Discrete Mathematics 62 (1986) 145–154
North-Holland

CONNECTED CUTSETS OF A GRAPH AND TRIANGLE BASES OF THE CYCLE SPACE

P. DUCHET, M. LAS VERGNAS and H. MEYNIEL
Université Pierre et Marie Curie (Paris VI), U.E.R. 48-Mathématiques, 75005 Paris, France

Received 19 November 1985

We investigate some properties of graphs whose cycle space has a basis constituted of triangles ('null-homotopic' graphs). We obtain characterizations in the case of planar graphs, and more generally, of graphs not contractible onto K_5. These characterizations involve separating subsets and decompositions into triangulations of discs.

The notion of *homotopy* in graphs was introduced by the authors [3]. Another notion of homotopy in graphs has been considered by Quilliot [4].

Cycles in graphs are viewed algebraically, i.e., they are considered as vectors in $\mathrm{GF}(2)^E$, where E is the edge-set of the graph. We say that two cycles C and C' are *homotopic* in a graph $G=(V, E)$ if there are triangles T_i $(i=1, \ldots, k)$ such that $C=C'+\sum_{i=1}^{k} T_i$.

The relationship between the null-homotopy property (i.e., 'any two cycles are homotopic') and properties relative to connectedness have been considered in the context of topological spaces (see Whyburn [6, chap. XI] for details). Inspired by these concepts we investigate in the present paper some properties of graphs of a similar flavour. We mention that the graph properties we obtain are not reducible to topological properties.

In the case of graphs not contractible onto K_5, we prove in Section 3 the equivalence between the null-homotopy property and the property that any two induced subgraphs whose union is the whole graph have a connected intersection. This last property is characterized in two ways in Section 1. The equivalence is not true in general (Section 2).

Graphs considered in this paper are finite, without loops or multiple edges. For a graph $G=(V, E)$ and $A \subseteq V$, G_A denotes the subgraph of G induced by $A: G_A=(A, E \cap 2^A)$. For $G=(V, E)$ and $G'=(V', E')$ we write $G \cup G'$ for the graph $(V \cup V', E \cup E')$. The graph $G \cap G'$ is defined similarly.

1. Well-connected graphs

Let $G=(V, E)$ be a connected graph. By a *minimal relative cutset* in G, or a m.r.-cutset for short, we mean a *cutset* C of G (i.e., a separating subset of V)

for which there exist two vertices x, y such that C is inclusion-minimal with the property of separating x and y.

An *inclusion-minimal cutset* is a m.r.-cutset, but the converse is not true in general (see Corollary 3.4.).

A subset S of V is *connected* if the subgraph G induced by S is connected.

For a cutset S of G, let $C_1 \cdots C_p$ be the connected components of $G_{V \setminus S}$. The induced subgraphs $G_{S \cup C_i}$ are *S-pieces* of G.

Definition 1.1. A graph G is said to be *well-connected* if G is connected and every minimal relative cutset of G is connected.

As easily seen, every contraction of a well-connected graph produces a well-connected graph. We recall that an *elementary contraction* in G is the identification of two adjacent vertices of G and that a *contraction* is a succession of elementary contractions. A graph obtained from G by some contraction is called a *contraction* of G.

More precisely we have

Theorem 1.2. *A graph G is well-connected if and only if G is connected and is not contractible onto the complete bipartite graph $K_{p,q}$, for any $p, q \geq 2$.*

Proof. In order to prove the necessity of the condition, we consider a contraction of $G = (V, E)$ onto $K_{p,q}$ ($p, q \geq 2$). Let $\{a_1 \cdots a_p\}$, $\{b_1 \cdots b_q\}$ be the bipartition of $K_{p,q}$. By the definition of a contraction V admits a partition $A_1, \ldots, A_p$, $B_1, \ldots, B_q$ into $p + q$ connected subsets such that:

no edge joins two A_i's or two B_j's, and
$A_i \cup B_j$ is connected for every i and j, $1 \leq i \leq p$, $1 \leq j \leq q$.

We see that $A_1 \cup \cdots \cup A_p$ is a cutset of G. This cutset contains a disconnected m.r.-cutset of G, separating for instance B_1 and B_2.

Conversely, suppose G is contraction-minimal with the property to have a disconnected m.r.-cutset. Let S be a disconnected m.r.-cutset with minimum cardinality.

The minimality of G with respect to contraction implies that the connected components of S are singletons $\{s_i\}$, $1 \leq i \leq p$. By the same argument, we see that the connected components $C_1 \cdots C_q$ of $G_{V \setminus S}$ are also singletons $\{c_j\}$. Finally, by the minimality of $|S|$, each c_j is adjacent to s_i for $1 \leq i \leq p$, $1 \leq j \leq q$. □

Theorem 1.3. *A necessary and sufficient condition for a connected graph G to be well-connected is that for every pair of connected induced subgraphs G_1, G_2 whose union is, G, the graph $G_1 \cup G_2$ is connected.*

Proof. Suppose G admits a pair of induced subgraphs G_A, G_B such that

$$G_{A\cap B} = G_A \cap G_B \text{ is not connected,} \tag{*}$$

$$G_A \text{ and } G_B \text{ are connected and } G = G_A \cup G_B. \tag{**}$$

Note that condition (**) implies that $A \cap B$ separates in G the sets $A \backslash B$ and $B \backslash A$.

We denote by $d_G(X, Y)$ the distance of two disjoint subsets X, Y in a graph G, i.e., the minimum length of a path connecting X and Y in G. We denote by d_A and d_B respectively the distance functions in G_A and G_B.

Among all connected components of $G_{A\cap B}$, we choose a pair X, Y of components minimizing the value of $d_A(X, Y) + d_B(X, Y)$. Let $\alpha = x_A, \ldots, y_A$ and $\beta = x_B, \ldots, y_B$ be shortest paths respectively in G_A and G_B with x_A, x_B in X and y_A, y_B in Y. α (resp. β) must contain a vertex a of $A \backslash B$ (resp. a vertex b of $B \backslash A$). Moreover, by the minimality of $d_A(X, Y) + d_B(X, Y)$, we have

$$\{x_A, \ldots, a, \ldots, y_A\} \cap B = \{x_A, y_A\}$$

and

$$\{x_B, \ldots, b, \ldots, y_B\} \cap A = \{x_B, y_B\}.$$

There is a path connecting x_A to x_B in G_x (resp. y_A to y_B in G_Y). Thus a and b are connected in G by two vertex-disjoint paths μ and v respectively included in $(A \backslash B) \cup X \cup (B \backslash A)$ and $(A \backslash B) \cup Y \cup (B \backslash A)$.

The existence of the paths μ and v implies that a minimal cutset relative to a and b included in the cutset $A \cap B$ necessarily meets X and meets Y. Hence such a m.r.-cutset is not connected and G is not well connected.

Conversely, suppose that G is not well-connected. By Theorem 1.2 above, G admits a partition into connected subsets $A_1, \ldots, A_p, B_1, \ldots, B_q$ $(p, q \geq 2)$ such that $A_i \cup B_j$ is connected but neither $A_i \cup A_{i'}$ nor $B_j \cup B_{j'}$ are, for $1 \leq i < i' \leq p$, $1 \leq j < j' \leq q$. Then, set

$$A = A_1 \cup A_2 \cup \cdots \cup A_p \cup B_2 \cup \cdots \cup B_q$$

and

$$B = A_1 \cup A_2 \cup \cdots \cup A_p \cup B_1.$$

The graphs G_A and G_B are connected and $G_A \cup G_B = G$. But $A \cap B = A_1 \cup \cdots \cup A_p$ does not induce a connected subgraph. $\square$

As corollaries, we mention two 'glueing lemmas' that will be useful in Section 3.

Corollary 1.4. *Suppose G_1 and G_2 are two induced subgraphs of a graph $G = G_1 \cup G_2$ such that $G_1 \cap G_2$ is connected. If G_1 and G_2 are well-connected, then G is well-connected.*

Proof. Let G, G_1, G_2 be graphs satisfying the hypothesis of Corollary 1.4. Every contraction G' of $G_1 \cup G_2$ is a union of a contraction G_1' of G_1 and a contraction G_2' of G_2. The graphs G', G_1', G_2' satisfy the hypothesis of Corollary 1.4. Hence, by Theorem 1.2, it suffices to show that $G_1 \cup G_2$ cannot be a complete bipartite graph $K_{p,q}$ $(p, q \geqslant 2)$. This verification is easy since every well-connected induced subgraph of $K_{p,q}$ is a star. □

Corollary 1.5. *Suppose G_1 and G_2 are two induced subgraphs of a graph $G = G_1 \cup G_2$ such that $G_1 \cap G_2$ is well-connected. Then G is well-connected if and only if both G_1 and G_2 are well-connected.*

Proof. The 'if' part follows from Corollary 1.4. Conversely suppose $G = G_1 \cup G_2$ is well-connected but G_1 is not. We assume $G_1 \cap G_2$ is connected, so that G_1 and G_2 are connected. We show that $G_1 \cap G_2$ is not well-connected.

Set $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$.

Contractions of edges in $G_2 \backslash V_1$ do not affect the situation. Thus we may suppose $G_2 \backslash V_1$ has no edge: the connected components of $G \backslash V_1$ are singletons.

By Theorem 1.3 G_1 is a union of two connected induced subgraphs G_A and G_B such that $A \cap B$ is not connected in G_1, hence in G. Let $S_1, \ldots, S_p$ be the connected components of $A \cap B$ in G_1. The contraction of an edge in some G_{S_i} does not modify our hypothesis. So we may assume that every S_i is a singleton $\{s_i\}$ (because if some contraction of $G_1 \cap G_2$ is not well-connected, the same holds for $G_1 \cup G_2$).

Let A_2 (resp. B_2) be the set of all vertices in $V_2 \backslash V_1$ that are adjacent to A (resp. to B). Put $A' = A \cup A_2$ and $B' = B \cup B_2$. The graphs $G_{A'}$ and $G_{B'}$ are connected and $G = G_{A'} \cup G_{B'}$. Since G is well-connected, the subgraph $G_{A' \cap B'}$ must be connected (Theorem 1.3). Every s_i is therefore connected to some vertex in $A_2 \cap B_2$. Since $V_1 \cap V_2$ separates $V_1 \backslash V_2$ and $V_2 \backslash V_1$, every s_i is in V_2: we have $A \cap B \subseteq V_2$.

We know that $G_A \cup G_{B \cup V_2} = G$. The graphs G_A and $B_{B \cup V_2}$ are connected (since both G_B and G_2 are connected. By Theorem 1.3 again, $G_{A \cap (B \cup V_2)}$ must be connected. Similarly $G_{B \cap (A \cup V_2)}$ is connected. From $A \cap B \subseteq V_2$ we get $A \cap (B \cup V_2) = A \cap V_2$ and $B \cap (A \cup V_2) = B \cap V_2$. Thus $G_{A \cap V_2}$ and $G_{B \cap V_2}$ are connected and $G_{A \cap V_2} \cup G_{B \cap V_2} = G_{V_1 \cap V_2}$ (since $G_A \cup G_B = G_1$). Hence $G_{A \cap V_2} \cap G_{B \cap V_2} = G_{A \cap B}$ is not connected. By Theorem 1.3, $G_1 \cap G_2 = G_{V_1} \cap G_{V_2}$ is not well-connected. □

Remark. Corollary 1.5 does not hold in general if the hypothesis that G_1 and G_2 are induced subgraphs of $G_1 \cup G_2$ is removed. But it can be shown that Corollary 1.4 is still valid under the weaker hypothesis: $G = G_1 \cup G_2$, $G_1 \cap G_2$ connected, G_1 and G_2 well-connected.

2. Null-homotopic graphs

Definition 2.1. A graph G is said to be *null-homotopic* if every cycle in G is homotopic to the zero cycle.

In other terms a graph is null-homotopic if and only if its cycle space admits a basis constituted of triangles.

A simple example of null-homotopic graphs is given by triangulated graphs (graphs such that every cycle of length 4 possesses a chord. A larger class is the class of *dismantelable graphs,* characterized by Quilliot [4] and Novakowski and Winkler [8] as graphs in which the pursuit game is won by the pursuer. We say that a vertex v of a graph G with at least two vertices is *removable* if there is a vertex w of G adjacent to v such that every vertex of G adjacent to v and different from v is also adjacent to w.

A *dismantelable* graph $G=(V, E)$ is defined by the property: there is a linear order $v_1 \cdots v_n$ of V such that v_i is removable from the subgraph induced by $\{v_1 \cdots v_i\}$ for $2 \leqslant i \leqslant n$.

The proof that every dismantelable graph is null-homotopic is left to the reader. The converse is false.

As in the topological context, we have the following property.

Theorem 2.2. *A connected null-homotopic graph is well-connected.*

Proof. Let S be a m.r.-cutset of a connected graph $G=(V, E)$ and suppose S is not connected.

By definition, there exist two vertices x and y separated by S such that S is inclusion minimal with this property.

Denote by C_x and C_y the connected components of $G_{V\setminus S}$ containing respectively x and y. By the minimality of S, C_x and C_y are adjacent to every vertex and hence to every connected component of S. So, choosing two such components S_1 and S_2, we are sure of the existence of a cycle C included in $S_1 \cup S_2 \cup C_x \cup C_y$ that passes successively through S_1, C_x, S_2, C_y.

Let T be the set of edges linking S_1 and C_x. We remark that every triangle of G has an *even* number of edges in T. On the contrary, C possesses an *odd* number of edges in T. Hence C cannot be an algebraic sum of triangles: G is not null-homotopic. □

Remark 2.3. The converse of Theorem 2.2 is not true in general. The graph C^p_{3p+q} $(1 \leqslant q \leqslant p)$ with vertex-set $\mathbb{Z}/(3p+q)\mathbb{Z}$ and edges xy for $y-x \in \{\pm 1, \ldots, \pm p\}$ (mod $3p+q$) is well connected but is not null-homotopic.

Proof. The graph C^p_{3p+q} is well connected since every separating subset must contain two disjoint 'circular interval' (i.e., subsets of the form $\{a, a+$

$1, \ldots, a+b\}$ (mod $3p+q$) with at least p vertices each. Two such intervals induce a connected graph which is also a dominating subset. Hence every separating subset of C_{3p+q}^{p} is connected. □

In order to see that C_{3p+q}^{p} is not null homotopic, consider the set T of edges xy with $x \in \{2p+q+1, 2p+q+2, \ldots, 3p+q\}$ and $y \in \{1, 2, \ldots, p\}$. Every triangle of the graph contains an even number of edges in T. Thus the cycle $1, 2, \ldots, 2p+q, 1$, containing only one edge in T, is not a sum of triangles.

We conclude this section with some lemmas on null-homotopic graphs.

Lemma 2.4. *Every contraction of a null-homotopic graph produces a null-homotopic graph. Every power*[1] *of a null-homotopic graph is a null-homotopic graph.*

Lemma 2.5. *Suppose G is a connected graph with a connected cutset S. If every S-piece is null homotopic, then G is null-homotopic.*

The proofs are left to the reader.

Lemma 2.6. *Let $G=(V, E)$ be a null-homotopic graph and S be a cutset of G. Assume the induced subgraph G_S is null-homotopic. Then every S-piece is null-homotopic.*

Proof. Let P be an S-piece and C a cycle of G_P. Let $\mathcal{T}$ be a set of triangles of G such that $C = \sum_{t\in\mathcal{T}} t \pmod 2$. Denote by $\mathcal{T}_1$ (resp. $\mathcal{T}_2$) the set of all triangles of $\mathcal{T}$ that are included in $P\backslash S$ (resp. that meet both S and $P\backslash S$). Every triangle in $\mathcal{T}_1 \cup \mathcal{T}_2$ is in P. Moreover all the edges of the cycle $C' = C + \sum_{\mathcal{T}_1} t + \sum_{\mathcal{T}_2} t$ are in S. Thus, the cycle C' is null-homotopic in G_S, and C is null-homotopic in G_P.

Remark 2.7. A graph obtained from a null-homotopic graph by adding a new edge closing at least one triangle is clearly also null-homotopic. We conjecture that all null-homotopic graphs are obtained by a sequence of such operations, starting from a tree. In other words

Conjecture 2.8. *In every null-homotopic graph G there is at least one edge e such that $G\backslash e$* is null-homotopic.

3. Planar graphs and graphs not contractible onto K_5

By Remark 2.3 the converse of Theorem 2.2 does not hold in general. However a converse holds for planar graphs and more generally for graphs not contractible onto K_5, the complete graph on 5 vertices.

[1] Two vertices are joined in the kth power of a graph when their distance in the graph is at most k.

The following definition is due to Wagner [5]: a *k-decomposition* of a graph G is a sequence of $p \geqslant 2$ graphs such that

$$G = G_1 \cup \cdots \cup G_P, \tag{3.1}$$

For $1 \leqslant i \leqslant p$, G_{i+1} and $G_1 \cup \cdots \cup G_i$ are two pieces of the graph $G_1 \cup \cdots \cup G_{i+1}$ with respect to some cutset which has at most k vertices. (3.2)

A k-decomposition is said to be *simplicial* (resp. *connected*) when all cutsets in (3.2) are complete (resp. connected) subgraphs.

A *disc-triangulation* is a connected planar graph such that every face is a triangle, except possibly one.

Theorem 3.3. *Let G be a connected planar graph. Then the following assertions are equivalent*:

(i) *G is null-homotopic,*

(ii) *G is well-connected,*

(iii) *G admits a simplicial* 3*-decomposition into disc-triangulations.*

Proof. (i)$\rightarrow$(ii) and (iii)$\rightarrow$(i) are implied by Theorem 2.2 and Lemma 2.5 respectively. We prove (ii)$\rightarrow$(iii) by induction on the number of vertices of G. The theorem is obvious for graphs having less than 4 vertices.

Let G be a planar well-connected graph with $n \geqslant 4$ vertices. If S is a minimal cutset of G with less than 4 vertices, G_S is connected hence must be well-connected. By Corollary 1.5, every S-piece is also well-connected. If G_S is K_1, K_2 or K_3 the conclusion follows immediately from the induction hypothesis. Otherwise G_S is a path of length 2. By Kuratowski's theorem there are only two S-pieces (otherwise G would contain an homeomorph of $K_{3,3}$). The vertices of S are on a non triangular face of each S-piece.

By the induction hypothesis, there are simplicial 3-decomposition of these two pieces into disc-triangulations. It is easily seen that the glueing by S of these decompositions yields a similar simplicial 3-decomposition as well.

So we may assume G is 4 connected. Suppose that G is not a disc-triangulation. Then G possesses at least two non-triangular faces F_1 and F_2.

By a well known Corollary of the Menger theorem, G contains four pairwise disjoint paths A, B, C, D joining four distinct vertices of F_1 to four distinct vertices of F_2. Let $a_1 \in F_1$ and $a_2 \in F_2$ be the endvertices of A and let b_1, b_2, c_1, c_2, d_1, d_2 be similarly defined. We may suppose that a_1, b_1, c_1, d_1 occur in that order on F_1. Clearly the vertices in $A \cup C$ separate b_1 and c_1.

We see that every cutset contained in $A \cup C$, and minimal with the property of separating b_1 and d_1, necessarily contains a_1 and c_1. Such a m.r.-cutset is therefore not connected. This fact contradicts the well-connectedness of G and concludes the proof. $\square$

Corollary 3.4. *A 3-connected planar graph G is null-homotopic if and only if: every minimal cutset of G is connected.*

Proof. If every minimal cutset of a 3-connected planar graph is connected, then every m.r.-cutset is connected. (Since no subdivided $K_{3,3}$ is planar).

The corollary fails for 2-connected graphs as shown by the graph drawn in Fig. 1.

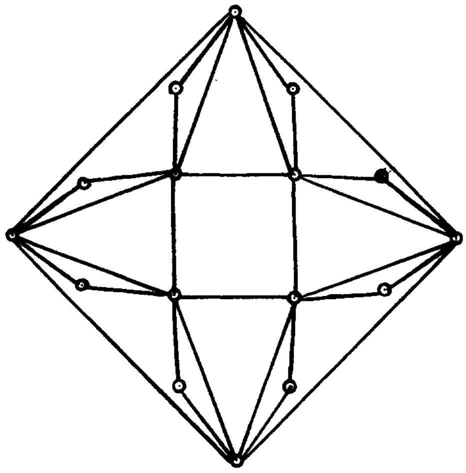

Fig. 1.

Theorem 3.3 generalizes to graphs not contractible onto K_5. The proof uses the decomposition theorem of Wagner [5] (short proof in [7]).

Theorem 3.5. *Let G be a connected graph not contractible onto K_5. The following assertions are equivalent.*

(i) *G is null-homotopic,*

(ii) *G is well-connected,*

(iii) *G admits a connected 3-decomposition whose members are disc-triangulations of discs.*

Proof. (i)→(ii) by Theorem 2.2 (iii)→(i) by Lemma 2.5. We prove (ii)→(iii) by induction on the number of vertices of G. This is obvious for small graphs. Suppose G is a well-connected graph not contractible onto K_5.

Adding edges to G, we eventually obtain a graph G' maximal with the property to be not contractible onto K_5. By Wagner's theorem, the graph G' either admits a simplicial 3-decomposition or is maximal planar or is L_8, the octagon with its four diameters.

If G' is not decomposable, since every well-connected subgraph of L_8 is planar, then G is planar and the conclusion follows from Theorem 3.3. If G' has a simplicial 3-decomposition $G' = G_1' \cup \cdots \cup G_p'$, $p \geq 2$, G admits a 3-decomposition $G = G_1 \cup \cdots \cup G_p$.

The graph $H = (G_1 \cup \cdots \cup G_{p-1}) \cap G_p$ is connected via Theorem 1.3. Since H

has at most 3 vertices H is also well-connected. Thus by Corollary 1.5, the graphs $G_1 \cup \cdots \cup G_{p-1}$ and G_p are well-connected. Applying the induction hypothesis to the graphs $G_1 \cup \cdots \cup G_{p-1}$ and G_p, we obtain the conclusion. □

A criterium for contractibility onto K_5 is the following result.

Theorem 3.6. *Let G be a connected graph with at least one edge. Suppose every edge of G is contained in at least 3 triangles. Then G is contractible onto K_5.*

Proof. We apply Wagner's theorem. Suppose G satisfies the hypothesis of Theorem 3.6 and is not contractible onto K_5. By the argument used in the proof of Theorem 3.5, G admits a 3-decomposition $G = G_1 \cup G_2 \cup \cdots \cup G_p$ whose members are planar. We may suppose that G_p is not 3-decomposable; The graph G_p contains an edge not in $G_1 \cup \cdots \cup G_{p-1}$. Necessarily there are 3 triangles of G_p containing this edge. In a planar graph if three triangles have an edge in common, one of them is a cutset: hence G_p is 3-decomposable, a contradiction. □

Remark 3.7. If every edge is contained in at least 2 triangles every vertex is of degree $\geqslant 3$. So, by a theorem of Dirac [2], the graph is contractible onto K_4.

This suggests the following problem: what is the maximal value of t such that if G has at least one edge and every edge of G contained in at least t triangles, then G is contractible onto K_{t+2}? We conjecture that this property holds for $t = 4$.

It does not hold for t sufficiently large as a consequence of the following theorem of [1]: for every fixed constant c, there exists an integer $t > 0$ and a graph G with n vertices and more than tn edges which is not contractible onto $K_{\lfloor ct \rfloor}$. If some edge e of G is in less than t triangles then the contraction of e produces a graph with $n-1$ vertices and more than $t(n-1)$ edges. Repeating this operation we obtain a graph G' not contractible onto $K_{\lfloor ct \rfloor}$ in which every edge belongs to at least t triangles. We do not know what happens when every edge of the graph is supposed to be in *exactly* t triangles.

Note added in proof

Our Conjecture 2.8 was recently disproved by C. Champetier who considered a triangulation of a certain 2-manifold.

References

[1] W. Fernandez De La Vega, On the maximum density of graphs with no subcontraction to K^s, Discrete Math. 46 (1983) 109–110.

[2] G.A. Dirac, A property of 4-chromatic graphs and some remarks on critical graphs, J. London Math. Soc. 27 (1952) 85–92.
[3] P. Duchet, M. Las Vergnas and H. Meyniel, Homotopy and convexity in graphs, Communication at the Coll. to the memory of K. Kuratowski, Łagow (1981), Poland.
[4] A. Quilliot, Thése d'Etat, Paris (1983).
[5] K. Wagner, Bemerkungen zu Hadwigers Vermutung, Math. Ann. 141 (1960) 433–451.
[6] G.T. Whyburn, Analytic Topology, AMS Coll. Pub. Vol. XXVIII (AMS, Providence, 1942).
[7] H.P. Young, A quick proof of Wagner's equivalence theorem, J. Lond. Mat. Soc. (2) 3 (1971) 661–664.