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Numerical simulation of fractional Cable equation of spiny neuronal dendrites

N.H. Sweilam [a,*], M.M. Khader [b], M. Adel [a]

[a] Department of Mathematics, Faculty of Science, Cairo University, Giza, Egypt
[b] Department of Mathematics, Faculty of Science, Benha University, Benha, Egypt

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ABSTRACT

In this article, numerical study for the fractional Cable equation which is fundamental equations for modeling neuronal dynamics is introduced by using weighted average of finite difference methods. The stability analysis of the proposed methods is given by a recently proposed procedure similar to the standard John von Neumann stability analysis. A simple and an accurate stability criterion valid for different discretization schemes of the fractional derivative and arbitrary weight factor is introduced and checked numerically. Numerical results, figures, and comparisons have been presented to confirm the theoretical results and efficiency of the proposed method.

Introduction

The Cable equation is one of the most fundamental equations for modeling neuronal dynamics. Due to its significant deviation from the dynamics of Brownian motion, the anomalous diffusion in biological systems cannot be adequately described by the traditional Nernst–Planck equation or its simplification, the Cable equation. Very recently, a modified Cable equation was introduced for modeling the anomalous diffusion in spiny neuronal dendrites [1]. The resulting governing equation, the so-called fractional Cable equation, which is similar to the traditional Cable equation except that the order of derivative with respect to the space and/or time is fractional.

Also, the proposed fractional Cable equation model is better than the standard integer Cable equation, since the fractional derivative can describe the history of the state in all intervals, for more details see [1,2] and the references cited therein.

The main aim of this work is to solve such this equation numerically by an efficient numerical method, fractional weighted average finite difference method (FWA–FDM).

In recent years, considerable interest in fractional calculus has been stimulated by the applications that this calculus finds in numerical analysis and different areas of physics and engineering, possibly including fractal phenomena. The applications range from control theory to transport problems in fractal structures, from relaxation phenomena in disordered

* Corresponding author. Tel.: +20 1003543201.
E-mail address: nsweilam@sci.cu.edu.eg (N.H. Sweilam).
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Table 1 The absolute error of the numerical solution of Eq. (35).

x	The absolute error
0.1	0.3063×10^{-3}
0.2	0.5826×10^{-3}
0.3	0.8019×10^{-3}
0.4	0.9427×10^{-3}
0.5	0.9912×10^{-3}
0.6	0.9427×10^{-3}
0.7	0.8019×10^{-3}
0.8	0.5826×10^{-3}
0.9	0.3063×10^{-3}

media to anomalous reaction kinetics of subdiffusive reagents [2,3]. Fractional differential equations (FDEs) have been of considerable interest in the literatures, see for example [4–13] and the references cited therein, the topic has received a great deal of attention especially in the fields of viscoelastic materials [14], control theory [15], advection and dispersion of solutes in natural porous or fractured media [16], anomalous diffusion, signal processing and image denoising/filtering [17].

In this section, the definitions of the Riemann–Liouville and the Grünwald–Letnikov fractional derivatives are given as follows:

Definition 1. The Riemann–Liouville derivative of order α of the function $y(x)$ is defined by

$$D_x^{\alpha}y(x)=\frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_0^x\frac{y(\tau)}{(x-\tau)^{\alpha-n+1}}d\tau,\quad x>0, \tag{1}$$

where n is the smallest integer exceeding α and $\Gamma(.)$ is the Gamma function. If $\alpha=n\in\mathbb{N}$, then (1) coincides with the classical n^{th} derivative $y^{(n)}(x)$.

Definition 2. The Grünwald–Letnikov definition for the fractional derivatives of order $\alpha>0$ of the function $y(x)$ is defined by

$$D^{\alpha}y(x)=\lim_{h\to0}\frac{1}{h^{\alpha}}\sum_{k=0}^{\left[\frac{x}{h}\right]}w_k^{(\alpha)}y(x-hk),\quad x\geqslant0, \tag{2}$$

where $\left[\frac{x}{h}\right]$ means the integer part of $\frac{x}{h}$ and $w_k^{(\alpha)}$ are the normalized Grünwald weights which are defined by $w_k^{(\alpha)}=(-1)^k\binom{\alpha}{k}$.

The Grünwald–Letnikov definition is simply a generalization of the ordinary discretization formula for integer order derivatives. The Riemann–Liouville and the Grünwald–Letnikov approaches coincide under relatively weak conditions; if $y(x)$ is continuous and $y'(x)$ is integrable in the interval $[0,x]$, then for every order $0<\alpha<1$ both the Riemann–Liouville and the Grünwald–Letnikov derivatives exist and coincide for any value inside the interval $[0,x]$. This fact of fractional calculus ensures the consistency of both definitions for most physical applications, where the functions are expected to be sufficiently smooth [15,18].

The plan of the paper is as follows: In the second section, some fractional formulae and some discrete versions of the fractional derivative are given. Also, the FWA–FDM is developed. In the third section, we study the stability and the accuracy of the presented method. In section ''Numerical results'' numerical solutions and exact analytical solutions of a typical fractional Cable problem are compared. The paper ends with some conclusions in section ''Conclusion and remarks.''

We consider the initial-boundary value problem of the fractional Cable equation which is usually written in the following way

$$\begin{aligned}u_t(x,t)=&D_t^{1-\beta}u_{xx}(x,t)-\mu D_t^{1-\alpha}u(x,t),\quad a<x<b,\\&0<t\leqslant T,\end{aligned} \tag{3}$$

where $0<\alpha,\beta\leqslant1$, μ is a constant and $D_t^{1-\gamma}$ is the fractional derivative defined by the Riemann–Liouville operator of order $1-\gamma$, where $\gamma=\alpha,\beta$. Under the zero boundary conditions

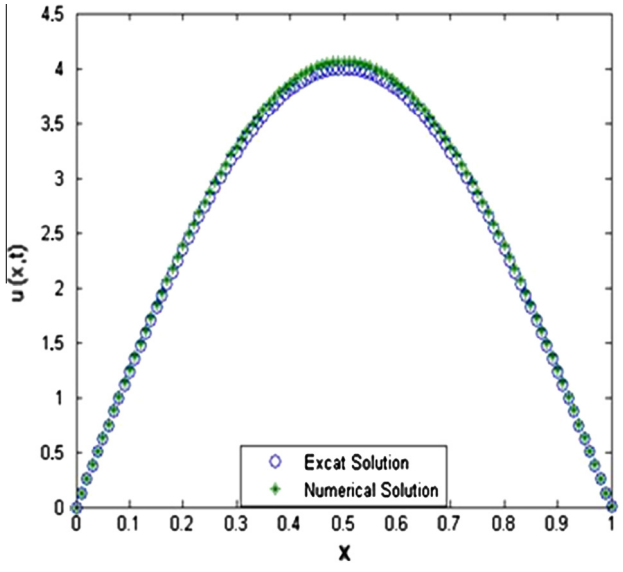

Fig. 1 The behavior of the exact solution and the numerical solution of (35) at $\lambda=0$ for $\alpha=0.2,\beta=0.7,\Delta x=\frac{1}{100},\Delta t=\frac{1}{40}$, with $T=2$.

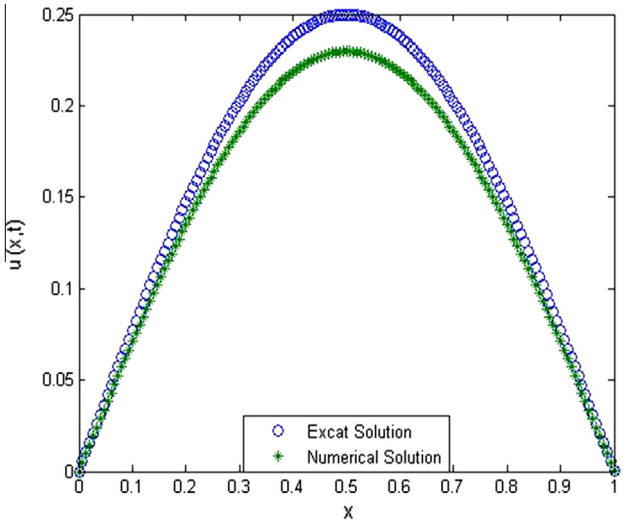

Fig. 2 The behavior of the exact solution and the numerical solution of (35) at $\lambda=0.5$ for $\alpha=0.1,\beta=0.3,\Delta x=\frac{1}{150},\Delta t=\frac{1}{10}$, with $T=0.5$.

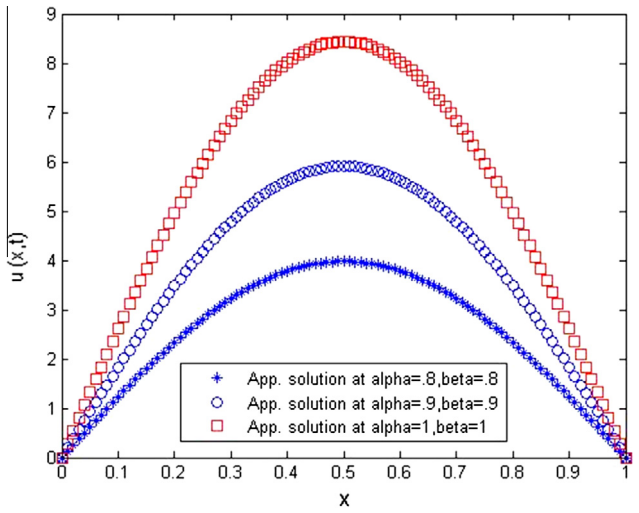

Fig. 3 The behavior of the approximate solution of (35) at $\lambda = 0.5$ for $\Delta x = \frac{1}{150}, \Delta t = \frac{1}{10}$, with $T = 0.5$, $\alpha = 0.8$, $\beta = 0.8$, $\alpha = 0.9$, $\beta = 0.9$, $\alpha = 1$, $\beta = 1$.

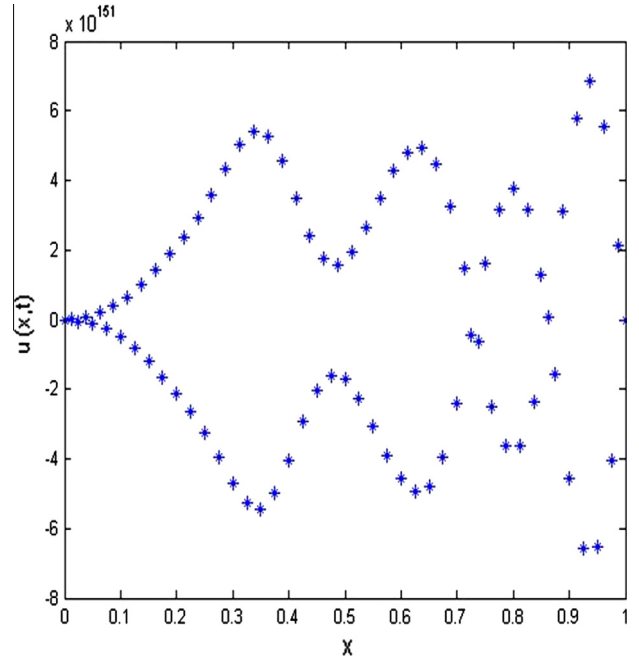

Fig. 4 The behavior of the unstable solution of (35) at $\lambda = 1$ for $\alpha = 0.1, \beta = 0.9, \Delta x = \frac{1}{80}, \Delta t = \frac{1}{140}$, with $T = 1$.

$$u(a,t) = u(b,t) = 0, \tag{4}$$

and the following initial condition

$$u(x,0) = g(x). \tag{5}$$

In the last few years, appeared many papers to study this model (3)–(5) [5,19–22], the most of these papers study the ordinary case of such system. In this paper, we study the fractional case and use the FWA–FDM to solve this model.

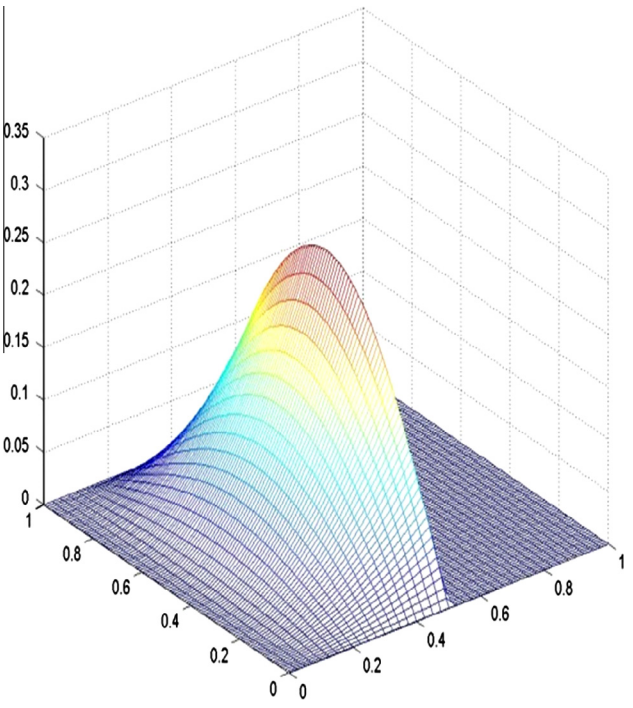

Fig. 5 The behavior of the numerical solution of (35) at $\lambda = 0$ for $\alpha = 0.2, \beta = 0.7, \Delta x = \frac{1}{100}, \Delta t = \frac{1}{40}$.

Finite difference scheme for the fractional Cable equation

In this section, we will use the FWA–FDM to obtain the discretization finite difference formula of the Cable Eq. (3). We use the notations Δt and Δx, at time-step length and space-step length, respectively. The coordinates of the mesh points are $x_j = a + j\Delta x$ and $t_m = m\Delta t$, and the values of the solution $u(x,t)$ on these grid points are $u(x_j, t_m) \equiv u_j^m \approx U_j^m$.

For more details about discretization in fractional calculus see [5].

In the first step, the ordinary differential operators are discretized as follows [23]

$$\left.\frac{\partial u}{\partial t}\right|_{x_j, t_m + \frac{\Delta t}{2}} = \delta_t u_j^{m+\frac{1}{2}} + O(\Delta t) \equiv \frac{u_j^{m+1} - u_j^m}{\Delta t} + O(\Delta t), \tag{6}$$

and

$$\left.\frac{\partial^2 u}{\partial x^2}\right|_{x_j, t_m} = \delta_{xx} u_j^m + O(\Delta x)^2 \equiv \frac{u_{j-1}^m - 2u_j^m + u_{j+1}^m}{(\Delta x)^2} + O(\Delta x)^2. \tag{7}$$

In the second step, the Riemann–Liouville operator is discretized as follows

$$D_t^{1-\gamma} u(x,t)\big|_{x_j, t_m} = \delta_t^{1-\gamma} u_j^m + O(\Delta t), \tag{8}$$

where

$$\begin{aligned} \delta_t^{1-\gamma} u_j^m &\equiv \frac{1}{(\Delta t)^{1-\gamma}} \sum_{k=0}^{\left[\frac{t_m}{\Delta t}\right]} w_k^{(1-\gamma)} u(x_j, t_m - k\Delta t) \\ &= \frac{1}{(\Delta t)^{1-\gamma}} \sum_{k=0}^{m} w_k^{(1-\gamma)} u_j^{m-k}, \end{aligned} \tag{9}$$

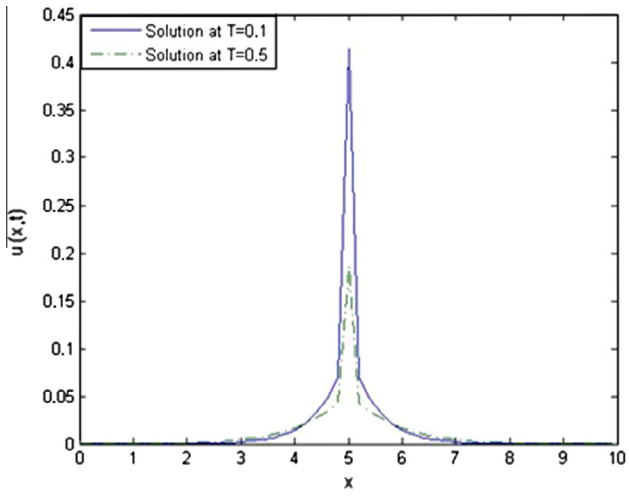

Fig. 6 The numerical solution of (37) where $\lambda = 0$, $\alpha = 0.5, \beta = 0.5, \Delta x = \frac{1}{50}, \Delta t = \frac{1}{30}$ with different values T.

Table 2 The maximum absolute error for different values of Δx and Δt.

Δx	Δt	Maximum error
$\frac{1}{20}$	$\frac{1}{30}$	0.00751
$\frac{1}{100}$	$\frac{1}{50}$	0.00716
$\frac{1}{150}$	$\frac{1}{100}$	0.00428
$\frac{1}{150}$	$\frac{1}{150}$	0.00234
$\frac{1}{150}$	$\frac{1}{200}$	0.00095
$\frac{1}{200}$	$\frac{1}{250}$	0.00010

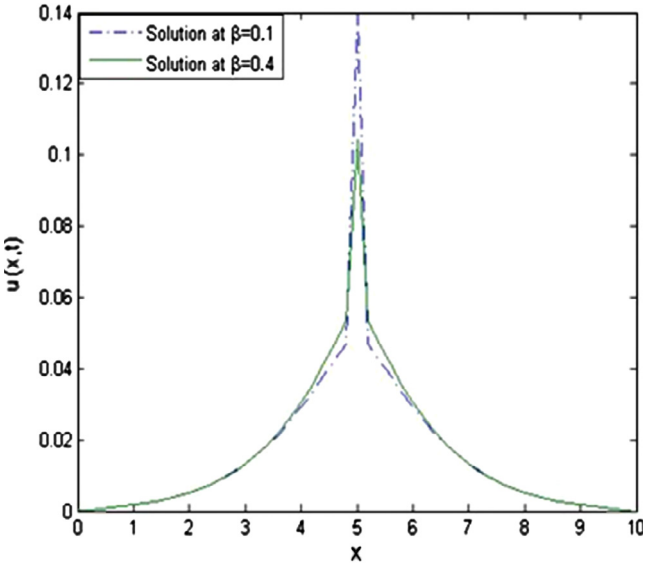

Fig. 7 The numerical solution of (37) where $\lambda = 0$, $\Delta x = \frac{1}{50}, \Delta t = \frac{1}{30}, \alpha = 0.5$, with different values of β at $T = 0.1$.

where $\left[\frac{t_m}{\Delta t}\right]$ means the integer part of $\frac{t_m}{\Delta t}$ and for simplicity, we choose $h = \Delta t$. There are many choices of the weights $w_k^{(\alpha)}$ [5,15], so the above formula is not unique. Let us de-

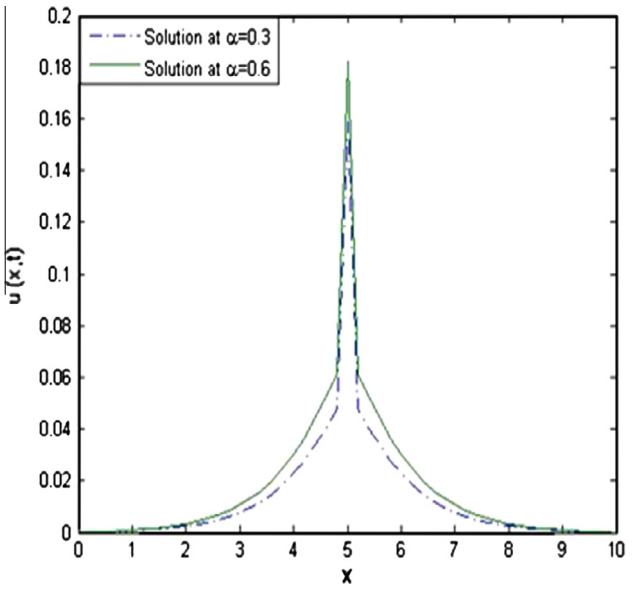

Fig. 8 The numerical solution of (37) where $\lambda = 0$, $\Delta x = \frac{1}{50}, \Delta t = \frac{1}{30}, \beta = 0.2$ with different values of α at $T = 0.1$.

note the generating function of the weights $w_k^{(\alpha)}$ by $w(z, \alpha)$, i.e.,

$$w(z, \alpha) = \sum_{k=0}^{\infty} w_k^{(\alpha)} z^k. \tag{10}$$

If

$$w(z, \alpha) = (1 - z)^{\alpha}, \tag{11}$$

then (9) gives the backward difference formula of the first order, which is called the Grünwald–Letnikov formula. The coefficients $w_k^{(\alpha)}$ can be evaluated by the recursive formula

$$w_k^{(\alpha)} = \left(1 - \frac{\alpha + 1}{k}\right) w_{k-1}^{(\alpha)}, \quad w_0^{(\alpha)} = 1. \tag{12}$$

For $\gamma = 1$ the operator $D_t^{1-\gamma}$ becomes the identity operator so that, the consistency of Eqs. (8) and (9) requires $w_0^{(0)} = 1$, and $w_k^{(0)} = 0$ for $k \geqslant 1$, which in turn means that $w(z,0) = 1$.

Now, we are going to obtain the finite difference scheme of the Cable Eq. (3). To achieve this aim, we evaluate this equation at the intermediate point of the grid $\left(x_j, t_m + \frac{\Delta t}{2}\right)$

$$\left[u_t(x,t) - D_t^{1-\beta} u_{xx}(x,t)\right]_{x_j, t_m + \frac{\Delta t}{2}} + \mu D_t^{1-\alpha} u(x_j, t_m) = 0. \tag{13}$$

Then, we replace the first order time-derivative by the forward difference formula (6) and replace the second order space-derivative by the weighted average of the three-point centered formula (7) at the times t_m and t_{m+1}

$$\delta_t u_j^{m+\frac{1}{2}} - \left\{\lambda \delta_t^{1-\beta} \delta_{xx} u_j^m + (1 - \lambda) \delta_t^{1-\beta} \delta_{xx} u_j^{m+1}\right\} + \mu \delta_t^{1-\alpha} u_j^m = TE_j^{m+\frac{1}{2}}, \tag{14}$$

with λ is being the weight factor and $TE_j^{m+\frac{1}{2}}$ is the resulting truncation error. The standard difference formula is given by

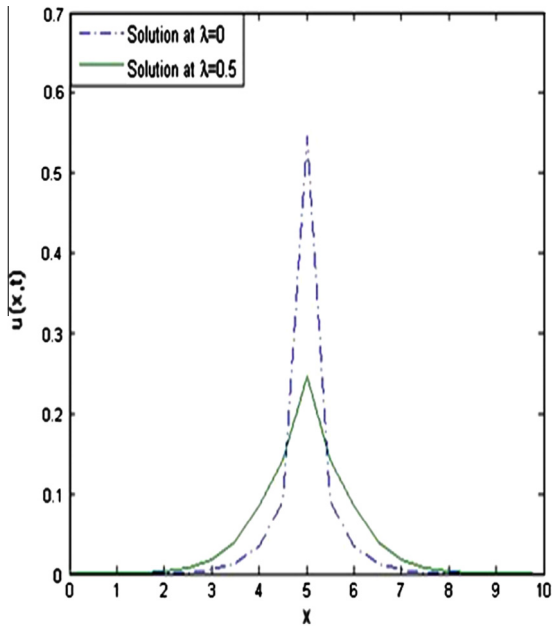

Fig. 9 The numerical solution of (37) where $\alpha = 0.5$, $\beta = 0.5, \Delta x = \frac{1}{50}, \Delta t = \frac{1}{30}$, with different values of λ at $T = 0.1$.

$$\delta_t U_j^{m+\frac{1}{2}} - \left\{\lambda \delta_t^{1-\beta} \delta_{xx} U_j^m + (1-\lambda)\delta_t^{1-\beta}\delta_{xx} U_j^{m+1}\right\} + \mu \delta_t^{1-\alpha} U_j^m = 0. \tag{15}$$

Now, by substituting from the difference operators given by (6), (7) and (9), we get

$$\begin{aligned}\frac{U_j^{m+1} - U_j^m}{\Delta t} &- \lambda \frac{1}{(\Delta t)^{1-\beta}} \sum_{r=0}^{m} w_r^{(1-\beta)} \left(\frac{U_{j-1}^{m-r} - 2U_j^{m-r} + U_{j+1}^{m-r}}{(\Delta x)^2}\right) \\ &- (1-\lambda)\frac{1}{(\Delta t)^{1-\beta}} \\ &\times \sum_{r=0}^{m} w_r^{(1-\beta)} \left(\frac{U_{j-1}^{m+1-r} - 2U_j^{m+1-r} + U_{j+1}^{m+1-r}}{(\Delta x)^2}\right) \\ &+ \mu \frac{1}{(\Delta t)^{1-\alpha}} \sum_{r=0}^{m} w_r^{(1-\alpha)} U_j^{m-r} = 0. \end{aligned} \tag{16}$$

Put $N_\beta = \frac{(\Delta t)^\beta}{(\Delta x)^2}, N_\alpha = (\Delta t)^\alpha, \phi = (1-\lambda)N_\beta$, and under some simplifications we can obtain the following form

$$-\phi U_{j-1}^{m+1} + (1+2\phi)U_j^{m+1} - \phi U_{j+1}^{m+1} = R, \tag{17}$$

where

$$\begin{aligned} R = U_j^m &+ N_\beta \sum_{r=0}^{m} \left[\lambda w_r^{(1-\beta)} + (1-\lambda) w_{r+1}^{(1-\beta)}\right]\left[U_{j-1}^{m-r} - 2U_j^{m-r} + U_{j+1}^{m-r}\right] \\ &- \mu N_\alpha \sum_{r=0}^{m} w_r^{(1-\alpha)} U_j^{m-r}. \end{aligned} \tag{18}$$

Eq. (17) is the fractional weighted average difference scheme. Fortunately, Eq. (17) is tridiagonal system that can be solved using conjugate gradian method. In the case of $\lambda = 1$ and $\lambda = \frac{1}{2}$, we have the backward Euler fractional quadrature method and the Crank–Nicholson fractional quadrature methods, respectively, which have been studied, e.g., in [24], but at $\lambda = 0$ the scheme is called fully implicit.

Stability analysis

In this section, we use the John von Neumann method to study the stability analysis of the weighted average scheme (17).

Theorem 1. *The fractional weighted average finite difference scheme (WADS) derived in (17) is stable at $0 \leqslant \lambda \leqslant \frac{1}{2}$ under the following stability criterion*

$$\frac{N_\alpha}{N_\beta} \geqslant \frac{(2\lambda - 1)2^{-\beta}}{1 - \mu 2^{-\alpha}}. \tag{19}$$

Proof. By using (18), we can write (17) in the following form

$$\begin{aligned} -\phi U_{j-1}^{m+1} + (1+2\phi)U_j^{m+1} - \phi U_{j+1}^{m+1} - U_j^m &= -\mu N_\alpha \sum_{r=0}^{m} w_r^{(1-\alpha)} U_j^{m-r} \\ &+ N_\beta \sum_{r=0}^{m} \left[\lambda w_r^{(1-\beta)} + (1-\lambda) w_{r+1}^{(1-\beta)}\right]\left[U_{j-1}^{m-r} - 2U_j^{m-r} + U_{j+1}^{m-r}\right]. \end{aligned} \tag{20}$$

In the fractional John von Neumann stability procedure, the stability of the fractional WADS is decided by putting $U_j^m = \xi_m e^{ijq\Delta x}$. Inserting this expression into the weighted average difference scheme (20) we obtain

$$\begin{aligned} -\phi \xi_{m+1} e^{iq(j-1)\Delta x} &+ (1+2\phi)\xi_{m+1} e^{iqj\Delta x} - \phi \xi_{m+1} e^{iq(j+1)\Delta x} - \xi_m e^{iqj\Delta x} \\ &= N_\beta \sum_{r=0}^{m} \left[\lambda w_r^{(1-\beta)} + (1-\lambda) w_{r+1}^{(1-\beta)}\right][e^{iq(j-1)\Delta x} \\ &- 2e^{iqj\Delta x} + e^{iq(j+1)\Delta x}]\xi_{m-r} - \mu N_\alpha \sum_{r=0}^{m} w_r^{(1-\alpha)} \xi_{m-r} e^{iqj\Delta x}, \end{aligned} \tag{21}$$

substitute by $\phi = (1-\lambda)N_\beta$ and divide (21) by $e^{iqj\Delta x}$ we get

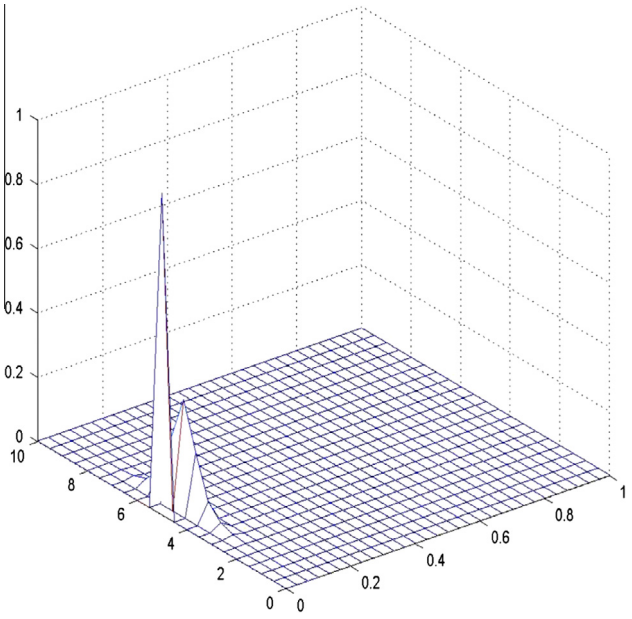

Fig. 10 The numerical solution of (37) where $\alpha = 0.5$, $\beta = 0.5, \Delta x = \frac{1}{50}, \Delta t = \frac{1}{30}$.

$$-(1-\lambda)N_\beta\xi_{m+1}e^{-iq\Delta x}+(1+2(1-\lambda)N_\beta)\xi_{m+1}-(1-\lambda)N_\beta\xi_{m+1}e^{iq\Delta x}-\xi_m-N_\beta\sum_{r=0}^{m}\left[\lambda w_r^{(1-\beta)}+(1-\lambda)w_{r+1}^{(1-\beta)}\right][e^{-iq\Delta x}-2+e^{iq\Delta x}]\xi_{m-r}+\mu N_\alpha\sum_{r=0}^{m}w_r^{(1-\alpha)}\xi_{m-r}=0. \quad (22)$$

Using the known Euler's formula $e^{i\theta}=\cos\theta+i\sin\theta$ we have

$$[1+2(1-\lambda)N_\beta-2(1-\lambda)N_\beta\cos(q\Delta x)]\xi_{m+1}-\xi_m-N_\beta\sum_{r=0}^{m}\left[\lambda w_r^{(1-\beta)}+(1-\lambda)w_{r+1}^{(1-\beta)}\right][-2+2\cos(q\Delta x)]\xi_{m-r}+\mu N_\alpha\sum_{r=0}^{m}w_r^{(1-\alpha)}\xi_{m-r}=0. \quad (23)$$

Under some simplifications, we can write the above equation in the following form

$$\left[1+4(1-\lambda)N_\beta\sin^2\left(\frac{q\Delta x}{2}\right)\right]\xi_{m+1}+\mu N_\alpha\sum_{r=0}^{m}w_r^{(1-\alpha)}\xi_{m-r}-\xi_m+4N_\beta\sin^2\left(\frac{q\Delta x}{2}\right)\sum_{r=0}^{m}\left[\lambda w_r^{(1-\beta)}+(1-\lambda)w_{r+1}^{(1-\beta)}\right]\xi_{m-r}=0. \quad (24)$$

The stability of the scheme is determined by the behavior of ξ_m. In the John von Neumann method, the stability analysis is carried out using the amplification factor η defined by

$$\xi_{m+1}=\eta\xi_m. \quad (25)$$

Of course, η depends on m. But, let us assume that, as in [13], η is independent of time. Then, inserting this expression into Eq. (24), one gets

$$\left[1+4(1-\lambda)N_\beta\sin^2\left(\frac{q\Delta x}{2}\right)\right]\eta\xi_m+\mu N_\alpha\sum_{r=0}^{m}w_r^{(1-\alpha)}\eta^{-r}\xi_m-\xi_m+4N_\beta\sin^2\left(\frac{q\Delta x}{2}\right)\sum_{r=0}^{m}\left[\lambda w_r^{(1-\beta)}+(1-\lambda)w_{r+1}^{(1-\beta)}\right]\eta^{-r}\xi_m=0, \quad (26)$$

divide by ξ_m to obtain the following formula of η

$$\eta=\frac{1-4N_\beta\sin^2\left(\frac{q\Delta x}{2}\right)\sum_{r=0}^{m}\left[\lambda w_r^{(1-\beta)}+(1-\lambda)w_{r+1}^{(1-\beta)}\right]\eta^{-r}-\mu N_\alpha\sum_{r=0}^{m}w_r^{(1-\alpha)}\eta^{-r}}{1+4(1-\lambda)N_\beta\sin^2\left(\frac{q\Delta x}{2}\right)}. \quad (27)$$

The scheme will be stable as long as $|\eta|\leqslant 1$, i.e.,

$$-1\leqslant\frac{1-4N_\beta\sin^2\left(\frac{q\Delta x}{2}\right)\sum_{r=0}^{m}\left[\lambda w_r^{(1-\beta)}+(1-\lambda)w_{r+1}^{(1-\beta)}\right]\eta^{-r}-\mu N_\alpha\sum_{r=0}^{m}w_r^{(1-\alpha)}\eta^{-r}}{1+4(1-\lambda)N_\beta\sin^2\left(\frac{q\Delta x}{2}\right)}\leqslant 1 \quad (28)$$

considering the time-independent limit value $\eta=-1$ and since $1+4(1-\lambda)N_\beta\sin^2\left(\frac{q\Delta x}{2}\right)>0$, then

$$-1-4(1-\lambda)N_\beta\sin^2\left(\frac{q\Delta x}{2}\right)\leqslant 1-4N_\beta\sin^2\left(\frac{q\Delta x}{2}\right)\sum_{r=0}^{m}(-1)^{-r}\left[\lambda w_r^{(1-\beta)}+(1-\lambda)w_{r+1}^{(1-\beta)}\right]-\mu N_\alpha\sum_{r=0}^{m}w_r^{(1-\alpha)}(-1)^{-r}.$$

From the above inequality, we can obtain

$$-2-4(1-\lambda)N_\beta\sin^2\left(\frac{q\Delta x}{2}\right)+\mu N_\alpha\sum_{r=0}^{m}w_r^{(1-\alpha)}(-1)^{-r}+4N_\beta\sin^2\left(\frac{q\Delta x}{2}\right)\sum_{r=0}^{m}\left[\lambda w_r^{(1-\beta)}+(1-\lambda)w_{r+1}^{(1-\beta)}\right](-1)^{-r}\leqslant 0.$$

Put $\theta=N_\beta\sin^2\left(\frac{q\Delta x}{2}\right)$, we find

$$-2-4(1-\lambda)\theta+\mu N_\alpha\sum_{r=0}^{m}w_r^{(1-\alpha)}(-1)^{-r}+4\theta\sum_{r=0}^{m}\left[\lambda w_r^{(1-\beta)}+(1-\lambda)w_{r+1}^{(1-\beta)}\right](-1)^{-r}\leqslant 0, \quad (29)$$

which can be written in the form

$$-2-4(1-\lambda)\theta+\mu N_\alpha\sum_{r=0}^{m}w_r^{(1-\alpha)}(-1)^{-r}+4\theta\left[(1-2\lambda)\sum_{r=1}^{m}(-1)^{r-1}w_r^{(1-\beta)}+\lambda+(-1)^m(1-\lambda)w_{m+1}^{(1-\beta)}\right]\leqslant 0.$$

Put

$$\frac{1}{M_m}=\frac{4\left\{(2\lambda-1)\left[1-\sum_{r=1}^{m}(-1)^{r-1}w_r^{(1-\beta)}\right]+(-1)^m(1-\lambda)w_{m+1}^{(1-\beta)}\right\}}{2-\mu N_\alpha\sum_{r=0}^{m}w_r^{(1-\alpha)}(-1)^{-r}}, \quad (30)$$

one finds that the mode is stable when

$$\frac{1}{\theta}\geqslant\frac{1}{M_m}. \quad (31)$$

Although, M_m depends on m, it turns out that M_m tends toward its limit value

$$\frac{1}{M}=\lim_{m\to\infty}\frac{1}{M_m}. \quad (32)$$

In this limit the stability condition is

$$\frac{1}{\theta}\geqslant\frac{1}{M}=\frac{4\left\{(2\lambda-1)\left[1-\sum_{r=1}^{\infty}(-1)^{r-1}w_r^{(1-\beta)}\right]+\lim_{m\to\infty}(-1)^m(1-\lambda)w_{m+1}^{(1-\beta)}\right\}}{2-\mu N_\alpha\sum_{r=0}^{\infty}w_r^{(1-\alpha)}(-1)^{-r}}, \quad (33)$$

but from Eqs. (10) and (11) with $z=-1$ one sees that $\sum_{r=0}^{\infty}(-1)^r w_r^{(1-\gamma)}=2^{1-\gamma}$, so that

$$\frac{1}{M}=\frac{4\left[(2\lambda-1)2^{1-\beta}+\lim_{m\to\infty}(-1)^m(1-\lambda)w_{m+1}^{(1-\beta)}\right]}{2-\mu N_\alpha 2^{1-\alpha}}, \quad (34)$$

since $\theta=N_\beta\sin^2\left(\frac{q\Delta x}{2}\right)$, replacing $\sin^2\left(\frac{q\Delta x}{2}\right)$ by its highest value and since $\lim_{m\to\infty}(-1)^m(1-\lambda)w_{m+1}^{(1-\beta)}=0$, therefore we find that the sufficient condition for the present method to be stable and this completes the proof of the theorem.

Remark 1. For $\frac{1}{2}<\lambda\leqslant 1$, the stability condition (19) can be satisfied under specific values of $N_\beta=\frac{(\Delta t)^\beta}{(\Delta x)^2}$. We can check this note from the results which presented in Table 1.

Numerical results

In this section, we present two numerical examples to illustrate the efficiency and the validation of the proposed numerical method when applied to solve numerically the fractional Cable equation.

Example 1. Consider the following initial-boundary problem of the fractional Cable equation

$$u_t(x,t) = D_t^{1-\beta}u_{xx}(x,t) - D_t^{1-\alpha}u(x,t) + f(x,t), \tag{35}$$

on a finite domain $0 < x < 1$, with $0 \leqslant t \leqslant T$, $0 < \alpha, \beta < 1$ and the following source term

$$f(x,t) = 2\left(t + \frac{\pi^2 t^{\beta+1}}{\Gamma(2+\beta)} + \frac{t^{\alpha+1}}{\Gamma(2+\alpha)}\right)\sin(\pi x), \tag{36}$$

with the boundary conditions $u(0,t) = u(1,t) = 0$, and the initial condition $u(x,0) = 0$.

The exact solution of Eq. (35) is $u(x,t) = t^2\sin(\pi x)$.

The behavior of the exact solution and the numerical solution of the proposed fractional Cable Eq. (35) by means of the FWA–FDM with different values of λ, α, β, Δt, Δx and the final time T are presented in Figs. 1–5.

In Table 1, we presented the behavior of the absolute error between the exact solution and the numerical solution of Eq. (35) at $\lambda = 1, \alpha = 0.9, \beta = 0.9, \Delta x = \frac{1}{10}, \Delta t = \frac{1}{3000}$ and $T = 0.01$.

Also, in Table 2, we presented the maximum error of the numerical solution for $\lambda = 0, \alpha = 0.2$, $\beta = 0.7$, $T = 0.1$ with different values of Δx and Δt.

Example 2. Consider the following initial-boundary problem of the fractional Cable equation

$$u_t(x,t) = D_t^{1-\beta}u_{xx}(x,t) - 0.5D_t^{1-\alpha}u(x,t), \quad 0 < x < 10,\ 0 < t \leqslant T, \tag{37}$$

with $u(0,t) = u(10,t) = 0$ and $u(x,0) = 10\delta(x-5)$, where $\delta(x)$ is the Dirac delta function.

The numerical solutions of this example are presented in Figs. 6–10 for different values of the parameters λ, α, β, Δx, Δt and the final time T.

Conclusion and remarks

This paper presented a class of numerical methods for solving the fractional Cable equations. This class of methods is very close to the weighted average finite difference method. Special attention is given to study the stability of the FWA-FDM. To execute this aim, we have resorted to the kind of fractional John von Neumann stability analysis. From the theoretical study, we can conclude that this procedure is suitable and leads to very good predictions for the stability bounds. The presented stability of the fractional weighed average finite difference scheme depends strongly on the value of the weighting parameter λ. Numerical solutions and exact solutions of the proposed problem are compared and the derived stability condition is checked numerically. From this comparison, we can conclude that the numerical solutions are in excellent agreement with the exact solutions. All computations in this paper are running using Matlab programming 8.

Conflict of interest

The authors have declared no conflict of interest.

References

[1] Henry BI, Langlands TAM, Wearne SL. Fractional Cable models for spiny neuronal dendrites. Phys Rev Lett 2008; 100(12):4 [Article ID 128103].

[2] Hilfer R. Applications of Fractional Calculus in Physics. Singapore: World Scientific; 2000.

[3] Metzler R, Klafter J. The random walk's guide to anomalous diffusion: a fractional dynamics approach. Phys Rep 2000;339:1–77.

[4] Kilbas AA, Srivastava HM, Trujillo JJ. Theory and Applications of Fractional Differential Equations. San Diego: Elsevier; 2006.

[5] Lubich C. Discretized fractional calculus. SIAM J Math Anal 1986;17:704–19.

[6] Miller KS, Ross B. An Introduction to the Fractional Calculus and Fractional Differential Equations. John Wiley and Sons; 1993.

[7] Oldham KB, Spanier J. Fractional Calculus: Theory and Applications, Differentiation and Integration to Arbitrary Order. New York: Academic Press; 1974.

[8] Sweilam NH, Khader MM, Al-Bar RF. Numerical studies for a multi-order fractional differential equation. Phys Lett A 2007; 371:26–33.

[9] Sweilam NH, Khader MM, Nagy AM. Numerical solution of two-sided space-fractional wave equation using finite difference method. J Comput Appl Math 2011;235:2832–41.

[10] Sweilam NH, Khader MM, Adel M. On the stability analysis of weighted average finite difference methods for fractional wave equation. J Fract Differential Cal 2012;2(2):17–25.

[11] Sweilam NH, Khader MM, Mahdy AMS. Crank–Nicolson finite difference method for solving time-fractional diffusion equation. J Fract Cal Appl 2012;2(2):1–9.

[12] Sweilam NH, Khader MM. A Chebyshev pseudo-spectral method for solving fractional order integro-differential equations. ANZIAM 2010;51:464–75.

[13] Yuste SB, Acedo L. An explicit finite difference method and a new von Neumann-type stability analysis for fractional diffusion equations. SIAM J Numer Anal 2005;42:1862–74.

[14] Bagley RL, Calico RA. Fractional-order state equations for the control of viscoelastic damped structures. J Guid Control Dyn 1999;14(2):304–11.

[15] Podlubny I. Fractional Differential Equations. San Diego: Academic Press; 1999.

[16] Benson DA, Wheatcraft SW, Meerschaert MM. The fractional-order governing equation of Le′vy motion. Water Resour Res 2000;36(6):1413–24.

[17] Cuesta E, Finat J. Image processing by means of a linear integro-differential equations. IASTED 2003:438–42.

[18] Liu F, Anh V, Turner I, Zhuang P. Time fractional advection–dispersion equation. Appl Math Comput 2003:233–46.

[19] Langlands TAM, Henry BI, Wearne SL. Fractional Cable equation models for anomalous electrodiffusion in nerve cells: infinite domain solutions. J Math Biol 2009;59(6):761–808.

[20] Liu F, Yang Q, Turner I. Stability and convergence of two new implicit numerical methods for the fractional Cable equation. J Comput Nonlinear Dyn 2011;6(1):7 [Article ID 01109].

[21] Quintana-Murillo J, Yuste SB. An explicit numerical method for the fractional Cable equation. Int J Differential Equations 2011;2011:12 [Article ID 231920].

[22] Rall W. Core conductor theory and Cable properties of neurons. In: Poeter R, editor. Handbook of physiology:the nervous system, vol. 1. Bethesda (MD): American Physiological Society; 1977. p. 39–97 [chapter 3].

[23] Morton KW, Mayers DF. Numerical solution of partial differential equations. Cambridge University Press; 1994.

[24] Palencia E, Cuesta E. A numerical method for an integro-differential equation with memory in Banach spaces: qualitative properties. SIAM J Numer Anal 2003;41:1232–41.