Normal Solvability of Elliptic Boundary Value Problems on Asymptotically Flat Manifolds

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Normal solvability is shown for a class of boundary value problems on Riemannian manifolds with noncompact boundary using a concept of weighted pseudodifferential operators and weighted Sobolev spaces together with Lopatinski-Shapiro type boundary conditions. An essential step is to show that the standard normal derivative defined in terms of the Riemannian metric is in fact a weighted pseudodifferential operator of the considered class provided the metric is compatible with the symbols. © 1992 Academic Press, Inc.

INTRODUCTION AND REVIEW OF THE $SG$-CALCULUS

It is well known that an elliptic pseudodifferential operator of order $m$,

$$P : H^m(X) \to L^2(X),$$
defined on the usual Sobolev spaces over a compact manifold $X$ is a Fredholm operator. Moreover, localization and parametrix construction furnish a Fredholm inverse which is a pseudodifferential operator, of order $-m$. If $P$ is invertible, then $P^{-1}$ is again pseudodifferential. For related topics see [G, S, U].

Another familiar case is that of a differential operator of order $2m$ in a bounded domain in $\mathbb{R}^n$ with smooth boundary, together with $m$ boundary

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operators \( B^1, \ldots, B^m \). A standard result (cf. Wloka \([W, \text{Hauptsatz 13.1}], \) or \([\text{LOP, SHA}]\)) says that the system \((P, B^1, \ldots, B^m)\) is a Fredholm operator between the right Sobolev spaces if and only if \(P\) is elliptic and \((P, B^1, \ldots, B^m)\) satisfies a condition of Lopatinski and Shapiro that can be expressed in various equivalent ways.

This result also holds for certain classical pseudodifferential boundary value problems, cf. Seeley \([\text{SEE, p. 279}]\). Moreover, one can construct the Boutet de Monvel algebra and again gets the Fredholm property linked to a Lopatinski–Shapiro type condition \([\text{RS, Sect. 3.1.1}], \) and a Fredholm inverse via a parametrix within the class \([\text{RS, Sect. 3.1.1}], \) in \([\text{GRU, Sects. 3.1, 3.2}]\) even for additional parameter dependence.

Things change substantially for noncompact manifolds.

\[ A : H_2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n) \]

is not Fredholm, and even the simple elliptic boundary value problem

\[ (A, \gamma_1) : H_2(\mathbb{R}_+) \rightarrow L^2(\mathbb{R}_+) \oplus \mathbb{C}, \]

\( \gamma_1 f = f'(0) \), has infinite codimensional, nonclosed range, so is not Fredholm. This shows that one will have to use different methods in order to recover Fredholm results for the noncompact case.

Cordes \([C]\), cf. \([\text{Pf1, SHU}]\), considered a space of weighted symbols on \(\mathbb{R}^n\). They are "globally" defined \(C^\infty\) functions on \(\mathbb{R}^n \times \mathbb{R}^n\) with double order \(m = (m_1, m_2) \in \mathbb{R} \times \mathbb{R}: SG^m(\mathbb{R}^n)\) consisting of all functions such that

\[
D_\xi^a D_\eta^b a(x, \xi) = O(\langle \xi \rangle^{a_1} \langle x \rangle^{b_1}),
\]

where \(\langle x \rangle = (1 + |x|^2)^{1/2}\). For the associated pseudodifferential operators one has a calculus closed under compositions and adjoints and the notion of an asymptotic expansion. They naturally act on the weighted Sobolev spaces

\[
H_m(\mathbb{R}^n) = \{ u \in \mathcal{S}'(\mathbb{R}^n) : \langle D \rangle^{m_1} \langle x \rangle^{m_2} u \in L^2(\mathbb{R}^n) \},
\]

\(m = (m_1, m_2)\). The class \(\mathcal{K}\) of of regularizing elements in the \(SG\)-calculus—those with symbols in the intersection of all the \(SG^m\) spaces—coincides with the set of all integral operators with kernel functions in \(\mathcal{S}(\mathbb{R}^{2n})\). The embedding \(H_{\mu}(\mathbb{R}^n) \subseteq H_m(\mathbb{R}^n)\) is compact, provided \(\mu_1 > m_1\) and \(\mu_2 > m_2\). This allows a parametrix construction modulo \(\mathcal{K}\) and Fredholm results for operators with "md-elliptic" symbols, i.e., symbols \(a \in SG^m(\mathbb{R}^n)\) such that \(a(x, \xi)\) is invertible for large \(|x| + |\xi|\) and

\[
[a(x, \xi)]^{-1} = O(\langle \xi \rangle^{-m_1} \langle x \rangle^{-m_2}).
\]
A typical example for an \( md \)-elliptic operator on \( \mathbb{R}^n \) is \( 1 - \Delta \) with the symbol \( \langle \xi \rangle^2 \in SG^{(2, 0)}(\mathbb{R}^n) \). In fact, \( 1 - \Delta : H^{(2, 0)}(\mathbb{R}^n) \to L^2(\mathbb{R}^n) \) is an isomorphism. This was extended to \( md \)-elliptic boundary value problems on \( \mathbb{R}^{n+1}_+ \) by Cordes and Erkip [CE, E, EH, ERN]. For related results on \( \mathbb{R}^{n+1}_+ \) also see Eskin’s book [ESK] and [P1, P2].

A different approach was taken by Lockhart and McOwen. They considered manifolds with finitely many ends (for a definition cf. Example 1.2(e)). Making extensive use of the particular coordinates suggested by the manifold they studied perturbations of so-called “translation invariant” (classically) elliptic differential operators on weighted Sobolev spaces. The Fredholm property was shown to depend on the choice of the weight. The technique also applied to boundary value problems of similar type, the boundary being the (compact) cross-section on the end ([LOM, LO]; cf. also [LO, p. 2] for a motivation of weighted spaces).

Here, we want to give a coordinate invariant approach for certain Riemannian manifolds with boundary. It differs from other concepts in this direction, say Melrose’s [M], by taking additionally advantage of the global effects due to the weighted symbol class techniques. Examples treated here include cases where the manifold or its boundary has finitely many ends, cf. also [SCL].

The basic idea is to use finitely many coordinate charts only. In addition, the changes of coordinates are subject to restrictive conditions on their derivatives. Then one introduces weighted symbols and Sobolev spaces.

It was shown in [SCS] that the above concept of \( SG \) symbol classes on \( \mathbb{R}^n \) transfers to all \( n \)-dimensional manifolds (without boundary) where the following holds (“\( SG \)-compatible manifolds”)

\[
\begin{align*}
\text{(SG1)} & \quad \text{the manifold has a finite atlas} \\
\text{(SG2)} & \quad \text{the atlas has a good shrinking} \\
\text{(SG3)} & \quad \text{all the changes of coordinates } \chi \text{ satisfy } D_x^2 \chi(x) = O(\langle x \rangle^{1-|\alpha|}).
\end{align*}
\]

The existence of a “good shrinking” means that one can cover the manifold, say \( \Omega \), by open sets in two ways

\[
\Omega = \bigcup_{j=1}^J \Omega_j = \bigcup_{j=1}^J \Omega_j', \quad \Omega_j \subseteq \Omega_j'.
\]

such that the coordinate charts map \( \Omega_j', \Omega_j \) onto open sets \( U_j', U_j \) in \( \mathbb{R}^n \), and there is a fixed positive \( \varepsilon_\Omega \) such that for each \( x \in U_j' \),

\[
B(x, \varepsilon_\Omega \langle x \rangle) \subseteq U_j.
\]

Moreover, it was shown in [SCS] that there is a partition of unity \( \{ \varphi_1, \ldots, \varphi_J \} \) and cut-off functions \( \{ \psi_1, \ldots, \psi_J \} \) with \( \varphi_j, \psi_j \) supported in \( \Omega_j' \), \( \varphi_j \psi_j = \varphi_j \), and

\[
D_x^2 \varphi_j(x), D_x^2 \psi_j(x) = O(\langle x \rangle^{-|\alpha|}).
\]
One obtains the classes $SG^m(\Omega)$ by asking that condition (0.1) be satisfied in local coordinates, and similarly the spaces $H^m(\Omega)$ via the partition of unity. $SG$-compatible manifolds include the compact manifolds, the Euclidean space $\mathbb{R}^n$, and manifolds with finitely many cylindrical ends, cf. [SCS]. A less elementary example is the "infinite-holed torus," cf. [SPI, p. 1-29], a two-dimensional surface in $\mathbb{R}^3$ with a countable number of holes:

![Infinite-holed torus diagram]

What makes this set-up particularly useful is that one can apply many techniques from the compact manifold case. In [SCP] for example, the existence of complex powers, zeta and eta functions was shown for elliptic operators of positive order $m = (m_1, m_2)$, $m_1, m_2 > 0$.

$SG$-compatible manifolds with boundary will be introduced in Section 1: take those (embedded) submanifolds of $SG$-compatible ones where all coordinate charts map the interior exactly to $\mathbb{R}^{n+1}_+$ and the boundary exactly to $\partial \mathbb{R}^{n+1}_+$. Weighted Sobolev spaces are obtained by restriction of the corresponding spaces on the full domain.

It is obvious that such a concept only makes sense, if it is compatible with the "usual" one: we would like to have the standard notions and methods within the new framework.

One of these essential notions on manifolds with boundary is that of the normal derivative. It is the actual analytical problem in Section 2 to show that the normal derivative in fact is a differential operator with a symbol satisfying the estimates in (0.1).

The normal derivative is given in a neighborhood of the boundary as the differential operator induced by the vector field of tangent vectors to the geodesics starting at the boundary with derivative equal to the unit normal vector. Clearly, it will require additional assumptions on the metric to make this an $SG$-operator. It turns out that it is sufficient to ask growth conditions of the type (0.1) for the Christoffel symbols connected to the metric and for the normal vector at the boundary. The assumptions imply that, in a suitable neighborhood of the boundary, the Levi-Civita connection and all its derivatives tend to zero as $|x| \to \infty$. In this sense the manifold is asymptotically flat near the boundary.

There is an interesting connection to differential geometry: On an $SG$-compatible manifold, one trivially has a metric of the above type. This follows from the existence of a partition of unity with the properties (0.4). The curvature tensor for such a metric decays like $|x|^{-2-\varepsilon}$ near infinity. If the decay were only slightly stronger, say $|x|^{-2-\varepsilon}$ for some $\varepsilon > 0$, then a result
of Abresch [A, Main Theorem] shows that, in dimension 2, the manifold has finite Betti numbers.

This underlines the importance of an example like the “infinite-holed torus. More details on manifolds of this linear periodic type will be treated in a forthcoming paper [SCL].

An application of the calculus gives normal solvability for a family of boundary value problems: Calling the manifold $X$ and the boundary $Y$ consider

$$Pu = f \quad \text{in } X,$$

$$B^k u \mid_Y = g^k, \quad k = 1, \ldots, m$$

(BVP)

where $P$, $B^1$, ..., $B^m$ are differential operators with $SG$-symbols, and $f$, $g^1$, ..., $g^m$ are given data in weighted Sobolev spaces over $X$ and $Y$, respectively. The number of boundary conditions imposed is—as usually—the number of zeroes the principal symbol of $P$ has in the upper half plane when considered a polynomial in the conormal variable. In Section 3, we show the Fredholm property for the boundary value problem (BVP), provided $P$ is $md$-elliptic in the interior and the residues of the principal symbols of the boundary operators $B^k$ modulo the part of the principal symbol of $P$ containing the zeroes with positive imaginary part satisfy a uniform version of the Lopatinski–Shapiro condition at the boundary.

It seems that the class of these operators is closed under a Fréchet topology like that introduced by Guillemin [GUI, p.141] or Schulze [SCU].

The proof uses a combination of techniques of Hörmander [H, Sect.20.1, HOR, Chap.II] and Erkip [EH, ERN]. The basic idea is the construction of the Calderon projector, reduction to the boundary, and localization so that the half space methods of [ERN] can be applied.

Changing the set-up slightly, it is possible to define the Boutet de Monvel algebra also for weighted operators and to obtain similar results on manifolds with noncompact boundaries, cf. [SCB].

1. $SG$-COMPATIBLE MANIFOLDS WITH BOUNDARY:
   DEFINITIONS, EXAMPLES, BASIC RESULTS

   Write $\mathbb{R}^{n+1} = \{x = (t, x') : t, x \in \mathbb{R}, x' = (x_2, \ldots, x_{n+1})\}$, $\mathbb{R}_{+}^{n+1} = \{(t, x') : t > 0\}$, $\mathbb{R}_{-}^{n+1} = \{(t, x') : t < 0\}$, $\partial \mathbb{R}_{+}^{n+1} = \{(t, x') : t = 0\}$.

   Suppose $\Omega$ is an $(n+1)$-dimensional $SG$-compatible manifold without boundary as in the Introduction. To fix notation, suppose $\Omega = \bigcup_{j=1}^{m} \Omega_j = \bigcup_{j=1}^{m} \Omega_j'$, with open sets $\Omega_j \subseteq \Omega_j'$ and coordinate homeomorphisms $\chi_{j} : \Omega_j \rightarrow U_j$, $\chi_j' : \Omega_j' \rightarrow U_j'$, $U_j, U_j' \subseteq \mathbb{R}^{n+1}$, such that the axioms $(SG1)$–$(SG3)$ are satisfied.
1.1. DEFINITION. $X$ is an $(n+1)$-dimensional $SG$-compatible manifold with boundary $Y$ in $\Omega$, if the following holds:

(i) There is an $(n+1)$-dimensional $SG$-compatible manifold $\Omega$ with $X$ an embedded $(n+1)$-dimensional submanifold of $\Omega$, $Y$ an embedded $n$-dimensional submanifold of $\Omega$, and $Y$ is the boundary of $X$ in $\Omega$.

(ii) $\chi_j(X \cap \Omega_j) \subseteq \mathbb{R}^{n+1}_+; \chi_j(\Omega_j \setminus (X \cup Y)) \subseteq \mathbb{R}^{n+1}_-; \chi_j(Y \cap \Omega_j) \subseteq \partial \mathbb{R}^{n+1}_+.$

1.2. EXAMPLES. In the following cases, $X$ is an $SG$-compatible manifold with boundary $Y$ in $\Omega$.

(a) $X$ is any compact manifold with boundary $Y$ in the usual sense, $\Omega$ is the “double” of $X$.

(b) $X = \mathbb{H}^{n+1}_+, Y = \partial \mathbb{H}^{n+1}_+, \Omega = \mathbb{H}^{n+1}.$

(c) $X = \{(x, y) \in \mathbb{R}^2 : x > 0, y > 1/x\}, Y = \{(x, y) \in \mathbb{R}^2, x > 0, y = 1/x\}, \Omega = \mathbb{R}^2.$

(d) (Noncompact Manifold with Compact Boundary) Suppose $\Omega$ is any $(n+1)$-dimensional $SG$-compatible manifold, and there is a real-valued $C^\infty$ function $\omega$ on $\Omega$ with nowhere vanishing differential such that $Y = \{x \in \Omega : \omega(x) = 0\}$ is compact. Then let $X = \{x \in \Omega : \omega(x) > 0\}.$

(e) Suppose $Y$ is an $n$-dimensional manifold with finitely many cylindrical ends, then $X = Y \times \mathbb{R}_+$ is an $SG$-compatible manifold with boundary $Y$ in $\Omega = Y \times \mathbb{R}$.

A $n$-dimensional manifold with finitely many cylindrical ends is one that outside a compact set consists of finitely many components $L_i \cong c_j \times (1, \infty)$, and each $c_j$ is a connected compact $(n-1)$-dimensional submanifold of $Y$.

(f) (Noncompact Boundary, Finitely Many Ends) Suppose $\Omega$ is an $(n+1)$-dimensional manifold with finitely many cylindrical ends, i.e., outside some compact set each of the finitely many components of $\Omega$ may be identified with $c_j \times (1, \infty)$, $c_j n$-dimensional, as above. Now suppose $c_j$ has an $n$-dimensional connected submanifold $d_j$ with boundary $e_j$. If $X$ is an $(n+1)$-dimensional submanifold with boundary in $\Omega$ and, outside the above compact set, $X$ is the subset of $\Omega$ identified with $d_j \times (1, \infty)$ and $Y$ is the subset identified with $e_j \times (1, \infty)$, then $X$ is $SG$-compatible with boundary $Y$ in $\Omega$.

(g) (Interior of Certain Manifolds with Finitely Many Cylindrical Ends) Let $\Omega = \mathbb{R}^{n+1}, X$ a connected open subset of $\mathbb{R}^{n+1}$ with boundary $Y$. Moreover suppose that outside some compact set, $X$ consists of finitely many components $L_j \cong C_j \times (1, \infty)$, where each $C_j \subseteq X$ is an $n$-dimensional submanifold of $\mathbb{R}^{n+1}$ with boundary $c_j$. Then $X$ is $SG$-compatible with boundary $Y$ in $\Omega = \mathbb{R}^{n+1}.$
(More precisely assume there is a diffeomorphism \( \varphi \) mapping a neighborhood of \( C_j \) in \( \mathbb{R}^{n+1} \) to a neighborhood of the origin in \( \mathbb{R}^{n+1} \) with \( \varphi(C_j \cup c_j) \subseteq \partial \mathbb{R}^{n+1} \), and \( \varphi(C_j) \) open in \( \partial \mathbb{R}^{n+1} \) with boundary \( \varphi(c_j) \).

**Proof.** Cases (a) and (b) are obvious. For the rest see the Appendix.

1.3. **Corollary.** It follows from Definition 1.1 that \( Y \) is an SG-compatible manifold of dimension \( n \).

1.4. **Definition (Weighted Sobolev Spaces on \( X \)).** For \( s \in \mathbb{R}^2 \) let

\[
H^+_s(X) = \{ u \in \mathcal{D}'(X) : \text{There is } v \in H_s(\Omega) \text{ such that } v = u \text{ on } X \}.
\]

Equip \( H^+_s(X) \) with the norm \( \| u \|_s^+ = \inf \{ \| v \|_s : v \in H_s(\Omega) \text{ and } v = u \text{ on } X \} \).

1.5. **Theorem.** \( H^+_s(X) \) is a Hilbert space. For \( \sigma_1 > s_1, \sigma_2 > s_2 \), \( H^+_{\sigma_1}(X) \) is compact in \( H^+_{\sigma_2}(X) \).

In fact, using the partition of unity \{ \varphi_1, \ldots, \varphi_J \} subordinate to the cover \{ \Omega_j \} and considering the distributions in local coordinates (indicated by the subscript \( j \)) one can show that the norm on \( H^+_s(X) \) is equivalent to

\[
\| u \|_*^s = \left( \sum (\varphi_j u)_{s_j}^{s_j^2} \right)^{1/2},
\]

with the norms on the right hand side the standard weighted half-space Sobolev norms on \( \mathbb{R}^{n+1}_+ \), cf. Erkip [ERN] (or Lions and Magenes [LM], if one replaces conditions (7.10) and (7.11) in [LM] by the conditions in Definition 1.1 of this text). The assertion then is a corollary of the corresponding result for \( X = \mathbb{R}^{n+1}_+ \).

2. **Normal Derivatives**

Given a manifold \( X \) with boundary \( Y \), it often is an important simplification to identify a neighborhood of \( Y \) in \( X \) with the collar \( Y \times [0, 1) \). This corresponds to a change of coordinates making the normal derivative an ordinary derivative with respect to a new variable.

On SG-compatible manifolds changes of coordinates are subject to growth conditions on their derivatives and shrinking conditions on their domains. It will be shown in this section that under additional assumptions on the metric (cf. 2.6 and 2.9) one can find a change of coordinates preserving the SG-compatible structure and making the geometrically defined normal derivative an ordinary derivative.
For SG-compatible manifolds one has to replace the concept of \( \varepsilon \)-neighborhoods by that of conic \( \varepsilon \)-neighborhoods: The idea is to have a ball of radius \( \varepsilon \langle x \rangle \) (instead of radius \( \varepsilon \)) about each \( x \) contained in the neighborhood. Formally, given \( x \in \Omega \) and \( \varepsilon > 0 \), define the ball of radius \( \varepsilon \langle x \rangle \) about \( x \) via local coordinates:

\[
B(x, \varepsilon \langle x \rangle) = \bigcup_{j=1}^{J} \chi_j^{-1}(B(\chi_j(x), \varepsilon \langle \chi_j(x) \rangle)).
\]

Due to the existence of \( \varepsilon_\Omega \) (cf. condition (SG2)) this is an open set for \( \varepsilon \leq \varepsilon_\Omega \).

2.1. DEFINITION. For a subset \( M \) of \( \Omega \), \( \varepsilon > 0 \) small, denote by \( M_{\text{con; } \varepsilon} \) (conic \( \varepsilon \)-neighborhood) the union of all \( B(x, \varepsilon \langle x \rangle) \), \( x \in M \). Call a neighborhood conic, if it contains a conic \( \varepsilon \)-neighborhood.

2.2. DEFINITION. (a) A linear operator \( A: C^\infty_c(X) \to C^\infty(X) \) is a pseudodifferential operator on \( X \) of order \( \leq m \), \( m = (m_1, m_2) \), if there is a pseudodifferential operator \( A^* \in SG^m(\Omega) \), such that \( rA^*e = A \), with \( r \) and \( e \) denoting restriction and extension by zero, respectively. In most cases, we will write \( A \) for both, \( A \) and \( A^* \).

(b) The operator \( A \in SG^m(X) \) is called \( md \)-elliptic of order \( m \) on \( X \), if there is a conic neighborhood of \( X \) in \( \Omega \) such that the symbol \( a \) of \( A \) is \( md \)-elliptic there, i.e., \( a(x, \xi) \) is invertible for large \( |x| + |\xi| \) and satisfies \( a(x, \xi)^{-1} = O(\langle \xi \rangle^{-m_1} \langle x \rangle^{-m_2}) \).

2.3 (cf. [SWS, Corollary 3.8 and Theorem 3.9]). Given a subset \( M \) of \( \Omega \) and \( \varepsilon > 0 \), there is a function \( \Psi \in C^\infty(\Omega) \) supported in \( M_{\text{con; } \varepsilon} \) and equal to 1 in a conic neighborhood of \( M \); moreover, \( \Psi \) can be chosen such that its derivatives satisfy

\[
D^x \Psi(x) = O(\langle x \rangle^{-|x|})
\]

for all multi-indices \( x \).

2.4. The \( md \)-ellipticity of \( A \) on \( X \) implies the existence of a parametrix \( B \) "on \( X \)"; There is a pseudodifferential operator \( B \) on \( \Omega \) such that \( \Psi(BA - Id) \theta \) is a regularizing operator for all functions \( \Psi \), \( \theta \) supported in the conic neighborhood of \( X \), where the symbol of \( A \) is invertible (with the standard conditions on the derivatives of \( \Psi \) and \( \theta \), cf. 2.3).

The restrictions of \( md \)-elliptic operators on \( \Omega \) are \( md \)-elliptic on \( X \).

The following statement is obvious:
2.5. Definition and Lemma. On $\Omega$ define the function

$$\lambda(x) = \sum_{j=1}^{t} \varphi_j(x) \langle \chi_j(x) \rangle$$

with the partition of unity $\{\varphi_j\}$ and the coordinate charts $\chi_j$. Then for $t \in \mathbb{R}$, the function $\lambda'$ is $md$-elliptic on $\Omega$ of order $(0, t)$, and multiplication with $\lambda'$ is an isomorphism from $H_s(\Omega)$ to $H_{s-(0, t)}(\Omega)$ for all $s \in \mathbb{R}^2$.

2.6. The Metric: Strong Assumptions. On $\Omega$ choose a Riemannian metric $g$. Denote by $\nabla$ the Levi-Civita connection. Assume for the moment that, in local coordinates, $g = ((g_{ij}))$ and

$$D^a g_{ij}(x) = O(\langle x \rangle^{-|a|}) \quad (2.1a)$$

and

$$[g_{ij}(x)]^{-1} = O(1). \quad (2.1b)$$

Weaker assumptions on the metric will suffice for our purpose, cf. 2.10. However, it should be noted that these strong assumptions can always be met on an $SG$-compatible manifold:

2.7. Example and Lemma. Suppose $\Omega$ is $SG$-compatible. Then there is a Riemannian metric on $\Omega$ such that (2.1a), (2.1b) hold: In all local coordinate systems take the Euclidean metrics and patch them together with a partition of unity satisfying (0.4).

One might define Sobolev spaces $H^s_{(s, t)}$ associated with the measure $(\det g)^{-1/2}$ connected with the metric. But because of properties (2.1.a) and (2.1.b) one has $(\det g)^{1/2} = O(1)$, $(\det g)^{-1/2} = O(1)$, and one can identify $H^s_{(s, t)}(\Omega) \cong H_{(s, t)}(\Omega)$, so that this does not lead to something new.

2.8. Corollary. Estimates (2.1.a) and (2.1.b) and the formula

$$\sum_{l=1}^{n+1} g_{ik} \Gamma^l_{ij} = \frac{1}{2}(\partial g_{ik}/\partial x_i + \partial g_{ki}/\partial x_j - \partial g_{ij}/\partial x_k)$$

(cf. [GKM, p. 84]) imply that $D^a \Gamma^k_{ij}(x) = O(\langle x \rangle^{-1-|a|})$ in local coordinates. In particular, $\Gamma$ tends to zero as $|x| \to \infty$: in this sense the manifold is asymptotically flat.

We will now study the vector field induced in a neighborhood of $Y$ by the tangent vectors to the geodesics starting in $Y$ with the unit normal vectors to $Y$ as initial tangents. The corresponding differential operator is the normal derivative.
2.9. **Lemma.** Let $n$ be the inward unit normal field at $Y$. Then in local coordinates

$$D^2n(x') = O(\langle x' \rangle^{-|x'|})$$

and

$$n_1(x') \geq c_0$$

for some constant $c_0 > 0$ and $n_1$ the first component of the vector $n$.

Here we are using the notation of Definition 1.1, writing $x = (t, x')$ and identifying the point $(0, x') \in \mathbb{R}^{n+1}$ with the point $x' \in \mathbb{R}^n$.

**Proof.** In local coordinates, $g$ is a matrix, and the inward normal vector is defined by

$$\langle e_k, g(0, x') n(x') \rangle = 0, \quad k = 2, \ldots, n+1,$$

$$n_1(x') > 0,$$

$$1 = |n(x')|_g = \langle n(x'), g(0, x') n(x') \rangle,$$

where $e_k$ is the $k$th unit vector, and $\langle \cdot, \cdot \rangle$ is the scalar product in $\mathbb{R}^{n+1}$.

Let $h$ denote the matrix obtained by replacing the first row in $g$ by $e_1$. Then $h$ also satisfies the estimates (2.1.a), (2.1.b). Define $n'(x') = h^{-1}(0, x') e_1$.

This immediately yields $n_1'(x') = 1$, $\langle e_k, g(0, x') n'(x') \rangle = \langle e_k, h(0, x') n'(x') \rangle = 0, \quad k = 2, \ldots, n+1$, $D^2n'(x') = O(\langle x' \rangle^{-|x'|})$, and

$$\langle n'(x'), g(0, x') n'(x') \rangle = \|g^{1/2}(0, x') n'(x')\|^2$$

$$\geq \{\|g^{-1/2}(0, x')\|^{-1} \|n'(x')\|^2\}^2 \geq \text{const.} > 0,$$

where the norms are taken in $\mathbb{R}^{n+1}$. We have used that $g$ is positive symmetric and estimates (2.1.a), (2.1.b). Then the normal field is

$$n(x') = \{n'(x') g(0, x') n'(x')\}^{-1/2} n'(x').$$

2.10. **Reduced Hypotheses on the Metric.** From now on, we shall only assume that, for the metric on $\Omega$, the results of Corollary 2.8 and Lemma 2.9 hold, i.e.,

(i) $D^2\Gamma^k_{\theta}(x) = O(\langle x \rangle^{-1-|x'|})$, $x \in \Omega$

(it suffices to ask this for $x$ in a conic neighborhood of the boundary), and

(ii) $D^2n(x') = O(\langle x' \rangle^{-|x'|})$ and $n_1(x') \geq c_0$, $x' \in Y$,

for some constant $c_0 > 0$ and $n_1$ the first component of $n$. 
2.11. Definition. Let $c$ denote the geodesic flow induced by the normal field at $Y$. This means (cf. [GKM, p. 56]) that $c$ satisfies

$$\nabla_c \dot{c} = 0; \quad c(0) \in Y; \quad \dot{c}(0) = n(c(0)).$$

2.12. Preparations. In local coordinates, the equations in 2.11 give a non-linear second order ODE

$$DD(x^k \circ c) = \sum_{i,j=1}^{n+1} \Gamma_{ij}^k(x \circ c) D(x^i \circ c) D(x^j \circ c) \quad (2.2.a)$$

$k = 1, \ldots, n+1$, with the coordinate functions $x^i$ and initial values

$$x \circ c(0) = (0, x_0) \quad (2.2.b)$$

$$D(x \circ c(0)) = n(x_0). \quad (2.2.c)$$

We will write the solution of (2.2.a)-(2.2.c) as $c(\tau; x_0)$, indicating the dependence on the initial value.

The concept now is to show that the geodesic equation (2.2.a)-(2.2.c) is solvable for sufficiently long times $\tau$, and that one can make a change of coordinates $x = c(\tau; x_0)$, which satisfies the $SG$-axioms ($SG1$)–($SG3$) of the Introduction.

The following proposition says that, given a starting point $x_0 \in Y$, there exists a solution $c(\tau; x_0)$ for times, $|\tau| < \delta(x_0)$, $\delta > 0$. Moreover, it gives estimates on $c$ and $\dot{c}$.

There arises a minor difficulty. The local equation comes from the manifold and only is defined in the open sets $U_j \subseteq \mathbb{R}^{n+1}$ of Definition 1.1. Therefore, one additionally wants the solution to stay in $U_j$ for these times. A priori we can assume the differential equation to be extended to $\mathbb{R}^{n+1}$ with the properties of the $\Gamma_{ij}^k$ (cf. 2.10) preserved, e.g., by using a cut-off function of the type in 2.3. A posteriori it follows from the estimates on $\dot{c}$ in Proposition 2.13 that the solution will stay in $U_j$ at least for a reduced $\delta$, if we start in a conic $\varepsilon_{\alpha}/2$-neighborhood of $U'_j$ (remember the concept of conic neighborhoods in Definition 2.1 and the existence of a positive $\varepsilon_{\alpha}$ such that $(U'_j)_{con; \varepsilon_{\alpha}}$ is contained in $U_j$).

2.13. Proposition. Let $G_j := (U'_j)_{con; \varepsilon_{\alpha}/2} \cap \{t = 0\}$ with the $\varepsilon_{\alpha}$ of (0.3.b) and $U_j$ of Definition 1.1.

For $x_0 \in G'_j$ the geodesic equation (2.2.a)-(2.2.c) has a solution $c(\tau; x_0)$ for $\tau \leq \delta(x_0)$ with a $\delta > 0$ independent of $x_0$. During that time

$$c(\tau; x_0) = O(\langle x_0 \rangle) \quad \text{and} \quad \dot{c}(\tau; x_0) = O(1).$$
Moreover, there is a positive constant such that for $x_0$ bounded away from zero
\[ \|c(\tau; x_0)\| \geq \text{const.} \langle x_0 \rangle. \]

Proof. It is sufficient to prove this proposition for large $\|x_0\|$: On a compact set, there is for each $x_0$ a solution for times $|\tau| < \varepsilon = \varepsilon(x_0)$, where $\varepsilon$ can be chosen to depend continuously on $x_0$. Hence there is a uniform $\varepsilon$ on the compact part.

In particular, we may assume that $\|x_0\|$ is so large that
\[ \langle x_0 \rangle \geq \|x_0\| \geq 2\langle x_0 \rangle. \]

By standard ODE theory there is a solution for times $|\tau| < \varepsilon$, $\varepsilon$ depending on $x_0$. The solution will be a $C^\infty$ function of $\tau$, so we can confine ourselves to positive times $\tau$.

Let $c^k(\tau) = x^k \circ c(\tau)$, $k = 1, \ldots, n + 1$, so $c$ is the $(n + 1)$-vector with components $c^k$. Write $r(\tau) = \|c(\tau)\| = (\sum |c^k(\tau)|^2)^{1/2}$ and $v(\tau) = \|\dot{c}(\tau)\| = (\sum |\dot{c}^k(\tau)|^2)^{1/2}$. Then
\[ \dot{r}(\tau) = \partial_r c(\tau) = \langle c(\tau)/\|c(\tau)\|, \dot{c}(\tau) \rangle \]
and
\[ \dot{v}(\tau) = \partial_r \|\dot{c}(\tau)\| = \langle \dot{c}(\tau)/\|\dot{c}(\tau)\|, \dot{c}(\tau) \rangle, \]
hence
\[ |\dot{r}(\tau)| \leq v(\tau) \quad (2.3.a) \]
and
\[ |\dot{v}(\tau)| \leq \|\dot{c}(\tau)\|. \]
Moreover, Eq. (2.2a) together with the fact that $\Gamma_{ij}^k(x) = O(\langle x \rangle^{-1})$, cf. 2.10, implies that
\[ |\dot{v}| \leq \|\dot{c}\| \leq dv^2/r, \quad (2.3b) \]
for some $d > 0$. In addition, 2.10 gives
\[ v(0) = \|\dot{c}(0; x_0)\| = \|n(x_0)\| \leq C_0 \quad \text{for all} \quad x_0 \in G_j. \quad (2.4) \]
At $\tau = 0$, the velocity is nonzero. From the positivity of the metric and the fact that the velocity for the solution of a geodesic equation is a constant in time when measured with respect to the metric, cf. [SPI, Theorem 9.12], one concludes that $v(\tau) \neq 0$ as long as the solution exists. Thus
\[ \frac{r}{v^2} = \frac{\dot{r}}{v} - \frac{\dot{v}^2}{v^2} = O(1) \]
in view of (2.3a) and (2.3b) with a constant independent of $\tau$ and $x_0$. Integrating,
\[
|r(\tau)/v(\tau) - r(0)/v(0)| \leq c_1 \tau.
\]
Restricting the time to $0 \leq \tau \leq \delta \|x_0\|$, $\delta = 1/(2C_0c_1)$, one gets
\[
r(\tau)/v(\tau) \geq (2C_0)^{-1} \|x_0\|, \quad \tau \leq \delta \|x_0\|. \tag{2.5}
\]
Using (2.3a) and (2.3b)
\[
|\dot{v}| \leq dv^2/r \leq 2dC_0 \|x_0\|^{-1} v.
\]
Integrating again,
\[
v(\tau) \leq v(0) \exp(2dC_0 \|x_0\|^{-1} \tau) \leq c_2 \tag{2.6}
\]
and
\[
|r(\tau) - r(0)| = |r(\tau) - \|x_0\| | \leq c_3 \tau \leq c_3 \delta \|x_0\|, \tag{2.7}
\]
where everything holds, provided the solution exists and $0 \leq \tau \leq \delta \|x_0\|$.

On the other hand, $c$ is the solution of an everywhere defined ODE. The only way for it to cease to exist is to blow up. This in turn forces $r$ and $v$ to blow up. By the above estimates, this will not happen for times $0 \leq \tau \leq \delta \|x_0\|$.

Reducing $\delta$, we may replace $\delta \|x_0\|$ by $\delta \langle x_0 \rangle$. Then (2.6) and (2.7) give the statement of the proposition.

2.14. Change of Coordinates. We want to introduce additional charts on $\Omega$ in a neighborhood of $Y$ using the geodesic flow given by Eqs. (2.2.a)-(2.2.c) on $U_j' \subseteq \mathbb{R}^n + 1$. Change coordinates $c(\tau; x_0) = x$, where now $c$ is considered a map
\[
c: \{(\tau, x_0) : x_0 \in G_j, |\tau| < \delta \langle x_0 \rangle \} \rightarrow \mathbb{R}^n + 1
\]
with the set $G_j$ and the constant $\delta$ of Proposition 2.13. We may assume that $\delta$ is already chosen so small that the solution is contained in a conic $\varepsilon_\Omega$-neighborhood of $U'_j$, and thus in $U_j$.

Will the new coordinates preserve the SG-compatible structure? One needs (cf. the Introduction) a finite atlas, a "good" shrinking, and growth conditions on the derivatives. The finiteness of the atlas is no problem. The estimates on the derivatives will be proven in Theorem 2.15, Lemma 2.16 for the sets $V_j' = \{(\tau, x_0) : x_0 \in G_j, |\tau| < \delta_0 \langle x_0 \rangle \}$ (with a possibly reduced $\delta_0 < \delta$; for the sake of simplicity assume $\delta = \delta_0$). For a "good" shrinking of $V_j$ let $G'_j = (U'_j)_{\text{con:}\varepsilon_\Omega/4} \cap \{t = 0\}$, and take the set
\[
V'_j = \{(\tau, x_0) : x_0 \in G'_j, |\tau| < \delta/2 \langle x_0 \rangle \}.
\]
Therefore it only remains to show the estimates on the flow $c$ and its inverse.

2.15. Theorem. On $U_j$ consider the solution $c(\tau; x_0)$ of Eqs. (2.2.a)–(2.2.c) which exists by Proposition 2.13 for $x_0 \in G_j = (U')_{\text{con.} : \omega/2} \cap \{t = 0\}$, $|\tau| \leq \delta \langle x_0 \rangle$, $\delta$ small. Then

$$D^x_\tau D^x_{x_0} c(\tau; x_0) = O(\langle x_0 \rangle^{1-k-|z|}).$$

Since $|\tau| \leq \delta \langle x_0 \rangle$, estimating the right hand side by $\langle x_0 \rangle$ is equivalent to estimating it by $\langle (\tau, x_0) \rangle$.

Proof (by induction on $|x|$ and, for fixed $|x|$, on $k$). In order to avoid additional indices for the derivatives we will write $\partial_x$ meaning any derivative with respect to one of the components of $x_0$.

For $|x| = 0, k = 0, 1$, Proposition 2.13 shows that $c(\tau; x_0) = O(\langle x_0 \rangle)$ and $\dot{c}(\tau; x_0) = O(1)$ for $|\tau| \leq \delta \langle x_0 \rangle$.

Moreover, since $\|c(\tau; x_0)\| \geq \text{const.} \langle x_0 \rangle$ for $|\tau| \leq \delta \langle x_0 \rangle$ and $x_0$ is bounded away from zero, with a positive constant,

$$\|\dot{c}(\tau; x_0)\| \leq \text{const.} \|I'(c(\tau; x_0))\| \|\dot{c}(\tau; x_0)\|^2 \leq O(\langle c(\tau; x_0) \rangle^{-1} \langle \dot{c}(\tau; x_0) \rangle^2) = O(\langle x_0 \rangle^{-1})$$

for large $\|x_0\|$, and the case $|x| = 0, k = 2$ is proven. For $|x| = 0, k \geq 2$, consider the geodesic equation (2.2.a),

$$\ddot{c}(\tau; x_0) = \sum_{ij} \Gamma^l_{ij}(c(\tau; x_0)) \dot{c}(\tau; x_0) \dot{c}(\tau; x_0), \quad l = 1, \ldots, n + 1. \quad (2.8)$$

Differentiation shows that $\partial_x^{k+1} c$ is a linear combination of terms that have already been shown to depend of order $O(\langle x_0 \rangle^{-k})$ on $x_0$ and $\tau$. This completes the case $|x| = 0$.

In order to get the idea for the argument in general, consider the case $|x| = 1$. Take derivatives with respect to $x_0$ in Eq. (2.8):

$$\partial^2_{x0} \partial_x c^l = \sum_{ij} \left[ \Gamma^l_{ij}(c) \right]' \partial_x c \partial_x c^l \partial_x c^l + \sum_{ij} \Gamma^l_{ij}(c) \partial_x \partial_x c^l \partial_x c^l + \sum_{ij} \Gamma^l_{ij}(c) \partial_x c^l \partial_x \partial_x c^l. \quad (2.9)$$

Write $\phi = \partial_x c$, $\psi = \phi = \partial_x \partial_x c$. Then (2.9) is equivalent to the first order system

$$\begin{pmatrix} \phi \\ \psi \end{pmatrix} = \begin{pmatrix} 0 & I \\ B & C \end{pmatrix} \begin{pmatrix} \phi \\ \psi \end{pmatrix}$$
with
\[ B(\tau, x_0) = \left( \sum_{j} \left[ \Gamma^k_{ij}(c) \right]_{(\ell)} \partial_x c(j) \partial_x c^i \right)_{k, l}, \]
\[ C(\tau, x_0) = \left( \sum_i \left( \Gamma^k_{ij} + \Gamma^k_{ij} \right) \partial_x c^i \right)_{k, j}, \]
and \( I \) the \((n+1) \times (n+1)\) identity matrix.

For the estimate on \( \varphi \) and \( \psi \) use Gronwall’s lemma, applied to the function \( R(\tau) = \| \varphi(\tau) \| + \| \psi(x_0) \psi(\tau) \| \), noting that for \( \tau = 0 \)
\[ \varphi(0) = \partial_x c(0; x_0) = \partial_x x_0 = O(1), \]
\[ \psi(0) = \partial_x \partial_x c(0; x_0) = \partial_x n(x_0) = O(\langle x_0 \rangle^{-1}). \]
by what has already been proven for \(|a| = 0\).

Suppose one has shown that \( \partial_x^k \partial_x c(\tau; x_0) = O(\langle x_0 \rangle^{-k}) \). Differentiating Eq. (2.8), each component of the vector \( \partial_x^{k+1} c(\tau; x_0) \) is a linear combination of terms of the form
\[ [\Gamma^k_{ij}(c)]^{(r_0)} \partial_x^{r_0} c \cdots \partial_x^1 c, \]
with \( r_j \leq k \) and \( \sum_{j=0}^l r_j = k + l - 1 \), so that by induction the whole term has order \((-1 - r_0) + (1 - r_1) + \cdots + (1 - r_l) = -k\) in \( \langle x_0 \rangle \). Writing—similarly as before—\( \varphi = (\partial_x c, \ldots, \partial_x^k \partial_x c) \) and differentiating the equation for \( \partial_x^{k+1} c \) with respect to \( x_0 \), one obtains a linear ODE \( \dot{\varphi} = A^\#(\tau, x_0) \varphi \) with a matrix \( A^\# \). The same method as above then shows that
\[ \partial_x \partial_x^{k+1} c(\tau; x_0) = O(\langle x_0 \rangle^{-k-1}). \]
This completes the case \(|a| = 1\).

For general \( a \), differentiation of the geodesic equation with respect to \( x_0 \) gives a non-homogeneous linear first order ODE. The concept is the same as before.

2.16. Lemma (Existence and Estimates for the Inverse Mapping). Denote by \( c(\tau; x_0) \) the solution to Eqs. (2.2.a)–(2.2.c). Then there is a \( \delta_0 > 0 \) such that
\[ c: \{ (\tau, x_0) : x_0 \in G_j, |\tau| < \delta_0 \langle x_0 \rangle \} \to U_j \]
is injective. Writing \((\tau, x_0) = F(c(\tau; x_0))\) one has
\[
D^*F(y) = O(\langle y \rangle^{-1})
\]
for all \(y\) in the image of \(c\).

**Proof.** First show injectivity. Except for the "global" aspect, the argument is standard. Consider the total differential of \(c\) for \(\tau = 0\).

\[
Dc(0; x_0) = \begin{pmatrix} n(x_0) & 0 & \ldots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \ldots & n(x_0) & 0 \\ I & \cdots & I & I \end{pmatrix}
\]

is an invertible matrix (\(I\) denotes the \(n \times n\) identity matrix). As a function of \(x_0\) its inverse is bounded by assumption 2.10(ii).

By Theorem 2.15, there exist \(d, C, C' > 0\) such that \(\|Dc(\tau; x_0)\|^{-1} \leq C\), \(\|Dc(\tau, x_0)\| \leq C'\), and \(\|D^2c(\tau; x_0)\| \leq d\langle x_0 \rangle^{-1}\), for all \(x_0 \in G_j, |\tau| \leq \delta \langle x_0 \rangle\).

First let \(\delta_1 = (\delta/C)(1 + 4d)^{-1}\). Whenever \(|\tau_1 - \tau_2| < \delta_1 \langle x_0 \rangle\) and \(|x_1 - x_0| < \delta_1 \langle x_0 \rangle\), with \(x_0, x_1 \in G_j, |\tau_0| < \delta \langle x_0 \rangle, |\tau_1| < \delta \langle x_1 \rangle\), one has
\[
\|Dc(\tau_1; x_1) - Dc(\tau_0; x_0)\| \leq 2d\delta_1 \leq \frac{1}{2C}.
\]

Now let \(\delta_0 = \delta_1/(2C' + 2)\), and suppose \(c(\tau_1; x_1) = c(\tau_2; x_2)\) for \(x_1, x_2 \in G_j, |\tau_1| < \delta_0 \langle x_1 \rangle, |\tau_2| < \delta_0 \langle x_2 \rangle\), and (w.l.o.g.) \(\langle x_1 \rangle \leq \langle x_2 \rangle\). Then
\[
\|x_1 - x_2\| \leq \|x_1 - c(\tau_1; x_1)\| + \|x_2 - c(\tau_2; x_2)\|
\]
\[
\leq \delta_0 C'(\langle x_1 \rangle + \langle x_2 \rangle) + \delta_1 \langle x_2 \rangle,
\]
and \(|\tau_1 - \tau_2| < \delta_1 \langle x_2 \rangle\). Hence
\[
0 = \|c(\tau_2; x_2) - c(\tau_1; x_1)\|
\]
\[
= \left\| \int_0^1 Dc(\tau_2 + t(\tau_1 - \tau_2); x_2 + t(x_1 - x_2)) \, dt(\tau_1 - \tau_2, x_1 - x_2) \right\|
\]
\[
\geq \|Dc(\tau_2; x_2)(\tau_1 - \tau_2, x_1 - x_2)\| - \int_0^1 \|Dc(\cdots) - Dc(\tau_2, x_2)\| \, dt
\]
\[
\times \|(\tau_1 - \tau_2, x_1 - x_2)\|
\]
\[
\geq \|Dc(\tau_2; x_2)\|^{-1} \cdot \|c(\tau_1 - \tau_2, x_1 - x_2)\| - \frac{1}{2C} \cdot \|c(\tau_1 - \tau_2, x_1 - x_2)\|
\]
by (2.10). Since \(\|Dc(\tau_2, x_2)\|^{-1} \leq C, \tau_1 = \tau_2\) and \(x_1 = x_2\),

Now for the estimates on \(F\). By Proposition 2.13 one has for large \(x_0\)
\[
\text{const.} \langle x_0 \rangle \leq \|c(\tau; x_0)\| \leq \text{const.} \langle x_0 \rangle
\]
for all $|\tau| \leq \delta \langle x_0 \rangle$ and positive constants. Therefore

$$F(y) = O(\langle y \rangle),$$

proving the case $\alpha = 0$. For $|\alpha| = 1$, the identity $F(c(\tau; x_0)) = (\tau, x_0)$ implies

$$DF(c(\tau; x_0)) = [Dc(\tau; x_0)]^{-1}$$

the $D$ denoting the total differential. We have just seen that $Dc(\tau; x_0)$ is invertible for $|\tau| \leq \delta_0 \langle x_0 \rangle$, and $[Dc(\tau; x_0)]^{-1} = O(1)$. Hence

$$DF(y) = O(1).$$

For $|\alpha| = n$, one shows with the chain rule and induction that $D^*F(y) |_{y = c(\tau; x_0)}$ is a linear expression of terms $h(\tau, x_0)$ such that

$$\partial^\beta h(\tau, x_0) = O(\langle x_0 \rangle^{1-n-|\beta|}), \quad |\tau| \leq \delta_0 \langle x_0 \rangle,$$

where $\partial^\beta$ denotes any mixed derivative with respect to $x_0$ or $\tau$. This gives

$$D^*F(y) = O(\langle y \rangle^{1-|\alpha|}).$$

2.17. Summary. In a conic $\delta$-neighborhood of $Y$ we can change coordinates in an SG-compatible way: $x = c(\tau; x_0)$. In the new coordinates, the normal derivative is $\partial_\nu$. Choose a function $\psi$ on $\Omega$ with $D^*\psi(z) = O(\langle z \rangle^{-|\alpha|})$, $z = (\tau, x_0)$ such that $\psi \equiv 1$ in a conic $\delta/2$-neighborhood of $Y$ and $\psi \equiv 0$ outside the conic $\delta$-neighborhood. Then $\partial_\nu := \psi \partial_\nu$ is a globally defined differential operator of order $(1, 0)$ on $\Omega$, coinciding with the normal derivative near $Y$. Its symbol is $\psi(\tau, x_0) \zeta \in SG^{(1, 0)}(\Omega)$, where $\zeta$ is the cotangent variable associated with $\tau$.

A consequence of this is the following weighted version of the trace theorem.

2.18. Theorem. Let $X$ be an SG-compatible manifold with boundary $Y$, embedded in the SG-compatible manifold $\Sigma$. Suppose there is a metric $g$ on $\Omega$ with the properties in 2.10. Denote by $\partial_\nu$ the normal derivative defined by $g$. Then, for $s > \frac{1}{2}$, $t \in \mathbb{R}$, and $\mu \in \mathbb{N}_0$ with $s - \mu > \frac{1}{2}$, the mapping

$$u \rightarrow (u |_{Y}, \partial_\nu u |_{Y}, \ldots, \partial^\mu_\nu u |_{Y}) = (\gamma^j u)_{j=1, \ldots, \mu},$$

from $C^\infty_c(\bar{X})$ to $[C^\infty_c(Y)]^{\mu+1}$ extends to a continuous linear map

$$H^{\infty}_{(s,t)}(X) \rightarrow \prod_{j=0}^{\mu} H^{(s-j-1/2,t)}(Y).$$
Proof. By definition, \( H_{(s, t)}(\Omega) = \{ f \in \mathcal{D}'(\Omega) : (\varphi_j f)_\ast \in H_{(s, t)}(\mathbb{R}^{n+1}) \}, \) where \( \varphi_j \) are the functions of the partition of unity and \((\cdot)_{\ast}\) denotes the function in local coordinates (i.e., on \( \mathbb{R}^{n+1} \)). From the standard trace theorem on \( \mathbb{R}^{n+1} \) and the fact that \( Y \) corresponds to \( \{ \tau = 0 \} \) on \( \mathbb{R}^{n+1} \) one obtains that

\[
f \in H_{(s, t)}(\Omega), \ s > \frac{1}{2} \text{ implies } f \mid _Y \in H_{(s - 1/2, t)}(Y).
\]

By 2.17, the operator \( \partial \nu \) is a (pseudo-)differential operator of order \((1, 0)\) on \( \Omega \). Together with the first statements we get the desired result.

2.19. Corollary. It also follows from Proposition 2.13, Remark 2.14 that there is a conic neighborhood \( U_Y \) of \( Y \) such that

\[
U_Y \cong \{ (\tau, y) : y \in Y, |\tau| < \lambda(y) \}
\]

with the function \( \lambda \) defined in 2.5, and it makes sense for small \( \tau \) to speak about the surfaces \( \{ \tau = \text{const.} \} \).

A simple argument shows that the geodesics are perpendicular to the surfaces \( \{ \tau = \text{const.} \} \). In the neighborhood \( U_Y \) of \( Y \), the tangent space \( TU_Y \) has an orthogonal decomposition

\[
TU_Y = T\{ c(\tau; x_0) : x_0 \in Y \text{ fixed}, |\tau| < \delta(x_0) \} \oplus T\{ c(\tau; x_0) : x_0 \in Y, \ \tau \text{ fixed} \}.
\]

This furnishes a similar decomposition of the cotangent space \( T^* U_Y \): use a metric \( g \) like in Example 2.7 to identify \( T^* U_Y \) and \( TU_Y \). In particular, there is a decomposition \( \xi = \zeta + \xi' \) of the cotangent variable \( \xi \), where \( \zeta \) is the cotangent variable connected with the variable \( \tau \). Moreover, \( g^{-1} \) then defines the scalar product in the fibers of \( T^* U_Y \), and the function

\[
\mu(x, \xi) = \left( 1 + \sum \xi_i (g(x)^{-1})_{ij} \xi_j \right)^{1/2}
\]

is well-defined on \( T^* U_Y \); \( D^2_{\xi} D^\beta_x \mu(x, \xi) = O(\langle \xi \rangle^{-1} \langle x \rangle^{-|\beta|}) \), and \( \mu \geq 1 \).

3. The Boundary Value Problem

Suppose \( X \) is \( SG \)-compatible with boundary \( Y \), contained in an \( SG \)-compatible manifold without boundary, \( \Omega \), cf. Definition 1.1. On \( \Omega \) choose a metric with the properties in 2.10. It defines a unit normal vector field at the boundary and a geodesic flow \( c \). It was shown in Section 2 that this flow induces an \( SG \)-compatible change of coordinates \( x = c(\tau; y) \),
$y \in \mathcal{Y}, \tau \in \mathbb{R}, |\tau| < \delta \langle y \rangle$, in a conic neighborhood (cf. 2.1) $U_Y$ of $Y$. $Y$ may be identified with $\{\tau = 0\}$; $\tau$ is the normal coordinate, and the normal derivative $\partial_\nu$ is $\partial_\nu$ near $Y$.

Each differential operator on $\Omega$ with an $SG$-symbol can be written in $U_Y$ as a differential operator with respect to $D_\nu$ and $D_{\nu} = -i\partial_\nu$ within the $SG$-calculus. We consider the boundary value problem

$$
Pu = f \quad \text{in } X
$$

$$
B^k u = g^k \quad \text{on } Y, \quad k = 1, \ldots, m,
$$

where $P$ is a differential operator in $SG^{(2m, \ell)}(\Omega)$, $B^1, \ldots, B^m$ are differential operators in $SG^{(r_k, l)}(\Omega)$, and $r_k < 2m, k = 1, \ldots, m$.

3.1. Theorem. If $P$ is $md$-elliptic on $X$, cf. Definition 2.2, and if the system $(P, B^1, \ldots, B^m)$ satisfies the uniform variant of the Lopatinski—Shapiro condition stated in the two assumptions, below, then the operator $(P, B^1, \ldots, B^m): H^s, t,(X) \rightarrow H^{s-2, m-2, t),(X) \times H^{s-1, m-1, t}(Y), k = 1 \ldots, m (s \geq 2m)$ is Fredholm. In particular, the boundary value problem (BVP) is normally solvable. We then call $(P, B^1, \ldots, B^m)$ an $md$-elliptic system.

For the formulation of the Lopatinski—Shapiro condition we need some notation. In the conic neighborhood $U_Y$ of $Y$ write $P = \sum_{j=0}^{2m} P_j(\tau) D_{\nu}$, where $(\tau, y)$ are normal coordinates, $D_{\nu} = D_\nu = -i\partial_\nu$, $D_{\nu}$ is the normal derivative, and $P_j(\tau) = P_j(\tau, y, D_\nu)$ are differential operators in $SG^{(2m-j, l)}(\Omega)$. Let $p^0(\tau, y, \xi) = \sum_{j=0}^{2m} p_j(\tau, y, \xi') \xi'$ be the $SG$-principal symbol of $P$, where $\xi'$ is the cotangent variable associated with $y, \xi$ with $\tau$ and $P_j$ is the $SG$-principal symbol of $P_j$, modulo $SG^{(2m-j-1, l-1)}(\Omega)$. Consider the polynomial

$$
p_{(y, \xi)}(z) = p^0((0, y), (\xi', z)).
$$

Since $P$ is $md$-elliptic, $p_{(y, \xi)}$ has no real zeroes for $|y| + |\xi'|$ large.

Assumption 1. For all large $|y| + |\xi'|$, $p_{(y, \xi)}$ has exactly $m$ zeroes with positive imaginary part $\tau_1(y, \xi'), \ldots, \tau_m(y, \xi')$.

Choose a function $\mu(y, \xi')$ replacing the notion of $\langle \xi' \rangle$ on $T^*Y$, cf. 2.19. Let

$$
p^+(z) = p^+_{(y, \xi')}(z) = \prod_{j=1}^{m} (z - \mu(y, \xi')^{-1} \tau_j(y, \xi')).
$$
Similarly, write the boundary operators \( B_k = \sum_{j=0}^{\tau_k} B_j^k(y, D_y) D_t^j \) (we can assume they are independent of \( \tau \) near the boundary) with \( \text{SG-principal symbols} \ \sum_{j=0}^{\tau_k} b_j^k(y, \xi') \xi^j, b_j^k \) being an \( \text{SG-principal symbol of} \ B_k^t. \) Scaling down the coefficient operators to order zero, let
\[
 b_{j(y, \xi')}^k(z) = \sum_{j=0}^{\tau_k} b_j^k(y, \xi') \mu(y, \xi')^{-\tau_k + j} \lambda(y)^{-\delta_k} z^j
\]
(for the definition of \( \lambda \) cf. 2.5), and define the polynomials
\[
r_{j(y, \xi')}^k = \sum_{j=0}^{m-1} r_j(y, \xi') z^j \quad \text{as the residues of} \ b_{j(y, \xi')}^k \ \text{modulo} \ p^+_{(y, \xi')}.
\]

**Assumption 2.** The determinant \( \det((r_{j(y, \xi')}^k))_{k,j} \) is bounded and bounded away from zero.

### 3.2. Corollary

Suppose \((P, B_1, \ldots, B_m)\) is an \( md \)-elliptic system like before. Let \((R, R', \ldots, R_m): C^\infty_c(X) \to C^\infty_c(X)\) be a system of operators such that for some \( \varepsilon > 0 \) and all \( (s, t) \) with \( s \geq 2m - \varepsilon \)

(i) \( R: H_{(s, t)}^+(X) \to H_{(s - 2m + \varepsilon, \ v - 1, \ t - \varepsilon)}^+(X) \) is bounded and

(ii) \( R^k: H_{(s, t)}^+(X) \to H_{(s - \varepsilon, \ v - 1, \ t - \varepsilon)}^+(X) \) is bounded for \( k = 1, \ldots, m \).

(This is certainly true for \( R, R', \ldots, R_m \) pseudodifferential operators of the corresponding order satisfying a weighted version of the standard "transmission conditions," e.g., of Grubb [GRU, Definition 2.1.1] or Rempel and Schulze [RS, Sect. 2.2.2], cf. [SCB].) Then for \( s \geq 2m \),

\[
(P + R, B_1 + R', \ldots, B_m + R_m): H_{(s, t)}^+(X) \to H_{(s - 2m, \ v - 1, \ t - \varepsilon)}^+(X) \times \prod_{k=1}^{m} H_{(s - \varepsilon, \ v - 1, \ t - \varepsilon)}^+(Y)
\]

is Fredholm. This follows from the invariance of the Fredholm property under compact perturbations.

### 3.3. Example

Let \( g \) be a metric satisfying the assumptions of 2.6, \( A_g \) the associated Laplacian. \( 1 - A_g \) is \( md \)-elliptic on \( \Omega \) of order (2, 0). The principal \( \text{SG-principal symbol,} \ p^0 = 1 + \sum_g \ g^{ij} \xi_j \xi_j \) has, for fixed \( (x', \xi') \), one root in the upper half plane and one in the lower half plane, so one can impose one boundary condition. Then the analogues of the classical results hold: \( 1 - A_g \) with Dirichlet or Neumann boundary conditions is an \( md \)-elliptic system.

**Proof of 3.3.** For the Dirichlet case, the boundary operator is the identity, so everything is obvious. For the Neumann problem, the boundary
operator is $\partial_y$ with principal symbol $-i\xi$, and $p^+(z) = z - z^+\tau_+$, where $\mu(y, \xi')\tau_+$ is the zero of $p^0$ in the upper half plane. Thus the residue of the boundary symbol modulo $p^+$ is $-i\tau_+$. The $md$-ellipticity of $p^0$ implies that $|\tau_+|$ is bounded and bounded away from zero, thus the system is $md$-elliptic.

3.4. Example (The Dirichlet problem on a strip in $\mathbb{R}^{n+1}$ is not $md$-elliptic, but still a Fredholm operator, even bijective). Let $X = \mathbb{R}^n \times (0, 1) = \{(x, y) : x \in \mathbb{R}^n, 0 < y < 1\}$, $Y = \mathbb{R}^n \times \{0\} \cup \mathbb{R}^n \times \{1\}$. Consider the boundary value problem

$$Au = f \text{ in } X,$$

$$\gamma_0(u) = u|_Y = g.$$ 

Then $(A, \gamma_0) : H^+_2(X) \to L^2(X) \times [H^{3/2}_3(\mathbb{R}^n) \times \{0, 1\}]$ is an isomorphism. $H^+_2$ is the standard Sobolev space on $X$. The operator $A$, however, is not $md$-elliptic.

Proof. First show that the problems

$$Au = f \text{ in } X, \quad u|_Y = 0 \quad (3.1.a)$$

and

$$Au = 0 \text{ in } X, \quad u|_{y=0} = g_0; \quad u|_{y=1} = g_1 \quad (3.1.b)$$

where $f \in L^2$, $g = (g_0, g_1) \in H^{3/2}_3(\mathbb{R}^n) \times \{0, 1\}$ both have a solution.

To (3.1.a) apply the Fourier transform $\mathcal{F}_{x \rightarrow \xi}$ with respect to $x \in \mathbb{R}^n$, denoting it as usual by $\hat{\cdot}$. The equation becomes $\hat{u}_{yy} - |\xi|^2 \hat{u} = \hat{f}$, $\hat{u}|_{y=0} = 0$. Using Fourier sine series,

$$\hat{u}(\xi, y) = \sum_{n=1}^{\infty} \frac{1}{n^2\pi^2 + |\xi|^2} \hat{f}_n(\xi) \sin(n\pi y),$$

where

$$\hat{f}_n(\xi) = \int_0^1 f(\xi, y) \sin(n\pi y) \, dy,$$

gives a solution $u(x, y)$. One checks that $u \in H^+_2(X)$ by establishing the estimate

$$\|D_x^a D_y^k u\|_{L^2} \leq \text{const. } \|f\|^{+}_{|a| + k - 2}.$$ 

For problem (3.1.b), the Fourier transform gives

$$\hat{u}_{yy} - |\xi|^2 \hat{u} = 0, \quad \hat{u}|_{y=0} = \hat{g}_0, \quad u|_{y=1} = \hat{g}_1.$$ 

$^1$ Note that, with cartesian coordinates, the strip is not an $SG$-compatible manifold with boundary. So one should not conclude from this example that $md$-ellipticity is unnecessary for the Fredholm property of a boundary value problem on an $SG$-compatible manifold.
This has the unique solution

\[ \hat{u}(y, \xi) = \frac{\sinh[(1 - y) |\xi|]}{\sinh |\xi|} \hat{g}_0(\xi) + \frac{\sinh[y |\xi|]}{\sinh |\xi|} \hat{g}_1(\xi). \]

Again \( u \in H_2(X), \) in fact

\[ \|D_x^k D_y^k u\|_{L^2} \leq \text{const.} \|g\|_{|x| + k - 1/2}. \]

Thus \((\Delta, \gamma_0)\) is surjective. It also is injective, because from what we just saw, problem (3.1.b) only has the trivial solution for \( g_0 = g_1 = 0. \) Thus \((\Delta, \gamma_0)\) is a topological isomorphism, and

\[ \|u\|_{\mu_{m+2}(X)} \leq \text{const.} \left\{ \|\Delta u\|_{\mu_m(X)} + \|u\|_{\mu_{m+2}(Y)} \right\}, \quad m \in \mathbb{N}_0. \]

3.5. Definition. (a) Define a distribution \( \delta \in \mathcal{S}'(\Omega) \) by \( \delta f = \int_Y f \, d\sigma(y) \) with the surface measure \( d\sigma(y) \) on \( Y \) induced by the Riemannian metric, and let \( \delta^i = D_i^k \delta, \) cf. [HOR, p. 193]. In \((\tau, x')\) coordinates, \( \delta = \delta_0 \) with \( \delta_0 \) the Dirac measure in \( \tau = 0. \)

(b) For a function \( u \) on \( X \) or \( \Omega \) let \( u^0 \) denote the function equal to \( u \) on \( X \) and equal to zero on \( \Omega \setminus X. \)

(c) For \( u \in H^+_{k, (1)}(X), \ s > \mu + 1/2, \ \mu \in \mathbb{N}_0, \) let \( u_\mu = (\mu)i^\mu \gamma_\mu(u) = D_i^\mu u |_Y, \) cf. 2.18.

3.6. Proof of Theorem 3.1. We will combine Erkip's weighted \( \mathbb{R}^n \) results (cf. [ERN]) with the compact manifold techniques of Hörmander starting from identity (2.2.1) of [HOR], cf. also Eqs. (20.1.4) and (20.1.5) in [H]:

\[ Pu^0 = (Pu)^0 - i \sum_{\mu = 0}^{2m-1} \sum_{j = 0}^{2m-\mu-1} P_{j+\mu+1}(0)(u_\mu \delta^j), \quad u \in \mathcal{S}(X). \quad (3.2) \]

By \( \mathcal{S}(X) \) we denote all those functions on \( X \) which satisfy the standard estimates for rapidly decreasing functions in all local coordinates.

3.7. Because of the \( md \)-ellipticity of \( P \) there is an \( SG \)-parametrix \( Q \) for \( P \) such that \( PQ = I_\epsilon + K_1, \ QP = I_\epsilon + K_2, \) where \( I_\epsilon \in SG^{10, 0}(\Omega) \) is the identity on functions supported in a small conic neighborhood of \( X, \) and \( K_1, K_2 \in \mathcal{K}. \) Applying \( Q \) to identity (3.2)

\[ u^0 + K_2(u^0) = Q((Pu)^0) - i \sum_{\mu = 0}^{2m-1} \sum_{j = 0}^{2m-\mu-1} QP_{j+\mu+1}(0)(u_\mu \delta^j). \quad (3.3) \]

Let us collect a few results. The proofs are the same as in the Euclidean case, using local coordinates.
Lemma 1.2 of [ERN] says that, given \((s, t)\) and \((s', t')\), \(s' \geq 0\),
\[
\|K_2(u^0)\|_{(s,t)}^+ = O(\|u\|_{(s',t')}^+), \quad u \in \mathcal{S}(X). \quad (3.4)
\]
with a constant depending on \(s, s', t, t'\). By Lemma 1.3 of [ERN], given
\((s, t) \in \mathbb{R}^2, s \geq 0, \|Q(v^0)\|_{(s+2m,t+t')}^+ = O(\|v\|_{(s,t)}), \) for \(v \in \mathcal{S}(X)\). Hence
\[
\|Q(Pu^0)\|_{(s+2m,t+t')}^+ = O(\|Pu\|_{(s',t')}^+), \quad u \in \mathcal{S}(X). \quad (3.5)
\]
Finally, the following estimate is an analog of [HOR, Lemma 2.1.3] or
[E, Eqs. (3.33), (3.34), (3.36)]: For \(s \geq 2m, t \in \mathbb{R}\)
\[
\|Q \mathcal{P}_{j+\mu+1}(u_\mu \delta^j)\|_{(s,t)}^+ = O(\|u_\mu\|_{(s-\mu-1/2,t)}^Y), \quad u \in \mathcal{S}(X). \quad (3.6)
\]

3.8. COROLLARY (Estimate without Boundary Condition). Combining
(3.2)–(3.6) one obtains that for \(s \geq 2m, t \in \mathbb{R}, s', t' \in \mathbb{R}\)
\[
\|u\|_{(s,t)}^+ = O\left(\|Pu\|_{(s-2m,t-\mu)}^+ + \|Pu\|_{(s',t')}^+ + \sum_{\mu=0}^{2m-1} \|u_\mu\|_{(s-\mu-1/2,t)}^Y\right),
\]
for all \(u \in \mathcal{S}(X)\).

In order to attack the boundary value problem, we need the following
lemma, cf. [ERN, L. 1.4] or [EH, L. 1]

3.9. LEMMA. Let \(A\) be a pseudodifferential operator of order \(-(m_1, m_2)\)
on \(\Omega\) such that in a conic neighborhood of \(X\), the symbol \(q\) of \(A\) has—in
local coordinates an asymptotic expansion \(\sum_{j=0}^{\infty} a_j\) modulo \(SG(-\infty, -\infty)\)
satisfying
\[
\begin{align*}
& (i) \quad a_j(x, (\zeta, \xi')) \text{ is a rational function of } \zeta \text{ for large } \langle x \rangle + \langle \xi' \rangle. \\
& (ii) \quad \text{For large } \langle x \rangle + \langle \xi' \rangle \text{ the poles of } a_j(\zeta) = a_j(x, (\zeta, \xi')) \text{ are not on} \\
& \quad \text{the real axis, and they all lie in some ball of radius } O(\langle \xi' \rangle). 
\end{align*}
\]
Then the operators \(A^{kj}: \mathcal{S}(Y) \rightarrow \mathcal{S}(Y)\), defined by \(A^{kj}u = \gamma_k A(u \delta^j)\)
are pseudodifferential operators in \(SG^{j+k+1-m_1, -m_2}(Y)\). The leading term of
the asymptotic expansion of the symbol of \(A^{kj}\) is, in local coordinates,
\[
\frac{1}{2\pi} \int_{W(\xi')} a_0(0, x', \eta, \xi') \eta^{k+j} d\eta, \quad (3.7)
\]
where \(W(\xi')\) is any contour in the upper half plane enclosing the poles of
\(a_0(\eta)\) there. Due to the estimates on the poles, one can find a path of length
\(O(\langle \xi' \rangle)\), causing the shift in the order.
3.10. The symbol of $Q$ has an asymptotic expansion $\sum q_j$ with each $q_j$ a linear combination of terms of the form

$$[p^0(x, \xi)]^{-1} p^{(\alpha_1)}(x, \xi) \cdots [p^0(x, \xi)]^{-1} p^{(\alpha_l)}(x, \xi) [p^0(x, \xi)]^{-1}$$

with $j_k \in \{0, 1\}$, and $\alpha_k, \beta_k$ multi-indices for derivatives with respect to $x$ and $\xi$, resp. The poles of $q_j(\xi)$ are the zeroes of $p^0(\xi)$. These are of order $O(\langle \xi \rangle^r)$. Lemma 3.9 shows that one can take traces in Eq. (3.3) and gets (cf. Eq. (3.5) in [ERN]) for $k = 0, \ldots, 2m - 1$

$$u_k = \gamma_k [Q(Pu)^0 - K_2 u^0] - i \sum_{\mu = 0}^{2m - 1} \sum_{j = 0}^{2m - \mu - 1} (QP_{j+\mu + 1})^{kj} u_{\mu}. \quad (3.8)$$

3.11. **Proposition.** There is an $c > 0$ such that for $s \geq 2m, t \in \mathbb{R}$

$$\|u\|_{(s, t)}^+ = O(\|Pu\|_{(s - 2m, t - 1)} + \|u\|_{(s - e, t - e)} + \sum_{k=1}^{m} \|\gamma_0 B^k u\|_{(s-r_k - 1/2, t - k)})$$

for $u \in H_{(s, t)}^+(X)$.

**Proof.** In view of Corollary 3.8 it suffices to estimate $\|u_{\mu}\|_{(s - 2m, t - 1/2)}$ in terms of the right hand side. Consider the $(m + 2m) \times 2m$ system of pseudodifferential equations given by the $2m$ equations from (3.8) and the $m$ boundary conditions. Scale it down, using on $Y$ an operator $A = (1 - \Delta)^{1/2}$ with a metric like in 2.6. For $\mu = 0, \ldots, 2m - 1$, let $U_{\mu} = A^{-\mu} u_{\mu}, F^k = A^{-k} \gamma_k [Q(Pu)^0 - Ku^0], G^k = \lambda(y)^{-k} A^{-\mu} \gamma_0 [B^k u]$. Then the problem can be written

$$\begin{pmatrix} I - \bar{Q} \\ \bar{B} \end{pmatrix} \begin{pmatrix} U_0 \\ \vdots \\ U_{r-1} \end{pmatrix} = \begin{pmatrix} F^k \\ \vdots \\ G^k \end{pmatrix}$$

with matrices $\bar{Q}$ and $\bar{B}$

$$\bar{Q} = \begin{pmatrix} -i \sum_{j=0}^{2m - \mu - 1} A^{-k} (QP_{j+\mu + 1})^{kj} A^\mu \end{pmatrix}_{k\mu},$$

$$\bar{B} = ((\lambda(y)^{-k} A^{-\mu} B^k(y, D_y) A^\mu))_{k\mu},$$

with $B^k = 0$ for $r_k < \mu < 2m$. The entries of $\bar{Q}$ and $\bar{B}$ are pseudodifferential operators of order $(0, 0)$. By (3.7), the principal $SG$-symbol $\sigma_0(x', \xi')$ of the matrix is

$$\sigma_0(y, \xi') = \begin{pmatrix} I - \bar{q}_0(y, \xi') \\ \bar{b}_0(y, \xi') \end{pmatrix}$$
for large $|y| + |\xi'|$, with

$$
\tilde{q}_0(y, \xi') = \left( \sum_{j=0}^{2m-\mu-1} \lambda(y, \xi')^{\mu-k} p_{j+\mu+1}(0, y, \xi') \right. \\
\left. \times \frac{1}{2\pi i} \int_{\gamma(\xi')} q_0(0, y, \xi', \xi') \xi^{j+\mu} \text{ d}\xi' \right)_{k\mu}
$$

($\lambda(y, \xi')$ is the principal $SG$-symbol of $A$) and

$$
\tilde{b}_0(y, \xi') = ((\lambda(y)^{-k} \lambda(y, \xi')^{\mu-r_k} b^k(y, \xi'))_{k\mu}.
$$

For large $\langle x \rangle + \langle \xi \rangle$, $q_0(x, \xi) = [p^0(x, \xi)]^{-1}$. For $y, \xi'$ fixed, one notices that $\sigma_0(y, \xi')$ is just the matrix constructed canonically for a constant coefficient boundary value problem on the half-line, cf. [ERN, Sect. 2].

From the theory there one can show that there exist pseudodifferential operators $S, T$ of order $(0, 0)$ on $Y$ such that

$$
S \left( \frac{I - \overline{Q}}{B} \right) = I + D_1, \quad [I - \overline{Q}] T = D_2, \quad \overline{B} T = I + D_3 \quad (3.9)
$$

with operators $D_1, D_2, D_3$ of order $(-\epsilon, -\epsilon)$ for some $\epsilon > 0$.

Applying the operator $S$ to the above equation one obtains

$$
\begin{pmatrix}
U_0 \\
\vdots \\
U_{r-1}
\end{pmatrix} = S \begin{pmatrix}
F^k \\
\vdots \\
G^k
\end{pmatrix} - D_1 \begin{pmatrix}
U_0 \\
\vdots \\
U_{r-1}
\end{pmatrix}
$$

and from this

$$
\|u_k\|_{(s-\mu-1/2, r)} = \|u_k\|_{(s-\mu-1/2, r)} = O \left( \|p_0\|_{(s-2m-1, l-1)} + \sum_{k=1}^{m} \| \gamma^k B^k u \|_{(s-r_k-1/2, l-\mu)} \right. \\
+ \left. \|u\|_{(s-\mu, l-\mu)} \right).
$$

3.12. Corollary. The estimate in 3.11 together with the compactness of the imbedding of $H^{+}_{(s, e)}(X)$ in $H^{+}_{(s-\epsilon, 1-e)}(X)$ shows that the kernel of the operator $(P, B^1, \ldots, B^m)$ is finite dimensional for any $s \geq 2m$, $t \in \mathbb{R}$.

3.13. Now show that the range is finite codimensional: For $\psi = (\psi_1, \ldots, \psi_m)$, $\psi_j \in \mathcal{S}(Y)$, let

$$
W \psi = -t \sum_{\mu=0}^{2m-1} \sum_{j=0}^{2m-\mu-1} Q P_{j+\mu+1} (A^\mu(T\psi)_\mu \delta^j),
$$
\((T\psi)_\mu\) denoting the \(\mu\)th component of \(T\psi\), \(T\) as in (3.9). From (3.6) one concludes that for \(s \geq 2m\), \(t \in \mathbb{R}\)

\[
\|W\psi\|_{(s, t)}^+ = O\left(\sum_{\mu = 0}^{2m-1} \|A^\mu(T\psi)_\mu\|_{(s-\mu-1/2, t)}\right)
\]

\[
= O\left(\sum_{k = 1}^{m} \|\psi_k\|_{(s-1/2, t)}^+\right).
\]

The operator \(P_{j+l+1}\) only contains \(y\)-derivatives, therefore

\[
P_{j+l+1}(A^\mu(T\psi)_\mu \delta^j)
\]

in \(X\). Computing the traces with (3.8) we obtain

\[
\begin{pmatrix}
B^1W\psi \\
\vdots \\
B^mW\psi
\end{pmatrix} = ((A^\alpha(y)^h \delta_{k\mu})) B\bar{Q}T\psi.
\]

Note that \(\bar{B}T = B(T-D_2) = I+D\) for some \(D \in SG^{(\ldots, \ldots)}\).

3.14. Conclusion. Define the operator

\[
A: H^+_{(s-2m, t)}(X) \times \prod_{k=1}^{m} H_{(s-r_k-1/2, t-l_k)}(Y) \rightarrow H^+_{(s, t)}(X)
\]

by

\[
A(f, g^1, \ldots, g^m) = Qf^0 + W\psi
\]

with

\[
\psi_k = \hat{\lambda}(y)^{-l_k} A^\alpha(g^k - \gamma_0(B^kQf^0)).
\]

Then \(A\) is bounded. Since \(K_1\) is compact, \(PA(f, g^1, \ldots, g^m) = f + K(f, g^1, \ldots, g^m)\) in \(X\) with compact \(K\). On \(Y\), (3.10) shows that \(B^kA(f, g^1, \ldots, g^m) = g^k + L^k(f, g^1, \ldots, g^m)\) with compact \(L^k\).

Thus \((P, B^1, \ldots, B^m)A\) is of the form \(I+\text{compact}\) on the right hand side of (3.11), and the corange of \((P, B^1, \ldots, B^m)\) is finite dimensional. Together with Corollary 3.12, this concludes the proof of Theorem 3.1.
A.1. Proof for Example 1.2(c). On $\mathbb{R}^2$ choose coordinate neighborhoods in the following way. Let $\delta = \epsilon > 0$ be small and define

$$\Omega_1 = \{(x, y): \|(x, y)\| > \delta, y > -\delta x, y < (1 + \epsilon)x\},$$

$$\Omega_2 = \{(x, y): \|(x, y)\| > \delta, x > -\delta y, y > (1 - \epsilon)x\},$$

$$\Omega_3 = \{(x, y): \|(x, y)\| < 5\epsilon \text{ or } x < 0 \text{ or } y < 0\}.$$

Then let $\Omega'_1$, $\Omega'_2$, $\Omega'_3$ be the same sets with $\epsilon$ replaced by $\epsilon/2$ and $\delta$ by $2\delta$. For coordinate maps choose

$$\chi_1: \Omega_1 \to \mathbb{R}^2, \quad \chi_1(x, y) = \left(y - \frac{1}{x}, x\right), \quad \chi_2: \Omega_2 \to \mathbb{R}^2, \quad \chi_2(x, y) = \left(x - \frac{1}{y}, y\right).$$

For $\chi_3: \Omega_3 \to \mathbb{R}^2$ take a linear transformation followed by a translation mapping $\Omega_3$ into $\{(x, y): x < -1\} \subseteq \mathbb{R}^2$.

A.2. Proof for Example 1.2(d). Introduce additional coordinate neighborhoods in a small neighborhood of the compact set $\{\varphi(x) = 0\}$, mapping $\{\varphi(x) > 0\}$ to $\mathbb{R}^{n+1}_+$, $\{\varphi(x) < 0\}$ to $\mathbb{R}^{n+1}_-$. The image of the other coordinate neighborhoods can then be transferred to either $\mathbb{R}^{n+1}_+$ or $\mathbb{R}^{n+1}_-$ by composing the original maps with linear maps and translations.

A.3. Example 1.2(e) works similarly. Replace the original coordinate maps in a neighborhood of $e_j \times (1, \infty)$ and proceed as in A.2.

A.4. Proof for Example 1.2(f). Choose $\Omega = Y \times \mathbb{R}$. Since $Y \times \mathbb{R}_+$ is an $(n+1)$-dimensional submanifold of $\Omega$ with boundary $Y$, we only have to find the “right” $SG$-compatible structure on $\Omega$. For simplicity assume that $Y$ has one end only, i.e.,

$$Y = Y_0 \cup K \cup K \times (1, \infty) \quad \text{(disjoint)},$$

where $Y_0$ is a relatively compact submanifold with boundary $K$; $K$ is compact. Choose open covers $\{K_j\}$, $\{K_j\}$ of $K$, coordinate maps $\kappa_j: K_j \to \mathbb{R}^{n-1}$ such that for some fixed $\epsilon > 0$, an $\epsilon$-neighborhood of the image of $K_j$ is contained in the image of $K_j$. Define coordinate maps

$$\beta_j: K_j \times (1, \infty) \times \mathbb{R} \to \mathbb{R}^{n+1}$$

by

$$\beta_j(x, r, t) = (t, \langle (r, t) \rangle, \kappa_j(x) \langle (r, t) \rangle).$$
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(Here \( \langle r, t \rangle = (1 + r^2 + t^2)^{1/2} \) as above.) Call \( \Omega_{ij} \) the image of \( K_j \times (2, \infty) \times \mathbb{R} \) and \( \Omega_{ij} \) the image of \( K_j \times (1, \infty) \times \mathbb{R} \). This will take care of the "ends."

As for the rest, choose open sets \( L'_j, L_j \) \((L'_j \subseteq L_j)\) in \( Y \) covering \( Y \cup K \times [1, 2) \) and \( K \times [1, 3) \), respectively. There are coordinate charts \( \lambda_j \) mapping \( L_j \) onto a bounded open set in \( \mathbb{R}^n \). Now define

\[
\lambda_j : L_j \times \mathbb{R} \rightarrow \mathbb{R}^{n+1}
\]

by

\[
\lambda_j(l, t) = (t, \lambda_j(l) \langle t \rangle),
\]

and denote by \( \Omega_{j2} \) and \( \Omega_{j3} \) the images of \( L_j \times \mathbb{R} \) and \( L_j \times \mathbb{R} \) under \( \lambda_j \).

Clearly, the \( \lambda_j \) and \( \beta_j \) map \( Y \times \mathbb{R}_+ \) to \( Y \times \mathbb{R}_+ \) and \( Y \times \{0\} \) to \( \partial \mathbb{R}^{n+1} \). It is a straightforward computation similar to that in the proof of Example 3.4 in [SCSI] that, for some fixed \( \varepsilon_{42} > 0 \), one has an \( \varepsilon_{42} \langle x \rangle \)-neighborhood of each \( x \in \Omega_{j1} \) or \( \Omega_{j2} \) contained in \( \Omega_{j1} \), \( \Omega_{j2} \), respectively. The changes of coordinates are

\[
\alpha_j, \alpha_j^{-1}(t, x') = \left( t, \lambda_j^{-1} \left( \frac{x'}{\langle t \rangle} \right) \right),
\]

\[
\beta_j, \beta_j^{-1}(t, s, y) = \left( t, s, \kappa_j \kappa_j^{-1} \left( \frac{y}{s} \right) \right).
\]

The mixed changes are slightly more complicated. The subset \( K \times (1, 2) \) of \( Y \) is covered twice. Writing

\[
\lambda_j^{-1}(x) = ([\lambda_j^{-1}(x)], [\lambda_j^{-1}(x)]), \in K \times (1, 2),
\]

one gets

\[
\beta_j, \alpha_j^{-1}(t, x')
\]

\[
= \left( t, \left( \lambda_j^{-1} \left( \frac{x'}{\langle t \rangle} \right) \right) \right), \kappa_j \left( \lambda_j^{-1} \left( \frac{x'}{\langle t \rangle} \right) \right), \kappa_j^{-1} \left( \frac{x'}{\langle t \rangle} \right), \in K \times (1, 2),
\]

and

\[
\alpha_j, \beta_j^{-1}(t, s, y) = \left( t, \langle t \rangle \cdot \lambda_j \left( \frac{y}{s}, (s^2 - 1 - t^2)^{1/2} \right) \right),
\]

where \( (s^2 - 1 - t^2)^{1/2} \in (1, 2) \). Since the \( \kappa_j \) and \( \lambda_j \) are \( C^\infty \) maps bounded in all derivatives it is no problem to check that

\[
D^2 \chi(x) = O(\langle x \rangle^{1 - |\alpha|}).
\]
A.5. Proof for Example 1.2(g). Without loss of generality the compact set is the closed unit ball, and the submanifolds \( C_j, c_j \) are identified with the intersection of \( X \) and \( Y \), resp., with the unit sphere. In the interior of the unit ball use any coordinates (as long as they have all derivatives bounded). In order to handle the outside, or, more precisely, a neighborhood of the outside, cover the sphere by discs \( D_k \) of sufficiently small radius such that the intersections of the boundary \( c_j \) with each \( D_k \) can be trivialized, i.e., there exist diffeomorphisms \( \chi_k \) mapping \( D_k \) to \( \mathbb{R}^n \) with the image of \( C_j \) contained in \( \mathbb{R}^n_+ \), the image of \( c_j \) in \( \partial \mathbb{R}^n_+ \), and the image of the rest contained in \( \mathbb{R}^n_+ \). Then simply take conic coordinates for \( \mathbb{R}^{n+1} \setminus B(0, 1 - \varepsilon) \cong \bigcup D_k \times (1 - \varepsilon, \infty) \cup D_k \times (1 - \varepsilon, \infty) \to \{ (x_t, t) : x \in \chi_k(D_k), t \in (1, \infty) \}. \)

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References

[C] H. O. Cordes, A global parametrix for pseudo-differential operators over \( \mathbb{R}^n \), with applications, preprint No. 90, SFB 72, Bonn, 1976.
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