*$, hence $T = Z$ (lemma 3).

9. Now we are going to prove that if a line intersects two sides of a triangle (not at their common point), it also intersects the third side. But before doing so we shall prove a useful lemma.

LEMMA. *Let θ be a subset of* **S** *with the following properties:*

1. *If x and $y \in \theta$ and $x \vee y$ exists, the latter is also in θ.*
2. *There exist two points x and y in θ such that $x \vee y$ does not exist and every point of xy belongs to θ.*
3. *If $x \in \theta$ and $y \leqslant x$, then $y \in \theta$.*
4. *If $u \in$ **S** *with $r(u) > 0$, then there exists a $v \leqslant u$ of rank $r(v) \geqslant r(u)-1$ such that $v \in \theta$.*

*Then θ is flat and hence $\theta =$ **S** *or $\theta \in \mathfrak{P}^*$ (owing to property 4).*

Proof: We have only to show that if x' and y' are two points in θ such that $x' \vee y'$ does not exist, every point of $x'y'$ belongs to θ.

The general case can easily be reduced to the case that xy and $x'y'$ have a point in common (in the same way as in part *b*. of the proof of III, 9). So we may suppose $x = x'$.

a. $y \vee y'$ exists.

Then $y \vee y' \in \theta$. Let $z < y \vee y'$ be the point such that $x \vee z$ exists; $z \in \theta$.

If t' is an arbitrary point of $x'y'$, then $t' \vee z$ exists and intersects xy in a point t (III, 8). $t \in \theta$ and $z \in \theta$, hence $t \vee z \in \theta$ and consequently $t' \in \theta$.

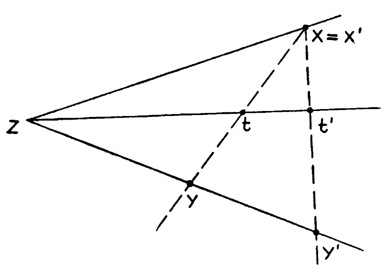

Fig. 7

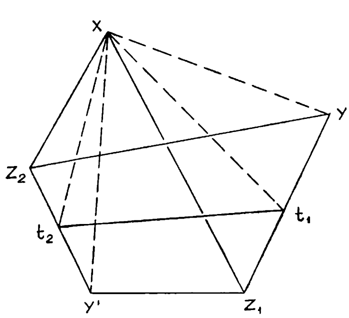

Fig. 8

b. $y \vee y'$ does not exist. If there is a point $y'' \in \theta$ such that $y \vee y''$ and $y' \vee y''$ exist but $x \vee y''$ does not, then we apply a. twice: to xy and xy'' and then to xy'' and xy'.

c. $y \vee y'$ does not exist and there is no point $y'' \in \theta$ as in b.

There are points z_1 and z_2 in θ such that $z_i \vee y$ and $z_i \vee y'$ exist and $z_1 \vee z_2$ does not. Then $z_i \vee x$ must exist because of the above hypothesis.

Choose points $t_1 < y \vee z_1$ and $t_2 < y' \vee z_2$, $t_i \neq z_i$, such that $t_1 \vee t_2$ exists. Then $x \vee t_i$ does not exist. t_1 and t_2 are in θ. Hence we can apply the line of reasoning of a. thrice: first we project xy onto xt_1, then xt_1 onto xt_2 and finally xt_2 onto xy'. This completes the proof.

10. PROPOSITION. *If A, B, C, P and Q are points in $\mathfrak{P}$, $P \in A+B$, $Q \in A+C$, $A \notin B+C$, $A \notin P+Q$, then the line $B+C$ intersects $P+Q$ at a point R.*

Proof: For maximal elements u of $\mathbf{S}$, we define the following subsets of $\mathbf{S}$ by induction:

$(1, u)$ contains all $x \leqslant u$ with $x \in B^* \cap C^*$ or $x \in P^* \cap Q^*$.

$(2, u)$ contains all $z \leqslant x \vee y$ where x and $y \in (1, u)$.

If $n \geqslant 2$, then

$(n+1, u)$ contains all $z \leqslant x \vee y$ where $x \in (n, u)$, $y \leqslant u$ and $y \in (n, v)$ for some maximal v such that $r(u \wedge v) = i(\mathbf{S}) - 1$.

Notice that $(n, u) \subset (n+1, u)$ for every n and that $x \in (n, u)$, $x \leqslant v$ and $r(u \wedge v) = i(\mathbf{S}) - k$ implies $x \in (n+k, v)$.

Now we define

$R^* = \bigcup (n, u)$ where the union is taken over all $n \geqslant 1$ and all maximal $u \in \mathbf{S}$.

From the above remark it is clear that $x \vee y \in R^*$ if $x \in R^*$, $y \in R^*$ and $x \vee y$ exists.

From the definition of (n, u) it follows that $y \in R^*$ if $y \leqslant x$ such that $x \in R^*$.

Now we observe that in $B^* \cap C^*$ there exist two points x and y such that $x \vee y$ does not exist; but then all points of the imaginary line xy belong to $B^* \cap C^*$ and hence to R^*.

In the next sections we shall prove:

(1) If $u \in \mathbf{S}$ and $r(u) > 0$, then there exists a $v \leqslant u$, $r(v) \geqslant r(u) - 1$ such that $v \in R^*$.

(2) $R^* \neq \mathbf{S}$.

Hence we can apply the preceding lemma and conclude $R^* \in \mathfrak{P}^*$. Therefore there exists a point $R \in \mathfrak{P}$ such that $R \in B+C$ and $R \in P+Q$; for $R^* \supset B^* \cap C^*$ and $R^* \supset P^* \cap Q^*$. This proves the proposition.

11. First we examine the sets $(2, u)$. It is not hard to see that there must be a $u' \leqslant u$ such that $u' \in (2, u)$ and $x \leqslant u'$ for every $x \in (2, u)$. (We shall call u' the *maximal element* of $(2, u)$; for any maximal $x \in \mathbf{S}$, the maximal element of $(2, x)$ is denoted by x'.)

We distinguish four cases:

α. $r(u') = i(\mathbf{S})$, i.e. $u' = u$.

β. $r(u') = i(\mathbf{S}) - 1$ and $(1, u)$ contains an element of rank $i(\mathbf{S}) - 1$.

γ. $r(u') = i(\mathbf{S}) - 1$ and $(1, u)$ does not contain any element of rank $i(\mathbf{S}) - 1$.

δ. $r(u') = i(\mathbf{S}) - 2$. Then $u' \in B^* \cap C^*$ and $u' \in P^* \cap Q^*$ and for all $x \leqslant u$ such that $x \in B^* \cap C^*$ or $x \in P^* \cap Q^*$ we have $x \leqslant u'$.

Note that there is always a $v \leqslant u$ such that $v \in B^* \cap C^*$ and $r(v) \geqslant i(\mathbf{S}) - 2$ and analogous for $P^* \cap Q^*$. Moreover, that

$$A^* \cap B^* \cap C^* = A^* \cap P^* \cap Q^*.$$

In the next section we shall prove the existence of a $u \in \mathbf{S}$ of type γ; moreover, that R^* has the propery mentioned as (1) in the preceding section. In sections 13 and 14 we shall finally show that if u is of type γ, $(n+1, u) = (n, u)$ for $n \geqslant 2$. Hence $R^* \neq \mathbf{S}$, as we required in (2) of the preceding section.

12. We start with taking a point $x \in P^*$, $\notin Q^*$ (remember that$P \neq Q$); let X be the element of $\mathfrak{P}$ corresponding to x.

If $P^* \supset X^*$, then $P = X$ (lemma 3). In that case we can find a point $y \in P^*$, $\notin Q^*$, $y \neq x$ (e.g. on a line in P^* passing through x); then $P \neq Y$ and hence $P^* \not\supset Y^*$.

If $P^* \not\supset X^*$, we take $y = x$.

In both cases there exists a point in Y^* that is not in P^*. Hence we can find a $u_1 \in \mathbf{S}$ of rank $i(\mathbf{S})$ such that $y < u_1$ and $u_1 \notin P^*$.

Let $v_1 < u_1$ be an element of P^* of rank $i(\mathbf{S}) - 1$ and $w_1 < u_1$ an element of Q^* of rank $i(\mathbf{S}) - 1$. Owing to the fact that $y \in P^*$, $\notin Q^*$, $v_1 \neq w_1$.

Hence $r(v_1 \wedge w_1) = i(\mathbf{S}) - 2$. Remark that if $x \leqslant u_1$ and $x \in P^*$, then $x \leqslant v_1$; and similar for Q^*.

The same can be done with B^* and C^* instead of P^* and Q^* in some maximal element u_2 of $\mathbf{S}$; then we get $v_2 < u_2$ of rank $i(\mathbf{S}) - 1$, $v_2 \in B^*$, and $w_2 < u_2$ of rank $i(\mathbf{S}) - 1$, $w_2 \in C^*$, such that $v_2 \neq w_2$.

If $u_1 \neq u_2$, we select a u_3 of rank $i(\mathbf{S})$ such that $r(u_1 \wedge u_3) = i(\mathbf{S}) - 1$, $r(u_2 \wedge u_3) = r(u_1 \wedge u_2) + 1$ and such that $u_3 \not\geqslant v_1 \wedge w_1$. Now $u_1 \notin P^*$, $\notin Q^*$ and hence the same is true for u_3, for $r(u_1 \wedge u_3) = i(\mathbf{S}) - 1$. We can find $v_3 < u_3$ and $w_3 < u_3$, of rank $i(\mathbf{S}) - 1$, $v_3 \in P^*$ and $w_3 \in Q^*$. As $v_3 \wedge u_1 = v_1 \wedge u_3$ and $w_3 \wedge u_1 = w_1 \wedge u_3$, $v_3 \neq w_3$. Now we repeat this reasoning with u_3 instead of u_1, etc.

Finally we come, for instance, to u_4 of rank $i(\mathbf{S})$ such that $r(u_4 \wedge u_2) = i(\mathbf{S}) - 1$ and such that there exist $v_4 < u_4$, $v_4 \in P^*$, $r(v_4) = i(\mathbf{S}) - 1$, and $w_4 < u_4$, $w_4 \in Q^*$, $r(w_4) = i(\mathbf{S}) - 1$, with $v_4 \neq w_4$.

If $u_4 \wedge u_2 \not\geqslant v_2 \wedge w_2$, we can reason as above to show that there exist $v_4^* < u_4$, $v_4^* \in B^*$, $r(v_4^*) = i(\mathbf{S}) - 1$, and $w_4^* < u_4$, $w_4^* \in C^*$, $r(w_4^*) = i(\mathbf{S}) - 1$, with $v_4^* \neq w_4^*$.

If $u_4 \wedge u_2 > v_2 \wedge w_2$, we take u_5 and u_6 of rank $i(\mathbf{S})$ in the way that $r(u_4 \wedge u_5) = r(u_5 \wedge u_6) = r(u_6 \wedge u_2) = i(\mathbf{S}) - 1$, $u_4 \wedge u_5 \not\geqslant v_4 \wedge w_4$, $u_5 \wedge u_6 \not\geqslant v_5 \wedge \wedge w_5$, $u_6 \wedge u_2 \not\geqslant v_2 \wedge w_2$. In that case we have in u_6 similar v_6, w_6, v_6^*, w_6^*, $\in P^*$, Q^*, B^* and C^* respectively, as we had in u_4 in the case $u_4 \wedge u_2 \not\geqslant v_2 \wedge w_2$.

So we have constructed a $u \in \mathbf{S}$ of rank $i(\mathbf{S})$ that must be of type γ or δ; $u = u_4$ or $u = u_6$.

If u is of type δ, we proceed as follows to construct an element of type γ:

Consider $v \leqslant u$, $r(v) = i(\mathbf{S}) - 2$, $v \in B^* \cap C^*$ and $v \in P^* \cap Q^*$.

If there is a point $x \in B^* \cap C^*$, $\notin P^* \cap Q^*$, such that $x \vee v$ does not exist, then we take a w such that $x < w$, $r(w) = i(\mathbf{S})$, $r(u \wedge w) = i(\mathbf{S}) - 1$. It is very easy to show that w is of type γ.

If for every $x \in B^* \cap C^*$, $\notin P^* \cap Q^*$, $x \vee v$ exists, we choose such a point x.

Select a point $y \in B^* \cap C^*$ such that $y \vee v$ does not exist; then $y \in P^* \cap Q^*$. Take $w_1 > y$ such that $r(w_1) = i(\mathbf{S})$, $r(w_1 \wedge u) = i(\mathbf{S}) - 1$. (Note that, in what follows, the characters w_1 and w_2 have not the same meaning as above). Then w_1 must be of type δ; for $y \in B^* \cap C^* \cap P^* \cap Q^*$ and $w_1 \wedge v \in B^* \cap \cap C^* \cap P^* \cap Q^*$ is of rank $i(\mathbf{S}) - 3$ and disjoint from y; the elements, for instance, of P^* that are $\leqslant w_1$ are all contained in one of them, which is the join of y and the element $\in P^*$ of rank $i(\mathbf{S}) - 2$ that is $\leqslant w_1 \wedge u$.

Now we select a point $z < x \vee v$, $z \not\leqslant v$, such that $z \vee y$ does not exist. Take $w_2 > z$ of rank $i(\mathbf{S})$ such that $r(w_2 \wedge w_1) = i(\mathbf{S}) - 1$. As $z \in B^* \cap C^*$ but $\notin P^* \cap Q^*$, w_2 must be of type γ.

So we have constructed an element of $\mathbf{S}$ of type γ.

Finally we shall prove the property indicated as (1) in section 10.

Let u be an arbitrary maximal element of $\mathbf{S}$. If u is of type α, β or γ, $r(u') \geqslant i(\mathbf{S}) - 1$; so $(2, u)$ contains an element $\leqslant u$ of rank $\geqslant i(\mathbf{S}) - 1$ and so does R^*.

If u is of type δ, we can construct, as we did before, a maximal $w_2 \in \mathbf{S}$ that is of type γ. It is easy to see that $(3, w_1)$ must contain an element of rank $i(\mathbf{S}) - 1$ and hence the same is true of $(4, u)$.

Hence there exists an element $\leqslant u$ of rank $\geqslant i(\mathbf{S}) - 1$ that is $\in R^*$.

13. In this section we are going to prove:

If u is of type γ, then $(3, u) = (2, u)$. Taking into account that $A^* \supset \supset P^* \cap B^*$ [1]) and hence $P^* \supset A^* \cap B^*$ and $B^* \supset P^* \cap A^*$ one can easily verify that we must have in u a situation such as indicated in the figure, which shows the case that u is a plane $(r(u) = 3)$; in the general case we have subspaces of rank $i(\mathbf{S}) - 1$ instead of lines, etc.; then

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[1]) We have to prove proposition 10 only in the case that $P \neq B$ and $Q \neq C$, the other cases being trivial.

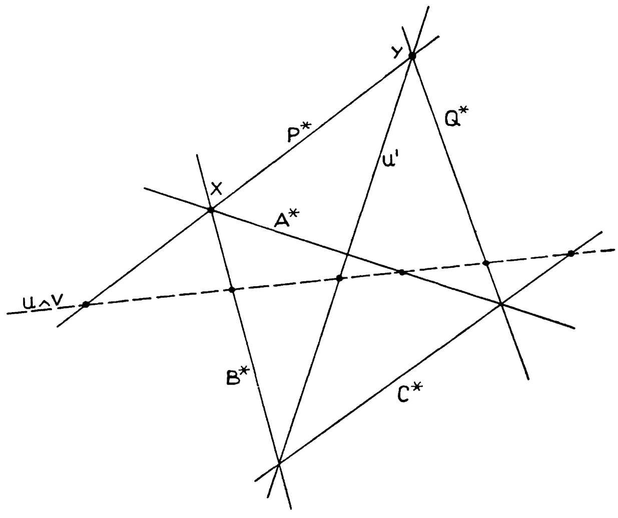

Fig. 9

$A^* \cap B^* \cap C^* = A^* \cap P^* \cap Q^*$ contains an element of rank $i(\mathbf{S})-3$ that is $\leqslant u$.

Now let v be such that $r(v)=i(\mathbf{S})$ and $r(u \wedge v)=i(\mathbf{S})-1$.

First suppose v such that there is no $w<u \wedge v$ of rank $i(\mathbf{S})-2$ that is $\in B^* \cap P^*$, $B^* \cap C^*$, $C^* \cap Q^*$ or $P^* \cap Q^*$.

Then it is clear that v is of type γ or δ; in the latter case $(2, v)$ does not add anything to $(3, u)$. In the former one we proceed as follows: we define a projectivity π of u upon v by taking: $\pi=$identity on $u \wedge v$, $x^\pi = x'$, $y^\pi = y'$ where x and y are points $<u$ which belong to $B^* \cap P^*$ and $P^* \cap Q^*$ respectively and x' and y' are similar in v. It is not very difficult to verify that π maps elements of B^* onto elements of B^* and does similarly with C^*, P^*, Q^* and A^*.

But then u'^{π} must be equal to v'. As π is the identity on $u \wedge v$, $u' \wedge v = v' \wedge u$. Hence $(2, v)$ cannot add anything to $(3, u)$ that was not previously in $(2, u)$.

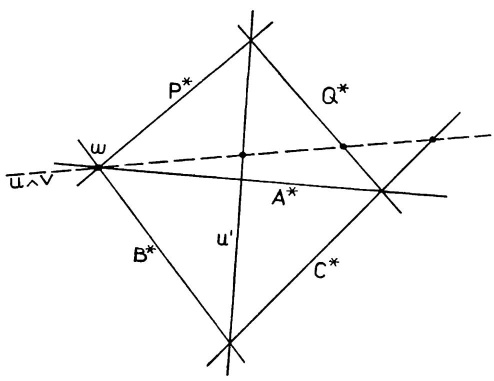

Fig. 10

Now let v be such that $u \wedge v \notin A^*$ and that there is a $w \leqslant u \wedge v$ of rank $i(\mathbf{S})-2$ such that $w \in P^* \cap B^*$.

We have only to consider the case that v is of type γ, for otherwise $(2, v)$ can add no points to $(3, u)$ that are not previously in $(2, u)$. We shall prove the existence of a projectivity π of u upon v that transforms elements of B^* into elements of B^* and similarly for C^*, P^*, Q^* and A^* and that leaves $v' \wedge u$ pointwise invariant. As π leaves w and $v' \wedge u$ invariant, it also leaves $u \wedge v$ invariant, as $w \neq v' \wedge u$. It is clear that π must transform u' into v'. Hence $u' \wedge v$ is transformed into $v' \wedge u$. But the latter is invariant under π (see the above characterisation of π), hence $u' \wedge v = v' \wedge u$. Hence again $(2, v)$ does not add any new point to $(3, u)$.

To prove the existence of the required projectivity π we reason as follows:

We choose maximal w_1 and $w_2 \in \mathbf{S}$ with the following properties:

a. $w_{1,2} \geqslant v' \wedge u$.

b. $r(w_1 \wedge u) = r(w_1 \wedge w_2) = r(w_2 \wedge v) = i(\mathbf{S}) - 1$.

c. $w_1 \wedge u$ is in the general position such as described above, i.e. it does not contain elements $\leqslant u$ of $B^* \cap C^*$, $B^* \cap P^*$, $C^* \cap Q^*$ or $P^* \cap Q^*$ of rank $i(\mathbf{S})-2$; the same is true of $w_2 \wedge v$.

Then w_1 and w_2 are of type γ or δ; if they are both of type δ, we may, moreover, suppose that they do not contain the same element of rank $i(\mathbf{S})-2$ of $B^* \cap C^* \cap P^* \cap Q^*$.

If, for instance, w_1 is of type γ, then there is a projectivity of u upon w_1 transforming elements of B^*, C^*, P^*, Q^* and A^* respectively into similar elements that is the identity on $u \wedge w_1$ and hence leaves invariant $v' \wedge u$; this has been proved above.

Now if w_2 is also of type γ, there exists a similar projectivity of w_1 upon w_2 and of w_2 upon v (for v was supposed to be of type γ). Thus we find the projectivity of u upon v that we looked for.

It is also possible that w_1 or w_2 or both are of type δ. Let us suppose, for instance, w_1 of type δ and w_2 of type γ.

We choose a point $x < w_1$, $x \nleqslant u \wedge w_1$, $x \nleqslant w_1 \wedge w_2$, $x \in B^* \cap C^* \cap P^* \cap Q^*$. Then we project $u \wedge w_1$ upon $w_1 \wedge w_2$ from x; this projection transforms elements of A^*, B^*, C^*, P^* and Q^* into similar elements and can hence be extended to a projectivity of u upon w_2 of the same property.

In the other possible cases we follow a similar line of reasoning. We find always a projectivity of u upon v with the required properties.

The case $u \wedge v \in A^*$ is treated likewise.

Finally we have to consider the case that $u \wedge v$ contains, for instance, the element $\leqslant u$ of $B^* \cap C^*$ of rank $i(\mathbf{S})-2$. But then, again, $(2, v)$ can add no points to $(3, u)$ that were not already contained in $(2, u)$.

14. After we have proved in the preceding section that $(2, u) = (3, u)$ if u is of type γ, we shall now suppose

$(2, u)=(3, u)= \dots =(n, u)$ for every maximal $u \in \mathbf{S}$ of type γ, $(n \geqslant 3)$, and show that

$$(n, u)=(n+1, u) \text{ if } u \text{ is of type } \gamma.$$

Suppose v_1 maximal $\in \mathbf{S}$ such that $r(u \wedge v_1)=i(\mathbf{S})-1$ and such that there is no $w \leqslant u \wedge v_1$ of rank $i(\mathbf{S})-2$ that is $\in B^* \cap C^*$, $B^* \cap P^*$, $C^* \cap Q^*$ or $P^* \cap Q^*$.

Then v_1 must be of type γ or δ; in the former case $(2, v_1)=(n, v_1)$. But we have seen in the preceding section that if $x \in (2, v_1)$ and $x \leqslant u \wedge v_1$, then $x \in (2, u)$. Hence (n, v_1) does not add anything to $(n+1, u)$ that is not in (n, u).

Now suppose that v_1 is of type δ and that we have maximal $v_2, \dots, v_k$ such that $r(v_i, v_{i+1})=i(\mathbf{S})-1$, $v_2, \dots, v_{k-2}$ and v_{k-1} are of type δ and v_k is of type γ.

If there is any $i<k-1$ such that $v_i \wedge v_{i+1} \geqslant w$ of rank $i(\mathbf{S})-2$, $w \in B^* \cap C^* \cap P^* \cap Q^*$, then we can add some maximal $v_{i,1}, \dots, v_{i,l}$ such that $r(v_i \wedge v_{i,1})=r(v_{i,1} \wedge v_{i,2})= \dots =r(v_{i,l} \wedge v_{i+1})=i(\mathbf{S})-1$ and such that neither $v_i \wedge v_{i,1}$, $v_{i,1} \wedge v_{i,2}, \dots,$ nor $v_{i,l} \wedge v_{i+1}$ has a similar property as $v_i \wedge v_{i+1}$. Hence we may suppose about $v_1, \dots, v_k$: for every $i<k-1$, $v_i \wedge v_{i+1}$ does not contain an element of $B^* \cap C^* \cap P^* \cap Q^*$ of rank $i(\mathbf{S})-2$.

Then we can select points $x_i < v_i$ $(i = 1, \dots, k-1)$ such that $x_i \in B^* \cap C^* \cap P^* \cap Q^*$, $x_i \nleqslant v_{i-1} \wedge v_i$ and $x_i \nleqslant v_i \wedge v_{i+1}$. We project $v_{i-1} \wedge v_i$ upon $v_i \wedge v_{i+1}$ from x_i. Note that we take $v_0=u$. Thus we get a projectivity of $u \wedge v_1$ upon $v_{k-1} \wedge v_k$ that transforms elements of A^*, B^*, C^*, P^* and Q^* respectively into similar elements; this application can be extended to a projectivity π of u upon v_k of the same property. It is not hard to verify that π maps u' upon v_k'.

Now we know that $(n+1-k, v_k)=(2, v_k)$, as v_k is of type γ. The only points that $(n+1-k, v_k)$ can add to $(n+1, u)$ must hence be $\leqslant (v_k' \wedge v_{k-1})^{\pi^{-1}}$; but this is $u' \wedge v_1$ and therefore $(n+1-k, v_k)$ does not add anything to $(n+1, u)$ that was not already in (n, u).

Another situation that we have to consider is the one where $v_1, \dots, v_k$ are as above with the only difference that v_k is of type α, β or δ. The former one can be reduced to the case we have just treated and in the latter two cases $(n+1-k, v_k)$ does not add anything to $(n+1, u)$ that was not previously in (n, u).

The cases that $u \wedge v_1$ is not quite as we supposed at the beginning of this section can be treated in a similar way as in section 13.

Hence $(n+1, u)=(n, u)$, which had to be proved.

This achieves the proof of proposition 10.

15. Now we consider the smallest subset $\mathfrak{P}'$ of $\mathfrak{P}$ with the property:

If X and Y are two points of $\mathbf{S}$, then every point $Z \in X+Y$ belongs to $\mathfrak{P}'$. (Here the points of $\mathbf{S}$ are considered as elements of $\mathfrak{P}$; cf. section 2.) We shall prove:

PROPOSITION. *If P and Q are elements of $\mathfrak{P}'$ and $R \in P+Q$, then R is also in $\mathfrak{P}'$.*

Proof:

a. $P \in \mathbf{S}$, $Q \in \mathbf{S}$. Then $R \in \mathfrak{P}'$ owing to the definition.

b. $P \in \mathbf{S}$, $Q \notin \mathbf{S}$.

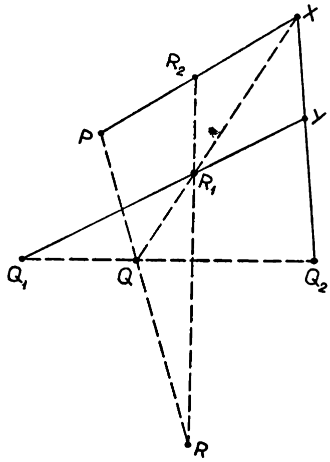

Fig. 11

There exist points Q_1 and Q_2, both $\in \mathbf{S}$, such that $Q \in Q_1+Q_2$. We may suppose $P \notin Q_1+Q_2$, for otherwise the proof is trivial. Hence there exists a point $X \in \mathbf{S}$ such that $X \vee P$ and $X \vee Q_2$ exist but $X \vee Q_1$ does not. Let Y be the point $\in X \vee Q_2$ such that $Y \vee Q_1$ exists. Let R_1 be the intersection of the lines $Q_1 \vee Y$ and $Q+X$ and R_2 that of $P \vee X$ and $R+R_1$, then $R \in R_1+R_2$ where R_1 and R_2 are points of $\mathbf{S}$ (because of 8).

c. $P \notin \mathbf{S}$, $Q \notin \mathbf{S}$.

Then $P \in P_1+P_2$ and $Q \in Q_1+Q_2$ where the points P_i and Q_i belong to $\mathbf{S}$.

On the line $R+P_1$ there must be a point X such that $X+P_2$ contains a point $Y \in Q_1+Q_2$ (follows from proposition 10). From *b.* it follows that $X \in X_1+X_2$ where X_1 and X_2 belong to $\mathbf{S}$. But then $R \in R_1+R_2$ where R_1 and $R_2 \in \mathbf{S}$, as follows again from *b.*

This completes the proof.

16. PROPOSITION. *If P is an arbitrary element of $\mathfrak{P}'$ and Q a point $\in \mathbf{S}$, $Q \notin P^*$, then there exists a point $R \in \mathbf{S}$, $R \neq Q$, $R \in P+Q$.*

Proof: The case $P \in \mathbf{S}$ is trivial. So we suppose $P \notin \mathbf{S}$.

On account of the definition of $\mathfrak{P}'$ we can find two different points Q_1 and R_1 in $\mathbf{S}$ such that $P \in Q_1+R_1$. If $Q \in Q_1+R_1$, nothing remains to be proved.

If $Q \notin Q_1+R_1$ but, for instance, $Q \vee Q_1$ exists, then we consider the point $X < Q \vee Q_1$ such that $X \vee R_1$ exists. As $Q \notin P^*$, $Q \neq X$. Hence the line $P+Q$ intersects $X \vee R_1$ in a point $R \neq Q$. Owing to proposition 8 $R \in \mathbf{S}$.

If neither $Q \vee Q_1$ nor $Q \vee R_1$ exists, we take a point $X \in \mathbf{S}$ such that $X \vee Q_1$ and $X \vee R_1$ exist but $X \vee Q$ does not. (Bear in mind that $Q \notin Q_1 + R_1$.) Then we can find different points $Q_2 < Q_1 \vee X$ and $R_2 < R_1 \vee X$ such that $Q \vee Q_2$ exists and $P \in Q_2 + R_2$. Then we proceed as above with Q_2 and R_2 in stead of Q_1 and R_1.

17. Proposition. *On every line in $\mathfrak{P}'$ there are at least three points.*

Proof: Consider a line $P+Q$. It is not very difficult to find a point $t \in \mathbf{S}$ that is neither in P^* nor in Q^*. Let t be identified with $T \in \mathfrak{P}'$. From proposition 16 it follows that there are points P_1 and $Q_1 \in \mathbf{S}$ such that

$$P \in T + P_1, \; Q \in T + Q_1.$$

From proposition 10 it follows that $P+Q$ and $P_1 + Q_1$ have a point R in common. It may easily be verified that $R \neq P$ and $R \neq Q$. This proves the proposition.

18. From propositions 6, 10, 15 and 17 it follows that $\mathfrak{P}'$ is a projective space. (See G. Birkhoff [2], pg. 116.)

We define a flat in $\mathfrak{P}'$ as a set of points that contains with any two points P and Q the line $P+Q$. It follows from proposition 8 that the elements of $\mathbf{S}$ can be considered as flats in $\mathfrak{P}'$.

It is clear that the intersection of $\mathbf{S}$ with a flat in $\mathfrak{P}'$ is a flat subset of $\mathbf{S}$. Now we apply axiom VIII (see III, 12). Let $X_1, \ldots, X_p$ be a set of points such that $\mathbf{S}$ is the only flat subset that contains them all. Let V be the smallest flat in $\mathfrak{P}'$ that contains $X_1, \ldots, X_p$. Then every point of $\mathbf{S}$ must belong to V. Hence V contains all points of $\mathfrak{P}'$.

If $X \in \mathfrak{P}'$, the symbol X will also denote the flat consisting of the element X only. Flats in $\mathfrak{P}'$ will be denoted by great Italic characters.

The partially ordered (by inclusion) set of all flats of $\mathfrak{P}'$ will be called $\mathbf{P}$; the inclusion will be denoted by $\leqslant$.

The smallest flat containing two flats U and V is called $U+V$; their intersection is also a flat and will be called $U \cap V$. Analogous for an arbitrary number of flats: $\sum U$ or $\bigcap U$.

Every element of $\mathbf{S}$ can be identified in a unique way with an element of $\mathbf{P}$. If x and y are $\in \mathbf{S}$ such that $x \vee y$ exists and they are identified with X and Y in $\mathbf{P}$ respectively, then we shall often write $X \vee Y$ in stead of $X+Y$.

The maximal element of $\mathbf{P}$ will be called A. A has finite rank n, as we saw above.

19. Now we shall introduce a polarity σ in the space $\mathbf{P}$ such that $\mathbf{S}$ will be the set of strictly isotropic subspaces with respect to σ or, in the

case that the space $\mathbf{P}$ is represented by a linear space over a field of characteristic 2, is contained in such a set.

If P is a point, P^σ is defined as the smallest flat that contains P^*.

If V is a flat, we define $V^\sigma = \bigcap P^\sigma$ where the intersection is taken over all points $P \leqslant V$.

Finally $0^\sigma = A$.

The proof that σ is a polarity and that $\mathbf{S}$ is (part of) the polar geometry corresponding to σ will be given in the next sections.

20. Lemma. *If P is a point, $r(P^\sigma) \geqslant n-1$ (where n is the rank of* $\mathbf{P}$).

Proof: Suppose $r(P^\sigma) < n-1$. We choose a point $Q \in \mathbf{S}$, $Q \nleqslant P^\sigma$ and then a flat $H \geqslant Q + P^\sigma$ of rank $n-1$.

The elements $X \in \mathbf{S}$ such that $X \leqslant H$ form a flat subset θ of $\mathbf{S}$; $\theta \in \mathfrak{P}^*$.

$\theta \supset P^*$ and $Q \in \theta$, $Q \notin P^*$, hence $\theta \neq P^*$. But this is a contradiction with lemma 3.

Hence $r(P^\sigma) \geqslant n-1$.

21. Lemma. *If P is a point of* $\mathbf{S}$, *then $r(P^\sigma) = n-1$.*

Proof: We shall prove that every point in P^σ depends on two points in P^*. From this it follows that $X < P^\sigma$, $X \in \mathbf{S}$, implies: $X \in P^*$. As $P^* \neq \mathbf{S}$, P^σ cannot be equal to A; hence $r(P^\sigma) = n-1$.

It suffices to prove: if both Q and R depend on two points in P^* and $T < Q + R$, then T depends on two points in P^*.

a. $Q \in P^*$, $R \in P^*$: trivial case.

b. $Q \in P^*$, $R \notin P^*$.

Then $R < R_1 + R_2$ where the points R_1 and R_2 belong to P^*.

In this case $R \nleqslant P \vee Q$; hence there exists a point $X \in P^* \cap Q^*$, $X \notin R^*$. Applying proposition 16 we find a point $Y \in \mathbf{S}$, $Y \neq X$, $Y < X + R$.

$P \in R_1^* \cap R_2^*$, hence $P \in R^*$. Therefore $P \in X^* \cap R^*$, hence $P \in Y^*$. $P \in \mathbf{S}$, so $Y \in P^*$.

As $T < Q + X + Y$, the line $T + Y$ meets $Q \vee X$ in a point Z. Hence $T < Y + Z$ where Y and $Z \in P^*$.

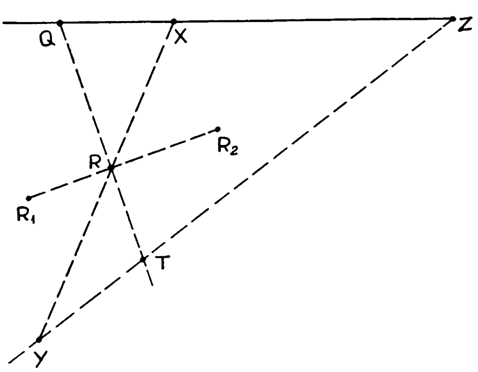

Fig. 12

c. The general case can be reduced to *b.* in the same way as in 15, *c.*

22. PROPOSITION. *If P is an arbitrary point of* $\mathbf{P}$, $r(P^\sigma)=n-1$.

Proof. In view of lemmas 20 and 21 we have only to prove: $r(\mathbf{P}^\sigma)\leqslant \leqslant n-1$ if $P\notin \mathbf{S}$.

Take points A, B, C and D in $\mathbf{S}$ such that $P=(A+B)\cap(C+D)$ and such that $A\vee C$ and $B\vee D$ exist and have a point Q in common.

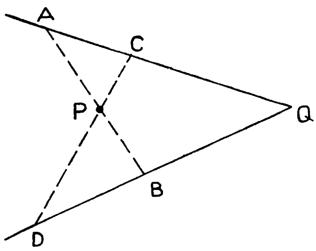

Fig. 13

Then $P^*\supset A^*\cap B^*$ and $P^*\supset C^*\cap D^*$.

As $A^*\cap B^*\not\subset C^*\cap D^*$, there is a point $\in A^*\cap B^*$ that is not $\in C^*\cap D^*$. It follows from lemma 7 that P^* is the smallest flat subset of $\mathbf{S}$ that contains both $A^*\cap B^*$ and $C^*\cap D^*$.

Hence $X\leqslant A^\sigma\cap B^\sigma+C^\sigma\cap D^\sigma$ if $X\in P^*$.

Therefore $P^\sigma\leqslant A^\sigma\cap B^\sigma+C^\sigma\cap D^\sigma$.

A and $B\in\mathbf{S}$ and $A\neq B$. From the proof of lemma 21 it follows that A^* contains a basis of A^σ and similar for B. Hence $A^\sigma\neq B^\sigma$. As $r(A^\sigma)= =r(B^\sigma)=n-1$, $r(A^\sigma\cap B^\sigma)\leqslant n-2$. Similarly $r(C^\sigma\cap D^\sigma)\leqslant n-2$.

It is easy to verify that $A^*\cap B^*\cap Q^*=C^*\cap D^*\cap Q^*$. We shall prove that $A^*\cap B^*\cap Q^*$ contains a basis of $A^\sigma\cap B^\sigma\cap Q^\sigma$ and similar for $C^\sigma\cap D^\sigma\cap Q^\sigma$. Hence $A^\sigma\cap B^\sigma\cap Q^\sigma=C^\sigma\cap D^\sigma\cap Q^\sigma$. As the latter flat has rank $\geqslant n-3$, $A^\sigma\cap B^\sigma$ and $C^\sigma\cap D^\sigma$ have a flat of rank $\geqslant n-3$ in common. Therefore $r(A^\sigma\cap B^\sigma+C^\sigma\cap D^\sigma)\leqslant n-1$, hence $r(P^\sigma)\leqslant n-1$, which had to be proved.

We have still to show that $A^*\cap B^*\cap Q^*$ contains a basis of $A^\sigma\cap B^\sigma\cap Q^\sigma$. Keep in mind that A, B and $Q\in\mathbf{S}$, $A\notin B^*$, $Q\not\leqslant A+B$.

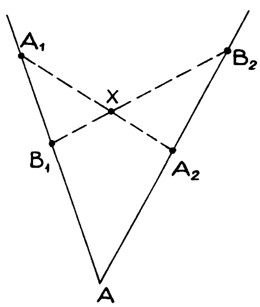

Fig. 14

Let X be a point, $X \leqslant A^\sigma \cap B^\sigma \cap C^\sigma$.

$X \leqslant A^\sigma$, hence $X \leqslant A_1 + A_2$ where A_1 and A_2 are points in A^* (See the proof of lemma 21).

We can find a point $B_1 < A \vee A_1$, $B_1 \in B^*$. As $A \notin B^*$, $B_1 \neq A$. Hence a point $B_2 < A \vee A_2$ exists such that $X < B_1 + B_2$. $X < B^\sigma$ and $B_1 < B^\sigma$, hence $B_2 < B^\sigma$; B_2 also $\in \mathbf{S}$, hence $B_2 \in B^*$ (see proof of lemma 21). Hence B_1 and $B_2 \in A^* \cap B^*$.

One can easily prove the existence of a point $Y \in \mathbf{S}$ such that $B_1 \vee Y$ and $B_2 \vee Y$ exist and are $\in A^* \cap B^*$.

If we can find a point $Q_1 \in Q^*$, $Q_1 < B_1 \vee Y$, $Q_1 \neq Y$, we apply an analogous reasoning as in the case of B_1 and B_2 to find $Q_2 \in A^* \cap B^* \cap Q^*$ such that $X < Q_1 + Q_2$.

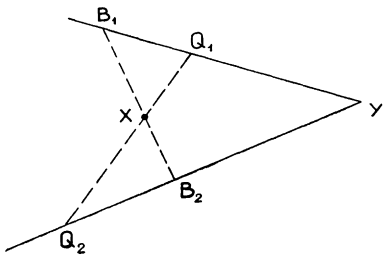

Fig. 15

If such a point Q_1 does not exist, $Y \in A^* \cap B^* \cap Q^*$, for there must exist a point $\in Q^*$ on the line $B_1 \vee Y$.

As $Q^* \not\supset A^* \cap B^*$, we can find a point $Z \in A^* \cap B^*$, $Z \notin Q^*$.

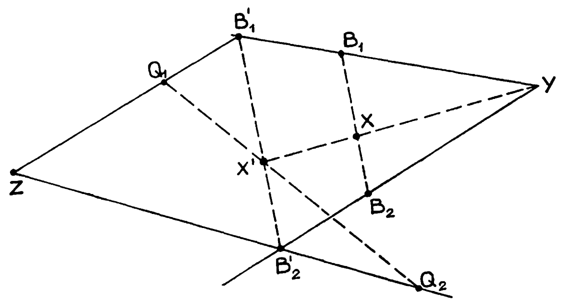

Fig. 16

Let $B_i' < B_i \vee Y$ be such a point that $B_i' \vee Z$ exists. Then $B_i' \vee Z \in$ $\in A^* \cap B^*$.

Take $X' = (X + Y) \cap (B_1' + B_2')$.

Then we can find points Q_1 and $Q_2 \in A^* \cap B^* \cap Q^*$, $Q_i < B_i' + Z$, such that $X' < Q_1 + Q_2$ (see above), for $Z \notin Q^*$.

Then $X < Q_1 + Q_2 + Y$ where Q_1, Q_2 are Y belong to $A^* \cap B^* \cap Q^*$.

23. Lemma. *Let P be an arbitrary point and Q a point $\in \mathbf{S}$, $Q < P^\sigma$. Then $Q \in P^*$.*

Proof: If $Q \notin P^*$, the elements of **S** which are $\leqslant P^\sigma$ would form a flat subset that contains P^* and a point $Q \notin P^*$, hence all the elements of **S**. This would imply $P^\sigma = A$, which is not true.

24. Lemma. *If P, Q and R are points, $P < Q + R$, $Q \in P^*$ and $R \in \mathbf{S}$, then $P \in \mathbf{S}$.*

Proof: We may suppose $P \neq Q$. Then $Q^* \not\subset P^*$ (see lemma 3). Choose a line $L \in \mathbf{S}$ through Q, $L \notin P^*$. Let $Q' < L$ be the point such that $Q' \vee R$ exists. Then $Q' \in Q^* \cap R^*$, hence $Q' \in P^*$.

As $L \notin P^*$, $Q' = Q$. But then $Q \vee R$ exists and $P < Q \vee R$, hence $P \in \mathbf{S}$.

25. Proposition. *If P and Q are points and $P < Q^\sigma$, then $Q < P^\sigma$.*

Proof: We may suppose $P \neq Q$.

a. $P \in Q^*$.

We choose a point $Q_1 \in P^*$, $Q_1 \notin Q^*$. There is a point $Q_2 < Q_1 + Q$, $Q_2 \in \mathbf{S}$, $Q_2 \neq Q_1$ (proposition 16). If $Q_2 \vee P$ exists, Q_1 and $Q_2 \in P^*$ and hence $Q < P^\sigma$.

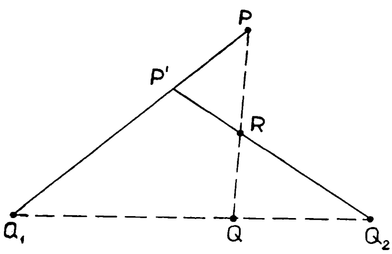

Fig. 17

If $Q_2 \vee P$ does not exist, we take the point $P' < P \vee Q_1$ such that $P' \vee Q_2$ exists. Then $P + Q$ intersects $P' \vee Q_2$ at a point $R \in \mathbf{S}$ different from P. Applying lemma 24 we see that $Q \in \mathbf{S}$.

$P \in Q^*$ implies: $P \vee Q$ exists; hence $Q \in P^*$.

b. $P \notin Q^*$.

From proposition 16 and lemma 23 we infer the existence of points P_1 and P_2, both $\in Q^*$, such that $P < P_1 + P_2$. We may suppose $Q \not\leqslant P_1 + P_2$. Now choose a point $Q_1 \in P_1^* \cap P_2^*$, $Q_1 \notin Q^*$. As $Q \in P_1^\sigma \cap P_2^\sigma$ (follows from *a.*), we can find a point $Q_2 \in P_1^* \cap P_2^*$, $Q_2 \neq Q_1$, such that $Q < Q_1 + Q_2$ (proposition 16 and lemma 23). $Q_i \in P_1^* \cap P_2^*$ implies $Q_i \in P^*$. Hence $Q < P^\sigma$.

26. Proposition. *If $P_1, \ldots, P_k$ are points, then $(P_1 + \ldots + P_k)^\sigma = P_1^\sigma \cap \ldots \cap P_k^\sigma$.*

Proof: We shall prove that

$P^\sigma \geqslant P_1^\sigma \cap P_2^\sigma$ if $P \leqslant P_1 + P_2$ (P, P_1 and P_2 points).

From this it follows by induction that

$P \leqslant P_1 + \ldots + P_k$ implies $P^\sigma \geqslant P_1^\sigma \cap \ldots \cap P_k^\sigma$ and this proves the assertion if we take account of the definition of σ (see section 19).

Now suppose $P \leqslant P_1 + P_2$.

Then $P^* \supset P_1^* \cap P_2^*$.

Making use of proposition 16 and lemma 23 one can prove that every point $\neq P_1$, $\neq P_2$ of $P_1^\sigma \cap P_2^\sigma$ depends on at most three points of $P_1^* \cap P_2^*$ in a way analogous to that of the end of section 22 where it was proved that $A^* \cap B^* \cap Q^*$ contains a basis of $A^\sigma \cap B^\sigma \cap Q^\sigma$. Difficulties arise if, for instance, $P_1 < P_1^\sigma \cap P_2^\sigma$. Then choose a point $X \in P_1^* \cap \cap P_2^*$ such that $P_2 \nleqslant P_1 + X$ and next a point $Y < P_1 + X$, $Y \neq X$, $\neq P_1$. As $Y < P_1^\sigma \cap P_2^\sigma$, it depends upon at most three points of $P_1^* \cap P_2^*$. As $X \in P_1^* \cap P_2^*$, P_1 depends on at most four points of $P_1^* \cap P_2^*$.

So $P_1^* \cap P_2^*$ contains a basis of $P_1^\sigma \cap P_2^\sigma$. But then it follows from

$$P^* \supset P_1^* \cap P_2^* \text{ that}$$

$$P^\sigma \geqslant P_1^\sigma \cap P_2^\sigma,$$

which had to be proved.

27. Proposition. *σ is a polarity.*

Proof: (a′). For every $V \in \mathbf{P}$, V^σ is uniquely defined. See section 19.

(b′). If $V \leqslant W$, then $V^\sigma \geqslant W^\sigma$. Follows from the definition of σ.

To prove the converse it suffices to show that $\sigma^2 = 1$.

From proposition 25 it follows that $V \leqslant V^{\sigma\sigma}$.

Now choose a W such that $V \oplus W = A$. (Bear in mind that A is the maximal element of $\mathbf{P}$.) From proposition 26 it follows that $r(V^\sigma) \geqslant n - - r(V)$. Suppose $r(V^\sigma) > n - r(V)$.

As $r(W^\sigma) \geqslant n - r(W)$, we find

$$r(V^\sigma) + r(W^\sigma) > n - r(V) + n - r(W) = n.$$

From proposition 26 again it follows that $V^\sigma \cap W^\sigma = A^\sigma$.

$A \leqslant A^{\sigma\sigma}$, hence $A^{\sigma\sigma} = A$. Hence $A^\sigma = 0$, for if a point $P \leqslant A^\sigma$, we should have $A^{\sigma\sigma} \leqslant P^\sigma \neq A$.

Hence $V^\sigma \cap W^\sigma = 0$. But then it is impossible that $r(V^\sigma) + r(W^\sigma) > n$. So we see that

$$r(V^\sigma) = n - r(V).$$

Hence $r(V^{\sigma\sigma}) = r(V)$ and this implies $V^{\sigma\sigma} = V$, i.e. $\sigma^2 = 1$.

(c′). To every $Z \in \mathbf{P}$ there exists an X such that $X^\sigma = Z$. For take $X = Z^\sigma$.

So we see that σ is a duality. As we have already proved that $\sigma^2 = 1$, σ is a polarity.

28. PROPOSITION. *If* $X \in \mathbf{S}$, *then* $X \leqslant X^\sigma$.

Proof: If P is a point $\in \mathbf{S}$, then $P \in P^*$ and hence $P < P^\sigma$.

If $X = P_1 + \ldots + P_k \in \mathbf{S}$, where $P_1, \ldots, P_k$ are points, then $P_i \in P_j^*$ for every i and j and hence $P_i < P_j^\sigma$.

Therefore $X^\sigma = P_1^\sigma \cap \ldots \cap P_k^\sigma \geqslant P_1 + \ldots + P_k = X$.

29. The maximal elements of $\mathbf{S}$ are projective spaces of rank $\geqslant 3$; hence they can be represented by linear spaces over a field F. As the characteristic of F does not depend on the special choice of F, we can define:

characteristic of $\mathbf{S}$ = *characteristic of* F.

Now we shall prove:

PROPOSITION. *If* $\mathbf{S}$ *is of characteristic* $\neq 2$, $X \in \mathbf{S}$ *if, and only if,* $X \leqslant X^\sigma$.

Proof: "only if" has been proved in proposition 28.

"if":

It suffices to prove: if P is a point and $P \leqslant P^\sigma$, then $P \in \mathbf{S}$.

For suppose $X = P_1 + \ldots + P_k$ and $X \leqslant X^\sigma$. Then $P_1 + \ldots + P_k \leqslant \leqslant P_1^\sigma \cap \ldots \cap P_k^\sigma$. $P_i \leqslant P_i^\sigma$ implies $P_i \in \mathbf{S}$. Then $P_i \leqslant P_j^\sigma$ implies $P_i \in P_j^*$ (lemma 23), i.e. $P_i \vee P_j$ exists. Hence $P_1 \vee \ldots \vee P_k$ exists, i.e. $X \in \mathbf{S}$.

So we suppose $P < P^\sigma$, P a point.

There exist points Q and $R \in \mathbf{S}$ such that $P < Q + R$. Hence it suffices to show:

If there are two different points of $\mathbf{S}$ on a line L, every isotropic point on L is in $\mathbf{S}$.

Taking account of proposition III, 9 it is readily seen that we have to prove the above assertion for *one* such line L only; realize that if two points of $\mathbf{S}$ are conjugated with respect to σ, they are joined in $\mathbf{S}$ (lemma 23).

Suppose $\mathbf{P}$ represented as the lattice of subspaces of the linear space A and σ represented by the semi-bilinear form f that is supposed to be hermitian or skew-symmetric.

Now choose four independent points $x_\#$, $y_\#$, $u_\#$ and $v_\#$ such that $x_\# + y_\#$, $x_\# + u_\#$, $y_\# + v_\#$ and $u_\# + v_\#$ belong to $\mathbf{S}$ but $x_\# + v_\#$ and $y_\# + u_\#$ do not.

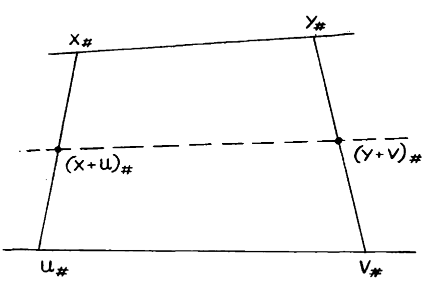

Fig. 18

Take x, u, y and v such that $f(x, v) = f(u, y) = 1$; this is possible because $x_{\#} + v_{\#}$ is not in $\mathbf{S}$, hence $f(x, v) \neq 0$ (lemma 23), and the same for u and y.

$$f(x+u,\ y+v) = 0+1+1+0 = 2 \neq 0.$$

Hence $(x+u)_{\#} + (y+v)_{\#} \notin \mathbf{S}$; we choose L equal to this line.

Now suppose $z_{\#} < L$, $z_{\#} \leqslant z_{\#}^{\sigma}$.

$$z = (x+u) + \lambda(y+v).$$

If f is a hermitian α-form, then

$$f(z, z) = 2(\lambda + \lambda^{\alpha}) = 0.$$

Hence
$$\lambda + \lambda^{\alpha} = 0.$$

$$z_{\#} < (x+\lambda y)_{\#} + (u+\lambda v)_{\#} = M.$$

$f(x+\lambda y,\ u+\lambda v) = \lambda + \lambda^{\alpha} = 0$. Hence $(x+\lambda y)_{\#}$ and $(u+\lambda v)_{\#}$, which are $\in \mathbf{S}$, are joined in $\mathbf{S}$. Therefore $M \in \mathbf{S}$. As $z_{\#} < M$, $z_{\#} \in \mathbf{S}$.

If f is skew-symmetric, the proof is even simpler:

$f(x+\lambda y,\ u+\lambda v) = -\lambda + \lambda = 0$. Hence $M \in \mathbf{S}$ and therefore $z_{\#} \in \mathbf{S}$.

This completes the proof.

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