Global attractor of the Cahn–Hilliard equation in $H^k$ spaces✩

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In this paper, by using an iteration procedure, regularity estimates for the linear semigroups and a classical existence theorem of global attractor we prove that the Cahn–Hilliard equation $u_t = -\Delta^2 u + \Delta g(u)$ possesses a global attractor in Sobolev space $H^k$ for all $k \geq 0$, which attracts any bounded subset of $H^k(\Omega)$ in the $H^k$-norm.

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1. Introduction


However, when a binary solution is cooled sufficiently, phase separation may occur and then proceed in two ways: either by nucleation in which nuclei of the second phase appear randomly and grow, or the whole solution appears to nucleate at once and then periodic or semiperiodic structures appear in the so-called spinodal decomposition. The pattern formation resulting from phase transition has been observed in alloys, glasses, and polymer solutions. The model governing the phase separation of binary systems is the following Cahn–Hilliard equation:

$$
\begin{align*}
\frac{\partial u}{\partial t} &= -\Delta^2 u + \Delta g(u), \quad x \in \Omega, \\
\frac{\partial u}{\partial n} \Big|_{\partial \Omega} &= 0, \quad \frac{\partial u}{\partial n} \Big|_{\partial \Omega} = 0, \quad \int_{\Omega} u \, dx = 0, \\
u(x, 0) &= \varphi(x),
\end{align*}
$$

(1.1)

where the unknown function $u$ represents the concentration of material $B$ (see Remarks 1.1). $\Delta$ is the Laplace operator. $\Omega \subset R^n \quad (1 \leq n \leq 3)$ is a $C^\infty$ bounded domain. $g(s)$ is a polynomial on $s \in R^1$, which is given by

$$
g(s) = \sum_{k=1}^{p} a_k s^k.
$$

(1.2)
Due to the劳卢德 average field theory, we have \( p \) should be an odd number, i.e., \( p = 2m + 1 \ (m \geq 1) \), and

\[
a_p > 0.
\]

**Remarks 1.1.** The sum of the concentrations of materials \( A \) and \( B \) in a binary solution is a constant, i.e., \( u_A + u_B = C \), where \( u_A \) and \( u_B \) denote the concentration of materials \( A \) and \( B \), respectively. So, it is enough to describe the phase separation only by a single function \( u = u_B \). Thus, the Cahn–Hilliard equation (1.1) only contains one unknown function.

The dynamical properties of the Cahn–Hilliard equation (1.1) such as the global asymptotical behaviors of solutions and existence of global attractors, are important for the study of phase separation of binary systems, which ensure the stability of phase transition and provide the mathematical foundation for the study of phase transition dynamics. So, many authors are interested in the existence of global attractors for general nonlinear dissipative dynamical systems such as [4,6,7,14–16]. As for the Cahn–Hilliard equation, the existence of global solutions and global attractors in \( L^2(\Omega) \) have been proved by B. Nicolaenko, B. Scheurer, and R. Temam [11,14]. Further, under certain assumptions the existence of global attractors in \( H^1(\Omega) \) and \( H^2(\Omega) \) has been given in [2,3,5]. For convenience, we introduce the main results as follows.

**Lemma 1.1.** Under the conditions (1.2) and (1.3), for \( \varphi \in H \) the following three claims hold.

1. Eq. (1.1) has a unique global weak solution \( u \in L^\infty((0, \infty), H) \cap L^2((0, \infty), H_1^2) \) for any \( p = 2m + 1 \geq 3 \).
2. For any \( p \geq 3 \) as \( n = 1,2 \) and \( p = 3 \) as \( n = 3 \), Eq. (1.1) has a unique strong solution \( u \in L^2((0, T), H_1) \cap H^1((0, T), H) \) for any \( T > 0 \).
3. Eq. (1.1) has a global attractor \( A \subset H \) which attracts any bounded set of \( H \) in the \( H \)-norm.

Here the spaces \( H, H_1^2 \) and \( H_1 \) are defined as follows:

\[
H = \left\{ u \in L^2(\Omega) \mid \int_\Omega u \, dx = 0 \right\},
\]

\[
H_2 = \left\{ u \in H^2(\Omega) \cap H \mid \frac{\partial u}{\partial n} \big|_{\partial \Omega} = 0 \right\},
\]

\[
H_1 = \left\{ u \in H^4(\Omega) \cap H \mid \frac{\partial u}{\partial n} \big|_{\partial \Omega} = \frac{\partial \Delta u}{\partial n} \big|_{\partial \Omega} = 0 \right\}.
\]

In this paper, we shall use the regularity estimates for the linear semigroups, combining with the classical existence theorem of global attractors, to prove that the Cahn–Hilliard equation possesses, in any \( k \)th differentiable function spaces \( H^k(\Omega) \), a global attractor, which attracts any bounded set of \( H^k(\Omega) \) in \( H^k \)-norm. The basic idea is an iteration procedure which is from Ma and Wang’s recent books [8–10].

**2. Preliminaries**

Let \( X \) and \( X_1 \) be two Banach spaces, \( X_1 \subset X \) a compact and dense inclusion. Consider the abstract nonlinear evolution equation defined on \( X \), given by

\[
\begin{aligned}
\frac{du}{dt} &= Lu + G(u), \\
u(0) &= \varphi,
\end{aligned}
\]

where \( u(t) \) is an unknown function, \( L : X_1 \to X \) a linear operator, and \( G : X_1 \to X \) a nonlinear operator.

A family of operators \( S(t) : X \to X \) \((t \geq 0)\) is called a semigroup generated by (2.1) provided \( S(t) \) satisfies the properties:

1. \( S(t) : X \to X \) is a continuous mapping for any \( t \geq 0 \).
2. \( S(0) = \text{id} : X \to X \) the identity,
3. \( S(t + s) = S(t) \cdot S(s), \forall t, s \geq 0 \),

and the solution of (2.1) can be expressed as

\[
u(t, \varphi) = S(t)\varphi.
\]

Next, we introduce the concepts and definitions of invariant sets, global attractors, \( \omega \)-limit sets for the semigroup \( S(t) \).
**Definition 2.1.** Let \( S(t) \) be a semigroup defined on \( X \). A set \( \Sigma \subset X \) is called an invariant set of \( S(t) \) if \( S(t) \Sigma = \Sigma \), \( \forall t \geq 0 \). An invariant set \( \Sigma \) is an attractor of \( S(t) \) if \( \Sigma \) is compact, and there exists a neighborhood \( U \subset X \) of \( \Sigma \) such that for any \( \varphi \in U \),
\[
\inf_{\psi \in \Sigma} \| S(t)\varphi - \psi \|_X \to 0, \quad \text{as} \quad t \to \infty.
\]
In this case, we say that \( \Sigma \) attracts \( U \). Especially, if \( \Sigma \) attracts any bounded set of \( X \), \( \Sigma \) is called a global attractor of \( S(t) \).

For a set \( D \subset X \), we define the \( \omega \)-limit set of \( D \) as follows:
\[
\omega(D) = \bigcap_{s \geq 0} \bigcup_{t \geq s} S(t)D,
\]
where the closure is taken in the \( X \)-norm.

The following Lemma 2.1 is the classical existence theorem of global attractor by R. Temam [14].

**Lemma 2.1.** Let \( S(t) : X \to X \) be the semigroup generated by (2.1). Assume the following conditions hold:

1. \( S(t) \) has a bounded absorbing set \( B \subset X \), i.e., for any bounded set \( A \subset X \) there exists a time \( t_A \geq 0 \) such that \( S(t_A) \varphi \in B \), \( \forall \varphi \in A \) and \( t > t_A \);
2. \( S(t) \) is uniformly compact, i.e., for any bounded set \( U \subset X \) and some \( T > 0 \) sufficiently large, the set \( \bigcup_{t \geq T} S(t)U \) is compact in \( X \).

Then the \( \omega \)-limit set \( A = \omega(B) \) of \( B \) is a global attractor of (2.1), and \( A \) is connected providing \( B \) is connected.

Note that we used to assume that the linear operator \( L \) in (2.1) is a sectorial operator which generates an analytic semigroup \( e^{L} \). It is known that there exists a constant \( \lambda \geq 0 \) such that \( L - \lambda I \) generates the fractional power operators \( L^\alpha \) and fractional order spaces \( X_\alpha \) for \( \alpha \in \mathbb{R}^1 \), where \( L = -(L - \lambda I) \). Without loss of generality, we assume that \( L \) generates the fractional power operators \( L^\alpha \) and fractional order spaces \( X_\alpha \) as follows:
\[
L^\alpha = (-L)^\alpha : X_0 \to X \quad \alpha \in \mathbb{R}^1,
\]
where \( X_\alpha = D(L^\alpha) \) is the domain of \( L^\alpha \). By the semigroup theory of linear operators (see Pazy [13]), we know that \( X_\beta \subset X_\alpha \) is a compact inclusion for any \( \beta > \alpha \).

Thus, Lemma 2.1 can be equivalently expressed in the following Lemma 2.2.

**Lemma 2.2.** Let \( u(t, \varphi) = S(t)\varphi \ (\varphi \in X, \ t \geq 0) \) be a solution of (2.1) and \( S(t) \) be the semigroup generated by (2.1). Let \( X_\alpha \) be the fractional order space generated by \( L \). Assume

1. for some \( \alpha \geq 0 \) there is a bounded set \( B \subset X_\alpha \), for any \( \varphi \in X_\alpha \) there exists \( t_\varphi > 0 \) such that \( u(t, \varphi) \in B \), \( \forall t > t_\varphi \);
2. there is a \( \beta > \alpha \), for any bounded set \( U \subset X_\beta \) there are \( T > 0 \) and \( C > 0 \) such that \( \| u(t, \varphi) \|_{X_\beta} \leq C, \quad \forall t > T \text{ and } \varphi \in U \).

Then (2.1) has a global attractor \( A \subset X_\alpha \) which attracts any bounded set of \( X_\alpha \) in the \( X_\alpha \)-norm.

For sectorial operators, we also have the following properties which can be found in A. Pazy [13].

**Lemma 2.3.** Let \( L : X_1 \to X \) be a sectorial operator which generates an analytic semigroup \( T(t) = e^{Lt} \). If all eigenvalues \( \lambda \) of \( L \) satisfy \( \Re \lambda < -\lambda_0 \) for some real number \( \lambda_0 > 0 \), then for \( L^\alpha \ (L = -L) \) we have

1. \( T(t) : X \to X_\alpha \) is bounded for all \( \alpha \in \mathbb{R}^1 \) and \( t > 0 \),
2. \( T(t)L^\alpha x = L^\alpha T(t)x, \forall x \in X_\alpha \),
3. for each \( t > 0 \), \( L^\alpha T(t) : X \to X \) is bounded, and
\[
\| L^\alpha T(t) \| \leq C_\alpha t^{-\alpha} e^{-\delta t},
\]
where some \( \delta > 0 \), \( C_\alpha > 0 \) is a constant only depending on \( \alpha \),
4. the \( X_\alpha \)-norm can be defined by
\[
\| x \|_{X_\alpha} = \| L^\alpha x \|_X.
\]
3. Main results

Let $H$ and $H_1$ be the spaces defined as in (1.4). We define the operators $L : H_1 \rightarrow H$ and $G : H_1 \rightarrow H$ by

\[
\begin{align*}
Lu &= -\Delta^2 u, \\
Gu &= \Delta g(u),
\end{align*}
\] (3.1)

where $g(u)$ is the same as one of (1.2). Thus, the Cahn–Hilliard equation (1.1) can be written into the abstract form (2.1).

It is well known that the linear operator $L : H_1 \rightarrow H$ given by (3.1) is a sectorial operator and the fractional power operator $(-L)^{\frac{1}{2}}$ is given by

\[
(-L)^{\frac{1}{2}} = -\Delta.
\] (3.2)

The space $H_1^2$ is the same as (1.4), $H_1^4$ is given by $H_1^4 = \text{closure of } H_1^2$ in $H_1(\Omega)$ and $H_k = H^4_k(\Omega) \cap H_1$ for $k \geq 1$.

The main result in this paper is given by the following theorem, which provides the existence of global attractors of the Cahn–Hilliard equation (1.1) in any $k$th order space $H_k$.

**Theorem 3.1.** Let the function $g$ be a polynomial of order $p$

\[
g(u) = \sum_{k=1}^{p} a_k u^k, \quad p = 2m + 1 \quad (m \geq 1, \quad m \in \mathbb{N}),
\]

with leading coefficient

\[a_p > 0.\]

Assume $p = 3$ for $n = 3$. Then, for any $\alpha \geq 0$ Eq. (1.1) has a global attractor $A$ in $H_\alpha$, and $A$ attracts any bounded set of $H_\alpha$ in the $H_\alpha$-norm.

**Proof.** From Lemma 1.1, we know that the solution of system (1.1) is a strong solution for any $\varphi \in H$. Hence, the solution $u(t, \varphi)$ of system (1.1) can be written as

\[
\begin{align*}
u(t, \varphi) &= \text{e}^{tL} \varphi + \int_0^t \text{e}^{(t-\tau)L} G(u) \, d\tau.
\end{align*}
\] (3.3)

By (3.1) and (3.2), we rewrite (3.3) into

\[
\begin{align*}
u(t, \varphi) &= \text{e}^{tL} \varphi - \int_0^t (-L)^{\frac{1}{2}} \text{e}^{(t-\tau)L} g(u) \, d\tau.
\end{align*}
\] (3.4)

Next, according to Lemma 2.2, we prove Theorem 3.1 in the following six steps.

**Step 1.** We prove that for any bounded set $U \subset H_\frac{3}{2}$ there is a constant $C > 0$ such that the solution $u(t, \varphi)$ of system (1.1) is uniformly bounded by the constant $C$ for any $\varphi \in U$ and $t \geq 0$. To do that, we firstly check that system (1.1) has a global Lyapunov function as follows:

\[
F(u) = \int_\Omega \left( \frac{1}{2} |\nabla u|^2 + f(u) \right) \, dx,
\] (3.5)

where

\[
f(z) = \int_0^z g(z) \, dz = \sum_{k=1}^{p} \frac{1}{k+1} a_k z^{k+1}.
\]

In fact, if $u(t, \varphi)$ is a strong solution of system (1.1), we have

\[
\frac{d}{dt} F(u(t, \varphi)) = \left( DF(u), \frac{du}{dt} \right)_H.
\] (3.6)

By (3.2), we get

\[
\frac{du}{dt} = Lu + G(u) = -(L)^{\frac{1}{2}} DF(u).
\] (3.7)
Since $L$ is symmetric, for any $\alpha, \beta \in \mathbb{R}^1$ we have
\[
\langle (-L)^\alpha u, v \rangle_H = \langle (-L)^{\alpha - \beta} u, (-L)^\beta v \rangle_H.
\]
Hence, it follows from (2.2) and (3.6)–(3.7) that
\[
\frac{dF(u)}{dt} = \langle DF(u), -(-L)^\alpha DF(u) \rangle_H = -\|DF(u)\|^2_{H^\frac{1}{2}},
\]
which implies that (3.5) is a Lyapunov function.
Integrating (3.8) from 0 to $t$ gives
\[
F(u(t, \varphi)) = -\int_0^t \|DF\|^2_{H^\frac{1}{2}} dt + F(\varphi).
\] (3.9)

Using (1.3) and (3.5), we have
\[
F(u) = \int_\Omega \left( \frac{1}{2} |\nabla u|^2 + f(u) \right) dx = \int_\Omega \left( \frac{1}{2} |\nabla u|^2 + \frac{1}{p+1} a_p u^{p+1} + \sum_{k=1}^{p-1} \frac{1}{k+1} a_k u^{k+1} \right) dx
\]
\[
\geq \int_\Omega \left( \frac{1}{2} |\nabla u|^2 + \frac{1}{p+1} a_p u^{p+1} - \sum_{k=1}^{p-1} \frac{1}{k+1} |a_k| |u|^{k+1} \right) dx
\]
\[
\geq \int_\Omega \left( \frac{1}{2} |\nabla u|^2 + \frac{1}{p+1} a_p u^{p+1} - \sum_{k=1}^{p-1} \frac{1}{k+1} |a_k| \left( |u|^p + \varepsilon^{-\frac{k+1}{p-1}} \right) \right) dx
\]
\[
= \int_\Omega \left( \frac{1}{2} |\nabla u|^2 + \frac{1}{p+1} a_p u^{p+1} - \varepsilon \left( \sum_{k=1}^{p-1} \frac{1}{k+1} |a_k| \right) |u|^{p+1} - \sum_{k=1}^{p-1} \frac{1}{k+1} |a_k| \varepsilon^{-\frac{k+1}{p-1}} \right) dx.
\]
Choosing $\varepsilon$ such that $\varepsilon \left( \sum_{k=1}^{p-1} \frac{1}{k+1} |a_k| \right) = \frac{1}{2(p+1)} a_p$, and noting that $p$ is an odd number, i.e., $p = 2m + 1$ ($m \geq 1$), we get
\[
F(u) \geq \int_\Omega \left( \frac{1}{2} |\nabla u|^2 + \frac{1}{p+1} a_p u^{p+1} - \frac{1}{2(p+1)} a_p |u|^{p+1} - \sum_{k=1}^{p-1} \frac{1}{k+1} |a_k| \varepsilon^{-\frac{k+1}{p-1}} \right) dx
\]
\[
= \int_\Omega \left( \frac{1}{2} |\nabla u|^2 + \frac{1}{p+1} a_p |u|^{p+1} - \frac{1}{2(p+1)} a_p |u|^{p+1} - \sum_{k=1}^{p-1} \frac{1}{k+1} |a_k| \varepsilon^{-\frac{k+1}{p-1}} \right) dx
\]
\[
= \int_\Omega \left( \frac{1}{2} |\nabla u|^2 + \frac{1}{2(p+1)} a_p |u|^{p+1} - \sum_{k=1}^{p-1} \frac{1}{k+1} |a_k| \varepsilon^{-\frac{k+1}{p-1}} \right) dx
\]
\[
\geq C_1 \int_\Omega (|\nabla u|^2 + |u|^{p+1}) dx - C_2.
\]
Combining with (3.9) yields
\[
C_1 \int_\Omega (|\nabla u|^2 + |u|^{p+1}) dx - C_2 \leq -\int_0^t \|DF\|^2_{H^\frac{1}{2}} dt + F(\varphi),
\]
\[
C_1 \int_\Omega (|\nabla u|^2 + |u|^{p+1}) dx + \int_0^t \|DF\|^2_{H^\frac{1}{2}} dt \leq F(\varphi) + C_2,
\]
\[
\int_\Omega (|\nabla u|^2 + |u|^{2m+2}) dx \leq C, \quad \forall t \geq 0, \quad \varphi \in U,
\]
which implies
\[
\|u(t, \varphi)\|_{H^\frac{1}{2}} \leq C, \quad \forall t \geq 0, \quad \varphi \in U \subset H^\frac{1}{2},
\] (3.10)
where $C_1, C_2$ and $C$ are positive constants. $C$ only depends on $\varphi$. 

Step 2. We prove that for any bounded set \( U \subset H_\alpha \) \((\frac{1}{4} \leq \alpha < \frac{1}{2})\) there exists \( C > 0 \) such that
\[
\|u(t, \varphi)\|_{H_\alpha} \leq C, \quad \forall t \geq 0, \quad \varphi \in U, \quad \alpha < \frac{1}{2}.
\] (3.11)

In fact, it follows from (2.2) and (3.4) that
\[
\|u(t, \varphi)\|_{H_\alpha} = \|e^{L_\alpha} \varphi - \int_0^t (L_\alpha)^{1+\alpha} g(u) \, dt\|_{H_\alpha} \leq \|\varphi\|_{H_\alpha} + \int_0^t \| (L_\alpha)^{1+\alpha} g(u) \|_{H_\alpha} \, dt.
\]

Hence, by (3.10) and Lemma 2.3 we deduce that
\[
\|u(t, \varphi)\|_{H_\alpha} \leq \|\varphi\|_{H_\alpha} + \int_0^t \|(L_\alpha)^{1+\alpha} g(u)\|_{H_\alpha} \, dt.
\]

We claim that \( g : H_{\frac{1}{4}} \to H \) is bounded.

By \( H_{\frac{1}{4}} \hookrightarrow L^{2p}(\Omega) \), we have
\[
\|g(u)\|_{H_{\frac{1}{4}}}^2 = \int_{\Omega} |g(u)|^2 \, dx = \int_{\Omega} \left| \sum_{k=1}^{\rho} a_k u^k \right|^2 \, dx \leq \int_{\Omega} \left( \sum_{k=1}^{\rho-1} |a_k| (|u|^p + \sum_{k=1}^{\rho-1} |a_k| (|u|^p + e^{-\frac{1}{p-1}}) \right)^2 \, dx
\]
\[
\leq C \left( \int_{\Omega} |u|^{2p} \, dx + 1 \right) \leq C (\|u\|_{H_{\frac{3}{4}}}^{2p} + 1),
\]

which implies that \( g : H_{\frac{1}{4}} \to H \) is bounded.

Hence, by (3.10) and Lemma 2.3 we deduce that
\[
\|u(t, \varphi)\|_{H_\alpha} \leq \|\varphi\|_{H_\alpha} + C \int_0^t \tau^{-\beta} e^{-\beta \tau} \, d\tau \leq C, \quad \forall t \geq 0, \quad \varphi \in U \subset H_\alpha,
\]
where \( \beta = \frac{1}{2} + \alpha \) \((0 < \beta < 1)\). Hence, (3.11) holds.

Step 3. We prove that for any bounded set \( U \subset H_\alpha \) \((\frac{1}{4} \leq \alpha < \frac{3}{4})\) there is a constant \( C > 0 \) such that
\[
\|u(t, \varphi)\|_{H_\alpha} \leq C, \quad \forall t \geq 0, \quad \varphi \in U \subset H_\alpha, \quad \alpha < \frac{3}{4}.
\] (3.12)

In fact, by the embedding theorems of fractional order spaces (see Pazy [13]):
\[
H_\alpha \hookrightarrow C^0(\Omega) \cap H^1(\Omega) \quad \text{as} \quad \alpha > \frac{3}{8},
\]
we have
\[
\|g(u)\|_{H_{\frac{3}{4}}}^2 = \int_{\Omega} |\nabla g(u)|^2 \, dx = \int_{\Omega} \left| \nabla \left( \sum_{k=1}^{\rho} a_k u^k \right) \right|^2 \, dx = \int_{\Omega} \left| \sum_{k=1}^{\rho} k a_k u^{(k-1)} \nabla u \right|^2 \, dx
\]
\[
\leq \int_{\Omega} \left( p |a_k| |u|^{p-1} + \sum_{k=1}^{\rho-1} k |a_k| (|u|^{p-1} + e^{-\frac{1}{p-1}}) \right)^2 \, dx \leq C \int_{\Omega} |\nabla u|^2 \, dx
\]
\[
\leq C \int_{\Omega} \left( \sup_{x \in \Omega} |u|^{2p-2} + 1 \right) |\nabla u|^2 \, dx \leq C (\|u\|_{H_\alpha}^{2p-2} + 1) \|u\|_{H_{\frac{3}{4}}}^2 \leq C (\|u\|_{H_\alpha}^{2p-2} + 1) \|u\|_{H_{\alpha}}^2,
\]

which implies
\[
g : H_\alpha \to H_{\frac{1}{4}} \quad \text{is bounded for} \quad \alpha > \frac{3}{8}.
\] (3.13)

Therefore, it follows from (3.11) and (3.13) that
\[
\|g(u(t, \varphi))\|_{H_{\frac{1}{4}}} < C, \quad \forall t \geq 0, \quad \varphi \in U, \quad \alpha < \frac{1}{2}.
\] (3.14)
Then, by using same method as that in Step 2, we get from (3.18) that
\[
\|u(t, \varphi)\|_{H_{\alpha}} = \left\| e^{t\varphi} - \int_0^t (-L)^{\frac{1}{2} + \alpha} e^{(t-\tau)L} g(u) \, d\tau \right\|_{H_{\alpha}} \leq \|\varphi\|_{H_{\alpha}} + \int_0^t \|(-L)^{\frac{1}{2} + \alpha} e^{(t-\tau)L} g(u)\|_{H^\frac{1}{4}} \, d\tau
\]
\[
\leq \|\varphi\|_{H_{\alpha}} + \int_0^t \|(-L)^{\frac{1}{2} + \alpha} e^{(t-\tau)L} \cdot g(u)\|_{H^\frac{1}{4}} \, d\tau \leq \|\varphi\|_{H_{\alpha}} + C \int_0^t \tau^{-\beta} e^{-\beta \tau} \, d\tau
\]
\[
\leq C, \quad \forall t > 0, \quad \varphi \in U \subset H_{\alpha},
\]
(3.15)

where \(\beta = \frac{1}{4} + \alpha (0 < \beta < 1)\). Hence, (3.12) holds.

**Step 4.** We prove that for any bounded set \(U \subset H_{\alpha} (\frac{3}{4} \leq \alpha < 1)\) there is a constant \(C > 0\) such that
\[
\|u(t, \varphi)\|_{H_{\alpha}} \leq C, \quad \forall t \geq 0, \quad \varphi \in U \subset H_{\alpha}, \quad \alpha < 1.
\]
(3.16)

In fact, by the embedding theorems of fractional order spaces (see Pazy [13]):
\[
H^2(\Omega) \hookrightarrow W^{1,6}(\Omega) \hookrightarrow W^{1,4}(\Omega), \quad H_{\alpha} \hookrightarrow C(\Omega) \cap H^2(\Omega), \quad \text{as } \alpha \geq \frac{1}{2},
\]
we deduce that
\[
\|g(u)\|_{H_{\frac{1}{2}}}^2 = \int_\Omega |\Delta g(u)|^2 \, dx = \int_\Omega \left| \Delta \left( \sum_{k=1}^p a_k u^k \right) \right|^2 \, dx = \int_\Omega \left| \sum_{k=1}^p \left[ k(k-1)a_k u^{k-2}(\nabla u)^2 + ka_k u^{k-1} \Delta u \right] + a_1 \Delta u \right|^2 \, dx
\]
\[
\leq \int_\Omega \left( p(p-1)a_p |u|^{p-2} |\nabla u|^2 + pa_p |u|^{p-1} \Delta u + \sum_{k=2}^{p-1} \left[ k(k-1)a_k (\varepsilon |u|^{p-2} + e^{-\frac{k-1}{p-2}}) |\nabla u|^2 \right. \right.
\]
\[
\left. + ka_k (\varepsilon |u|^{p-1} + e^{\frac{k-1}{p-2}}) |\Delta u| \right] + a_1 |\Delta u| \right|^2 \, dx
\]
\[
\leq C \int_\Omega (|u|^{p-2} |\nabla u|^2 + |\nabla u|^2 + |u|^{p-1} |\Delta u| + |\Delta u|)^2 \, dx
\]
\[
\leq C \int_\Omega (|\nabla u|^4 + |u|^{2p-4} |\nabla u|^4 + |\Delta u|^2 + |u|^{2p-2} |\Delta u|^2) \, dx
\]
\[
\leq C \int_\Omega (|\nabla u|^4 + \sup_{x \in \Omega} |u|^{2p-4} |\nabla u|^4 + |\Delta u|^2 + \sup_{x \in \Omega} |u|^{2p-2} |\Delta u|^2) \, dx
\]
\[
\leq C \left( \|u\|_{W^{1,4}}^2 + \|u\|_{H_{\alpha}}^2 \right) + \|u\|_{H_{\alpha}}^2 \leq C \left( \|u\|_{H_{\alpha}}^2 + \|u\|_{H_{\alpha}}^2 \right)
\]
(3.17)

It implies
\[
g : H_{\alpha} \rightarrow H_{\frac{1}{2}} \quad \text{is bounded for } \alpha \geq \frac{1}{2}.
\]

Therefore, by (3.12) and (3.17) we derive that
\[
\|g(u(t, \varphi))\|_{H_{\frac{1}{2}}} < C, \quad \forall t \geq 0, \quad \varphi \in U, \quad \frac{1}{2} \leq \alpha < \frac{3}{4}.
\]
(3.18)

Then, by using same methods as those in Steps 2 and 3, we get from (3.18) that
\[
\|u(t, \varphi)\|_{H_{\alpha}} = \left\| e^{t\varphi} - \int_0^t (-L)^{\frac{1}{2} + \alpha} e^{(t-\tau)L} g(u) \, d\tau \right\|_{H_{\alpha}} \leq \|\varphi\|_{H_{\alpha}} + \int_0^t \|(-L)^{\frac{1}{2} + \alpha} e^{(t-\tau)L} g(u)\|_{H^\frac{1}{4}} \, d\tau
\]
Step 5. We prove that for any bounded set $U \subset H_\alpha$ ($\alpha \geq 0$) there is a constant $C > 0$ such that
\[
\left\| u(t, \varphi) \right\|_{H_\alpha} \leq C, \quad \forall t \geq 0, \quad \varphi \in U \subset H_\alpha, \quad \alpha \geq 0.
\]
(3.20)

In fact, by the embedding theorems of fractional order spaces (see Pazy [13]), we obtain
\[
H^3(\Omega) \subset W^{2,6}(\Omega) \subset W^{1,6}(\Omega), \quad H_\alpha \subset C^0(\Omega) \cap H^3(\Omega), \quad \text{as } \alpha \geq \frac{3}{4}.
\]
(3.21)

Hence, it follows from (3.21) that
\[
g : H_\alpha \to H^\frac{3}{4} \text{ is bounded for } \alpha \geq \frac{3}{4}.
\]

Therefore, by (3.16) we derive that
\[
\left\| g(u(t, \varphi)) \right\|_{H^\frac{3}{4}} < C, \quad \forall t \geq 0, \quad \varphi \in U, \quad \frac{3}{4} \leq \alpha < 1.
\]
(3.22)

Then, it follows from (3.22) that
\[
\left\| u(t, \varphi) \right\|_{H_\alpha} = \left\| e^{t\alpha} \varphi - \int_0^t (-L)^{\frac{3}{4}} e^{-\tau L} g(u) d\tau \right\|_{H_\alpha} \leq \left\| \varphi \right\|_{H_\alpha} + \int_0^t \left\| (-L)^{\frac{3}{4}} e^{-(t-\tau) L} g(u) \right\|_{H^\frac{3}{4}} d\tau
\]
\[
\leq \left\| \varphi \right\|_{H_\alpha} + \int_0^t \left\| (-L)^{\frac{3}{4} - \frac{1}{4}} e^{-\tau L} \right\| \cdot \left\| g(u) \right\|_{H^\frac{3}{4}} d\tau \leq \left\| \varphi \right\|_{H_\alpha} + C \int_0^t \tau^{-\beta} e^{-\delta \tau} d\tau
\]
\[
\leq C, \quad \forall t \geq 0, \quad \varphi \in U \subset H_\alpha,
\]
(3.23)

where $\beta = \alpha - \frac{1}{4}$ ($0 < \beta < 1$). Hence, (3.20) is valid for $1 \leq \alpha < \frac{5}{4}$.

By doing the same procedures as Steps 1–4, we can prove that (3.20) holds for all $\alpha \geq 0$.

Step 6. We show that for any $\alpha \geq 0$, system (1.1) has a bounded absorbing set in $H_\alpha$. We first consider the case of $\alpha = \frac{1}{4}$.

It is well known that the Cahn–Hilliard equation possesses a global attractor in $H$ space, and the global attractor of this equation consists of equilibria with their stable and unstable manifolds. Thus, each trajectory has to converge to a critical point. From (3.20) and (3.8), we deduce that for any $\varphi \in H^\frac{1}{4}$ the solution $u(t, \varphi)$ of system (1.1) converges to a critical point of $F$. Hence, we only need to prove the following two properties:

\begin{enumerate}
\item $F(u) \to \infty \iff \left\| u \right\|_1 \to \infty$.
\item the set $S = \{ u \in H^\frac{1}{4} \mid DF(u) = 0 \}$ is bounded.
\end{enumerate}

Property (1) is obviously true, we now prove (2) in the following. It is easy to check if $DF(u) = 0$, $u$ is a solution of the following equation
\[
\begin{aligned}
-\Delta u + g(u) &= 0, \\
\frac{\partial u}{\partial n} \bigg|_{\partial \Omega} &= 0, \\
\int_{\Omega} u \, dx &= 0,
\end{aligned}
\]
(3.24)

where $g(u)$ is given by (1.2). Taking the scalar product of (3.24) with $u$, then we derive that
\[
\int_{\Omega} \left( |\nabla u|^2 + \sum_{k=1}^{2m+1} a_k u^{k+1} \right) dx = 0.
\]
(3.25)
By (1.3) and (3.25), we have
\[
\int_{\Omega} \left( |\nabla u|^2 + a_{2m+1} u^{2(m+1)} - \sum_{k=1}^{2m} |a_k| \cdot |u|^{k+1} \right) \, dx \leq 0.
\]
Using Hölder inequality and the above inequality, we have
\[
\int_{\Omega} (|\nabla u|^2 + u^{2(m+1)}) \, dx \leq C,
\]
where $C > 0$ is a constant. Thus, property (2) is proved.

Now, we show that system (1.1) has a bounded absorbing set in $H_{\alpha}$ for any $\alpha \geq \frac{1}{4}$, i.e., for any bounded set $U \subset H_{\alpha}$ there are $T > 0$ and a constant $C > 0$ independent of $\phi$ such that
\[
\|u(t, \phi)\|_{H_{\alpha}} \leq C, \quad \forall t \geq T, \ \phi \in U.
\]
(3.26)

From the above discussion, we know that (3.26) holds as $\alpha = \frac{1}{4}$. By (3.4) we have
\[
u(t, \phi) = e^{(t-T)H} u(T, \phi) - \int_{T}^{t} (-L)^{\frac{1}{2}} e^{(t-\tau)H} g(u) \, d\tau.
\]
(3.27)

Let $B \subset H_{\frac{1}{4}}$ be the bounded absorbing set of system (1.1), and $T_0 > 0$ such that
\[
u(t, \phi) \in B, \quad \forall t > T_0, \ \phi \in U \subset H_{\alpha} \left( \alpha \geq \frac{1}{4} \right).
\]
(3.28)

It is well known that
\[
\|e^{tL}\| \leq Ce^{-\lambda_1 t},
\]
where $\lambda_1 > 0$ is the first eigenvalue of the equation
\[
\begin{aligned}
- \Delta u &= \lambda u, \\
\frac{\partial u}{\partial n} |_{\partial \Omega} &= 0, \\
\int_{\Omega} u \, dx &= 0.
\end{aligned}
\]

Hence, for any given $T > 0$ and $\phi \in U \subset H_{\alpha} \left( \alpha \geq \frac{1}{4} \right)$ we have
\[
\|e^{(t-T)H} u(T, \phi)\|_{H_{\alpha}} = \|(-L)^{\frac{1}{2}} e^{(t-\tau)H} u(T, \phi)\|_{H} \to 0, \quad \text{as } t \to \infty.
\]
(3.29)

From (3.27)-(3.28) and Lemma 2.3, for any $\frac{1}{4} \leq \alpha < \frac{1}{2}$ we get that
\[
\|u(t, \phi)\|_{H_{\alpha}} \leq \|e^{(t-T_0)L} u(T_0, \phi)\|_{H_{\alpha}} + \int_{T_0}^{t} \|(-L)^{\frac{1}{2}} e^{(t-\tau)H} g(u)\|_{H} \, d\tau
\]
\[
\leq \|e^{(t-T_0)L} u(T_0, \phi)\|_{H_{\alpha}} + C \int_{T_0}^{t} \tau^{-(\alpha + \frac{1}{2})} e^{-\lambda_1 \tau} \, d\tau,
\]
(3.30)

where $C > 0$ is a constant independent of $\phi$.

Then, we infer from (3.29) and (3.30) that (3.26) holds for all $\frac{1}{4} \leq \alpha < \frac{1}{2}$. By the iteration method, we have that (3.26) holds for all $\alpha \geq \frac{1}{4}$.

Finally, this theorem follows from (3.20), (3.26) and Lemma 2.2. The proof is completed. \(\square\)

**Remarks 3.1.** The attractors $A_{\alpha} \subset H_{\alpha}$ in Theorem 3.1 are the same for all $\alpha \geq 0$, i.e., $A_{\alpha} = A$, $\forall \alpha \geq 0$. Hence, $A \subset C^\infty (\Omega)$.

Theorem 3.1 implies that for any $\phi \in H$, the solution $u(t, \phi)$ of (1.1)-(1.3) satisfies that
\[
\lim_{t \to \infty} \inf_{\nu \in A} \|u(t, \phi) - \nu\|_{C^k} = 0, \quad \forall k \geq 1.
\]


References
