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Valuation Rings in Finite-Dimensional Division Algebras

H. H. BRUNGS*

Department of Mathematics, University of Alberta, Edmonton, Alberta, Canada

AND

JOACHIM GRÄTER†

Institut für Algebra und Zahlentheorie, Technische Universität, Pockelsstrasse 14, D-3300 Braunschweig, West Germany

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A subring B' of a division algebra D is called a valuation ring of D if x or x^{-1} is contained in B' for every x in D, $x \neq 0$. Such a ring is called an extension of the valuation ring B of K, the centre of D, if $B' \cap K = B$. Let D be a division algebra finite-dimensional over its centre K, $[D:K] = n^2$, B a valuation ring of K and $\mathscr{B} = \{B_i \mid i \in I\}$ the set of all extension of B in D. Theorem 1. B possesses at most n extensions in D, i.e., $|\mathscr{B}| \leqslant n$. Theorem 2. Any two extensions of B in D are conjugate in D. Theorems 3 and 4. The set T of elements in D which are integral over B is a subring of D if and only if $|\mathscr{B}| \geqslant 1$. In this case $T = \bigcap B_i$, B_i in $\mathscr{B}$.

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Theorems about the existence and extensions of valuations are central in valuation theory in the commutative case. Unfortunately, these results can not be fully extended to non-commutative rings and fields. We say a subring B is a valuation ring of a division ring D if x or x^{-1} is contained in B for every x in D, $x \neq 0$. Such a ring is called invariant if $dBd^{-1} = B$ for every nonzero d in D. Schilling in [7] deals with invariant valuation rings and Mathiak in [6] investigates properties of valuations that correspond to valuation rings in skew fields.

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In this paper we deal with extensions of valuation rings B in the centre K of a finite-dimensional division algebra D. This means we want to obtain information about the set $\mathscr{B}$ of all valuation rings B' of D with $B' \cap K = B$. As the division algebra of quaternions over the field D of rational numbers shows, this set $\mathscr{B}$ can be empty for certain valuation rings B in D, see [2].

Valuation rings of finite-dimensional division algebras are not always invariant. However, Gräter in [4] shows that they are locally invariant in the following sense: A valuation ring B is called locally invariant if $xP(x) = P(x)x$ for all non-units x in B where $P(x)$ is the smallest completely prime ideal in B containing x. There it is also shown that locally invariant valuation rings B with only one proper nonzero prime ideal, i.e., of rank 1, are in fact invariant.

Wadsworth in [8] proves the following result, which extends a result of Cohn in [2]:

LEMMA 1. *A valuation ring B of the centre K of a finite-dimensional division algebra D has an invariant extension B' in D if and only if B has a unique extension in every subfield F with $K \subseteq F \subseteq D$.*

COROLLARY. *If B has an invariant extension B' in D then $|\mathscr{B}| = 1$.*

Proof. Let $B' = B_1$ and B_2 be elements in $\mathscr{B}$. Let a be an element in B_2, not in B_1 and let F be a subfield of D containing K and a. It follows that $B_1 \cap F \neq B_2 \cap F$.

LEMMA 2. *Let B be a valuation ring of K, the centre of the finite dimensional division algebra D. Then there exists only a finite number of extensions B' of B in D.*

Proof. Let $B_1, ..., B_k$ be extensions of B in D. We prove $k \leqslant [D:K]$. By [4, Hilfssatz 1], we know $B_i \not\subseteq B_j$ for $i \neq j$. Thus, there are $x_1, ..., x_k \in B_1 \cap \cdots \cap B_k$ such that the following hold (see [5, Hilfssatz 3.2]):

(*) $\qquad x_i$ is a unit in B_i, x_i is not a unit in B_j for $i \neq j$.

We show that $x_1, ..., x_k$ are linearly independent: Let $l_1, ..., l_k \in K$, not all zero, and let $l_1 x_1 + \cdots + l_k x_k = 0$. We can assume that $l_i \in B$ and that l_1 is a unit in B. It follows by (*) that $l_1 x_1 + \cdots + l_k x_k$ is a unit in B_1, a contradiction.

We conclude this section with the following observation:

LEMMA 3. *If B' is in $\mathscr{B}$ and M' is the maximal ideal of B', then $[B'/M' : B/M] \leqslant [D:K]$, where M is the maximal ideal of the valuation ring $B = B' \cap K$.*

Proof. Let $x_1, \ldots, x_m$ be in B' such that $\{x_1 + M', \ldots, x_m + M'\}$ is linearly independent over B/M. Then, $\{x_1, \ldots, x_m\}$ is linearly independent over K, since otherwise $\sum \alpha_i x^i = 0$ with α_i in K, not all zero, leads to $\sum a_i x^i = 0$ with $a_i \in B$, not all a_i in M.

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Throughout the rest of the paper let D be a division algebra finite-dimensional over its centre K. Let $B \neq K$ be a valuation ring of K and let $\mathscr{B}$ be the set of all extensions of B in D.

LEMMA 4. *Let $\mathscr{B} \neq \varnothing$. Then there exists a valuation ring $R \neq D$ in D with $B' \subseteq R$ for all B' in $\mathscr{B}$.*

Proof. We show first that B_i and $dB_i d^{-1}$ have a common completely prime ideal $P \neq (0)$ where B_i is in $\mathscr{B}$ and d in D, $d \neq 0$. If d is a unit in B_i then $B_i = dB_i d^{-1}$. If d is a non-unit in B_i and $P_i(d)$ is the completely prime ideal of B_i minimal with the property of containing d then $dP_i(d) d^{-1} = P_i(d)$ (see [4]), since B_i is locally invariant. We choose $P = P_i(d)$ in this case. If d is not contained in B_i, one considers d^{-1} instead of d.

By Lemma 2, $\mathscr{B}$ is finite and hence the number of valuation rings conjugate to B_i is finite. Thus, there exists a completely prime ideal P, $P \neq 0$, in B_i which is contained in all valuation rings conjugate to B_i. Form $R_i' = (B_i)_P$, the localization of B_i at P. Every valuation ring B_i' conjugate to B_i is contained in R_i': Let x be in B_i'. If x is in B_i, then x is in R_i. If x is not in B_i, then x^{-1} is in B_i and not in $P \subseteq B_i'$ and $x = (x^{-1})^{-1}$ is in R_i'. Let R_i be the subring of D minimal with the property of containing all valuation rings conjugate to B_i. This ring is an invariant valuation ring in D and satisfies $R_i \neq D$.

Let B_i, B_j be arbitrary elements in $\mathscr{B}$ and R_i, R_j the rings as defined above. We can assume that $K \cap R_i \subseteq K \cap R_j$ holds, since these intersections are overrings of the valuation ring B in K. The smallest subring $\tilde{R}_i = R_i(K \cap R_j)$ of D containing R_i and $K \cap R_j$ is an invariant valuation ring of D with $\tilde{R}_i \cap K = R_j \cap K$. Hence, $R_i \subseteq \tilde{R}_i = R_j$ by the corollary to Lemma 1. The set $\mathscr{B}$ is finite and there exists therefore a subring $R \neq D$ of D minimal with the property of containing all B_i in $\mathscr{B}$.

In the following, let R be the subring of D minimal with the property of containing all B_i in $\mathscr{B}$ and let N be its maximal ideal. R is an invariant valuation ring of D with $R \neq D$.

LEMMA 5. *Let $|\mathscr{B}| > 1$, let Z be the centre of R/N, and let S be the maximal separable extension of $(R \cap K)/(N \cap K)$ in Z. Then, the following hold*:

(1) *$(R \cap K)/(N \cap K)$ is a proper subfield of Z.*

(2) *S is a Galois extension of $(R \cap K)/(N \cap K)$ and each $(R \cap K)/(N \cap K)$-automorphism of S is induced by an inner automorphism of D.*

Proof. (1) We assume that $(R \cap K)/(N \cap K)$ is equal to Z. Then $\{B_i/N \mid B_i \in \mathscr{B}\}$ is the set of all extensions of $B/(N \cap K)$ to R/N, where $B/(N \cap K)$ is a valuation ring of the centre of R/N. Since $B/(N \cap K)$ has more than one extension in R/N, we have $B/(N \cap K) \neq Z$ and there exists a proper subring R' of R/N which contains all the B_i/N, $B_i \in \mathscr{B}$. Thus, there is a proper subring of R which contains all the B_i, contradicting the minimality of R.

(2) By the theorem of the primitive element for finite separable extensions there exists an element r in R with

$$S = [(R \cap K)/(N \cap K)](r + N).$$

Let $f(x) = \mathrm{Irr}(r, K)$ be the monic irreducible polynomial of r in $K[x]$ and $f(x) = (x - a_1)(x - a_2) \cdots (x - a_t) \in D[x]$ follows with $a_1 = r$ and $a_i = d_i r d_i^{-1}$ for $i = 1, ..., t$ and certain elements d_i in D, $d_i \neq 0$ (see [9, pp. 130–131]. Since r is in R and R is invariant, we have a_i in R for all i. If we denote by i_d the inner automorphism of D that sends x to dxd^{-1} we see that i_d induces an automorphism of R with $i_{d_j}(r) = a_j$ and an automorphism $\bar{i}_d$ of R/N, since $i_d(N) = N$. The element $r + N$ is the centre of R/N and hence all elements $a_i + N$ are in Z. We obtain

$$\bar{f}(x) = (x - \bar{a}_1)(x - \bar{a}_2) \cdot \cdots \cdot (x - \bar{a}_t) \in [(R \cap K)/(N \cap K)][x],$$

where $\bar{a}_i = a_i + N$. Let $\bar{g}(x) = \mathrm{Irr}(\bar{r}, (R \cap K)/(N \cap K))$ be the irreducible separable polynomial of $\bar{r} = r + N$ in $[(R \cap K)/(N \cap K)][x]$ and $\bar{g}(x) = (x - \bar{a}_{k_1})(x - \bar{a}_{k_2}) \cdot \cdots \cdot (x - \bar{a}_{k_m})$ follows for certain $\bar{a}_{k_1}, ..., \bar{a}_{k_m}$ in $\{\bar{a}_1, ..., \bar{a}_t\}$. We reorder the $\bar{a}_i$ if necessary and write

$$\bar{g}(x) = (x - \bar{a}_1) \cdot (x - \bar{a}_2) \cdot \cdots \cdot (x - \bar{a}_m).$$

Each of the $\bar{a}_i$, $i = 1, ..., m$ is in S and this shows that S is a Galois extension of $(R \cap K)/(N \cap K)$ with Galois group $G = \{\bar{i}_{d_j|S} \mid j = 1, ..., m\}$.

THEOREM 1. *Let $[D : K] = n^2$. Then B possesses at most n extensions in D, i.e., $|\mathscr{B}| \leqslant \sqrt{[D : K]}$.*

Proof. We prove the theorem by induction on n. The statement is trivial for $n=1$ or $|\mathscr{B}| \leqslant 1$. Let $n>1$ and $|\mathscr{B}|>1$. We use the notations of Lemma 5. It is well-known that there exists a maximal commutative subfield S' of R/N which is a separable extension of Z, the centre of R/N. Let $l=[S:(K\cap R)/(K\cap N)]$ and $d^2=[R/N:Z]$, i.e., $[S':Z]=d$. If S'' denotes the maximal separable extension of $(K\cap R)/(K\cap N)$ in S' then

$$[S'':(K\cap R)/(K\cap N)]=[S':Z]\cdot[S:(K\cap R)/(K\cap N)]$$

(see [1, 3.7.7]). Thus, $[S'':(K\cap R)/(K\cap N)]=d\cdot l$. Furthermore, there is an element $r\in R$ such that $(K\cap R)/(K\cap N)(\bar{r})=S''$ where $\bar{r}=r+N$. Since $K(r)$ is a commutative subfield of D, we get $[K(r):K]\leqslant n$ and $[(K\cap R)/(K\cap N)(\bar{r}):(K\cap R)/(K\cap N)]\leqslant n$, i.e., $d\cdot l\leqslant n$. Theorem 1 is proved if we show $|\mathscr{B}|\leqslant d\cdot l$. By Lemma 5, $(R\cap K)/((N\cap K)$ is a proper subfield of Z, i.e., $d<n$. Since Z is a purely inseparable extension of S, $B/(K\cap N)$ has at most l extensions to the commutative field Z (see [7, p. 57]). By induction, each of these extensions has at most d extensions in R/N. Altogether, $B/(K\cap N)$ possesses at most $d\cdot l$ extensions to R/N, i.e., $|\mathscr{B}|\leqslant d\cdot l$.

THEOREM 2. *Let D be a division algebra finite dimensional over its centre K and let $B\neq K$ be a valuation ring of K. If B_1, B_2 are two valuation rings of D extending B, then B_1 and B_2 are conjugate, i.e., $B_2=dB_1d^{-1}$ for some nonzero d in D.*

Proof. We prove the theorem by induction on $n=[D:K]$. The statement is trivial for $n=1$ and $|\mathscr{B}|\leqslant 1$. Let $n>1$ and $|\mathscr{B}|>1$. We use the notations of Lemma 5. Since S is a Galois extension of $(R\cap K)/(N\cap K)$, there exists a nonzero $d\in D$ such that $(dB_1d^{-1}/N)\cap S=(B_2/N)\cap S$ by Lemma 5(2). Z is a purely inseparable extension of S and (by [7, p. 57] we get $(dB_1d^{-1}/N)\cap Z=(B_2/N)\cap Z$. We use induction for the division algebra R/N with centre Z to conclude that there exists an element $r\in R$, $r\neq 0$, with $B_2/N=rdB_1d^{-1}r^{-1}/N$ and $B_2=rdB_1d^{-1}r^{-1}$ follows.

Lemma 5 and the proof of Theorem 2 show how to obtain all extensions of a given valuation ring of the centre using the fact that S is a Galois extension of $(R\cap K)/(N\cap K)$. An example of this is given at the end of this paper.

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The next results describe the integral closure of a valuation ring of the centre K in the division algebra D.

THEOREM 3. *Let D be a division algebra finite dimensional over its centre K. Let B be a valuation ring of K. Assume that there is at least one valuation ring B_1 of D with $B_1 \cap K = B$. Then $T = \bigcap B_i$, $B_i \in \mathscr{B}$, the intersection of all extensions of B, is the integral closure of B in D.*

Proof. Let t be any element in D, integral over B and let B_i be a valuation ring of D with $B_i \cap K = B$. Then $t^m + b_{m-1}t^{m-1} + \cdots + b_1 t + b_0 = 0$ for certain b_i in B. If t is not in B_i, then t^{-1} is in B_i and $1 = -(t^{-1}b_{m-1} + \cdots + t^{-m}b_0)$ is in the maximal ideal of B_i, a contradiction. Hence, any $t \in D \backslash T$ is not integral over B. It remains to show that every t in T is integral over B. Let $f(x) = \mathrm{Irr}(t, K)$ be the monic irreducible polynomial of t in $K[x]$. By [9, pp. 130–131] we have $f(x) = (x - a_1) \cdot \cdots \cdot (x - a_m)$ for $a_i = d_i t d_i^{-1}$ and d_i in D, $d_i \neq 0$. However, T is invariant and every a_i is therefore contained in T. This means that the coefficients of $f(x)$ are in $T \cap K = B$ and t is integral over B.

COROLLARY. *Let $\mathscr{B} \neq \varnothing$ and let L be a commutative subfield of D containing K. If B' is a valuation ring of L extending B then there exists a valuation ring B'' in D extending B'.*

Proof. Let T be the integral closure of B in D. Then, $T \cap L$ is the intersection of the finite number of extensions of B in L, since $T \cap L$ is the integral closure of B in L. By Theorem 3, T is the intersection of the finite number of extensions of B in D. Thus, there must exist a B'' in $\mathscr{B}$ with $B'' \cap L = B'$.

THEOREM 4. *Let D be a division algebra finite dimensional over its centre K. Let B be a valuation ring of K. The integral closure T of B in D is a subring of D if and only if B has an extension to D.*

Proof. If B has an extension to D then T is a subring of D by Theorem 3. Conversely, assume that the set T of elements in D integral over B is a subring of D.

We observe that T is invariant, since dxd^{-1} ($d \in D$, $d \neq 0$) is integral over B whenever x is integral over B. We prove: MT is a proper ideal of T where M is the maximal ideal of B. Assume that $1 = m_1 t_1 + \cdots + m_k t_k$ with $m_1, ..., m_k \in M$ and $t_1, ..., t_k \in T$. We can also assume that $m_i B \subseteq m_1 B$ for all $i \in \{1, ..., k\}$. Thus, $m_1^{-1} m_i \in B$ and $m_1^{-1} = t_1 + m_1^{-1} m_2 t_2 + \cdots + m_1^{-1} m_k t_k$, where $t_1, m_1^{-1} m_2 t_2, ..., m_1^{-1} m_k t_k \in T$. Since T is a ring, we get $m_1^{-1} \in T$, a contradiction.

There exists a maximal ideal N of T containing MT and $N \cap B = M$ follows. The set $T \backslash N$ is a left and right Ore-set in T, since T is invariant and $T_N = \{ts^{-1} \mid t \in T, s \in T \backslash N\}$, the localization of T on N, can be formed.

We show that $B' = T_N$ satisfies $B' \cap K = B$. Otherwise, there exists an element $u \in K \backslash B$ and $u = ts^{-1}$, $t \in T$, $s \in T \backslash N$. That implies $s = u^{-1}t$, $u^{-1} \in M$, $s \in N$—a contradiction.

It remains to prove that B' is a valuation ring of D. Let $x \in D$, $x \notin B'$ and consider the commutative subfield $K(x)$ of D. Then $T \cap K(x) = A$ is the integral closure of B in $K(x)$ and $L = B' \cap K(x)$ is a local overring of A which in turn is a Prüfer ring. Hence, L is a valuation ring and $x^{-1} \in L \subseteq T_N = B'$ follows.

The next theorem shows that not every valuation ring in the centre of a finite-dimensional division algebra D is extendible to D. At first, we need the following:

DEFINITION. Let F be a set of valuation rings in K. F is of finite character if for each $k \in K$ there is only a finite number of valuation rings B in F such that $k \notin B$.

Clearly, if F is of finite character and $k \in K$, $k \neq 0$, there is only a finite number of valuation rings B in F such that k is not a unit in B.

THEOREM 5. *Let D be a finite-dimensional division algebra with centre K, $(D \neq K)$ and let F be a set of valuation rings in K of finite character. If for each $B \in F$ the residue class field B/M is finite then only a finite number of B in F are extendible to D.*

Proof. Since $D \neq K$ there exist $a, b \in D$ with $ab - ba \neq 0$. Let $f_1(x) = \operatorname{Irr}(a, K)$, $f_2(x) = \operatorname{Irr}(b, K)$, $f_3(x) = \operatorname{Irr}(ab - ba, K)$ be the monic irreducible polynomials in $K[x]$ of a, b and $ab - ba$, respectively. Since F is of finite character, there exists a finite subset G of F such that for each B in $F \backslash G$ each coefficient of $f_1(x)$, $f_2(x)$, $f_3(x)$ is either equal to zero or a unit in B. We prove that each B in $F \backslash G$ is not extendible to D. Let $B \in F \backslash G$ and assume there is an extension B' of B to D with the maximal ideal M'. We are done if we show that a, b and $ab - ba$ are in $B' \backslash M'$, since this leads to the contradiction that B'/M' is a non-commutative finite field. We prove that $a \in B' \backslash M'$, the other proofs are similar. Let $f_1(x) = f_n x^n + \cdots + f_1 x + f_0$. We can assume $a \in B'$—otherwise consider a^{-1} instead of a. Since $a \neq 0$, we know that f_0 is a unit in B'. Thus, $a \in M'$ implies $0 = f_1(a) = f_n a^n + \cdots + f_1 a + f_0 \in f_0 + M' \nsubseteq M'$, a contradiction.

A global field K is either an algebraic number field (i.e., a finite extension of the rational number field D) or a function field (i.e., a finite extension of a field $k(x)$ of rational functions in one indeterminate x over a finite field k).

COROLLARY. *Let D be a finite-dimensional division algebra over a global field K. Only a finite number of valuation rings in K are extendible to D.*

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We consider examples in this section. Let L be a commutative field and $x_1, ..., x_n$ indeterminates over L. For each x_i let $L(x_i)(a_i)$ be a cyclic Galois extension of $L(x_i)$ of degree n_i with Galois group $\langle \sigma_i \rangle$ such that the following two conditions hold:

(i) If B is the x_i-adic valuation ring of $L(x_i)$ and B' any extension of B to $L(x_i)(a_i)$ with maximal ideal M', then $(B+M')/M' = B'/M'$, i.e., B and B' have the same residue class field.

(ii) Let e_i be the ramification index of B in $L(x_i)(a_i)$, then $(e_i, n_{i-1}) = 1$ for $i > L$.

Let E denote the field $L(x_1, ..., x_n)(a_1, ..., a_{n-1})$. Using (i) it follows that $[L(x_1, ..., x_n)(a_1, ..., a_{j+1}) : L(x_1, ..., x_n)(a_1, ..., a_j)] = n_{j+1}$ and each σ_j can be extended to an automorphism of E which maps every element of $L(x_1, ..., x_n)(a_1, ..., a_{j-1}, a_{j+1}, ..., a_{n-1})$ to itself. E is a Galois extension of $K = L(x_1, ..., x_n)$ such that its Galois group $G = G(E/K)$ is isomorphic to the direct product of the groups $\langle \sigma_i \rangle$:

$$G \cong \langle \sigma_1 \rangle \times \langle \sigma_2 \rangle \times \cdots \times \langle \sigma_{n-1} \rangle.$$

We define a factor set $f: G \times G \to E^*$ as follows: Let $\tau = \sigma_1^{r_1} \cdots \sigma_{n-1}^{r_{n-1}}$, $\mu = \sigma_1^{m_1} \cdots \sigma_{n-1}^{m_{n-1}}$ be elements in G. Then $f(\tau, \mu) = x_2^{\varepsilon_2} \cdots x_n^{\varepsilon_n}$ with $\varepsilon_{j+1} = 1$ if $r_j + m_j \geqslant n_j$ and $\varepsilon_{j+1} = 0$ otherwise.

One checks that f is indeed a factor set and we denote by $D = (E, G, f)$ the crossed product of E with its Galois group G and factor set f. The set of elements $\{u_\tau \mid \tau \in G\}$ forms a basis for D over E with $u_{\text{id}} = 1$. We also put $u_{j+1} = u_{\sigma_j}$ for $j = 1, ..., n-1$, and obtain $u_j e = \sigma_{j-1}(e)\, u_j$ for all e in E; $u_i u_j = u_j u_i$ and $u_j^{n_{j-1}} = x_j$. Using this notation we can write $D = K(a_1, ..., a_{n-1})(u_2, ..., u_n)$.

Proposition 1. *D is a division algebra with centre K.*

This proposition together with Proposition 2 (see below) will be proved later.

We now define in K, the centre of D, valuation rings $B_1 \subset B_2 \subset \cdots \subset B_n$ with the maximal ideals $M_n \subset M_{n-1} \subset \cdots \subset M_1$ in the following way: B_n is the x_n-adic valuation ring of $K = L(x_1, ..., x_n)$, $B_n/M_n \cong L(x_1, ..., x_{n-1})$. Next, B_{n-1} is the valuation ring in K, containing M_n, such that B_{n-1}/M_n is the x_{n-1}-adic valuation ring of B_n/M_n, hence $B_{n-1}/M_{n-1} \cong L(x_1, ..., x_{n-2})$. This process is continued until finally B_1/M_2 is the x_1-adic valuation ring of $B_2/M_2 \cong L(x_1)$, $B_1/M_1 \cong L$. It follows that B_i is a valuation ring of rank $n-i+1$.

PROPOSITION 2. *Every B_i has an extension B'_i in D with maximal ideal M'_i such that $B'_1 \subset B'_2 \subset \cdots \subset B'_n$ and*

$$B'_i/M'_i \cong L(x_1, ..., x_{i-1})(a_1, ..., a_{i-1})(u_2, ..., u_{i-1}).$$

We put $B = B_1$ and know by Proposition 2 that there exists at least one extension B'_1 of B in D. All the other extensions of B in D are conjugate to B'_1 in D by Theorem 2. The corresponding inner automorphisms of D are determined by the automorphisms of the centres $Z(B'_i/M'_i)$ of B'_i/M'_i. We have that $Z(B'_i/M'_i) \cong L(x_1, ..., x_{i-1})(a_{i-1})$ $(i > 1)$ is a Galois extension of $(B'_i \cap K)/(M'_i \cap K) \cong L(x_1, ..., x_{i-1})$, where the Galois group is generated by the automorphism that corresponds to the inner automorphism of D defined by u_i (Lemma 5). This implies that through

$$u_n^{i_n} \cdot u_{n-1}^{i_{n-1}} \cdot \cdots \cdot u_2^{i_2} \cdot B'_1 \cdot u_2^{-i_2} \cdot \cdots \cdot u_n^{-i_n}$$

all extensions of B in D are given.

The actual number r of these extensions of B in D depends on the numbers r_i, the number of extensions of the x_i-adic valuation ring in $L(x_i)$ to $L(x_i)(a_i)$. We have $r = r_1 \cdots r_{n-1}$. We put $r_n = 1$ and $m = \min\{i \mid i \in \{1, ..., n\}$ and $r_i = 1\}$ and see that B'_m is the smallest valuation ring of D which contains all extensions of B, i.e., $R = B'_m$.

As a concrete example choose $L = \mathbb{C}$, the field of complex numbers, and $n = 3$. Further, let $\operatorname{Irr}(a_1, L(x_1)) = Y^2 - (x_i + 1)$ and $n_1 = 2$, $e_1 = 1$ follows. For $\operatorname{Irr}(a_2, L(x_2)) = Y^3 - x_2$ we obtain $n_2 = 3$ and $e_2 = 3$. In this case we have $r_1 = 2$, $r_2 = 1$ and B has the two extensions B'_1 and $u_2 B'_1 u_2^{-1}$ with B'_2 the smallest valuation ring in D containing B'_1 and $u_2 B'_1 u_2^{-1}$. If we choose L and n as above with the same $\operatorname{Irr}(a_1, L(x_1))$, but change $\operatorname{Irr}(a_2, L(x_2))$ to $Y^2 - (x_2 + 1)$, we obtain $e_2 = 1$ and $r_2 = 2$ and four extensions B'_1, $u_2 B'_1 u_2^{-1}$, $u_3 B'_1 u_3^{-1}$, $u_2 u_3 B'_1 u_3^{-1} u_2^{-1}$ which are all contained in $R = B'_3$.

It remains to prove Propositions 1 and 2. Let S_j be an extension of the x_j-adic valuation of $L(x_1, ..., x_n)$ to E with N_j as its maximal ideal. We have $S_j/N_j \cong L(x_1, ..., x_{j-1}, x_{j+1}, ..., x_n)(a_1, ..., a_{j-1}, a_{j+1}, ..., a_{n-1})$, and $\sigma_k(S_j) = S_j$ for $k \neq j$, using condition (i).

We show that $E(u_2)$ is a division ring that contains an extension of S_2. Observe that S_2 is a discrete valuation ring with $N_2 = dS_2$ for some element d, $d^{e_2} S_2 = x_2 S_2$ and $u_2 S_2 = \sigma_1(S_2)\, u_2 = S_2 u_2$. We have $(e_2, n_1) = 1$ by condition (ii) and integers s and t exist with $se_2 + tn_1 = 1$. With $y = u_2^s d^t$ one obtains $y S_2 y^{-1} = S_2$ and $y^{n_1} S_2 = dS_2$. It follows that $S_2^{(2)} = S_2 + S_2 y + \cdots + S_2 y^{n_1 - 1}$ is a subring of $\mathrm{E}(u_2)$ with $N_2^{(2)} = N_2 + S_2 y + \cdots + S_2 y^{n_1 - 1}$ as a maximal completely prime ideal. Since $y^{n_1} S_2 = dS_2$, one can show that for each a in $E(u_2) \backslash S_2^{(2)}$ (a in $N_2^{(2)}$, $a \neq 0$) there exists a b in $N_2^{(2)}$ (b in $E(u_2) \backslash S_2^{(2)}$) with ab, ba in $S_2^{(2)} \backslash N_2^{(2)}$. (One can choose for b a power of y.) Using this property one shows that $E(u_2)$ is a division ring: Let a, b be non-zero elements in $E(u_2)$ with $ab=0$. Then there exist c, d in $E(u_2)$ with ca, bd in $S_2^{(2)}\backslash N_2^{(2)}$ and $0=cabd \in S_2^{(2)}\backslash N_2^{(2)}$, a contradiction.

$S_2^{(2)}$ is a valuation ring in $E(u_2)$. To prove this, let a be in $E(u_2)$ with a, a^{-1} not in $S_2^{(2)}$. There exist c, d in $N_2^{(2)}$ with ac, da^{-1} in $S_2^{(2)}\backslash N_2^{(2)}$, but $dc=da^{-1}ac \in S_2^{(2)}\backslash N_2^{(2)}$. In fact $S_2^{(2)}$ is discrete since $N_2^{(2)}=yS_2^{(2)}$, is invariant and

$$S_2^{(2)}/N_2^{(2)} \cong L(x_1, x_3, ..., x_n)(a_1, a_3, ..., a_{n-1}).$$

Every S_i $(i>2)$ has an extension in $E(u_2)$. To prove this, observe that $u_2 S_i = \sigma_1(S_i)\, u_2 = S_i u_2$, since $i \neq 1$.

We put $S_i^{(2)} = S_i + S_i u_2 + \cdots + S_i u_2^{n_1-1}$ and $N_i^{(2)} = N_i + N_i u_2 + \cdots + N_i u_2^{n_1-1}$. As before, $S_i^{(2)}$ is a subring of $E(u_2)$ with $N_i^{(2)}$ as an ideal which is maximal and completely prime, since

$$S_i^{(2)}/N_i^{(2)} \cong L(x_1, ..., x_{i-1}, x_{i+1}, ..., x_n)(a_1, ..., a_{i-1}, a_{i+1}, ..., a_{n-1})(u_2).$$

In fact, $S_i^{(2)}$ is a discrete valuation ring in $E(u_2)$, an extension of S_i and invariant.

One proceeds in this way and shows with the same type of arguments that $E(u_2, u_3)$,..., D are division rings and that the valuation ring $S_i^{(j)}$, $i>j$, in $E(u_2, ..., u_j)$ can be extended to a valuation ring $S_i^{(j+1)}$ in $E(u_2, ..., u_{j+1})$. Finally, $S_n^{(n)}$ with maximal ideal $M_n^{(n)}$ is an extension in D of the x_n-adic valuation ring with

$$S_n^{(n)}/M_n^{(n)} \cong L(x_1, ..., x_{n-1})(a_1, ..., a_{n-1})(u_2, ..., u_{n-1}).$$

We put $B'_n = S_n^{(n)}$. The extension B'_{n-1} in D of B_{n-1} is constructed similarly in B'_n/M'_n, and one repeats this process.

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