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The quasilinearization method for boundary value problems on time scales

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Abstract

In this paper, we apply the method of quasilinearization to a family of boundary value problems for second order dynamic equations $-y^{\Delta\nabla} + q(t)y = H(t, y)$ on time scales. The results include a variety of possible cases when H is either convex or a splitting of convex and concave parts and whether lower and upper solutions are of natural form or of natural coupled form.

Keywords: Time Scales; Quasilinearization; Boundary value problem; Coupled solutions

1. Introduction

Beginning from the work [3] of Atici and Guseinov, in the literature the homogeneous Sturm–Liouville equation on time scales is considered in the form

$$-y^{\Delta\nabla}(t) + q(t)y(t) = 0, \quad t \in [a, b].$$

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Here we apply the well-known quasilinearization method for the nonhomogeneous dynamic equation given by the form

$$-y^{\Delta\nabla}(t) + q(t)y(t) = H\big(t, y(t)\big), \quad t \in [a, b],$$

with the periodic boundary conditions (not separated boundary conditions)

$$y\big(\rho(a)\big) = y(b), \qquad y^{\Delta}\big(\rho(a)\big) = y^{\Delta}(b),$$

as well as with the separated boundary conditions

$$y\big(\rho(a)\big) = A, \qquad y\big(\sigma(b)\big) = B,$$

where $a \neq b$.

The paper is organized in the following manner. In Section 2, we provide preliminary material about the calculus on time scales. In Section 3, we introduce the definition of coupled upper and lower solutions for the given type of dynamic equations. Under certain assumptions on f and g which are placed in the decomposition of H as sum, the uniqueness result for the solutions of the periodic boundary value problem is given. In Section 4, we admit a decomposition of H into a sum of convex and concave functions, f and g, assuming g is strictly decreasing and f satisfies only a one sided Lipschitz condition and develop the method of quasilinearization. For the periodic boundary value problem we shall rely heavily on Topal's paper [18]. In Section 5, we only state without proof, the results obtained similar to those in Sections 3 and 4, for separated boundary value problem. Related papers on time scales are [2,8,12,14,16].

2. Calculus on time scales

For the details of basic notions connected to time scales we refer to [1,3,4,6, 9–11,13].

Let $\mathbb{T}$ be a nonempty closed subset (*time scale* or *measure chain*) of the real numbers $\mathbb{R}$. The forward and backward jump operators $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ are well defined, respectively, by

$$\sigma(t) = \inf\{s \in \mathbb{T}\colon\ s > t\} \quad \text{and} \quad \rho(t) = \sup\{s \in \mathbb{T}\colon\ s < t\}.$$

In this definition we put $\inf\emptyset = \sup\mathbb{T}$ and $\sup\emptyset = \inf\mathbb{T}$, where $\emptyset$ denotes the empty set. A point $t \in \mathbb{T}$ is called *left-dense* if $t > \inf\mathbb{T}$ and $\rho(t) = t$, *left-scattered* if $\rho(t) < t$, *right-dense* if $t < \sup\mathbb{T}$ and $\sigma(t) = t$, *right-scattered* if $\sigma(t) > t$. We define the sets $\mathbb{T}^k$, $\mathbb{T}^*$ and $\mathbb{T}_k$ which are derived from the time scale $\mathbb{T}$ as follows. If $\mathbb{T}$ has a left-scattered maximum t_1, then $\mathbb{T}^k = \mathbb{T} - \{t_1\}$, otherwise $\mathbb{T}^k = \mathbb{T}$. If $\mathbb{T}$ has a right-scattered minimum t_2, then $\mathbb{T}_k = \mathbb{T} - \{t_2\}$, otherwise $\mathbb{T}_k = \mathbb{T}$. Finally, $\mathbb{T}^* = \mathbb{T}^k \cap \mathbb{T}_k$. The forward graininess $\mu : \mathbb{T}^k \to {\mathbb{R}_0}^+$ and the backward graininess $\nu : \mathbb{T}_k \to {\mathbb{R}_0}^-$ are defined by

$$\mu(t) = \sigma(t) - t \quad \text{and} \quad \nu(t) = \rho(t) - t,$$

respectively.

If $f:\mathbb{T}\to\mathbb{R}$ is a function and $t\in\mathbb{T}^k$, then the "delta derivative" of f at the point t is defined to be the number $f^{\Delta}(t)$ (provided it exists) with the property that for each $\epsilon>0$ there is a neighborhood $U\subset\mathbb{T}$ of t such that

$$\left|f\big(\sigma(t)\big)-f(s)-f^{\Delta}(t)\big[\sigma(t)-s\big]\right|\leqslant\epsilon\left|\sigma(t)-s\right|\quad\text{for all } s\in U.$$

If $t\in\mathbb{T}_k$, then the "nabla derivative" of f at the point t is the number $f^{\nabla}(t)$ (provided it exists) with the property that for each $\epsilon>0$ there is a neighborhood $U\subset\mathbb{T}$ of t such that

$$\left|f\big(\rho(t)\big)-f(s)-f^{\nabla}(t)\big[\rho(t)-s\big]\right|\leqslant\epsilon\left|\rho(t)-s\right|\quad\text{for all } s\in U.$$

A function $F:\mathbb{T}^{\kappa}\mapsto\mathbb{R}$ is called a *delta-antiderivative* of $f:\mathbb{T}\mapsto\mathbb{R}$ provided $F^{\Delta}(t)=f(t)$ holds for all $t\in\mathbb{T}^{\kappa}$. In this case we define the integral of f by

$$\int_a^t f(s)\,\Delta s=F(t)-F(a)$$

for all $a,t\in\mathbb{T}$. A function $G:\mathbb{T}_{\kappa}\mapsto\mathbb{R}$ is called a *nabla-antiderivative* of $f:\mathbb{T}\mapsto\mathbb{R}$ provided $G^{\nabla}(t)=f(t)$ holds for all $t\in\mathbb{T}_{\kappa}$. In this case we define the integral of f by

$$\int_a^t f(s)\,\nabla s=G(t)-G(a)$$

for all $a,t\in\mathbb{T}$.

Note that in the case $\mathbb{T}=\mathbb{R}$ we have

$$f^{\Delta}(t)=f^{\nabla}(t)=f'(t),$$

$$\int_a^b f(t)\,\Delta t=\int_a^b f(t)\,\nabla t=\int_a^b f(t)\,dt.$$

In the case $\mathbb{T}=\mathbb{Z}$ we have

$$f^{\Delta}(t)=f(t+1)-f(t),\qquad f^{\nabla}(t)=f(t)-f(t-1),$$

$$\int_a^b f(t)\,\Delta t=\sum_{k=a}^{b-1}f(k)\quad\text{and}\quad\int_a^b f(t)\,\nabla t=\sum_{k=a+1}^{b}f(k)$$

where $a,b\in\mathbb{T}$ with $a\leqslant b$.

The following theorems either are in the reference [3] or are not difficult to verify.

Theorem 2.1. *For $f:\mathbb{T}\to\mathbb{R}$ and $t\in\mathbb{T}^k$ the following hold*:

(i) *If f is Δ-differentiable at t, then f is continuous at t.*
(ii) *If f is continuous at t and t is right-scattered, then f is Δ-differentiable at t and*

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t}.$$

(iii) *If t is right-dense, then f is Δ-differentiable at t if and only if the limit*

$$\lim_{s \to t} \frac{f(t) - f(s)}{t - s}$$

exists as a finite number. In this case $f^{\Delta}(t)$ is equal to this limit.
(iv) *If f is Δ-differentiable at t, then*

$$f\big(\sigma(t)\big) = f(t) + \big[\sigma(t) - t\big] f^{\Delta}(t).$$

Theorem 2.2. *For $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}_k$ the following hold*:

(i) *If f is ∇-differentiable at t, then f is continuous at t.*
(ii) *If f is continuous at t and t is left-scattered, then f is ∇-differentiable at t and*

$$f^{\nabla}(t) = \frac{f(\rho(t)) - f(t)}{\rho(t) - t}.$$

(iii) *If t is left-dense, then f is ∇-differentiable at t if and only if the limit*

$$\lim_{s \to t} \frac{f(t) - f(s)}{t - s}$$

exists as a finite number. In this case $f^{\nabla}(t)$ is equal to this limit.
(iv) *If f is ∇-differentiable at t, then*

$$f\big(\rho(t)\big) = f(t) + \big[\rho(t) - t\big] f^{\nabla}(t).$$

Theorem 2.3. *The following inequality holds*:

$$\left| \int_a^b f(t)g(t)\,\nabla t \right| \leqslant \int_a^b \big|f(t)g(t)\big|\,\nabla t \leqslant \Big(\max_{\sigma(a) \leqslant t \leqslant b} \big|f(t)\big| \Big) \int_a^b \big|g(t)\big|\,\nabla t.$$

3. Coupled lower and upper solutions

We shall study the periodic boundary value problem (PBVP)

$$-y^{\Delta\nabla}(t) + q(t)y(t) = f\big(t, y(t)\big) + g\big(t, y(t)\big), \quad t \in [a, b], \tag{3.1}$$

$$y\big(\rho(a)\big) = y(b), \qquad y^{\Delta}\big(\rho(a)\big) = y^{\Delta}(b), \tag{3.2}$$

where $a, b \in \mathbb{T}$, $[a, b] = \{t \in \mathbb{T}\colon\ a \leqslant t \leqslant b\}$ is closed and bounded subset and $|\nu(a)| \geqslant \mu(b)$. Assume that the following conditions are satisfied throughout this section:

(H1) $q(t)$ is continuous, $q(t) \geqslant 0$ and $q(t) \not\equiv 0$ for each $t \in [a, b]$.
(H2) $f, g \in \mathbb{C}([a, b] \times \mathbb{R})$ with respect to standard topology on $\mathbb{T} \times \mathbb{R}$.

We now present a method of upper and lower solutions which is well-known for ordinary differential equations [5,7,15,17], and recently developed by Akin [2] for separated boundary value problems on time scales.

We define the set

$$\mathbb{D} := \bigl\{y \in \mathbb{B}\colon\ y^{\Delta} \text{ is continuous on } \mathbb{T}^k \text{ and } \nabla\text{-differentiable on } \mathbb{T}^*,\\ y^{\Delta\nabla} \text{ is continuous on } \mathbb{T}^*\bigr\}$$

where the Banach space $\mathbb{B} = C([\rho(a), \sigma(b)])$ is the set of real-valued continuous (in the topology of $\mathbb{T}$) functions $y(t)$ defined on $[\rho(a), \sigma(b)]$ with the norm

$$\|y\| = \max_{t \in [\rho(a), \sigma(b)]} \bigl|y(t)\bigr|.$$

Definition 3.1. Real valued functions $\alpha(t), \beta(t) \in \mathbb{D}$ on $[\rho(a), \sigma(b)]$ are called *natural coupled lower and upper solutions* for (3.1), (3.2) if respectively the following hold:

$$-\alpha^{\Delta\nabla}(t) + q(t)\alpha(t) \leqslant f\bigl(t, \alpha(t)\bigr) + g\bigl(t, \alpha(t)\bigr)$$

for t in $[a, b]$, $\alpha(\rho(a)) = \alpha(b)$ and $\alpha^{\Delta}(\rho(a)) \geqslant \alpha^{\Delta}(b)$,

$$-\beta^{\Delta\nabla}(t) + q(t)\beta(t) \geqslant f\bigl(t, \beta(t)\bigr) + g\bigl(t, \beta(t)\bigr)$$

for t in $[a, b]$, $\beta(\rho(a)) = \beta(b)$ and $\beta^{\Delta}(\rho(a)) \leqslant \beta^{\Delta}(b)$.

Theorem 3.2 [18]. *Assume that α and β are coupled lower and upper solutions for* (3.1), (3.2) *and $\alpha(t) \leqslant \beta(t)$ on $[\rho(a), \sigma(b)]$. Then problem* (3.1), (3.2) *has a solution $y(t)$ with $\alpha(t) \leqslant y(t) \leqslant \beta(t)$ for t in $[\rho(a), \sigma(b)]$.*

Some results concerning monotone methods and the method of quasilinearization for second order dynamic equations require the use of second derivative test. The next lemma deals with the sign of the delta and the delta–nabla derivatives of a function at a point of local maximum. The proof of the lemma follows from a related result in [2] and Theorem 2.5 in [3] which gives the relationship between delta and nabla derivatives.

Lemma 3.1. *Assume $x \in \mathbb{D}$. Choose $c \in (a, b)$ such that*

$$x(c) = \max\bigl\{x(t)\colon\ t \in [a, b]\bigr\}$$

and

$$x(t) < x(c) \quad \textit{for } t \in (c, b].$$

Then

$$x^{\Delta}(c) \leqslant 0 \quad \textit{and} \quad (x^{\Delta})^{\nabla}(c) \leqslant 0.$$

Theorem 3.3. *Assume that*

(i) $\alpha(t), \beta(t)$ *are coupled lower and upper solutions of* (3.1), (3.2) *on* $[\rho(a), \sigma(b)]$;
(ii) *the function* g *is strictly decreasing in* y;
(iii) *for each* $t \in [a, b]$, $f(t, x) - f(t, y) \leqslant q(t)(x - y)$ *where* $y \leqslant x$.

Then $\alpha(t) \leqslant \beta(t)$ *on* $[\rho(a), \sigma(b)]$.

Proof. Define $h = \alpha - \beta$. For the sake of contradiction, let us assume the result is false. Let c be such that $h(t)$ has a positive maximum at some c in $[a, b]$ such that c satisfies the hypothesis of Lemma 3.1. Consequently, we have $(\alpha - \beta)^{\Delta}(c) \leqslant 0$ and $(\alpha - \beta)^{\Delta\nabla}(c) \leqslant 0$. On the other hand,

$$\begin{aligned}
(\alpha - \beta)^{\Delta\nabla}(c) &= \alpha^{\Delta\nabla}(c) - \beta^{\Delta\nabla}(c) \\
&\geqslant -f\big(c, \alpha(c)\big) - g\big(c, \alpha(c)\big) + q(c)\alpha(c) \\
&\quad + f\big(c, \beta(c)\big) + g\big(c, \beta(c)\big) - q(c)\beta(c) \\
&> 0.
\end{aligned}$$

This implies that $h^{\Delta\nabla} > 0$ which is a contradiction. In view of the boundary conditions that $h(t)$ satisfies, the result follows. □

Corollary 3.4. *Under the hypotheses of Theorem* 3.2*, the solution of the BVP* (3.1), (3.2) *is unique.*

4. Main results

First, we denote the sector for every $u, v \in \mathbb{D}$ such that

$$[u, v] := \big\{w \in \mathbb{D}\colon\ u(t) \leqslant w(t) \leqslant v(t),\ t \in \big[\rho(a), \sigma(b)\big]\big\}.$$

In the following results, note that f_x and f_{xx} are the usual partial derivatives of f over the time scale $\mathbb{R}$.

Theorem 4.1. *In addition to the hypotheses of Theorem* 3.3 *assume that*

(i) α_0, β_0 *are coupled lower and upper solutions of* (3.1), (3.2) *on* $[\rho(a), \sigma(b)]$,

(ii) $f, g \in C^2([a,b] \times [\alpha_0, \beta_0])$ *satisfy*

$$f_{yy}(t,y) \geqslant 0, \quad g_{yy}(t,y) \leqslant 0 \quad \textit{for } (t,y) \in [a,b] \times [\alpha_0, \beta_0].$$

Then there exist monotone sequences $\{\alpha_n\}, \{\beta_n\}$ *which converge uniformly to the unique solution of* (3.1), (3.2) *in* $[\alpha_0, \beta_0]$.

Proof. First we note that the assumptions $f_{yy}(t,y) \geqslant 0$ and $g_{yy}(t,y) \leqslant 0$ along with using mean value theorem on $\mathbb{R}$ yield the following inequalities

$$f(t,u) \geqslant f(t,v) + f_y(t,v)(u-v) \quad \text{and} \tag{4.1}$$

$$g(t,u) \geqslant g(t,v) + g_y(t,u)(u-v) \tag{4.2}$$

for $u \geqslant v$.

Define two linearizations F and G of $f+g$ as follows:

$$\begin{aligned} F(t,\alpha_0,\beta_0;y) &= f(t,\alpha_0) + f_y(t,\alpha_0)(y-\alpha_0) + g(t,\alpha_0) \\ &\quad + g_y(t,\beta_0)(y-\alpha_0), \\ G(t,\alpha_0,\beta_0;y) &= f(t,\beta_0) + f_y(t,\alpha_0)(y-\beta_0) + g(t,\beta_0) \\ &\quad + g_y(t,\beta_0)(y-\beta_0). \end{aligned}$$

In addition to the PBVP (3.1), (3.2) we also consider the following PBVP's

$$-y^{\Delta\nabla}(t) + q(t)y(t) = F(t,\alpha_0,\beta_0;y), \quad t \in [a,b], \tag{4.3}$$

with boundary conditions (3.2) and

$$-y^{\Delta\nabla}(t) + q(t)y(t) = G(t,\alpha_0,\beta_0;y), \quad t \in [a,b], \tag{4.4}$$

with boundary conditions (3.2).

The inequalities (4.1), (4.2) and hypothesis (i) imply

$$\begin{aligned} &-\alpha_0{}^{\Delta\nabla} + q(t)\alpha_0 \leqslant f(t,\alpha_0) + g(t,\alpha_0) \equiv F(t,\alpha_0,\beta_0;\alpha_0), \\ &-\beta_0{}^{\Delta\nabla} + q(t)\beta_0 \geqslant f(t,\beta_0) + g(t,\beta_0) \\ &\quad \geqslant f(t,\alpha_0) + f_y(t,\alpha_0)(\beta_0-\alpha_0) + g(t,\alpha_0) + g_y(t,\beta_0)(\beta_0-\alpha_0) \\ &\quad \equiv F(t,\alpha_0,\beta_0;\beta_0). \end{aligned}$$

Hence by Theorem 3.2 there exist a solution $\alpha_1(t)$ of (4.3), (3.2) such that $\alpha_0(t) \leqslant \alpha_1(t) \leqslant \beta_0(t)$ on $[\rho(a), \sigma(b)]$.

Similarly, using (4.1), (4.2) and hypothesis (i), we obtain

$$\begin{aligned} &-\alpha_0{}^{\Delta\nabla} + q(t)\alpha_0 \leqslant f(t,\alpha_0) + g(t,\alpha_0) \\ &\quad \leqslant f(t,\beta_0) + f_y(t,\alpha_0)(\alpha_0-\beta_0) + g(t,\beta_0) + g_y(t,\beta_0)(\alpha_0-\beta_0) \\ &\quad \equiv G(t,\alpha_0,\beta_0;\alpha_0), \\ &-\beta_0{}^{\Delta\nabla} + q(t)\beta_0 \geqslant f(t,\beta_0) + g(t,\beta_0) \equiv G(t,\alpha_0,\beta_0;\beta_0), \end{aligned}$$

and therefore, as before, there exists a solution $\beta_1(t)$ of (4.4), (3.2) such that $\alpha_0(t) \leqslant \beta_1(t) \leqslant \beta_0(t)$ on $[\rho(a), \sigma(b)]$.

Now since $-\alpha_1{}^{\Delta\nabla} + q(t)\alpha_1(t) = F(t, \alpha_0, \beta_0; \alpha_1)$, we get that

$$\begin{aligned}
&-\alpha_1{}^{\Delta\nabla} + q(t)\alpha_1 \\
&\quad = f(t, \alpha_0) + f_y(t, \alpha_0)(\alpha_1 - \alpha_0) + g(t, \alpha_0) + g_y(t, \beta_0)(\alpha_1 - \alpha_0) \\
&\quad \leqslant f(t, \alpha_1) + g(t, \alpha_1) + g_y(t, \alpha_1)(\alpha_0 - \alpha_1) + g_y(t, \beta_0)(\alpha_1 - \alpha_0) \\
&\quad \leqslant f(t, \alpha_1) + g(t, \alpha_1) + \big[g_y(t, \beta_0) - g_y(t, \alpha_1)\big](\alpha_1 - \alpha_0) \\
&\quad \leqslant f(t, \alpha_1) + g(t, \alpha_1)
\end{aligned}$$

in view of the fact $g_y(t, u)$ is nonincreasing in u and $\alpha_1 \leqslant \beta_0$. Similarly we obtain

$$\begin{aligned}
&-\beta_1{}^{\Delta\nabla} + q(t)\beta_1 \\
&\quad = f(t, \beta_0) + f_y(t, \alpha_0)(\beta_1 - \beta_0) + g(t, \beta_0) + g_y(t, \beta_0)(\beta_1 - \beta_0) \\
&\quad \geqslant f(t, \beta_1) + f_y(t, \beta_1)(\beta_0 - \beta_1) + f_y(t, \alpha_0)(\beta_1 - \beta_0) + g(t, \beta_1) \\
&\quad \geqslant f(t, \beta_1) + g(t, \beta_1) + \big[-f_y(t, \beta_1) + f_y(t, \alpha_0)\big](\beta_1 - \beta_0) \\
&\quad \geqslant f(t, \beta_1) + g(t, \beta_1)
\end{aligned}$$

because of the fact $f_y(t, u)$ is nondecreasing in u and $\alpha_0 \leqslant \beta_1$. We can conclude from the above estimates that α_1 and β_1 are coupled lower and upper solutions, respectively, for the PBVP (3.1), (3.2) and it then follows from Theorem 3.2 that $\alpha_1(t) \leqslant \beta_1(t)$ on $[\rho(a), \sigma(b)]$. Consequently these results yield

$$\alpha_0(t) \leqslant \alpha_1(t) \leqslant \beta_1(t) \leqslant \beta_0(t) \quad \text{on } \big[\rho(a), \sigma(b)\big].$$

Next we consider the following PBVP's

$$-y^{\Delta\nabla}(t) + q(t)y(t) = F(t, \alpha_1, \beta_1; y), \quad t \in [a, b], \tag{4.5}$$

with boundary conditions (3.2) and

$$-y^{\Delta\nabla}(t) + q(t)y(t) = G(t, \alpha_1, \beta_1; y), \quad t \in [a, b], \tag{4.6}$$

with boundary conditions (3.2).

Observe that

$$\begin{aligned}
&-\alpha_1{}^{\Delta\nabla} + q(t)\alpha_1 \leqslant f(t, \alpha_1) + g(t, \alpha_1) \equiv F(t, \alpha_1, \beta_1; \alpha_1), \\
&-\beta_1{}^{\Delta\nabla} + q(t)\beta_1 \geqslant f(t, \beta_1) + g(t, \beta_1) \\
&\quad \geqslant g(t, \alpha_1) + f_y(t, \alpha_1)(\beta_1 - \alpha_1) + g(t, \alpha_1) + g_y(t, \beta_1)(\beta_1 - \alpha_1) \\
&\quad \equiv F(t, \alpha_1, \beta_1; \beta_1)
\end{aligned}$$

in view of (4.1) and (4.2).

Consequently by Theorem 3.2, we obtain, as before, a solution $\alpha_2(t)$ of (4.5) exists such that

$$\alpha_1(t) \leqslant \alpha_2(t) \leqslant \beta_1(t) \leqslant \beta_0(t) \quad \text{on } \big[\rho(a), \sigma(b)\big].$$

Similarly, since

$$\begin{aligned}
&-{\alpha_1}^{\Delta\nabla} + q(t)\alpha_1 \leqslant f(t,\alpha_1) + g(t,\alpha_1)\\
&\quad \leqslant f(t,\beta_1) + f_y(t,\alpha_1)(\alpha_1-\beta_1) + g(t,\beta_1) + g_y(t,\alpha_1)(\alpha_1-\beta_1)\\
&\quad \equiv G(t,\alpha_1,\beta_1;\alpha_1),\\
&-{\beta_1}^{\Delta\nabla} + q(t)\beta_1 \geqslant f(t,\beta_1) + g(t,\beta_1) \equiv G(t,\alpha_1,\beta_1;\beta_1),
\end{aligned}$$

we conclude that there exists a solution $\beta_2(t)$ of (4.6), (3.2) satisfying

$$\alpha_1(t) \leqslant \beta_2(t) \leqslant \beta_1(t), \quad \text{on } \bigl[\rho(a),\sigma(b)\bigr].$$

Again one can see that α_2 and β_2 are coupled lower and upper solutions, respectively for PBVP (3.1), (3.2). The validity of the inequality $\alpha_2 \leqslant \beta_2$ follows from Theorem 3.3.

Following the similar arguments as before, we have that

$$\alpha_n(t) \leqslant \alpha_{n+1}(t) \leqslant \beta_{n+1}(t) \leqslant \beta_n(t), \quad t \in \bigl[\rho(a),\sigma(b)\bigr],\ n = 0,1,2,\dots,$$

where for each n, α_{n+1} is a solution of the PBVP

$$-y^{\Delta\nabla} + q(t)y = F(t,\alpha_n,\beta_n;y), \quad t \in [a,b],$$

and the boundary conditions (3.2), and β_{n+1} is a solution of the PBVP's

$$-y^{\Delta\nabla} + q(t)y = G(t,\alpha_n,\beta_n;y), \quad t \in [a,b],$$

with the boundary conditions (3.2).

Since $[a,b]$ is compact, it follows that the convergence of each sequence $\{\alpha_n\}$ or $\{\beta_n\}$ is uniform.

Now we show that each sequence $\{\alpha_n\}, \{\beta_n\}$ converges to the solution of the PBVP (3.1), (3.2). Topal [18] has constructed the Green's function $G(t,s)$, associated with the PBVP (3.1), (3.2). $G(t,s)$ can be employed to solve nonlinear dynamic equations through the following observation: y is a solution of the PBVP (3.1), (3.2) if and only if y is continuous on $\mathbb{T}$ and

$$y(t) = \int_{\rho(a)}^{b} G(t,s)\bigl[f\bigl(s,y(s)\bigr) + g\bigl(s,y(s)\bigr)\bigr]\,\nabla s, \quad t \in \bigl[\rho(a),\sigma(b)\bigr].$$

Hence for the solution $y(t)$, $t \in [\rho(a),\sigma(b)]$ of the PBVP (3.1), (3.2) the equation

$$y(t) = \int_{\rho(a)}^{b} G(t,s)\bigl[f\bigl(s,y(s)\bigr) + g\bigl(s,y(s)\bigr)\bigr]\,\nabla s, \quad t \in \bigl[\rho(a),b\bigr] \tag{4.7}$$

holds. Conversely, if a function $y(t)$, $t \in [\rho(a),b]$ is a solution of Eq. (4.7), then the extension $y(t)$, $t \in [\rho(a),\sigma(b)]$ of this function, where

$$y\big(\sigma(b)\big) = \int_{\rho(a)}^{b} G\big(\sigma(b), s\big)\big[f\big(s, y(s)\big) + g\big(s, y(s)\big)\big]\,\nabla s,$$

will be a solution of the PBVP (3.1), (3.2).

Thus between solutions of the PBVP (3.1), (3.2) and Eq. (4.7) there is a one-to-one correspondence. Consequently, the existence and uniqueness of solution of the PBVP (3.1), (3.2) is equivalent to that Eq. (4.7). As a result of this, we investigate Eq. (4.7) in the Banach space $\mathbb{B} = C([\rho(a), b])$ of real valued functions $y(t)$ defined on $[\rho(a), b]$ with the norm

$$\|y\| = \max_{t\in[\rho(a),b]} \big|y(t)\big|.$$

Now define

$$\alpha_{n+1}(t) = \int_{\rho(a)}^{b} G(t,s)F(s, \alpha_n, \beta_n; \alpha_{n+1})\,\nabla s, \quad t \in [a, b].$$

Note that $\{\alpha_n\}$ converges monotonically and uniformly to some function α and

$$F(s, \alpha_n, \beta_n; \alpha_{n+1}) \to f\big(s, \alpha(s)\big) + g\big(s, \alpha(s)\big)$$

where the convergence is uniform on $[a, b]$. Now by Theorem 2.3,

$$\max_{t\in[\rho(a),b]} \int_{\rho(a)}^{b} \big|G(t,s)\big|\,\nabla s \leqslant \max_{(t,s)\in[\rho(a),b]\times[\rho(a),b]} G(t,s)\big(b - \rho(a)\big).$$

We have desired result

$$\alpha(t) = \int_{\rho(a)}^{b} G(t,s)\big[f\big(s, \alpha(s)\big) + g\big(s, \alpha(s)\big)\big]\,\nabla s, \quad t \in [a, b]. \qquad \square$$

Corollary 4.2. *Assume that*

(i) α_0, β_0 *are coupled lower and upper solutions of* (3.1), (3.2) *on* $[\rho(a), \sigma(b)]$,
(ii) $f, g \in C^2([a, b] \times [\alpha_0, \beta_0])$,
(iii) $f_y(t, y) \leqslant 0$ *and* $g_y(t, y) < 0$ *on* $(t, y) \in [a, b] \times [\alpha_0, \beta_0]$,
(iv) $f_{yy}(t, y) \geqslant 0$ *and* $g_{yy}(t, y) \leqslant 0$ *on* $(t, y) \in [a, b] \times [\alpha_0, \beta_0]$.

Then the conclusion of the Theorem 4.1 *is true.*

The above theorem also contains the following result which is new because we get simultaneously lower and upper bounds by means of monotone sequences that converge to the unique solution of the PBVP

$$-y^{\Delta\nabla}(t) + q(t)y(t) = f\big(t, y(t)\big), \quad t \in [a, b], \tag{4.8}$$

$$y\big(\rho(a)\big) = y(b), \qquad y^{\Delta}\big(\rho(a)\big) = y^{\Delta}(b). \tag{4.9}$$

Corollary 4.3. *Assume that*

(i) α_0, β_0 *are lower and upper solutions of* (4.7), (4.8) *on* $[\rho(a), \sigma(b)]$,
(ii) $f \in C^2([a, b] \times [\alpha_0, \beta_0])$,
(iii) $f_y(t, y) < 0$ *and* $f_{yy}(t, y) \leqslant 0$ *on* $(t, y) \in [a, b] \times [\alpha_0, \beta_0]$.

Then there exist monotone sequences $\{\alpha_n\}, \{\beta_n\}$ *which converge uniformly to the unique solution* y *of* (4.7), (4.8) *in* $[\alpha_0, \beta_0]$.

We now prove the following lemma which plays an important role in the proof of the result concerning the order of convergence of the sequences $\{\alpha_n\}$ and $\{\beta_n\}$.

Lemma 4.1. *Let* $G_n(t, s)$ *be a Green's function for PBVP*

$$-y^{\Delta\nabla}(t) + q_n(t)y(t) = h(t), \quad t \in [a, b],$$
$$y\big(\rho(a)\big) = y(b), \qquad y^{\Delta}\big(\rho(a)\big) = y^{\Delta}(b),$$

where $q_n(t) \geqslant 0$, $q_n(t) \not\equiv 0$ *for all* $t \in \mathbb{T}$ *and* $n \in \mathbb{N}$. *If* $q_n(t) \leqslant q_{n+1}(t)$, *then* $G_n(t, s)$ *is decreasing as a sequence on* $[\rho(a), b] \times [\rho(a), b]$.

Proof. First one can observe that $G_n(t, s) - G_{n+1}(t, s)$ is sufficiently smooth and satisfies the PBVP

$$-y^{\Delta\nabla}(t) + q_n(t)y(t) = \big(q_{n+1}(t) - q_n(t)\big)G_{n+1}(t, s), \quad t \in [a, b],$$
$$y\big(\rho(a)\big) = y(b), \qquad y^{\Delta}\big(\rho(a)\big) = y^{\Delta}(b),$$

for fixed s. Hence we have

$$G_n(t, s) - G_{n+1}(t, s) = \int_{\rho(a)}^{b} G_n(t, r)\big(q_{n+1}(r) - q_n(r)\big)G_{n+1}(r, s)\,\nabla r, \quad t \in [a, b].$$

The integrand on the right hand side is positive in view of the positivity property of the Green's function (see [18]) $G_n(t, s)$ and monotonicity of the sequence $q_n(t)$. Then the result follows. $\square$

To get the quadratic convergence result, in addition to (iii) of Theorem 3.3, we assume that for each $t \in [a, b]$, $f(t, x) - f(t, y) \not\equiv q(t)(x - y)$, where $y \leqslant x$.

Corollary 4.4. *The convergence of each sequence* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *is quadratic.*

Proof. Set $u_n = y - \alpha_n$ and $v_n = \beta_n - y$.

We only show the quadratic convergence with u_n. Applying the mean value theorem, there exist s_1, s_2, s_3 such that

$$\alpha_n \leqslant s_1, s_2, s_3 \leqslant \beta_n$$

and

$$\begin{aligned}
&-{u_{n+1}}^{\Delta\nabla} + \big[q(t) - f_y(t, \beta_n)\big]u_{n+1} \\
&\quad = f(t, y) + g(t, y) - f(t, \alpha_n) - g(t, \alpha_n) \\
&\qquad - \big[f_y(t, \alpha_n) + g_y(t, \beta_n)\big](\alpha_{n+1} - \alpha_n) - f_y(t, \beta_n)u_{n+1} \\
&\quad = (f + g)_y(t, s_1)u_n - \big[f_y(t, \alpha_n) + g_y(t, \beta_n)\big](\alpha_{n+1} - \alpha_n) \\
&\qquad - f_y(t, \beta_n)u_{n+1} \\
&\quad = \big[f_y(t, s_1) - f_y(t, \alpha_n)\big]u_n + \big[g_y(t, s_1) - g_y(t, \beta_n)\big]u_n \\
&\qquad + \big[f_y(t, \alpha_n) + g_y(t, \beta_n)\big]u_{n+1} - f_y(t, \beta_n)u_{n+1} \\
&\quad \leqslant f_{yy}(t, s_2)(s_1 - \alpha_n)u_n - g_{yy}(t, s_3)(\beta_n - s_1)u_n \\
&\quad \leqslant f_{yy}(t, s_2){u_n}^2 - g_{yy}(t, s_3)(u_n + v_n)u_n.
\end{aligned}$$

Then using the inequality $u_n v_n \leqslant (1/2)[{u_n}^2 + {v_n}^2]$ and the hypothesis on f_y, we have that

$$-{u_{n+1}}^{\Delta\nabla} + \big[q(t) - f_y(t, \beta_n)\big]u_{n+1} \leqslant \left(M + \frac{3}{2}N\right)\|u_n\|^2 + \frac{N}{2}\|v_n\|^2$$

where $|f_{yy}| \leqslant M$, $|g_{yy}| \leqslant N$. Since

$$u_{n+1}(t) = \int_{\rho(a)}^{b} G_{q_n}(t, s)\big\{-{u_{n+1}}^{\Delta\nabla} + \big[q(s) - f_y(s, \beta_n)\big]u_{n+1}\big\}\,\nabla s, \quad t \in \mathbb{T},$$

and $G_{q_n}(t, s) \geqslant 0$ for each n, it follows that

$$\begin{aligned}
0 \leqslant u_{n+1}(t) \leqslant{} & \max_{(t,s)\in[\rho(a),b]\times[\rho(a),b]} G_{q_n}(t, s)\big(b - \rho(a)\big) \\
& \times \left[\left(M + \frac{3}{2}N\right)\|u_n\|^2 + \frac{N}{2}\|v_n\|^2\right]
\end{aligned}$$

where $G_{q_n}(t, s)$ is the Green's function for the dynamic equation

$$-y^{\Delta\nabla} + \big[q(t) - f_y(t, \beta_n)\big]y = 0, \quad t \in [a, b],$$

with the boundary conditions (3.2). Since $q_n(t) = q(t) - f_y(t, \beta_n)$ is a positive and monotone increasing sequence, as a result of Lemma 4.1 the sequence $\{G_{q_n}(t, s)\}$ is uniformly bounded on $[\rho(a), b] \times [\rho(a), b]$. Similar quadratic convergence result can be drawn for v_n as well. □

Example 4.1. Let $\mathbb{T}$ be any time scale. Consider the following PBVP:

$$-y^{\Delta\nabla}(t) + y(t) = \frac{1}{2}\left(y + \frac{2}{y}\right) - e^{y}, \quad t \in [a,b], \tag{4.10}$$

$$y\big(\rho(a)\big) = y(b), \qquad y^{\Delta}\big(\rho(a)\big) = y^{\Delta}(b). \tag{4.11}$$

Since

$$\frac{1}{2} = -\alpha_0^{\Delta\nabla}(t) + \alpha_0(t) \leqslant \frac{9}{4} - e^{1/2}$$

and $\alpha_0(\rho(a)) = \alpha_0(b)$, $\alpha_0^{\Delta}(\rho(a)) = \alpha_0^{\Delta}(b)$, $\alpha_0(t) = 1/2$ is a lower solution of PBVP (4.10), (4.11). Similarly, $\beta_0(t) = 1$ is an upper solution of PBVP (4.10), (4.11) since

$$1 = -\beta_0^{\Delta\nabla}(t) + \beta_0(t) \geqslant \frac{3}{2} - e$$

and $\beta_0(\rho(a)) = \beta_0(b)$, $\beta_0^{\Delta}(\rho(a)) = \beta_0^{\Delta}(b)$.

If we choose $f(t,y) = (1/2)(y + 2/y)$ and $g(t,y) = -e^{y}$, then these functions satisfy the hypotheses of Theorem 4.1 on $[a,b] \times [\alpha_0, \beta_0]$. Therefore there is a unique solution of the PBVP (4.10), (4.11) between $\alpha_0(t) = 1/2$ and $\beta_0(t) = 1$.

Example 4.2. Let $\mathbb{T}$ be any time scale. Consider the following PBVP:

$$-y^{\Delta\nabla}(t) + y(t) = -y^2 - 4y, \quad t \in [0,4], \tag{4.12}$$

$$y(0) = y(4), \qquad y^{\Delta}(0) = y^{\Delta}(4). \tag{4.13}$$

Then $\alpha_0(t) = -1$ and $\beta_0(t) = 1$ are lower and upper solutions for the PBVP (4.12), (4.13), respectively. As in the proof of the Theorem 4.1, for each n, α_{n+1} is a solution of the PBVP

$$-y^{\Delta\nabla} + y(t) = g(t,\alpha_n) + g_y(t,\beta_n)(y - \alpha_n), \quad t \in [0,4],$$

with the boundary conditions (4.13), and β_{n+1} is a solution of the PBVP's

$$-y^{\Delta\nabla} + y(t) = g(t,\beta_n) + g_y(t,\beta_n)(y - \beta_n), \quad t \in [0,4],$$

with the boundary conditions (4.13), where $g(t,y) = -y^2 - 4y$.

One can easily see that α_n and β_n are obtained by

$$\alpha_n = \frac{\alpha_{n-1}(2\beta_{n-1} - \alpha_{n-1})}{2\beta_{n-1} + 5}, \qquad \beta_n = \frac{{\beta_{n-1}}^2}{2\beta_{n-1} + 5}.$$

We see that the sequence β_n is monotone decreasing and its limit is zero and the sequence α_n is monotone increasing and its limit is zero. Therefore they both converge to the unique solution, which is $y = 0$, of this PBVP (4.12), (4.13).

5. Separated boundary conditions

In this section, we will study the separated boundary value problems and state results similar to those in Sections 3 and 4. Once again, we shall only state results whose proofs can be obtained using analogous arguments.

We are concerned with the boundary value problem

$$-y^{\Delta\nabla}(t) + q(t)y(t) = f\big(t, y(t)\big) + g\big(t, y(t)\big), \quad t \in [a, b], \tag{5.1}$$

$$y\big(\rho(a)\big) = A, \qquad y\big(\sigma(b)\big) = B, \tag{5.2}$$

where $a, b \in \mathbb{T}$, $[a, b] = \{t \in \mathbb{T}\colon\ a \leqslant t \leqslant b\}$. Assume that the following conditions are satisfied throughout this section:

(S1) $q(t)$ is continuous, $q(t) \geqslant 0$ for each $t \in [a, b]$.
(S2) $f, g \in \mathbb{C}([a, b] \times \mathbb{R})$ with respect to the standard topology of $\mathbb{T} \times \mathbb{R}$.

Definition 5.1. Real valued functions $\alpha(t), \beta(t) \in \mathbb{D}$ on $[\rho(a), \sigma(b)]$ are called *natural coupled lower and upper solutions* for (5.1), (5.2) if respectively the following hold:

$$-\alpha^{\Delta\nabla}(t) + q(t)\alpha(t) \leqslant f\big(t, \alpha(t)\big) + g\big(t, \alpha(t)\big)$$

for t in $[a, b]$, $\alpha(\rho(a)) \leqslant A$ and $\alpha(\sigma(b)) \leqslant B$,

$$-\beta^{\Delta\nabla}(t) + q(t)\beta(t) \geqslant f\big(t, \beta(t)\big) + g\big(t, \beta(t)\big)$$

for t in $[a, b]$, $\beta(\rho(a)) \geqslant A$ and $\beta(\sigma(b)) \geqslant B$.

Consider Banach space of a set of continuous functions on $[\rho(a), \sigma(b)]$,

$$\mathbb{B} = \mathbb{C}\big[\rho(a), \sigma(b)\big],$$

with the norm

$$\|y\| = \max_{t\in[\rho(a),\sigma(b)]} \big|y(t)\big|.$$

Define

$$L := \max_{t\in[\rho(a),\sigma(b)]} \int_{\rho(a)}^{b} G(t, s)\,\nabla s$$

where $G(t, s)$ is the Green's function for the nonhomogeneous dynamic equation (5.1) with following boundary conditions

$$y\big(\rho(a)\big) = 0, \qquad y\big(\sigma(b)\big) = 0$$

on a time scale $\mathbb{T}$. The positivity property of this Green's function has been obtained in [3].

Theorem 5.1. *If $M > 0$ satisfies $M \geqslant \max\{|A|, |B|\}$ and $LQ \leqslant M$, where $Q > 0$ satisfies*

$$Q \geqslant \max_{\|y\| \leqslant 2M} \left| f(t, y) + g(t, y) \right| \quad \textit{for } t \in \left[\rho(a), b\right],$$

then the BVP (5.1), (5.2) *has a solution.*

Theorem 5.2. *Assume that α and β are coupled lower and upper solutions for* (5.1), (5.2) *and $\alpha(t) \leqslant \beta(t)$ on $[\rho(a), \sigma(b)]$. Then problem* (5.1), (5.2) *has a solution $y(t)$ with $\alpha(t) \leqslant y(t) \leqslant \beta(t)$ for $t \in [\rho(a), \sigma(b)]$.*

Theorem 5.3. *Assume that*

(i) *$\alpha(t), \beta(t)$ are coupled lower and upper solutions of* (5.1), (5.2) *on $[\rho(a), \sigma(b)]$,*
(ii) *the function g is strictly decreasing in y,*
(iii) *for each $t \in [a, b]$, $f(t, x) - f(t, y) \leqslant q(t)(x - y)$ where $y \leqslant x$.*

Then $\alpha(t) \leqslant \beta(t)$ on $[\rho(a), \sigma(b)]$.

Corollary 5.4. *Under the hypotheses of Theorem* 5.3*, solution of the BVP* (5.1), (5.2) *is unique.*

Theorem 5.5. *In addition to the hypotheses* (ii) *and* (iii) *of Theorem* 5.3 *assume that*

(i) *α_0, β_0 are coupled lower and upper solutions of* (5.1), (5.2) *on $[\rho(a), \sigma(b)]$,*
(ii) *$f, g \in C^2([a, b] \times [\alpha_0, \beta_0])$ satisfy*

$$f_{yy}(t, y) \geqslant 0, \quad g_{yy}(t, y) \leqslant 0 \quad \textit{for } (t, y) \in [a, b] \times [\alpha_0, \beta_0].$$

Then there exist monotone sequences $\{\alpha_n\}, \{\beta_n\}$ which converge uniformly to the unique solution of (5.1), (5.2) *in $[\alpha_0, \beta_0]$.*

Corollary 5.6. *The convergence of each sequence $\{\alpha_n\}$ and $\{\beta_n\}$ is quadratic.*

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