On tricyclic graphs of a given diameter with minimal energy

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Abstract

The energy of a graph is defined as the sum of the absolute values of all the eigenvalues of the graph. Let \( \mathcal{G}(n, d) \) be the class of tricyclic graphs \( G \) on \( n \) vertices with diameter \( d \) and containing no vertex disjoint odd cycles \( C_p, C_q \) of lengths \( p \) and \( q \) with \( p + q \equiv 2 \pmod{4} \). In this paper, we characterize the graphs with minimal energy in \( \mathcal{G}(n, d) \).

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1. Introduction

Let \( G \) be a simple graph with \( n \) vertices. Let \( A(G) \) be the adjacency matrix of \( G \). The characteristic polynomial of \( G \) is

\[
\phi(G, \lambda) = \det(\lambda I - A) = \sum_{i=0}^{n} a_i \lambda^{n-i}.
\]

Sachs theorem states that [2] for \( i \geq 1 \),

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\[ a_i = \sum_{S \in L_i} (-1)^p(S) 2^c(S), \]

where \( L_i \) denotes the set of Sachs graphs of \( G \) with \( i \) vertices, that is, the graphs in which every component is either a \( K_2 \) or a cycle, \( p(S) \) is the number of components of \( S \) and \( c(S) \) is the number of cycles contained in \( S \). In addition \( a_0 = 1 \). The roots \( \lambda_1, \ldots, \lambda_n \) of \( \phi(G, \lambda) \) are called the eigenvalues of \( G \). Since \( A(G) \) is symmetric, all eigenvalues of \( G \) are real. Other undefined notation may refer to [1,2].

The energy of \( G \), denoted by \( E(G) \), is then defined as
\[
E(G) = \sum_{i=1}^{n} |\lambda_i|.
\]
In chemistry, the energy of a given molecular graph is of interest since it is clearly related to the total \( \pi \)-electron energy of the molecule represented by that graph. See Refs. [2,3,4] for more details on graph-energy concept and a survey of the mathematical properties.

It is known that [2] \( E(G) \) can be expressed as the Coulson integral formula
\[
E(G) = \frac{1}{\pi} \int_{0}^{+\infty} \frac{dx}{x^2} \ln \left[ \left( \sum_{i=0}^{\lfloor n/2 \rfloor} (-1)^i a_{2i} x^{2i} \right)^2 + \left( \sum_{i=0}^{\lfloor n/2 \rfloor} (-1)^i a_{2i+1} x^{2i+1} \right)^2 \right]. \tag{1.1}
\]

Let \( b_{2i}(G) = (-1)^i a_{2i} \) and \( b_{2i+1}(G) = (-1)^i a_{2i+1} \) for \( 0 \leq i \leq \lfloor n/2 \rfloor \). Clearly, \( b_0(G) = 1 \) and \( b_2(G) \) equals the number of edges of \( G \). Thus, by (1.1), \( E(G) \) is a strictly monotonically increasing function of \( b_i(G), i = 1, \ldots, \lfloor n/2 \rfloor \). A quasi-order is introduced (see [2]): if \( G_1 \) and \( G_2 \) are two graphs, then
\[ G_1 \succeq G_2 \iff b_i(G_1) \succeq b_i(G_2) \quad \text{for all } i \geq 0. \]
If \( G_1 \succeq G_2 \), and there exists one \( j \) such that \( b_j(G_1) > b_j(G_2) \), then we write \( G_1 > G_2 \). Therefore
\[ G_1 > G_2 \implies E(G_1) > E(G_2). \]
This increasing property of energy has been used in the study of extremal values of energy over some classes of graphs. For instance, Gutman [6] determined the trees with minimal and maximal energies. Yan and Ye [21] determined the trees of a given diameter with minimal energy. Li and Zhou [13] determined the unicyclic graphs of a given diameter with minimal energy. Recently, Yang and Zhou [26] determined the bicyclic graphs of a given diameter with minimal energy. More results in this direction can be found in Refs. [5,7,9–11,13,15,16].

Let \( C_n \) and \( P_n \) denote a cycle and a path of \( n \) vertices, respectively. A connected simple graph with \( n \) vertices and \( e = n + 2 \) edges is called a tricyclic graph. Let \( \mathcal{G}(n, d) \) be the class of tricyclic graph \( G \) with \( n \) vertices, diameter \( d \) and containing no disjoint two odd cycles \( C_p, C_q \) with \( p + q \equiv 2 \) (mod 4). Let \( G_{n,d}^0 \) be the graph formed by joining 3 pendent vertices to a vertex of degree one of the \( K_{1,n-1} \) (e.g., see Fig. 1), and \( G_{n,d}^1 \) be the graph formed by joining \( n - d - 3 \) pendent vertices and a path \( P_{d-2} \) of length \( d - 3 \) respectively, to two vertices of degree 4 of the complete bipartite graph \( K_{2,4} \) (e.g., see Fig. 1). In this paper, we show that \( G_{n,d}^1(0) \) (respectively) has minimal energy in \( \mathcal{G}(n, d) \) for \( n \geq 11, d \geq 3 \) (in \( \mathcal{G}(n, 2) \), respectively). Be aware of that \( b_{2i+1}(G_{n,d}^1) = 0, b_{2i+1}(G_{n,d}^0) = 0 \) for \( i \geq 1 \), it is sufficient to evaluate \( b_{2i}(G) \) for \( G \in \mathcal{G}(n, d) \).

The following lemmas are needed in the proof of our main theorem.

**Lemma 1.1** [25]. Let \( G \) be any graph. Then \( b_4(G) = m(G, 2) - 2s \), where \( m(G, 2) \) is the number of 2-matchings of \( G \) and \( s \) is the number of quadrangles in \( G \).

**Lemma 1.2** [15]. If \( G \in \mathcal{G}(n, d) \), then \( b_{2i} \geq 0 \) for \( 0 \leq i \leq \lfloor n/2 \rfloor \).
Lemma 1.3 [25]. Let $G$ be a graph with $n$ vertices and let $uv$ be a pendent edge of $G$ with pendent vertex $v$. Then for $2 \leq i \leq n$, $b_i(G) = b_i(G - v) + b_{i-2}(G - u - v)$.

Lemma 1.4 [21]. If $G \in \mathcal{G}(n, d)$ with $uv \in E(G)$ and $C_{sj}$ ($s_j \geq 0$) are cycles of $G$ containing edge $uv$, then

$$b_{2i}(G) = b_{2i}(G - uv) + b_{2i-2}(G - u - v) + \sum_{uv \in C_{sj}} f(C_{sj}) 2b_{2i-S_j}(G - C_{sj}),$$

where $f(C_{sj}) = -1$ if $s_j \equiv 0 \pmod{4}$, and $f(C_{sj}) = 1$ otherwise.

We call the edge $uv$ of $G$ in Lemmas 1.3 and 1.4 reduce-edge of graph $G$. Let $\mathcal{T}(n, d)$ be the set of trees with $n$ vertices and diameter $d$ and $T_{n,d}$ be the tree obtained by attaching $n - d$ pendent vertices to an end vertex of $P_d$. Let $\mathcal{U}(n, d)$ be the class of unicyclic graphs with $n$ vertices and diameter $d$, and let $U_{n,d}$ be the graph obtained by attaching $n - d - d$ pendent vertices and a path $P_{d-3}$ to two non-adjacent vertices of a quadrangle, respectively (see Fig. 1). Following results comes from [26].

Lemma 1.5. (1) If $T \in \mathcal{T}_{n,d}$, then $b_{2i}(T) \geq b_{2i}(T_{n,d})$.

(2) If $G \in \mathcal{U}(n, d)$, $b_{2i}(G) \geq b_{2i}(T_{n,d})$.

(3) $b_{2i}(T_{n,d}) \geq b_{2i}(T_{n,d_0})$ where $n - 2 \geq d \geq d_0 \geq 3$.

2. Lemmas and main results

In this section, we shall determine the tricyclic graphs in $\mathcal{G}(n, d)$ having the minimal energy. Our idea is to show $E(G) > E(G_{n,d}^1)$ for any $G \in \mathcal{G}(n, d)$.

Following two facts are immediate.

Fact 1. For any $G \in \mathcal{G}(n, d)$, there are at most three edge-disjoint cycles contained in $G$.

Fact 2. $b_{2i}(G_{n,3}^0) \geq b_{2i}(G_{n,3}^1)$. That is $E(G_{n,3}^0) \geq E(G_{n,3}^1)$.

Note that $b_{2i}(G_{n,3}^1) = 0$ for $(i \geq 3)$. By Lemma 1.3 and the facts that $b_2(G_{n,3}^1) = n + 2$, $b_4(G_{n,3}^1) = 4n - 24$ and $b_2(G_{n,3}^0) = n + 2$, $b_4(G_{n,3}^0) = 4n - 18$, we have our result.

Lemma 2.1. For $G_{n,d}^1$ and $T_{n,d}$ (see Fig. 1), $b_{2i}(G_{n,d}^1) \geq b_{2i}(T_{n,d})(3 \leq d \leq n - 2)$.
Proof. We prove it by induction on \( n \) and \( d \). Firstly, let \( d = 3, 4 \) since \( b_{2i}(G_{n,d}^{1}) = b_{2i}(T_{n,d}) = 0 \) for \( i \geq 3 \), we need to evaluate the values of \( b_{4}(G_{n,d}^{1}) \) and \( b_{4}(T_{n,d}) \) only. By direct calculation, we have \( b_{4}(G_{n,3}^{1}) = 3n - 15, b_{4}(T_{n,3}) = n - 3, b_{4}(G_{n,4}^{1}) = 4n - 21, b_{4}(T_{n,4}) = 2n - 7 \), so our statement is true for \( d = 3, 4 \) (\( n \geq 9 \)). Secondly, we assume that the statement is true for the number of vertices less than \( n \) and diameter is less than \( d \). At last, we are to show that the result is true for \( n \) and \( d \). We choose edge \( xy \) as the reduce-edge of \( G_{n,d}^{1} \) and by induction hypothesis, we have

\[
b_{2i}(G_{n,d}^{1}) = b_{2i}(G_{n-1,d-1}^{1}) + b_{2i-2}(G_{n-2,d-2}^{1}) \\
\geq b_{2i}(T_{n-1,d-1}) + b_{2i-2}(T_{n-2,d-2}) = b_{2i}(T_{n,d})
\]

Hence, the statement is true. \( \square \)

Let \( \mathcal{B}(n, d) \) be the class of bicyclic graphs with \( n \) vertices and diameter \( d \) and let \( B_{n,d} \) be the graph formed by joining \( n - d - 2 \) pendant vertices and a path \( P_{d-2} \) respectively, to two vertices of degree of 3 of the complete bipartite graph \( K_{2,3} \) (see Fig. 1). Denote the star with \( n \) edges by \( S_{n} \).

Lemma 2.2. For \( U_{n,d}(d \geq 3), B_{n,d}(d \geq 3) \) and \( G_{n,d}^{1} \), we have

1. \( b_{2i}(U_{n,d}) = b_{2i}(T_{n-1,d}) + b_{2i-2}(S_{n-d-1} \cup P_{d-3}) + b_{2i-2}(P_{d-2}) \).
2. \( b_{2i}(B_{n,d}) = b_{2i}(U_{n-1,d}) + b_{2i-2}(S_{n-d-2} \cup P_{d-3}) + b_{2i-2}(P_{d-2}) \).
3. \( b_{2i}(G_{n,d}^{1}) = b_{2i}(B_{n-1,d}) + b_{2i-2}(S_{n-d-3} \cup P_{d-3}) + b_{2i-2}(P_{d-2}) \).
4. \( b_{2i}(G_{n,d}^{1}) = b_{2i}(G_{n-1,d}) + b_{2i-2}(P_{d-2}) + 3b_{2i-4}(P_{d-3}) \).
5. \( b_{2i}(G_{n,d}^{1}) = b_{2i}(G_{n-1,d-1}) + b_{2i-2}(G_{n-2,d-2}) \).

Proof. (1) and (2) are directly obtained by Lemma 1.4.

3. We choose edges \( uv, vw \) as reduce-edges of graph \( G_{n,d}^{1} \) and graph \( G_{n,d}^{1} - uv \) respectively, then we have

\[
b_{2i}(G_{n,d}^{1}) = b_{2i}(G - uv) + b_{2i-2}(G - u - v) - 6b_{2i-4}(P_{d-3}) \\
= b_{2i}(G - uv) + b_{2i-2}(T_{d,d-2}) - 6b_{2i-4}(P_{d-4}) \\
= b_{2i}(B_{n-1,d}) + b_{2i-2}(S_{n-d} \cup P_{d-3}) + 3b_{2i-4}(P_{d-3}) \\
\quad + b_{2i-2}(P_{d-2}) - 6b_{2i-4}(P_{d-3}) \\
= b_{2i}(B_{n-1,d}) + b_{2i-2}(S_{n-d-1} \cup P_{d-3}) + b_{2i-2}(P_{d-2}) .
\]

4. We choose \( zu \) as the reduce-edge of \( G_{n,d}^{1} \),

\[
b_{2i}(G_{n,d}^{1}) = b_{2i}(G_{n-1,d}) + b_{2i-2}(T_{d+2,d-2}) \\
= b_{2i}(G_{n-1,d}) + b_{2i-2}(P_{d-2}) + 3b_{2i-4}(P_{d-3}) .
\]

5. It is straightforward if edge \( xy \) be chosen as a reduce-edge of \( G_{n,d}^{1} \). \( \square \)

Let \( \mathcal{A}(n, d) \) be the class of graphs of diameter \( d \) with \( n \) vertices whose components are (i) all trees, or (ii) all trees except one being unicyclic, or (iii) all trees except one being bicyclic.

Lemma 2.3. Let \( G \in \mathcal{A}(n, d) \), then \( b_{2i}(G) \geq b_{2i}(S_{n-d-3} \cup P_{d-3}) + b_{2i-2}(P_{d-2}) \).
Proof. It is obvious by Lemma 2.2. □

Let \( \mathcal{F} = \{R_n, W_n, S_n, Q_n\} \), where \( R_n, W_n, S_n, Q_n \) are depicted in Fig. 2.

Lemma 2.4. For each \( G \in \mathcal{G}(n, 2) \), then \( b_{2i}(G) \geq b_{2i}(G_n^0) \), that is \( G_n^0 \) has minimal energy in \( \mathcal{G}(n, 2) \).

Proof. By Theorem 2.8 in [15], if \( G \notin \mathcal{F} \), the result is true for \( n \geq \max\{11, d + 3\} \). Now we verify that if \( G \in \mathcal{F} \), the result is also true.

The values of \( b_i \) for each graph in \( \mathcal{F} \) are listed below:

\[
\begin{align*}
 b_3(R_n) &= 6, & b_3(W_n) &= 6, & b_3(S_n) &= 6, & b_3(Q_n) &= 8, \\
 b_4(R_n) &= 3n - 6, & b_4(W_n) &= 3n - 7, & b_4(S_n) &= 3n - 8, & b_4(Q_n) &= 3n - 9, \\
 b_5(R_n) &= 12, & b_5(W_n) &= 8, & b_5(S_n) &= 4, & b_5(Q_n) &= 2n - 8, \\
 b_6(R_n) &= 3n - 14, & b_6(W_n) &= 2n - 10, & & \\
 b_7(R_n) &= 6, & &
\end{align*}
\]

where each \( b_i(G_j) = 0 \) except the values listed above.

Note that \( b_3(G_n^0) = (-1)a_3 = -((-1)2^1 \times 2) = 4, b_4(G_n^0) = 3(n - 5) + 6 = 3n - 9, \)
\( b_i(G_n^0) = 0(l \geq 5) \). For \( G \in \mathcal{F} \), Let

\[
E(G) - E(G_n^0) = \frac{1}{\pi} \int_0^\infty \frac{dx}{x^2} \ln \frac{f_j(x)}{f_g(x)},
\]

where

\[
\begin{align*}
 f_g(x) &= [1 + (n + 3)x^2 + (3n - 9)x^4]^2 + [4x^3]^2, \\
 f_1(x) &= [1 + (n + 3)x^2 + (3n - 6)x^4 + (3n - 14)x^6]^2 + [6x^3 + 12x^5 + 6x^7]^2, \\
 f_2(x) &= [1 + (n + 3)x^2 + (3n - 7)x^4 + (2n - 10)x^6]^2 + [6x^3 + 8x^5]^2, \\
 f_3(x) &= [1 + (n + 3)x^2 + (3n - 8)x^4]^2 + [6x^3 + 4x^5]^2, \\
 f_4(x) &= [1 + (n + 3)x^2 + (3n - 9)x^4]^2 + [8x^3 + (2n - 8)x^5]^2.
\end{align*}
\]

Using case by case checking, it is easy to see that \( f_j(x) - f_g(x) \geq 0 (j = 1, \ldots, 8) \), where \( n \geq 9 \)
and \( x > 0 \). Hence \( E(G_n^0) < E(G) \) for \( G \in \mathcal{F} \). Therefore, the statement is true for each graph
in \( \mathcal{G}(n, 2) \). □

Lemma 2.5. For each \( G \in \mathcal{G}(n, 3) \), then \( b_{2i}(G) \geq b_{2i}(G_{n,3}^1)(n \geq 11) \).
**Proof.** It is obvious that \( G \notin \{ R_n, W_n, S_n, Q_n, G_n^0 \} \). By Theorem 2.8 in [15], \( b_4(G) > b_4(G_{n,3}^1) \) \((n \geq 11)\), that is \( b_{2i}(G) \geq b_{2i}(G_{n,3}^1) \) since \( b_{2i}(G) \geq 0 \) by Lemma 1.2 and \( b_{2i}(G_{n,3}^1) = 0 \) \((i \geq 3)\). \( \square \)

**Lemma 2.6.** For a graph \( G \in \mathcal{G}(n, 4) \) with no pendant vertices, then \( b_2(G) \geq b_2(G_{n,4}^1)(n \geq 8) \).

**Proof.** It is evident that \( G \notin \{ R_n, W_n, S_n, Q_n, G_n^0 \} \), by Lemmas 2.2–2.5 in [15], \( b_4(G) \geq \frac{n^2 + n}{2} - 23 \). Then, \( b_4(G) - b_{2i}(G_{n,4}^1) = \frac{n^2 + n}{2} - 23 - (5n - 31) = \frac{1}{2} (n - 4.5)^2 - \frac{17}{8} > 0 \) for \((n \geq 8)\). \( \square \)

For a graph \( G \), it will be convenient to denote the diameter of \( G \) by \( \text{diam}(G) \).

**Lemma 2.7.** For each graph \( G \in \mathcal{G}(n, 4) \) with at least one pendant vertex, then \( b_2(G) \geq b_2(G_{n,4}^1) \) where \( n \geq 8 \).

**Proof.** We are to prove it by induction on \( n \). Since \( G \) is tricyclic and has diameter 4, so \( n \geq 8 \). Firstly, we are to show the statement is true when \( n = 8 \). All the graphs in \( \mathcal{G}(8, 4) \) is isomorphic to one of the graphs in Fig. 3.

Using case by case checking, we have \( b_4(G) \geq b_4(G_{8,4}^1) = 4 \), and note that \( b_{2i}(G_{8,4}^1) = 0 \) \((i \geq 3)\). Thus \( E(G) \geq E(G_{8,4}^1) \). Secondly, we assume that the statement is true for \( n \geq 8 \). Let \( u \) be a pendant vertex and \( v \) be its neighbor and we choose \( uv \) (\( zu \), respectively) as the reduce-edge of \( G(G_{n,4}^1, \text{respectively}) \). By induction hypothesis, \( b_4(G - u) \geq b_4(G_{n-1,4}^1) \). So, to prove Lemma 2.7, it is sufficient to show that \( b_2(G - u - v) \geq b_2(G_{n,4}^1 - z - u) = b_2(K_{1,5}) = 5 \). Suppose to the contrary, \( G - u - v \) contains at most 4 edges. Note that \(|E(G)| = n + 2\) which implies that \( d(v) \geq n - 2 \). Note that diam(G) = 4 which leads us to that diam(G - u - v) \( \geq 2 \), thus \( G - u - v \) contains a \( P_3 \) as its a subgraph. Hence, \(|V(G)| \geq n - 2 + 1 + 2 = n + 1\), a contradiction. So we have \( b_4(G) = b_4(G - u) + b_2(G - u - v) \geq b_4(G_{n-1,4}^1) + 5 = b_4(G_{n,4}^1) \). \( \square \)

By Lemma 2.6 and Lemma 2.7, we obtain the following Lemma.

Fig. 3. Graphs in \( \mathcal{G}(8, 4) \) with some pendant vertices.
Lemma 2.8. For any graph $G$ in $\mathcal{G}(n, 4)$, $b_{2i}(G) \geq b_{2i}(G_{n,4}^1)(n \geq 8)$.

Lemma 2.9. For each graph $G \in \mathcal{G}(n, d)$ with at least one pendent vertex, then $b_{2i}(G) \geq b_{2i}(G_{n,d}^1)$ where $n = \max\{11, d + 4\}$.

Proof. We prove it by double induction on $n$ and diameter $d$. The base cases $d = 3$ and $d = 4$ have been done by Lemma 2.5 and Lemma 2.8, so we assume from now on that the result is true for $d \geq 5$.

Suppose that there is a pendent vertex $u$ of $G$ such that the degree of its neighbor $v$ is two. Then $G - u \in \mathcal{G}(n - 1, d - 1)$ and $G - u - v \in \mathcal{G}(n - 2, d - 2)$. By induction hypothesis, $b_{2i}(G - u) \geq b_{2i}(G_{n-1,d-1}^1)$, $b_{2i-2}(G - u - v) \geq b_{2i-2}(G_{n-2,d-2}^1)$. By Lemma 2.2(5), $b_{2i}(G_{n,d}^1) = b_{2i}(G_{n-1,d-1}^1) + b_{2i-2}(G_{n-2,d-2}^1)$, hence, we have $b_{2i}(G) \geq b_{2i}(G_{n,d}^1)$.

Next, we suppose that the neighbor of any pendent vertex has degree at least 3. Let $u$ be the pendent vertex and $v$ be its neighbor. First, we assume that $uv$ is not on all diametrical paths of $G$. Then $diam(G - u) = diam(G) = d$. By induction hypothesis, $b_{2i}(G - u) \geq b_{2i}(G_{n-1,d}^1)$. Now we need only to verify that $b_{2i-2}(G - u - v) \geq b_{2i-2}(T_{d+2,d-2}^1)$. Since $b_{2i}(G_{n-1,d}^1) + b_{2i-2}(T_{d+2,d-2}^1) = b_{2i}(G_{n,d}^1)$, we have $b_{2i}(G) \geq b_{2i}(G_{n,d}^1)$.

Next, we consider the case where $uv$ is on all diametrical paths of $G$. By Lemma 2.2(5), $b_{2i}(G_{n,d}^1) = b_{2i}(G_{n-1,d-1}^1) + b_{2i-2}(G_{n-2,d-2}^1)$, so we assume from now on that the result is true for $d \geq 5$.

Now we need only to verify that $b_{2i-2}(G - u - v) \geq b_{2i-2}(G_{n-2,d-2}^1)$. Note that $G$ is tricyclic and diam$(G - u - v) \geq d - 2$, so $|V(G - u - v)| \geq 2d - 1$. Note that $G - u - v$ is a forest, or unicyclic, or bicyclic. If $G - u - v$ is a forest, by Theorem 1.1 in [21], we have $b_{2i-2}(G - u - v) \geq b_{2i-2}(T_{d+2,d-2}^1)$. If $G - u - v$ is unicyclic, by the result in [26], $b_{2i-2}(G - u - v) \geq b_{2i-2}(U_{n-2,d-2}^1)$ and then by the result (1) in Lemma 2.2, we have $b_{2i-2}(U_{n-2,d-2}^1) \geq b_{2i-2}(T_{d+2,d-2}^1)$. If $G - u - v$ is a bicyclic, by Lemma 12 and Theorem 3 in [26], $E(G - u - v) \geq E(B_{n-2,d-2}^1)$, then by result (2) in Lemma 2.2, we again have $b_{2i-2}(G - u - v) \geq b_{2i-2}(T_{d+2,d-2}^1)$.

Secondly, we assume that edge $uv$ is on each diametrical path of $G$. Now $diam(G - u) = d - 1$. By induction hypothesis, $b_{2i}(G - u) \geq b_{2i}(G_{n-1,d-1}^1)$. By Lemma 2.2(5), $b_{2i}(G_{n,d}^1) = b_{2i}(G_{n-1,d-1}^1) + b_{2i-2}(G_{n-2,d-2}^1)$, to prove the result, it is sufficient for us to show that $b_{2i-2}(G - u - v) \geq b_{2i-2}(G_{n-2,d-2}^1)$. Let all neighbors of $v$ be $\{u, v_1, v_2, \ldots, v_i\}$ where $2 \leq i \leq 6$. Then degree of each $v_k (k \leq i)$ must be 2 and $v_k$ must be on some cycles, otherwise $uv$ is not on each diametrical path of $G$. Please note that $i \neq 5, 6$. Otherwise, there exist two edge disjoint cycles sharing one common vertex which is $v$ by the property of $uv$ being on each diametrical path. Then there exists a maximal path which does not pass through edge $uv$ but is longer than diametrical path with $u$ as an end vertex. We have a contradiction. And furthermore each vertex $v_k$ must be an end vertex of at least one diametrical path of graph $G - u - v$. So $G - u - v$ is a tree, or unicyclic, or bicyclic (see Fig. 4). We are to evaluate the $b_{2i-2}(G - u - v)$ by discussing following three cases according to the structure of $G - u - v$.

Case 1. Assume that $G - u - v$ is a tree. Since $uv$ is on each diametrical path of $G$, $G - u - v$ must be a graph formed by four edge-disjoint paths $P_{d-1}$ with one common vertex (see Fig. 4). Let $G_{fork}$ be the graph formed by joining

![Fig. 4. When $G - u - v$ is a forest, or is unicyclic.](image-url)
three end vertices of three edge-disjoint paths \( P_{d-1}. \) We choose edge \( x_0x_1 \) as the reduce-edge of \( G - u - v: \)

\[
b_{2i-2}(G - u - v) = b_{2i-2}(P_{d-2} \cup G_{\text{fork}}) + b_{2i-4}(P_{d-3} \cup 3P_{d-2})
\geq b_{2i-2}(P_{d-3} \cup G_{\text{fork}}) + b_{2i-4}(P_{d-4} \cup G_{\text{fork}}) + b_{2i-4}(P_{d-3} \cup 3P_{d-2})
\geq [b_{2i-2}(P_{d-3}) + b_{2i-2}(G_{\text{fork}}) + |E(G_{\text{fork}})|b_{2i-4}(P_{d-4})]
+ b_{2i-4}(P_{d-5} \cup G_{\text{fork}}) + b_{2i-6}(P_{d-6} \cup G_{\text{fork}})
+ b_{2i-4}(P_{d-3} \cup 3P_{d-2}).
\]

There are four vertex-disjoint \( P_{d-5} \) in graph \( P_{d-5} \cup G_{\text{fork}} \) and \( |E(P_{d-5} \cup G_{\text{fork}}) - P_{d-5}| \geq 3d - 3, \) thus \( b_{2i-4}(P_{d-5} \cup G_{\text{fork}}) \geq 4(3d - 3)b_{2i-6}(P_{d-5}). \) Similarly, graph \( P_{d-3} \cup 3P_{d-2} \) contains four vertex-disjoint paths \( P_{d-5}, \) then \( b_{2i-4}(P_{d-3} \cup P_{d-2}) \geq 4b_{2i-4}(P_{d-5}). \) Hence, we have

\[
b_{2i-2}(G - u - v) \geq b_{2i-2}(P_{d-3}) + (3d - 3)b_{2i-4}(P_{d-4})
+ 4(3d - 3)b_{2i-6}(P_{d-5}) + 3b_{2i-4}(P_{d-5})
\geq b_{2i-2}(P_{d-3}) + (n - d + 2)b_{2i-4}(P_{d-4})
+ 3b_{2i-4}(S_{n-d-1} \cup P_{d-5}) + 3b_{2i-4}(P_{d-5}). \tag{2.2}
\]

On other hand, by Lemma 2.2(3), (2) and (1), we have

\[
b_{2i-2}(G^1_{n-2,d-2}) = b_{2i-2}(B_{n-3,d-2}) + b_{2i-4}(S_{n-d-1} \cup P_{d-5}) + b_{2i-4}(P_{d-4})
\leq b_{2i-2}(U_{n-4,d-2}) + 2b_{2i-4}(S_{n-d-1} \cup P_{d-5}) + 2b_{2i-4}(P_{d-4})
\leq b_{2i-2}(T_{n-4,d-2}) + 3b_{2i-4}(S_{n-d-1} \cup P_{d-5}) + 3b_{2i-4}(P_{d-4})
\leq b_{2i-2}(P_{d-3}) + (n - d - 1)b_{2i-4}(P_{d-4}) + 3b_{2i-4}(S_{n-d-1} \cup P_{d-5})
+ 3b_{2i-4}(P_{d-5})
\leq b_{2i-2}(P_{d-3}) + (n - d + 2)b_{2i-4}(P_{d-4}) + 3b_{2i-4}(S_{n-d-1} \cup P_{d-5})
+ 3b_{2i-4}(P_{d-5}). \tag{2.3}
\]

Comparing (2.2) and (2.3), we have \( b_{2i-2}(G - u - v) \geq b_{2i-2}(G^1_{n-2,d-2}), \) thus the result holds.

Case 2. Assume that \( G - u - v \) is unicyclic.

\( uv \) is on each diametrical path of \( G, \) \( G - u - v \) must be a graph in Fig. 4. Let \( G_\Delta \) be the graph \( G - u - v - P_{d-2}. \) We choose edge \( x_0x_1 \) as the reduce-edge of \( G - u - v: \)

\[
b_{2i-2}(G - u - v) = b_{2i-2}(G - u - v - x_0x_1) + b_{2i-4}(G - u - v - x_0x_1)
= b_{2i-2}(P_{d-2} \cup G_\Delta) + b_{2i-4}(P_{d-3} \cup (G_\Delta - x_0))
= b_{2i-2}(P_{d-2} \cup G_\Delta) + b_{2i-4}(P_{d-4} \cup G_\Delta - x_0)
+ b_{2i-4}(P_{d-5} \cup G_\Delta - x_0).
\]

Since graph \( P_{d-2} \cup G_\Delta \) is unicyclic with diameter at least \( d - 2 \) and number of vertices \( n - 4, \) \( b_{2i-2}(P_{d-2} \cup G_\Delta) \geq b_{2i-2}(U_{n-4,d-2}). \) Please note that \( P_{d-2} \cup (G_\Delta - x_0) \) is acyclic and \( |G_\Delta - x_0| \geq n - d - 2 \) and there are three vertex-disjoint \( P_{d-5} \) in acyclic graph \( P_{d-5} \cup (G_\Delta - x_0), \) thus \( b_{2i-4}(P_{d-5} \cup G_\Delta - x_0) \geq 2b_{2i-4}(S_{n-d-2} \cup P_{d-5}). \) It is obvious that \( P_{d-4} \cup (G_\Delta - x_0) \) contains three vertex-disjoint paths \( P_{d-4}, \) then \( b_{2i-4}(P_{d-3} \cup (G_\Delta - x_0)) \geq 3b_{2i-4}(P_{d-4}). \) Hence, we have

\[
b_{2i-2}(G - u - v) \leq b_{2i-2}(P_{d-2} \cup G_\Delta) + 2b_{2i-4}(P_{d-3} \cup (G_\Delta - x_0))
+ 3b_{2i-4}(P_{d-4} \cup G_\Delta - x_0)
+ 3b_{2i-4}(P_{d-5} \cup G_\Delta - x_0).
\]
\[ b_{2i-2}(G - u - v) \geq b_{2i-2}(U_{n-4,d-2}) + 2(2d - 3)b_{2i-6}(P_{d-5}) + 2b_{2i-4}(P_{d-5}) + 2b_{2i-4}(P_{d-5}) \]
\[ = b_{2i-2}(U_{n-4,d-2}) + 2b_{2i-4}(S_{n-d-1} \cup P_{d-5}) + 2b_{2i-4}(P_{d-4}). \]  
(2.4)

On other hand, by Lemma 2.2(3), (2) and (1), we have
\[ b_{2i-2}(G_{n-2,d-2}) = b_{2i-2}(B_{n-3,d-2}) + b_{2i-4}(S_{n-d-3} \cup P_{d-5}) + b_{2i-4}(P_{d-4}) \leq b_{2i-2}(U_{n-4,d-2}) + 2b_{2i-4}(S_{n-d-2} \cup P_{d-5}) + 2b_{2i-4}(P_{d-4}). \]  
(2.5)

By comparing (2.4) and (2.5), we have \( b_{2i-2}(G - u - v) \geq b_{2i-2}(G_{n-2,d-2}). \)

Case 3. Assume that \( G - u - v \) is bicyclic.

Let \( v^*_1(\neq u) \) be the neighbor of \( v \), we have \( \text{diam}(G - u - v - v_1) = d - 2 \) and \( G - u - v_1 \) is bicyclic. Note that \( \text{diam}(G - u - v - v_1 - v^*_1) \geq d - 4 \) and \( v^*_1 \) is a pendent vertex of \( G - u - v - v_1 \). By Theorem 3 in [26], \( b_{2i-2}(G - u - v - v_1) \geq b_{2i-2}(B_{n-3,d-2}). \) By Lemma 2.3 and Lemma 1.5(2), we have
\[ b_{2i-4}(G - u - v - v_1 - v^*_1) \geq b_{2i-4}(U_{n-4,d-4}) \]
\[ \geq b_{2i-4}(T_{n-4,d-4}) \]
\[ = b_{2i-4}(S_{n-d} \cup P_{d-5}) + b_{2i-6}(P_{d-6}) \]
\[ = b_{2i-4}(S_{n-d-1} \cup P_{d-5}) + b_{2i-4}(P_{d-5}) + b_{2i-6}(P_{d-6}) \]
\[ = b_{2i-4}(S_{n-d-1} \cup P_{d-5}) + b_{2i-4}(P_{d-4}) \]
\[ b_{2i-2}(G - u - v) = b_{2i-2}(G - u - v - v_1) + b_{2i-4}(G - u - v - v_1 - v^*_1) \]
\[ \geq b_{2i-2}(B_{n-3,d-2}) + b_{2i-4}(S_{n-d-1} \cup P_{d-5}) + b_{2i-4}(P_{d-4}) \]
\[ = b_{2i-2}(G_{n-2,d-2}), \]

where the last equality comes from Lemma 2.2(3).

Now we complete the proof of \( b_{2i}(G) \geq b_{2i}(G_{n,d}) \) for \( G \in \mathcal{G}(n, d) \) and \( G \) has at least one pendent vertex. \( \square \)

Next, we list some properties that will be useful in the proof of our main result.

**Lemma 2.10.** (1) \( b_{2i}(T_{n,d}) = b_{2i}(P_d) + (n - d - 1)b_{2i-2}(P_{d-1}) \).

(2) \( b_{2i}(B_{n-1,d-1}) + 2b_{2i-2}(B_{n-3,d-2}) \geq b_{2i}(G_{n,d}) \).

(3) \( b_{2i-2}(U_{n-1,d-1}) + b_{2i-2}(U_{n-3,d-2}) \geq b_{2i-2}(S_{n-d-3} \cup P_{d-3}) + b_{2i-2}(P_{d-2}) \).

**Proof.** (1) By Lemma 1.3,
\[ b_{2i}(T_{n,d}) = b_{2i}(T_{n-1,d}) + b_{2i-2}(P_{d-1}) \]
\[ = b_{2i}(T_{n-2,d}) + 2 \times b_{2i-2}(P_{d-1}) \]
\[ = \ldots \]
\[ = b_{2i}(P_{d+1}) + (n - d - 1)b_{2i-2}(P_{d-1}). \]
(2) By Lemma 1.3, and (1), (2) in Lemma 2.2,
\[
b_{2i}(B_{n-1,d-1}) + 2b_{2i-2}(B_{n-3,d-2}) \\
= b_{2i}(B_{n-2,d-1}) + b_{2i-2}(T_{d-3}) + 2b_{2i-2}(B_{n-4,d-2}) + 2b_{2i-4}(T_{d-1,d-4}) \\
\geq b_{2i}(B_{n-2,d-1}) + b_{2i-2}(P_{d-3}) + 2b_{2i-4}(P_{d-4}) + 2b_{2i-2}(B_{n-4,d-2}) \\
+ 2b_{2i-4}(P_{d-3}) + 2(2)b_{2i-6}(P_{d-5}) \\
= S_1 + S_2,
\]
where
\[
S_1 = b_{2i}(B_{n-2,d-1}) + b_{2i-2}(P_{d-3}) + b_{2i-2}(B_{n-4,d-2}) + 2b_{2i-4}(P_{d-4}), \\
S_2 = b_{2i-2}(B_{n-4,d-2}) + 2b_{2i-4}(P_{d-4}) + 4b_{2i-6}(P_{d-5}).
\]

On other hand, by (3), (4) and (5) in Lemma 2.2,
\[
b_{2i}(G_{n,d}^1) = b_{2i}(G_{n-1,d-1}^1) + b_{2i-2}(G_{n-2,d-2}^1) \\
= b_{2i}(G_{n-1,d-1}^1) + b_{2i-2}(G_{n-3,d-2}) + b_{2i-4}(P_{d-4}) + 3b_{2i-4}(P_{d-5}) \\
= [b_{2i}(B_{n-2,d-1}) + b_{2i-2}(S_{n-d-3} \cup P_{d-4}) + b_{2i-2}(P_{d-3})] \\
+ [b_{2i-2}(B_{n-4,d-2}) + b_{2i-4}(S_{n-d-4} \cup P_{d-5}) + b_{2i-4}(P_{d-4})] \\
+ b_{2i-4}(P_{d-4}) + 3b_{2i-4}(P_{d-5}) \\
= S_1 + b_{2i-2}(S_{n-d-3} \cup P_{d-4}) + b_{2i-4}(S_{n-d-4} \cup P_{d-5}) + 3b_{2i-4}(P_{d-5}) \\
= S_1 + S_3,
\]
where
\[
S_3 = b_{2i-2}(S_{n-d-3} \cup P_{d-4}) + b_{2i-4}(S_{n-d-4} \cup P_{d-5}) + 3b_{2i-4}(P_{d-5}).
\]

Next we will evaluate $S_2$ and $S_3$. Applying Lemma 2.2, we have
\[
S_2 = b_{2i-2}(T_{n-6,d-2}) + 2b_{2i-4}(S_{n-d-4} \cup P_{d-5}) + 2b_{2i-4}(P_{d-4}) + 4b_{2i-6}(P_{d-5}) \\
\geq \sum_{k=d-5} b_{2i-4}(P_k) + [b_{2i-2}(P_{d-4}) + (n-d-3)b_{2i-4}(P_{d-4})] \\
+ b_{2i-4}(S_{n-d-4} \cup P_{d-5}) + 2b_{2i-4}(P_{d-4}) + 4b_{2i-6}(P_{d-5}) \\
\geq b_{2i-2}(S_{n-d-3} \cup P_{d-4}) + b_{2i-4}(S_{n-d-4} \cup P_{d-5}) + 3b_{2i-4}(P_{d-5}) \\
\geq S_3.
\]

(3) By (1) and Eq. (1) in Lemma 2.2,
\[
b_{2i-2}(U_{n-4,d-2}) = b_{2i-2}(T_{n-5,d-2}) + b_{2i-4}(S_{n-d-3} \cup P_{d-5}) + b_{2i-4}(P_{d-4}) \\
= b_{2i-2}(P_{d-2}) + (n-d-4)b_{2i-4}(P_{d-3}) \\
+ b_{2i-4}(S_{n-d-3} \cup P_{d-5}) + b_{2i-4}(P_{d-4}), \\
b_{2i-2}(U_{n-3,d-2}) = b_{2i-2}(P_{d-2}) + (n-d-3)b_{2i-4}(P_{d-3}) \\
+ b_{2i-4}(S_{n-d-2} \cup P_{d-5}) + b_{2i-4}(P_{d-4}).
\]
Hence,
\begin{align*}
&b_{2i-2}(U_{n-4,d-2}) + b_{2i-2}(U_{n-3,d-2}) \\
&\geq b_{2i-2}(P_{d-2}) + b_{2i-2}(P_{d-3}) + (n - d - 3)b_{2i-2}(P_{d-4}) \\
&= b_{2i-2}(P_{d-2}) + b_{2i-2}(S_{n-d-3} \cup P_{d-3}). \quad \square
\end{align*}

We are now in a position to prove the main result. We know, by [15], that a tricyclic graph \( G \) contains at least 3 cycles and at most 7 cycles, furthermore, there do not exist 5 cycles in \( G \). In the rest of this article, we are to prove \( b_{2i}(G) \geq b_{2i}(G_{n,d}^1) \) if \( G \) has no pendent vertices by following three lemmas.

**Lemma 2.11.** For each \( G \in \mathcal{G}(n, d) \) with exactly 3 or 4 cycles and no pendent vertices, then \( b_{2i}(G) \geq b_{2i}(G_{n,d}^1) \), that is \( E(G) \geq E(G_{n,d}^1) \).

**Proof.** For each graph \( G \) in Fig. 5, there exists a cycle, say \( C_a \), has at most one common vertex with other cycles. To limit the number of cases discussion in the proof due to the analogue, we consider the graphs (d) and (k) here.

First we consider the graph (d). Assume that cycle \( C_a \) is connected to cycle \( C_b \) by a path \( P_1 \), \( C_b \) is connected to \( C_c \) by a path \( P_k \). To make proof cleaner, we will discuss two cases. Case (I): at least one of cycles \( \{C_a, C_b\} \) is even cycle, say \( C_a \), and further assume that \( |C_a| = a \equiv 0 \) (mod 4), and Case (II): Two cycles \( C_a \) and \( C_c \) are odd.

**Proof of Case (I).** Using Lemma 1.4 repeatedly, and be aware in mind, we choose edges along cycle \( C_a \) in clockwise order as our reduce-edges respectively (see Fig. 5). Then we have
\begin{align*}
b_{2i}(G) &= [b_{2i}(G - u_2) + b_{2i-2}(G - \sum_1^3 u_1^k) + b_{2i-4}(G - C_a)] + [b_{2i-2}(G - \sum_1^3 u_1^k) + b_{2i-4}(G - C_a)] \\\n&\quad - 2b_{2i-4}(G - C_a) = b_{2i}(G - u_2) + 2b_{2i-2}(G - \sum_1^3 u_1^k).
\end{align*}
Note that \( \text{diam}(G - u_2) = d \), and \( \text{diam}(G - \sum_1^3 u_1^k) \geq d - 2 \) due to \( a \geq 4 \). Hence \( \text{diam}(G - u_1u_2 - u_2u_3) \geq \left\lceil \frac{d}{2} \right\rceil + (l - 1) + \left\lceil \frac{d}{2} \right\rceil + (k - 1) + \left\lceil \frac{d}{2} \right\rceil \geq d \). Similarly, \( \text{diam}(G - u_1u_2 - u_3u_4) \geq d \) and \( \text{diam}(G - u_2u_3 - u_4u_5) \geq d - 1 \) where \( d \geq 5 \). For more detail, see expression of \( b_{2i}(G) \) below. By Lemma 1.4 and Lemma 1.5,

\[ \text{Fig. 5.} \ (a)-(g) \text{ are } 7 \text{ possible cases for the arrangement of three cycles in } G \text{ and (h)-(k) are four possible cases for the arrangement of four cycles in } G. \]
\[ b_{2i}(G) = b_{2i}(G - u_1u_2 - u_2u_3) + b_{2i-2}(G - u_2 - u_3) + b_{2i-2}(G - u_1 - u_2) + f(C_\alpha)2b_{2i-a}(G - C_\alpha) \]

(where \( f(C_\alpha) = -1 \) if \( a \equiv 0 \) (mod 4) and \( f(C_\alpha) = 1 \) otherwise)

\[ = b_{2i}(G - u_1u_2 - u_2u_3) + b_{2i-2}(G - u_2 - u_3 - u_4u_5) + b_{2i-4}(G - \sum_{k=2}^{5} u_k) \]

\[ + b_{2i-2}\left(G - \sum_{k=1}^{3} u_k\right) + b_{2i-4}\left(G - \sum_{k=1}^{4} u_k\right) - 2b_{2i-a}(G - C_\alpha) \]

\[ \geq b_{2i}(G - u_1u_2 - u_2u_3) + b_{2i-2}(G - u_2 - u_3 - u_4u_5) + b_{2i-2}(G - \sum_{k=1}^{3} u_k) \]

\[ \geq b_{2i}(G_{n-1,d-1}) + b_{2i-2}(G_{n-3,d-2}) + b_{2i-2}(G_{n-3,d-2}) \]

The last inequality comes from Lemma 2.10.

Proof of Case II. Two cycles \( C_\alpha \) and \( C_\beta \) are odd.

The item \( f(C_\alpha)2b_{2i-a}(G - C_\alpha) \) in the expression of \( b_{2i}(G) \) in the proof of Case (I) equals \( 2b_{2i-a}(G - C_\alpha) \) here where \( a \) is odd. We are going to show that the value of \( 2b_{2i-a}(G - C_\alpha) \) is none negative. Hence the inequality in the proof of Case (I) still holds. We choose edge \( v_1v_2(z_1z_2, \text{respectively}) \) on the cycle \( C_\beta \) (\( C_\alpha \), respectively) as our reduce-edge. We denote \( G - C_\alpha(G - C_\alpha - C_\beta, \text{respectively}) \) by \( G^*(G^{**}, \text{respectively}) \) without confusion. So we have

\[ 2b_{2i-a}(G^*) \]

\[ = 2b_{2i-a}(G^* - v_1v_2) + 2b_{2i-a-2}(G^* - v_1 - v_2) + 4b_{2i-a-b}(G^* - C_\alpha) \]

\[ = 2b_{2i-a}(G^* - v_1v_2 - z_1z_2) + 2b_{2i-a-2}(G^* - v_1v_2 - z_1 - Z - 2) \]

\[ + 4b_{2i-a-2}(G^* - C_\alpha - v_1v_2) \]

\[ + 2b_{2i-a-2}(G^* - v_1 - v_2 - z_1z_2) + 2b_{2i-a-4}(G^* - v_1v_2 - z_1 - z_2) \]

\[ + 4b_{2i-a-4}(G^* - v_1v_2 - C_\alpha) \]

\[ + 4b_{2i-a-2}(G^* - z_1z_2) + 4b_{2i-a-b-2}(G^{**} - z_1 - z_2) + 8b_{2i-a-b-c}(G^{**} - C_\alpha). \]

Please note that, for any acyclic graph \( \overline{G} \), \( b_{2k+1}(\overline{G}) = 0 \) and \( b_{2k}((\overline{G})) \geq 0 \). So from expression above, we have \( 2b_{2i-a}(G - C_\alpha) \geq 0 \).

From the discussion on case (I) and (II), we can see that the expression of \( b_{2i}(G) \) may have the least value if \( |C_\alpha| = a \equiv 1 \) (mod 4) since \( f(C_\alpha)2b_{2i-a}(G - C_\alpha) = -2b_{2i-a}(G - C_\alpha) \). Hence from now on, in order to compare the value of \( b_{2i}(G) \) and \( b_{2i}(G^*_{n,d}) \) more efficiently, we only consider the the weakest case, that is, each cycle is of length of multiple of 4.

Graph (a) (or graph (b), (c), respectively) is a special case of graph (d) (or graph (d), graph (e), respectively) with one or two connecting paths having length of zero.

Next we consider graph (k) in Fig. 5. Let \( G \) be the graph \( (k) \). Since \( G \) has exactly four cycles, by Fact 1, there are two cycles, say \( C_\beta \) and \( C_\gamma \), having \( t \) \( \geq 1 \) common edges and there is a path \( P_t \) connecting \( C_\alpha \) and \( C_\beta, C_\gamma \) (e.g., see Fig. 5).

In order to consider the worst case, let \( |C_\alpha| = a \equiv 0 \) (mod 4), hence \( \text{diam}(G - u_1u_2 - u_2u_3) \geq (a - 1) + (l - 1) + \lceil \frac{b + c}{2} \rceil - t \geq d \). Similarly, \( \text{diam}(G - u_1 - u_2 - u_3u_4) \geq d \) and
Fig. 6. Three possible cases for the arrangement of six cycles in $G$.

Fig. 7. One possible case for the arrangement of seven cycles in $G$.

diam$(G - u_2 - u_3 - u_4 u_5) \geq d - 1$ where $d \geq 5$. We choose edges $u_1 u_2, u_2 u_3, u_3 u_4, u_4 u_5$ respectively as our reduce-edge of graphs $G, G - u_1 u_2, G - u_1 u_2 - u_2 u_3, \ldots$, $G - u_1 u_2 - \cdots - u_3 u_4$ respectively. The proof is similar to the previous case for graph (d), so we omit the proof here. Graphs (h)-(j) are special cases of graph (k). □

**Lemma 2.12.** If $G \in \mathcal{G}(n, d)$ has exactly six cycles with no pendent vertex, then $b_{2i}(G) > b_{2i}(G_{n,d}^1)$.

Since $G$ has six cycles, then it is straightforward to check that either any two of the six cycles have exactly two vertices in common, or there are two cycles either having exactly one vertex in common, or having no vertex in common; see Fig. 6. Since the proof of Lemma 2.12 is not only similar to but easier than the proof of Lemma 2.13, so we are to give the detail proof of Lemma 2.13 only.

**Lemma 2.13.** For each $G \in \mathcal{G}(n, d)$ with exactly 7 cycles and no pendent vertex, then $b_{2i}(G) \geq b_{2i}(G_{n,d}^1)$, that is $E(G) \geq E(G_{n,d}^1)$.

**Proof.** Let three elementary cycles are $C_a, C_b$ and $C_c$ and let $C_1 = C_a, C_2 = C_a \cup C_b, C_3 = C_a \cup C_c, C_4 = C_b \cup C_c, C_5 = C_a \cup C_b \cup C_c$; see Fig. 7. To make the proof simpler, we consider the worst case, that is $|C_i| = 0 \pmod{4}$.

To evaluate the $b_{2i}(G)$, we are to break up cycles. Note that each edge is on at least 4 cycles. Please note that each cycle of $\{C_a, C_b, C_c\}$ plays the same role as others since there is no pendent vertices. Without loss of generality, let $C_1 (= C_a)$ be the longest cycle and let $u_1 u_2$ be an edge of $C_1$. Edge $u_1 u_2$ is in four cycles, say $C_1, C_2, C_3$ and $C_5$. If $v_1, \ldots, v_i$ are the vertices of $G$, we denote $G - v_1 - v_2 - \cdots - v_i$ by $G - \sum_{1}^{s} v_i$ and similar notation will be used repeatedly through the proof.

Note that diam$(G) \geq 5$ since the base cases of diam$(G) = 3, 4$ have been done by Lemma 2.5 and Lemma 2.8, and by our assumption that $|C_a| = 0 \pmod{4}$, so $|C_a| = a \geq \max\{2 \times \text{diam}(G), 12\} = 12$. Hence notation $G - \sum_{1}^{s} u_k$ ($s \leq 10$) is meaningful.
We choose edges $u_iu_{i+1}$, $w_kw_{k+1}$, $v_kv_{k+1}$ as our reduce-edges of the according graphs. By Lemma 1.4,
\[
b_{2i}(G) \geq b_{2i}(G - u_1u_2 - u_2u_3) + b_{2i-2}(G - u_2 - u_3) + b_{2i-2}(G - u_1 - u_2)
- 2 \sum_{i=1,2,3,5} b_{2i-|C_i|}(G - C_i). \tag{2.6}
\]
During the processing to break $G - u_1 - u_2$ and $G - u_2 - u_3$ down into smaller pieces of graphs by choosing $u_kw_{k+1}$ as the reduce-edge of the according graphs of $G - \sum_{k=1}^{k-2} u_{k-1}$ and $G - \sum_{k=2}^{k-2} u_{k-1}$, either $G - \sum_{k=1}^{k-2} u_{k-1}$ or $G - \sum_{k=2}^{k-2} u_{k-1}$ will use vertex $w_1$ to break up cycle $C_c$ depending on $k$ being odd or even where $w_1 = u_k$ or $w_1 = u_{k+1}$. Without loss of generality, let $k(=2s)$ be even, so $G - u_2 - u_3$ will use the vertex $w_1(=u_{2s})$ to break the cycle $C_c$. Hence $b_{2i-2}(G - u_1 - u_2)$ will not be involved with the item of $-2b_{2i-|C_1|}(G - C_4)$, but $b_{2i-2}(G - u_2 - u_3)$ will do. We give the expression of $b_{2i-2}(G - u_1 - u_2)$ first
\[
b_{2i-2}(G - u_1 - u_2)
= b_{2i-2}(G - u_1 - u_2 - u_3u_4) + b_{2i-4}(G - \sum_{k=1}^{4} u_k)
= b_{2i-2} \left( G - \sum_{k=1}^{3} u_k \right) + b_{2i-4} \left( G - \sum_{k=1}^{4} u_k - u_5u_6 \right) + b_{2i-6} \left( G - \sum_{k=1}^{6} u_k \right)
\vdots
= b_{2i-2} \left( G - \sum_{k=1}^{3} u_k \right) + b_{2i-4} \left( G - \sum_{k=1}^{5} u_k \right) + b_{2i-6} \left( G - \sum_{k=1}^{7} u_k \right)
+ b_{2i-8} \left( G - \sum_{k=1}^{8} u_k \right)
\vdots
\geq b_{2i-2} \left( G - \sum_{k=1}^{3} u_k \right) + b_{2i-4} \left( G - \sum_{k=1}^{5} u_k \right) + 2b_{2i-|C_1|}(G - C_1). \tag{2.7}
\]
Denote $G - \sum_{k=1}^{3} u_k$, $G - \sum_{k=1}^{5} u_k$ by $G^*$, $G^{**}$, respectively, then
\[
b_{2i-2}(G^*) = b_{2i-2}(G^* - v_2v_3) + b_{2i-4}(G^* - v_2 - v_3)
= b_{2i-2}(G^* - v_2) + b_{2i-4}(G^* - v_2 - v_3 - v_4v_5) + b_{2i-6} \left( G^* - \sum_{k=1}^{5} v_k \right)
\vdots
\geq b_{2i-2}(G^* - v_2) + b_{2i-4} \left( G^* - \sum_{k=1}^{3} v_k \right) + 2b_{2i-|C_2|}(G - C_2). \tag{2.8}
\]
\[ b_{2i-4}(G^{**}) = b_{2i-4}(G^{**} - u_6u_7) + b_{2i-6}(G^{**} - u_6 - u_7) \]
\[ = b_{2i-4}(G^{**} - u_6u_7) + b_{2i-6}(G^{**} - u_6 - u_7 - u_8u_9) \]
\[ + b_{2i-8} \left( G - \sum_{l=1}^{9} u_k \right) \]
\[ \vdots \]
\[ \geq b_{2i-4}(G^{**} - u_6 - u_7) + 2b_{2i-|C_5|}(G - C_5). \]  

(2.9)

Denoted \( u_{2s} \) by \( w_1 \) (see Fig. 7), and denoted \( G - \sum_{k=2}^{2s-1} u_k \) by \( \bar{G} \), then

\[ b_{2i-2}(G - u_2 - u_3) \]
\[ = b_{2i-2}(G - u_2 - u_3 - u_4u_5) + b_{2i-4} \left( G - \sum_{k=2}^{5} u_k \right) - 2b_{2i-2-|C_4|}(\bar{G} - C_4) \]
\[ \vdots \]
\[ \geq b_{2i-2}(\bar{G} - w_1w_2) + b_{2i-4}(\bar{G} - w_1 - w_2) - 2b_{2i-2-|C_4|}(\bar{G} - C_4) \]
\[ = b_{2i-2}(\bar{G} - w_1w_2 - w_2w_3) + b_{2i-4}(\bar{G} - w_2 - w_3) \]
\[ + b_{2i-4}(\bar{G} - w_1 - w_2 - w_3w_4) \]
\[ + b_{2i-6}(\bar{G} - \sum_{k=1}^{4} w_k) - 2b_{2i-2-|C_4|}(\bar{G} - C_4) - 2b_{2i-2-|C_4|}(\bar{G} - C_4) \]
\[ = b_{2i-2}(\bar{G} - w_2) + b_{2i-4}(\bar{G} - w_2 - w_3 - w_4w_5) + b_{2i-6}(\bar{G} - \sum_{k=2}^{5} w_k) \]
\[ + b_{2i-4}(\bar{G} - \sum_{k=3}^{4} w_k - w_4w_5) + b_{2i-6}(\bar{G} - \sum_{k=1}^{5} w_k) \]
\[ + b_{2i-6}(\bar{G} - \sum_{k=1}^{4} w_k - w_5w_6) \]
\[ + b_{2i-8}(\bar{G} - \sum_{k=1}^{6} w_k) - 2b_{2i-2-|C_4|}(\bar{G} - C_4) - 2b_{2i-2-|C_4|}(\bar{G} - C_4) \]
\[ \vdots \]
\[ \geq b_{2i-2}(\bar{G} - w_2) + 2b_{2i-2-|C_3|}(\bar{G} - C_3) + 2b_{2i-2-|C_4|}(\bar{G} - C_4) \]
\[ + 2b_{2i-2-|C_4|}(\bar{G} - C_4) - 2b_{2i-2-|C_4|}(\bar{G} - C_4) - 2b_{2i-2-|C_4|}(\bar{G} - C_4) \]
\[ = b_{2i-2}(\bar{G} - w_2) + 2b_{2i-2-|C_3|}(\bar{G} - C_3). \]  

(2.10)
Hence, by inequalities (2.6)–(2.10), we have

\[ b_{2i}(G) \geq b_{2i}(G - u_1 u_2 - u_2 u_3) + b_{2i-2}(G^* - v_2) + b_{2i-2}(\overline{G} - w_2) \]
\[ \geq b_{2i}(B^1_{n-1,d}) + b_{2i-2}(U_{n-4,d-2}) + b_{2i-2}(U_{n-3,d-2}) \]
\[ \geq b_{2i}(B^1_{n-1,d}) + b_{2i-2}(P_{d-2}) + b_{2i-2}(S_{n-d-3} \cup P_{d-3}) \]  
(2.11)
\[ = b_{2i}(G^1_{n,d}), \]  
(2.12)

where (2.11) comes from Lemma 2.10 (3) and (2.12) comes from Lemma 2.2(3). □

Combining Lemma 2.9 and Lemmas 2.11–2.13, we have our main result.

**Theorem 2.14.** For each graph \( G \in \mathcal{G}(n, d) \), \( b_{2i}(G) \geq b_{2i}(G^1_{n,d}) \) where \( n \geq \max\{11, d + 4\} \).

**References**