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Convergence in Perfect MV-Algebras

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INTRODUCTION

MV-algebras were defined by C. C. Chang [6] as algebras corresponding to the Lukasiewicz infinite valued propositional calculus. In [12], D. Mundici established a categorical equivalence between the category of MV-algebras and the category of abelian l-groups with strong unit, and in [10] A. Di Nola and A. Lettieri proved that the category of perfect MV-algebras [2] is equivalent with the category of abelian l-groups.

Using the Mundici functor Γ, a concept of convergence in MV-algebras was studied in [11] related to the order-convergence in abelian l-groups [9].

The aim of this paper is to investigate how the Di Nola–Lettieri functors $\mathscr{D}$ and Δ [10] allow us to study the convergence in perfect MV-algebras.

We refer to [6], [10], and [12] for all the unexplained notions and results on MV-algebras.

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1. PRELIMINARIES ON MV-ALGEBRAS AND l-GROUPS

Let $\langle A, \oplus, \cdot, \bar{}, 0, 1\rangle$ be an MV-algebra. For $x, y \in A$ denote xy instead of $x \cdot y$ and define

$$x \vee y = x\bar{y} \oplus y, \qquad x \wedge y = (x \oplus \bar{y})y. \tag{1}$$

Thus $\langle A, \vee, \wedge, 0, 1\rangle$ is a bounded distributive lattice. The *distance function* $d\colon A^2 \to A$ is defined by $d(x, y) = x\bar{y} \oplus \bar{x}y$. The following properties of d can be found in [6]:

$$d(x, y) = 0 \qquad \text{iff } x = y, \tag{2}$$

$$d(x, 0) = x, \qquad d(x, 1) = \bar{x}, \tag{3}$$

$$d(\bar{x}, \bar{y}) = d(x, y), \tag{4}$$

$$d(x, z) \leq d(x, y) \oplus d(y, z), \tag{5}$$

$$d(x \oplus u, y \oplus v) \leq d(x, y) \oplus d(u, v), \tag{6}$$

$$d(xu, yv) \leq d(x, y) \oplus d(u, v), \tag{7}$$

for any $x, y, z, u, v \in A$. Using Chang's representation theorem for MV-algebras [7] it can be shown that

$$d(x, y) = x\bar{y} \vee \bar{x}y. \tag{8}$$

We say that $I \subseteq A$ is an *MV-ideal* if the following conditions are satisfied:

(i1) if $x \leq y$ and $y \in I$ then $x \in I$;

(i2) if $x \in I$ and $y \in I$ then $x \oplus y \in I$.

The *radical* Rad A of A is the intersection of the maximal ideals of A. An MV-algebra is *perfect* if $A = \text{Rad } A \cup \overline{\text{Rad } A}$, where $\overline{\text{Rad } A} = \{\bar{x} \mid x \in \text{Rad } A\}$ (see [2] and [10]). If $x \in \text{Rad } A$ and $y \in \text{Rad } A$ then $x < y$. If $x \in \overline{\text{Rad } A}$ and $y \in \overline{\text{Rad } A}$ then $xy \in \overline{\text{Rad } A}$.

Let $\langle G, u\rangle$ be an abelian l-group with strong unit u. Then the interval $\Gamma(G, u) = [0, u]$ has a canonical structure of an MV-algebra:

$$x \oplus y = (x + y) \wedge u, \qquad xy = (x + y - u) \vee 0, \qquad \bar{x} = u - x, \tag{9}$$

where $+$ is the group operation on G and $\vee$ and $\wedge$ correspond to the lattice structure of G.

For any MV-algebra A there exists an abelian l-group $\langle G, u\rangle$ with a strong unit u such that A is isomorphic to $\Gamma(G, u)$. In fact, the Mundici functor Γ is a categorical equivalence between the category of abelian l-groups with strong unit and the category of MV-algebras [12].

For an abelian l-group G consider the lexicographic product $Z \times G$ and the perfect MV-algebra $\Delta(G) = \Gamma(Z \times G, (1,0))$ [10]. Each element of $\Delta(G)$ has either the form $(0, g)$ with $g \geq 0$ in G, or the form $(1, g)$ with $g \leq 0$ in G. The operations on $\Delta(G)$ are defined as follows:

$$(z, g) \oplus (t, h) = (1,0) \wedge (z + t, g + h), \tag{10}$$

$$\overline{(z, g)} = (1,0) - (z, g) = (1 - z, -g), \tag{11}$$

$$(z, g) \cdot (t, h) = (0,0) \vee (z + t - 1, g + h), \tag{12}$$

where $+$ is the group operation on G.

If G is an l-group and $g \in G$, then denote $g^{+} = g \vee 0$, $g^{-} = (-g) \vee 0$, $|g| = g^{+} + g^{-}$. We recall that $g = g^{+} - g^{-}$.

With these definitions, the distance function in $\Delta(G)$ become's

$$\begin{aligned} d((z, g), (t, h)) & \\ &= (1,0) \wedge ((0,0) \vee (z - t, g - h) + (0,0) \vee (t - z, h - g)) \\ &= (1,0) \wedge (|z - t|, |g - h|). \end{aligned}$$

It follows that

$$d((z, g), (t, h)) = \begin{cases} (0, |g - h|), & \text{if } z = t, \\ (1,0), & \text{otherwise}. \end{cases} \tag{13}$$

We remark that $(1, 0)$ is a strong unit of $Z \times G$ and

$$\mathrm{Rad}\, \Delta(G) = \{(0, x) | x \geq 0\}, \qquad \overline{\mathrm{Rad}\, \Delta(G)} = \{(1, x) | x \leq 0\}.$$

If A is a perfect MV-algebra then $\langle \mathrm{Rad}\, A, \oplus, 0 \rangle$ is a cancellative abelian monoid [10]. Define the following congruence $\ominus$ on $\mathrm{Rad}\, A \times \mathrm{Rad}\, A$:

$$(x, y) \ominus (u, v) \qquad \text{iff } x \oplus v = u \oplus y; \tag{14}$$

and denote by $[x, y]$ the congruence class of $(x, y) \in \mathrm{Rad}\, A \times \mathrm{Rad}\, A$.

Denote $\mathscr{D}(A) = (\mathrm{Rad}\, A \times \mathrm{Rad}\, A)/_{\ominus}$ and define

$$[x, y] + [u, v] = [x \oplus u, y \oplus v], \tag{15}$$

$$[x, y] \leq [u, v] \qquad \text{iff } x \oplus v \leq u \oplus y. \tag{16}$$

Thus $\mathscr{D}(A)$ becomes an abelian l-group such that

$$[x, y] \wedge [u, v] = [(x \oplus v) \wedge (u \oplus y), y \oplus v], \tag{17}$$

$$[x, y] \vee [u, v] = [x \oplus u, (x \oplus v) \wedge (u \oplus y)]. \tag{18}$$

The Di Nola–Lettieri functors $\mathscr{D}: \mathscr{P} \to \mathscr{A}$, $\Delta: \mathscr{A} \to \mathscr{P}$ realize a categorical equivalence between the category $\mathscr{P}$ of perfect MV-algebras and the category $\mathscr{A}$ of abelian l-groups [10].

If $[x, y] \geq [0,0]$ in $\mathscr{D}(A)$ then $[x, y] = [x\bar{y}, 0]$, so $\mathscr{D}(A)^+ = \{[x, 0] \| x \in \text{Rad } A\}$. We also have $-[x, y] = [y, x]$.

LEMMA 1. *If A is a perfect MV-algebra and $x, y \in \text{Rad } A$ then* $[x, y]^+ = [x\bar{y}, 0]$, $[x, y]^- = [\bar{x}y, 0]$, *and* $\|[x, y]\| = [d(x, y), 0]$.

Proof. We have $[x, y]^+ = [x, y] \vee [0,0] = [x, x \wedge y] = [x\overline{(x \wedge y)}, 0] = [x\bar{y}, 0]$ because $[x, x \wedge y] \geq [0,0]$ and $x\overline{(x \wedge y)} = x(\bar{x} \vee \bar{y}) = x\bar{x} \vee x\bar{y} = x\bar{y}$.

In the same manner, $[x, y]^- = -[x, y] \vee [0,0] = [y, x] \vee [0,0] = [\bar{y}x, 0]$ and $\|[x, y]\| = [x, y]^+ + [x, y]^- = [d(x, y), 0]$.

In order to study the convergence in perfect MV-algebras, we shall prove some preliminary results that establish a connection between arbitrary suprema and infima in A and arbitrary suprema and infima in $\mathscr{D}(\mathscr{A})$. The following lemma is well known.

LEMMA 2. *Assume $X \subseteq \text{Rad } A$ such that there exists $x = \bigvee_{y \in X} y$. Then $x \in \text{Rad } A$.*

Proof. If $x \in \overline{\text{Rad } A}$ then $x \leq a$ for any $a \in \overline{\text{Rad } A}$ because $y < a$ for any $y \in X$. Thus $x \leq \bar{y}$, so $y \leq \bar{x}$ for any $y \in X$. This results in $x \leq \bar{x}$. But $\bar{x} \in \text{Rad } A$ and $x \in \overline{\text{Rad } A}$, so we obtain a contradiction.

LEMMA 3. *If $x, x_i \in \text{Rad } A$ for $i \in I$ then the following are equivalent*:

(a) $x = \bigvee_{i \in I} x_i$ *in* A;

(b) $[x, 0] = \bigvee_{i \in I} [x_i, 0]$ *in* $\mathscr{D}(A)$;

(c) $[0, x] = \bigwedge_{i \in I} [0, x_i]$ *in* $\mathscr{D}(A)$.

Proof. For (a) $\Rightarrow$ (b) we have $[x_i, 0] \leq [x, 0]$, $i \in I$. Assume $[x_i, 0] \leq [a, 0]$ for any $i \in I$, so $x_i \leq a$, for any $i \in I$; therefore $x \leq a$, i.e., $[x, 0] \leq [a, 0]$.

The implication (b) $\Rightarrow$ (a) follows similarly and (b) $\Leftrightarrow$ (c) follows by l-group theory.

Remark 4. In an l-group G we have $x = \bigvee_{i \in I} x_i$ iff $x^+ = \bigvee_{i \in I} x_i^+$ and $x^- = \bigwedge_{i \in I} x_i^-$.

PROPOSITION 5. *Let A be a perfect MV-algebra and let $x, y, x_i, y_i \in \text{Rad } A$, for any $i \in I$. The following are equivalent*:

(a) $[x, y] = \bigvee_{i \in I} [x_i, y_i]$ *in* $\mathscr{D}(A)$;

(b) $x\bar{y} = \bigvee_{i \in I} x_i\overline{y_i}$ *and* $y\bar{x} = \bigwedge_{i \in I} y_i\overline{x_i}$ *in* A.

Proof. By the previous remark, $[x, y] = \bigvee_{i \in I} [x_i, y_i]$ iff $[x, y]^+ = \bigvee_{i \in I} [x_i, y_i]^+$ and $[x, y]^- = \bigwedge_{i \in I} [x_i, y_i]^-$. Using Lemmas 1 and 3, $[x, y]^+ = \bigvee_{i \in I} [x_i, y_i]^+$ in $\mathscr{D}(A)$ iff $x\bar{y} = \bigvee_{i \in I} x_i\overline{y_i}$ in A. Similarly, by the dual of Lemma 3, $[x, y]^- = \bigwedge_{i \in I} [x_i, y_i]^-$ iff $y\bar{x} = \bigwedge_{i \in I} y_i\overline{x_i}$.

Let $f\colon A \to B$ be a morphism of perfect MV-algebras. We say that f preserves the countable suprema in Rad A if for any countable subset S of Rad A we have $f(\bigvee_{s\in S} s) = \bigvee_{s\in S} f(s)$. Similarly, we define when f preserves the countable infima in Rad A.

PROPOSITION 6. *Let $f\colon A \to B$ be a morphism of perfect MV-algebras. The following are equivalent*:

(a) *f preserves the countable suprema in A*;

(b) *f preserves the countable infima in A*;

(c) *f preserves the countable suprema and infima in* Rad A;

(d) *$\mathscr{D}(f)$ preserves the countable suprema in $\mathscr{D}(A)$*;

(e) *$\mathscr{D}(f)$ preserves the countable infima in $\mathscr{D}(A)$*.

Proof. The equivalences (a) $\Leftrightarrow$ (b) and (d) $\Leftrightarrow$ (e) are obvious.

(c) $\Rightarrow$ (d) Recall from [10] that $\mathscr{D}(f)([x, y]) = [f(x), f(y)]$ for any $[a, b] \in \mathscr{D}(A)$. Assume countable suprema $[x, y] = \bigvee_{i\in I} [x_i, y_i]$ in $\mathscr{D}(A)$, so, by Proposition 5, $x\bar{y} = \bigvee_{i\in I} x_i\overline{y_i}$ and $y\bar{x} = \bigwedge_{i\in I} y_i\overline{x_i}$ in A. Thus $f(x)\overline{f(y)} = f(x\bar{y}) = f(\bigvee_{i\in I} x_i\bar{y}_i) = \bigvee_{i\in I} f(x_i)\overline{f(y_i)}$ and, similarly, $f(y)\overline{f(x)} = \bigwedge_{i\in I} f(y_i)\overline{f(x_i)}$. Again using Proposition 5, $[f(x), f(y)] = \bigvee_{i\in I} [f(x_i), f(y_i)]$, so $\mathscr{D}(f)$ preserves countable suprema in $\mathscr{D}(A)$.

(d) $\Rightarrow$ (c) Straightforward, using Lemma 3 and its dual.

(c) $\Rightarrow$ (a) Consider $x = \bigvee_{i\in I} x_i$. If $(x_i)_{i\in I} \subseteq$ Rad A then, by Lemma 2, the thesis follows from (c). If $(x_i)_{i\in I} \subseteq \overline{\text{Rad } A}$ then $\bar{x} = \bigwedge_{i\in I} \overline{x_i}$, so we again apply (c).

Assume $J = \{i \in I \mid x_i \in \text{Rad } A\} \neq \varnothing$ and $K = \{i \in I \mid x_i \in \overline{\text{Rad } A}\} \neq \varnothing$. Because $x_j \le x_k$ for every $j \in J$ and $k \in K$ we obtain $x = \bigvee_{i\in K} x_i$, so the thesis follows by the above remark.

The rest of the proof is obvious.

2. CONVERGENCE IN PERFECT MV-ALGEBRAS

In [11], the functor Γ is the main tool in obtaining a convergence theory for arbitrary MV-algebras. However, using the functors $\mathscr{D}$ and Δ we obtain a new look at convergence in perfect MV-algebras. In particular, a characterization of the Cauchy completion of a perfect MV-algebra in terms of these functors is obtained.

The concepts and results on order-convergence in abelian l-groups [9] will be used without mention. Let (x_n) be a sequence in an MV-algebra A. Denote by $x_n \uparrow$ (respectively, $x_n \downarrow$) that (x_n) is an increasing (respectively, decreasing) sequence. By $x_n \downarrow 0$ is meant that (x_n) is decreasing and $\bigwedge x_n = 0$. The same notation will be used for sequences in l-groups.

LEMMA 7. *If (x_n) is a sequence in a perfect MV-algebra A and $x_n \downarrow 0$ then there is a natural number n_0 such that $x_n \in \mathrm{Rad}\, A$ for $n \geq n_0$.*

Proof. Because Rad A is an MV-ideal, it suffices to show that there is n such that $x_n \in \mathrm{Rad}\, A$. Assume $x_n \notin \mathrm{Rad}\, A$ for all n. If $\mathrm{Rad}\, A \neq \{0\}$ then there is a nonzero element $a \in \mathrm{Rad}\, A$, so $a \leq x_n$ for all n, which contradicts $\bigwedge x_n = 0$. If $\mathrm{Rad}\, A = \{0\}$ then $\overline{\mathrm{Rad}\, A} = \{1\}$, so $x_n = 1$ for all n, which is again a contradiction.

LEMMA 8. *If A is a perfect MV-algebra and $(x_n) \subseteq \mathrm{Rad}\, A$ then $x_n \downarrow 0$ in A iff $[x_n, 0] \downarrow [0, 0]$ in $\mathscr{D}(A)$.*

Proof. Assume (x_n) is decreasing and $\bigwedge x_n = 0$, so $([x_n, 0])$ is a decreasing sequence in $\mathscr{D}(A)$. If $[0, 0] \leq u \leq [x_n, 0]$ for any n then $u = [x, 0]$ for some $x \in \mathrm{Rad}\, A$, hence $x \leq x_n$ for any n. Thus $x = 0$ and $u = [0, 0]$, hence $\bigwedge [x_n, 0] = [0, 0]$. The converse assertion follows similarly.

DEFINITION 9 [11]. Let (x_n) be a sequence in an arbitrary MV-algebra A. Then (x_n) converges to $x \in A$ $(x_n \to a)$ if there is a sequence (c_n) such that $c_n \downarrow 0$ and $d(x_n, x) \leq c_n$ for any n.

LEMMA 10 [11]. *If $x_n \to x$ and $y_n \to y$ in A then $\overline{x_n} \to \bar{x}$, $x_n \oplus y_n \to x \oplus y$, $x_n y_n \to xy$, $x_n \vee y_n \to x \vee y$, and $x_n \wedge y_n \to x \wedge y$. If $x_n \leq y_n$ for all n, then $x_n \to x$ and $y_n \to y$ implies $x \leq y$.*

LEMMA 11. *Let A be an arbitrary MV-algebra and let $(x_n) \subseteq \mathrm{Rad}\, A$. If $x_n \to x$ then $x \in \mathrm{Rad}\, A$.*

Proof. Recall that $x \in \mathrm{Rad}\, A$ iff $kx \leq \bar{x}$ for all k. Let k be an arbitrary nonzero integer. Thus $kx_n \leq \overline{x_n}$ for all n so, by Lemma 10, $kx \leq \bar{x}$, hence $x \in \mathrm{Rad}\, A$.

COROLLARY 12. *If A is a perfect MV-algebra, $(x_n) \subseteq \overline{\mathrm{Rad}\, A}$, and $x_n \to x$ then $x \in \overline{\mathrm{Rad}\, A}$.*

Proof. We have $(\overline{x_n}) \subseteq \mathrm{Rad}\, A$ and apply the previous lemma.

LEMMA 13. *Let A be a perfect MV-algebra and let $x_n \to x$ in A. If $x \in \mathrm{Rad}\, A$ then there is n_0 such that $x_n \in \mathrm{Rad}\, A$ for $n \geq n_0$.*

Proof. Assume for any natural number n there is $k_n \geq n$ such that $x_{k_n} \notin \mathrm{Rad}\, A$. In this way one can find a sequence (x_{k_n}) in $\overline{\mathrm{Rad}\, A}$ which is convergent to $x \in \mathrm{Rad}\, A$, contradicting the previous corollary.

COROLLARY 14. *If A is a perfect MV-algebra, $x_n \to x$ in A, and $x \in \overline{\mathrm{Rad}\, A}$ then there is n_0 such that $x_n \in \overline{\mathrm{Rad}\, A}$ for $n \geq n_0$.*

Let G be an l-group, let $(x_n) \subseteq G$, and let $x \in G$. We shall denote $x_n \xrightarrow{\circ} x$ the fact that the sequence (x_n) order-converges to x.

PROPOSITION 15. *If A is a perfect MV-algebra and $(x_n) \subseteq \mathrm{Rad}\, A$ then the following are equivalent*:

(a) $x_n \to x$ *in* A;

(b) $([x_n, 0]) \xrightarrow{\circ} [x, 0]$ *in* $\mathscr{D}(A)$.

Proof. (a) ⇒ (b) Suppose $d(x_n, x) \leq c_n$ for all n, where (c_n) is a sequence in A such that $c_n \downarrow 0$. By Lemma 7 one can assume that $(c_n) \subseteq \mathrm{Rad}\, A$, so $[c_n, 0] \downarrow [0, 0]$ in accordance with Lemma 8. Applying Lemma 1 we obtain $\|[x_n, 0] - [x, 0]\| = \|[x_n, x]\| = [d(x_n, x), 0] \leq [c_n, 0]$ for all n. Hence $([x_n, 0])$ order-converges to $[x, 0]$.

(b) ⇒ (a) The proof is similar to that for (a) ⇒ (b).

DEFINITION 16 [11]. Let (x_n) be a sequence in an arbitrary MV-algebra. We say that (x_n) is a *Cauchy sequence* if there is a sequence (c_n) such that $c_n \downarrow 0$ and $d(x_n, x_{n+p}) \leq c_n$ for all n, p.

PROPOSITION 17. *If A is a perfect MV-algebra and $(x_n) \subseteq \mathrm{Rad}\, A$ then the following are equivalent*:

(a) (x_n) *is a Cauchy sequence in* A;

(b) $([x_n, 0])$ *is an order-Cauchy sequence in* $\mathscr{D}(A)$.

Proof. The proof is similar to the proof of Proposition 15.

LEMMA 18. *If (x_n) is a Cauchy sequence in a perfect MV-algebra then there is no such that $\{x_n | n \geq n_0\} \subseteq \mathrm{Rad}\, A$ or $\{x_n | n \geq n_0\} \subseteq \overline{\mathrm{Rad}\, A}$.*

Proof. By definition, there is $c_n \downarrow 0$ such that $d(x_n, x_{n+p}) \leq c_n$ for all n, p and, by Lemma 7, there is n_0 such that $c_n \in \mathrm{Rad}\, A$ for $n \geq n_0$. Assume there is $n \geq n_0$ and p such that $x_n \in \overline{\mathrm{Rad}\, A}$ and $x_{n+p} \in \mathrm{Rad}\, A$ so $x_{n+p} \leq x_n$. Thus $x_n = x_{n+p} \oplus d(x_n, x_{n+p})$ with $x_{n+p}, d(x_n, x_{n+p}) \in \mathrm{Rad}\, A$. This results in $x_n \in \mathrm{Rad}\, A$ and the contradiction is obvious. If $x_n \in \mathrm{Rad}\, A$ and $x_{n+p} \in \overline{\mathrm{Rad}\, A}$, then $x_n \leq x_{n+p}$ and $x_{n+p} = x_n \oplus d(x_n, x_{n+p}) \in \mathrm{Rad}\, A$, which is a contradiction.

An MV-algebra is *Cauchy complete* if any Cauchy sequence is convergent.

PROPOSITION 19. *If A is a perfect MV-algebra then the following are equivalent*:

(a) A *is Cauchy complete*;

(b) $\mathscr{D}(A)$ *is an order-Cauchy complete l-group.*

Proof. (a) ⇒ (b). Assume $([x_n, y_n])$ is an order-Cauchy sequence in $\mathscr{D}(A)$ so $([x_n, y_n]^+)$ and $([x_n, y_n]^-)$ are order-Cauchy. By Lemma 1 and Proposition 17, $(x_n \overline{y_n})$ and $(y_n \overline{x_n})$ are Cauchy sequences in A, so, by Lemma 2, $x_n \overline{y_n} \to c_1$ and $y_n \overline{x_n} \to c_2$ for some $c_1, c_2 \in \text{Rad}\, A$. By Proposition 15 and Lemma 1, $[x_n, y_n]^+ \to [c_1, 0]$ and $[x_n, y_n]^- \to [c_2, 0]$. Since $[c_1, c_2] = [c_1, 0] - [c_2, 0]$ one gets $[x_n, y_n] \to [c_1, c_2]$ and $\mathscr{D}(A)$ is order complete.

(b) ⇒ (a) Let x_n be a Cauchy sequence in A. By Lemma 18, one can assume that $(x_n) \subseteq \text{Rad}\, A$ or $(x_n) \subseteq \overline{\text{Rad}\, A}$. If $(x_n) \subseteq \text{Rad}\, A$ then $([x_n, 0])$ is order-Cauchy in $\mathscr{D}(A)$ so there is $x \in \text{Rad}\, A$ such that $([x_n, 0])$ order-converges to $[x, 0]$. From Proposition 15, $x_n \to x$ in A.

If $(x_n) \subseteq \overline{\text{Rad}\, A}$ then $(\overline{x_n}) \subseteq \text{Rad}\, A$ and, by Definition 16 and (4), $(\overline{x_n})$ is a Cauchy sequence in A.

Definition 20 [11]. Consider an embedding $A \hookrightarrow B$ of MV-algebras. We say that B is a *Cauchy completion* of A if

(a) B is a Cauchy complete MV-algebra,

(b) $A \hookrightarrow B$ preserves the countable suprema,

(c) for each $b \in B$ there exist two sequences $(a_n), (c_n)$ in A such that $c_n \downarrow 0$ and $d(a_n, b) \le c_n$ for all n.

Lemma 21. *If $A \hookrightarrow B$ is an embedding of perfect MV-algebras then condition* (c) *in Definition* 20 *can be replaced by*:

(c′) *for each $b \in \text{Rad}\, B$ there exist two sequences $(a_n), (c_n)$ in* Rad A *such that $c_n \downarrow 0$ and $d(a_n, b) \le c_n$ for all n.*

Proof. (c) ⇒ (c′). Assume $b \in \text{Rad}\, B$ and two sequences $(a_n), (c_n)$ in A such that $c_n \downarrow 0$ and $d(a_n, b) \le c_n$ for all n. By Lemma 7 one can assume that $c_n \in \text{Rad}\, A$ for all n. If there exists an $a_n \in \overline{\text{Rad}\, A} \subseteq \overline{\text{Rad}\, B}$ then $b < a_n$ so $a_n = b \oplus d(a_n, b)$ with b and $d(a_n, b)$ in Rad B. Results $a_n \in A \cap \text{Rad}\, B = \text{Rad}\, A$ [1]. The contradiction is obvious so $(a_n) \subseteq$ Rad A.

(c′) ⇒ (c) If $b \in \overline{\text{Rad}\, A}$ then $\overline{b} \in \text{Rad}\, A$ so there are $(a_n), c_n \subseteq$ Rad A such that $c_n \downarrow 0$ and $d(a_n, b) \le c_n$ for all n. It follows that $d(\overline{a_n}, b) = d(a_n, \overline{b}) \le c_n$ for all n.

By [11] any MV-algebra A has a unique Cauchy completion. The following result gives a description of the Cauchy completion of a perfect MV-algebra in terms of functors $\mathscr{D}$ and Δ.

Let A be a perfect MV-algebra. By [9] there is a unique order-Cauchy completion G of the abelian l-group $\mathscr{D}(A)$. G is abelian so $G = \mathscr{D}(B)$ for some perfect MV-algebra $B = \Delta(G)$. We have an embedding of perfect MV-algebras $A \hookrightarrow B$.

THEOREM 22. *B is the unique Cauchy completion of A.*

Proof. We must prove that $A \hookrightarrow B$ satisfies the above conditions (a), (b), and (c′). The condition (a) follows from Proposition 19 and condition (b) from Proposition 6. In order to prove (c′) consider $b \in \operatorname{Rad} B$ so $h = [b, 0] \geq [0, 0]$ in $G = \mathscr{D}(B)$. Then there exist the sequences (g_n), (u_n) in $\mathscr{D}(A)$ such that $u_n \downarrow 0$ and $|g_n - h| \leq u_n$ for all n. Hence $|g_n^+ - h| \leq |g_n - h| \leq u_n$ for all n. Since $g_n^+, u_n \geq [0, 0]$ in $\mathscr{D}(A)$ there are $a_n, c_n \in \operatorname{Rad} A$ such that $g_n^+ = [a_n, 0]$ and $u_n = [c_n, 0]$. From Lemma 8, $c_n \downarrow 0$. Applying Lemma 1 we obtain $|g_n^+ - h| = [\![a_n, b]\!] = [d(a_n, b), 0] \leq [c_n, 0]$ for all n, so $d(a_n, b) \leq c_n$ for all n.

THEOREM 23. *Let $A \hookrightarrow B$ be an embedding of perfect MV-algebras, let $A = \Delta(G)$, let $B = \Delta(H)$, and let $G \hookrightarrow H$ be the corresponding abelian l-groups embedding. The following are equivalent*:

(a) *B is the Cauchy completion of A*;

(b) *H is the order-Cauchy completion of G.*

Proof. (a) ⇒ (b) From Proposition 19, H is order-Cauchy complete and by Proposition 6 the embedding $G \hookrightarrow H$ preserves countable suprema. Assume $h \geq 0$ in H. Then $(0, h) \in \operatorname{Rad} B$ so there are $(0, a_n), (0, c_n) \in \operatorname{Rad} A$ such that $(0, c_n) \downarrow (0, 0)$ and $d((0, a_n), (0, h)) \leq (0, c_n)$ for all n. From the definition of the functor Δ and (13) it follows that $d((0, a_n), (0, h)) = (0, |a_n - h|)$. Then there are $a_n, c_n \in G$, $a_n \geq 0$, $c_n \downarrow 0$ such that $|a_n - h| \leq c_n$.

Now assume h is an arbitrary element of H, so $h = h^+ - h^-$. Applying the previous construction for $h^+, h^- \geq 0$ one gets the sequences (a_n), (b_n), (c_n), and (d_n) in G^+ such that $c_n \downarrow 0$, $d_n \downarrow 0$, $|a_n - h^+| \leq c_n$, and $|b_n - h^-| \leq d_n$. If we denote $x_n = a_n - b_n$ and $u_n = c_n + d_n$ we have $u_n \downarrow 0$ and $|x_n - h| = |(a_n - h^+) + (h^- - b_n)| \leq |a_n - h^+| + |b_n - h^-| \leq c_n + d_n = u_n$ for all n. The last condition from the definition of the order-Cauchy completion is verified.

(b) ⇒ (a) The implication results immediately from Theorem 22.

The previous results show that there exists a strong relation between the convergence in a perfect MV-algebra A and the order-convergence in the l-group $\mathscr{D}(A)$. This relation seems to be more direct than the connection established in [11] for arbitrary MV-algebras.

3. THE CONNECTION BETWEEN MV-ALGEBRAS AND PERFECT MV-ALGEBRAS

Let $A = \langle A, \oplus, \cdot, \bar{\ }, 0, 1 \rangle$ be an MV-algebra and let $a \in A$.

On $A_a = \{x \in A | 0 \leq x \leq a\}$ we define the following operations: $x \oplus_a y = a \wedge (x \oplus y)$, $\neg x = a\bar{x}$, $x \odot_a y = \neg(\neg x \oplus_a \neg y)$. Then $\langle A_a, \oplus_a,$

$\odot_a, \neg, 0, a\rangle$ is an MV-algebra [3]. We say that an MV-algebra B is a pseudo-subalgebra of A if there is $a \in A$ such that B is isomorphic with A_a. In [3] it is proved that for every MV-algebra B there is a perfect MV-algebra A and an element $a \in A$ such that a is a generator for Rad A and B is isomorphic to A_a.

A general problem is to investigate how the properties of the perfect MV-algebra B can be transferred to the MV-algebra A.

In the sequel we shall establish some connections between convergence on an MV-algebra A and convergence on A_a, where $a \in A$.

Following (1), we define $x \vee_a y = (x \odot_a \neg y) \oplus_a y$ and $x \wedge_a y = (x \oplus_a \neg y) \odot_a y$.

LEMMA 24. *Let $x, y \in A_a$. Then*

(a) $x \odot_a y = a(\bar{a} \oplus x)(\bar{a} \oplus y)$,

(b) $x \vee_a y = x \vee y$, $x \wedge_a y = x \wedge y$,

(c) $x \leq_a y$ *iff* $x \leq y$,

(d) $\bigvee_a \{x_i | i \in I\} = \bigvee_{i \in I} x_i$, $\bigwedge_a \{x_i | i \in I\} = \bigwedge_{i \in I} x_i$, *for every family* $(x_i)_{i \in I} \subseteq A_a$,

(e) $d_a(x, y) = d(x, y)$, *where* d *is the distance in* A *and* d_a *is the distance in* A_a.

Proof. (a) $x \odot_a y = \neg(\neg x \oplus_a \neg y) = a(\bar{a} \vee (\bar{a} \oplus x)(\bar{a} \oplus y)) = a\bar{a} \vee a(\bar{a} \oplus x)(\bar{a} \oplus y) = a(\bar{a} \oplus x)(\bar{a} \oplus y)$.

(b) We first observe that, by (a) and (1):

$$\neg x \odot_a y = a(\bar{a} \oplus y)(\bar{a} \oplus a\bar{x}) = (a \wedge y)(\bar{a} \vee \bar{x}) = \bar{x}y$$

because $x, y \leq a$. We obtain $x \vee_a y = x \oplus_a (\neg x \odot_a y) = a \wedge (x \oplus \bar{x}y) = a \wedge (x \vee y) = x \vee y$ and similarly $x \wedge_a y = x \wedge y$.

(c) It is obvious from (b).

(d) Because $x_i \leq a$ for every $i \in I$, we have that $\bigvee_{i \in I} x_i \leq a$, so $\bigvee_{i \in I} x_i \in A_a$. From (c) it results that $\bigvee_{i \in I} x_i$ is an upper bound for the family $(x_i)_{i \in I}$ in A_a. Let z be in A_a such that $x_i \leq_a z$ for all $i \in I$. From (c), we obtain that $x_i \leq z$ for all $i \in I$, so $\bigvee_{i \in I} x_i \leq z$. Again using (c) we have $\bigvee_{i \in I} x_i \leq_a z$, so $\bigvee_{i \in I} x_i$ is the supremum in A_a. The other equality can be proved by duality.

(e) $d_a(x, y) = (\neg x \odot_a y) \oplus_a (x \odot_a \neg y) = \bar{x}y \vee_a x\bar{y} = \bar{x}y \vee x\bar{y} = d(x, y)$.

In the sequel we shall denote by $\rightarrow$ convergence in A and by $\rightarrow_a$ convergence in A_a.

LEMMA 25. *Let* $(x_n) \subseteq A_a$, $x_n \to x$. *Then* $x \in A_a$.

Proof. We have $x_n \leq a$ for all n and $x_n \to x$. By Lemma 10 it follows that $x \leq a$.

LEMMA 26. *Let* $(x_n) \subseteq A_a$ *and* $x \in A_a$. *The following are equivalent*:

(a) $x_n \to x$;

(b) $x_n \to_a x$.

Proof. (a) ⇒ (b). If $x_n \to x$ then there is $(c_n) \subseteq A$ such that $c_n \downarrow 0$ and $d(x_n, x) \leq c_n$ for all n. We observe that $d(x_n, x) \leq a$ because $x_n, x \leq a$. Denote $c'_n = c_n \wedge a$. Then we have $d_a(x_n, x) = d(x_n, x) = d(x_n, x) \wedge a \leq c_n \wedge a = c'_n$ and $\bigwedge c'_n = a \wedge \bigwedge c_n = 0$.

(b) ⇒ (a) It is obvious from Lemma 24(c) and (d).

LEMMA 27. *Let* $(x_n) \subseteq A_a$. *The following are equivalent*:

(a) (x_n) *is a Cauchy sequence in* A;

(b) (x_n) *is a Cauchy sequence in* A_a.

Proof. (a) ⇒ (b) If (x_n) is a Cauchy sequence in A then there is $(c_n) \subseteq A$ such that $c_n \downarrow 0$ and $d(x_n, x_{n+p}) \leq c_n$ for all n, p. If we denote $c'_n = c_n \wedge a$ we obtain $d(x_n, x_{n+p}) \leq c'_n$ for all n, p and $\bigwedge c'_n = 0$.

(b) ⇒ (a) It is obvious from Lemma 24(c) and (d).

COROLLARY 28. *If* A *is Cauchy complete then* A_a *is Cauchy complete.*

Proof. The proof follows from Lemmas 26 and 27.

THEOREM 29. *Let* A *be an MV-algebra, let* $a \in A$, *and let* B *be the Cauchy completion of* A. *Then* B_a *is the Cauchy completion of* A_a.

Proof. Because A is embedded in B and $A \hookrightarrow B$ preserves countable suprema it is obvious that A_a is embedded in B_a and that $A_a \hookrightarrow B_a$ preserves countable suprema. By the previous corollary, B_a is Cauchy complete. It is easy to see that the embedding preserves distance. We have to prove that for any $b \in B_a$ there are $(b_n), (e_n) \subseteq A_a$ such that $e_n \downarrow 0$ and $d(b_n, b) \leq e_n$ for all n.

Let $b \in B_a$. Because $B_a \subseteq B$, there are $(a_n), (c_n) \subseteq A$ such that $c_n \downarrow 0$ and $d(a_n, b) \leq c_n$ for all n. We take $b_n = a_n \wedge a$ and $e_n = c_n \wedge a$. It is obvious that $(b_n), (e_n) \subseteq A_a$ and $e_n \downarrow 0$. We observe that $\bar{a}b = 0$ because

$b \leq a$. Then we have

$$\begin{aligned} d(b_n, b) &= d(a_n \wedge a, b) \\ &= (\overline{a_n \wedge a})b \vee \bar{b}(a_n \wedge a) \\ &= (\overline{a_n} \vee \bar{a})b \vee (a_n \wedge a)\bar{b} \\ &= \overline{a_n}b \vee \bar{a}b \vee \left(a_n\bar{b} \wedge a\bar{b}\right) \\ &= \left(\overline{a_n}b \vee a_n\bar{b}\right) \wedge \left(\overline{a_n}b \vee a\bar{b}\right) \\ &= d(a_n, b) \wedge \left(\overline{a_n}b \vee a\bar{b}\right). \end{aligned}$$

From $\overline{a_n}b \leq b \leq a$, $a\bar{b} \leq a$, and $d(a_n, b) \leq c_n$ we obtain $d(b_n, b) \leq c_n \wedge a = e_n$. The last condition from the definition of Cauchy completion is verified.

If we denote by A^* the unique Cauchy completion of the MV-algebra A then the previous result can be expressed in the following manner:

COROLLARY 30. $(A^*)_a \simeq (A_a)^*$.

Let A be an arbitrary MV-algebra. We denote by $A^\#$ the perfect MV-algebra associated with A [3]. The above results show that the study of convergence in A can be reduced to the study of convergence in the perfect MV-algebra $A^\#$.

Let a be an element in A such that $A \simeq A^\#_a$. From Corollary 30 it follows immediately that:

COROLLARY 31. $(A^{\#*})_a \simeq (A^\#_a)^* \simeq A^*$.

In order to investigate some properties of $*$ and # we have to recall the construction of the Cauchy completion for abelian l-groups with strong unit and for MV-algebras. See [11] and [9] for more details.

Remark 32. Let (G, u) be an abelian l-group with strong unit. For two order-Cauchy sequences (g_n) and (h_n) in G we define

$$(g_n) \sim (h_n) \qquad \text{iff } |g_n - h_n| \overset{\circ}{\to} 0 \text{ in } G.$$

The relation $\sim$ is an equivalence relation on the set $\mathscr{C}(G)$ of all order-Cauchy sequences of G.

Let $G^* = \mathscr{C}(G)/_\sim$ and $[(g_n)]$ be the equivalence class of the sequence (g_n). Then G^* is an abelian l-group, $G \hookrightarrow G^*$ via the embedding $g \to [(g)]$ with $[(g)]$ denoting the class of $(g, g, g, \dots)$, and $u^* = [(u)]$ is a strong unit in G^*. The abelian l-group with strong unit (G^*, u^*) is the order-Cauchy completion of (G, u) ([9]).

Remark 33. Let A be an MV-algebra. For two Cauchy sequences (a_n) and (b_n) in A we define

$$(a_n) \sim (b_n) \qquad \text{iff } d(a_n, b_n) \to 0 \text{ in } A.$$

The relation $\sim$ is an equivalence relation on the set $\mathscr{C}(A)$ of all Cauchy sequences of A.

Let $A^* = \mathscr{C}(A)/_\sim$ and let $[(a_n)]$ be the equivalence class of the sequence (a_n). Then A^* is an MV-algebra with respect to the operations $[(a_n)] \oplus [(b_n)] = [(a_n \oplus b_n)]$, $[(a_n)] \cdot [(b_n)] = [(a_n \cdot b_n)]$, $\overline{[(a_n)]} = [(\overline{a_n})]$.

If $[a]$ is the class of $(a, a, a, \dots)$, then the map $a \to [(a)]$ is an embedding of MV-algebras $A \hookrightarrow A^*$.

Remark 34. Let A be an MV-algebra and let (G, u) be an abelian l-group with strong unit such that $A \simeq \Gamma(G, u)$. Then $A^* \simeq \Gamma(G^*, u^*)$ [11].

Let $\mathscr{M}$ be the category of all MV-algebras and let $\mathscr{M}_c$ be the subcategory of $\mathscr{M}$ which has the same objects but the morphisms are only those which preserve the countable infima.

PROPOSITION 35. *$*: \mathscr{M}_c \to \mathscr{M}$ is a functor.*

Proof. We must define $*$ on morphisms. Let A, B be two MV-algebras and let $f: A \to B$ be a morphism that preserves the countable infima. We define $f^*: A^* \to B^*$ by $f^*([(x_n)]) = [(f(x_n))]$. We have to prove that f^* is well defined. Let $[(x_n)] = [(y_n)]$ for two Cauchy sequences $(x_n), (y_n) \subseteq A$. It follows that $d(x_n, y_n) \to 0$, which means that there is a sequence $(c_n) \subseteq A$, $c_n \downarrow 0$ such that $d(x_n, y_n) \le c_n$ for all n. It follows that $d(f(x_n), f(y_n)) \le f(c_n)$ and $f(c_n) \downarrow 0$ because f preserves the countable infima. So, $[(f(x_n))] = [(f(y_n))]$ and f^* is well defined.

The rest of the proof consists of easy verifications.

We denote by $\mathscr{P}_s$ the category of perfect MV-algebras with principal radical [3]. The objects are pairs (A, a) with A a perfect MV-algebra and $a \in A$ such that a generates Rad A. It follows that Rad A is the set of the elements $x \in A$ with the property that there is a natural number n such that $x \le na$, where

$$na = \underbrace{a \oplus \cdots \oplus a}_{n}$$

The morphisms in $\mathscr{P}_s$ are $f: (A, a) \to (B, b)$, where f is a morphism of MV-algebras with $f(a) = b$.

In [3] it is proved that the category $\mathscr{P}_s$ is equivalent to the category $\mathscr{M}$ of all MV-algebras. One of the two functors that define the equivalence is

$\#: \mathscr{M} \to \mathscr{P}_s$, which is obtained in the following manner. Let A be an MV-algebra and let (G, u) be an abelian l-group with strong unit such that $A \simeq \Gamma(G, u)$. Then $A^\# = \langle \Delta(G), a = (0, u) \rangle$. The definition of functor # on morphisms is straightforward. Moreover, $A \simeq A^\#_a$.

LEMMA 36. *Let f: $A \to B$ be a morphism of MV-algebras. The following are equivalent*:

(a) *f preserves the countable infima in A*;

(b) *$f^\#$ preserves the countable infima in $A^\#$.*

Proof. Let (G, u) and (H, v) be abelian l-groups with strong unit such that $A \simeq \Gamma(G, u)$ and $B \simeq \Gamma(H, v)$. Let $\tilde{f}: G \to H$ be a morphism such that $\Gamma(\tilde{f}) = f$. From [11, Lemma 12] we have that f preserves the countable infima in A iff $\tilde{f}$ preserves the countable infima in G. From Proposition 6 it follows that $\tilde{f}$ preserves the countable infima in G iff $f^\#$ preserves the countable infima in $\Delta(G) = A^\#$. Now, the intended result is obvious.

LEMMA 37. *Let A be a perfect MV-algebra and let $a \in A$ be an element which generates* Rad A. *Then $a^* = [(a)]$ generates* Rad A^*.

Proof. Let $[(x_n)] \in \text{Rad } A^*$. We have to prove that there is a natural number k such that $[(x_n)] \le k[(a)]$. From $[(x_n)] \in \text{Rad } A^*$ it follows that $[(x_n)][(x_n)] = [(0)]$ [1], so $x_n x_n \to 0$ in A. From Lemma 13 it follows that there is n_1 such that $x_n x_n \in \text{Rad } A$ for $n \ge n_1$. From the definition of A^* we obtain that (x_n) is a Cauchy sequence in A, so, by Lemma 18, there is n_2 such that $\{x_n | n \ge n_2\} \subseteq \text{Rad } A$ or $\{x_n | n \ge n_2\} \subseteq \overline{\text{Rad } A}$. If we choose the second option it follows that, for $n \ge \max(n_1, n_2)$, $x_n \in \overline{\text{Rad } A}$ and $x_n x_n \in \text{Rad } A$. It follows that $x_n x_n$ is in Rad A and in $\overline{\text{Rad } A}$, which is a contradiction. So, for every $n \ge n_2$, we have $x_n \in \text{Rad } A$. Because (x_n) is a Cauchy sequence, there is a sequence (c_n) in A, $c_n \downarrow 0$ and $d(x_n, x_{n+p}) \le c_n$ for every n, p. From Lemma 13 we have that there is n_3 such that $c_n \in \text{Rad } A$ for $n \ge n_3$, so there is a natural number r_n such that $c_n \le r_n a$ for $n \ge n_3$. Define $n_0 = \max(n_2, n_3)$. We have that, for every $n \ge n_0$, $x_n \in \text{Rad } A$ and $d(x_{n_0}, x_n) \le c_{n_0} \le r_{n_0} a$. On the other hand, there is a natural number k_{n_0} such that $x_{n_0} \le k_{n_0} a$. For $n \ge n_0$, it follows that $x_n \le (x_n \vee x_{n_0}) \oplus x_{n_0} \overline{x_n} = x_{n_0} \oplus d(x_{n_0}, x_n) \le k_{n_0} a \oplus r_{n_0} a$. Define $k = k_{n_0} + r_{n_0}$. For every $n \ge n_0$, $x_n \le ka$. It follows that $(x_n \overline{ka}) \to 0$, so $[(x_n)]\overline{[(ka)]} = [(0)]$, which means that $[(x_n)] \le k[(a)]$. The thesis is proved.

Denote by $\mathscr{P}_{sc}$ the subcategory of $\mathscr{P}_s$ which has the same objects but the morphisms are only those which preserve countable infima. Lemma 36 shows that the functor $\#: \mathscr{M}_c \to \mathscr{P}_{sc}$ is well defined. Lemma 37 shows that the functor $*: \mathscr{P}_{sc} \to \mathscr{P}_s$ is well defined. We have

$$\mathscr{M}_c \xrightarrow{*} \mathscr{M} \xrightarrow{\#} \mathscr{P}_s, \tag{19}$$

$$\mathscr{M}_c \xrightarrow{\#} \mathscr{P}_{sc} \xrightarrow{*} \mathscr{P}_s. \tag{20}$$

THEOREM 38. *The functors* $*\#$ *and* $\#*$ *from the previous diagrams are isomorphic.*

Proof. We have to define a natural transformation between $*\#$ and $\#*$. Let A be an MV-algebra and (G, u) an abelian l-group such that $A \simeq \Gamma(G, u)$. By Remark 34, $A^* = \Gamma(G^*, u^*)$, where G^* is the order-Cauchy completion of G and $u^* = [(u)]$. It follows that $A^{*\#} = (\Delta(G^*), (0, u^*))$.

On the other hand, $A^{\#} = (\Delta(G), (0, u))$ and $A^{\#*} = (\Delta(G)^*, (0, u)^*)$, where $(0, u)^* = [((0, u))]$. We recall that $\Delta(G^*) = \Gamma(Z \times G^*, (1, [(0)]))$ and $\Delta(G)^* = \Gamma(Z \times G, (1, 0))^*$.

Define φ: $\Delta(G^*) \to \Delta(G)^*$ by $\varphi((z, [(g_n)_n])) = [((z, g_n))_n]$, for every order-Cauchy sequence (g_n) in G and for every $z \in \{0, 1\}$. We shall prove that φ is an isomorphism of MV-algebras.

— *φ is well defined*. Let $(g_n)_n, (h_n)_n$ be two order-Cauchy sequences in G such that $[(g_n)_n] = [(h_n)_n]$. It follows that $|g_n - h_n| \overset{o}{\to} 0$ in G, so, using (13), $d((z, g_n), (z, h_n)) = (0, |g_n - h_n|) \to 0$ in $Z \times G$. It follows that $[((z, g_n))_n] = [((z, h_n))_n]$.

— *φ is a morphism*. It is straightforward from the definition of the operations on the Cauchy completion.

— *φ is injective*. Let $\varphi((z, [(g_n)_n])) = \varphi((t, [(h_n)_n]))$, so $[((z, g_n))_n] = [((t, h_n))_n]$ in $\Delta(G)^*$. It follows that $d((z, g_n), (t, h_n)) \to (0, 0)$ in $\Delta(G)$. From (13) and Lemma 13 it is easy to see that $z = t$ and $d((z, g_n), (t, h_n)) = (0, |g_n - h_n|)$, so $|g_n - h_n| \overset{o}{\to} 0$ in G. It follows that $[(g_n)_n] = [(h_n)_n]$, so φ is injective.

— *φ is surjective*. Let $[((z_n, g_n))_n]$ in $\Delta(G)^*$. It follows that $((z_n, g_n))_n$ is a Cauchy sequence in $\Delta(G)$, so there is a sequence $((t_n, c_n))_n \downarrow (0, 0)$ such that $d((z_n, g_n), (z_p, g_p)) \le (t_n, c_n)$ for every n and $p \ge n$. By Lemma 13, there is n_0 such that $(t_n, c_n) \in \operatorname{Rad} \Delta(G)$ for every $n \ge n_0$, so $t_n = 0$ for every $n \ge n_0$. It follows that $c_n \downarrow 0$ in G. For $p \ge n \ge n_0$ we have $d((z_n, g_n), (z_p, g_p)) \le (0, c_n)$, so, by (13), $z_n = z_p$ and $|g_n - g_p| \le c_n$. It is easy to see that $z_n = z_{n_0} = z$ for every $n \ge n_0$ and that $(g_n)_n$ is a order-Cauchy sequence in G. For $n \ge n_0$, $(z, g_n) = (z_n, g_n)$, so $d((z, g_n), (z_n, g_n)) \to (0, 0)$ and $[((z_n, g_n))_n] = [((z, g_n))_n]$. Now we can observe that $\varphi((z, [(g_n)_n])) = [((z_n, g_n))_n]$, so φ is surjective.

We proved that $A^{*\#} \simeq A^{\#*}$.

From the definition, $\varphi((0,[(u)])) = [(0,u)]$, so φ preserves the generator of the radical.

To complete the definition of the natural transformation, it remains to prove the condition for morphisms. This is straightforward.

Remark 39. In the conditions of the previous theorem, we denote $a = (0,u)$ and $a^{\#} = (0,u^*)$. From the definition of #, we have that $A^* \simeq (A^*)^{\#}_{a^{\#}}$. By Corollary 31, it follows that

$$\left(A^{\#}_{a}\right)^* \simeq \left(A^*\right)^{\#}_{a^{\#}}.$$

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