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Dual Variational Methods in Critical Point Theory and Applications

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This paper contains some general existence theorems for critical points of a continuously differentiable functional I on a real Banach space. The strongest results are for the case in which I is even. Applications are given to partial differential and integral equations.

This paper develops dual variational methods to prove the existence and estimate the number of critical points possessed by a real valued continuously differentiable functional I on a real Banach space E. Our strongest results are obtained for the case in which I is even. I need neither be bounded from above nor below. This study was motivated by existence questions for nonlinear elliptic partial differential equations and several applications in this direction will be given as well as to integral equations.

To illustrate the sort of situation treated here, suppose I is even with $I(0) = 0$ and $I > 0$ near 0 in some uniform fashion. Then $u = 0$ is a local minimum for I. If I is also negative near ∞ when restricted to finite dimensional subspaces of E and satisfies a version of the

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Palais–Smale condition, then I possesses infinitely many distinct pairs of critical points. In fact for each $m \in \mathbf{N}$ we find a pair of critical values $0 < c_m \leqslant b_m$ of I which are obtained by taking respectively the max–min and the min–max of I over certain dual families of subsets of E. The sets are dual in the sense that each member of the first family has nonempty intersection with each member of the second family and conversely. A precise description of the sets is given in Section 2. When the critical values are degenerate, i.e., $b_m = \cdots = b_j = b$ for $j > m$, multiplicity statements guarantee that I has an infinite number of distinct critical points corresponding to the critical value b.

Some results are also obtained for the noneven case. The more delicate situations where I is not positive near 0 or is not negative near ∞ are also studied. The methods used to prove these results are modifications of those occuring in the Ljusternlik–Schnirelman theory of critical points.

Our abstract results were motivated by investigating the existence of solutions of the nonlinear elliptic partial differential equation

$$\begin{aligned} Lu &\equiv -\sum_{i,j=1}^{n} (a_{ij}(x)\, u_{x_i})_{x_j} + c(x)u = p(x, u), \qquad x \in \Omega, \\ u &= 0, \qquad x \in \partial\Omega, \end{aligned} \tag{0.1}$$

where p is odd in u and Ω is a smooth bounded domain contained in $\mathbf{R}^n$. The solutions of (0.1) are critical points of the functional

$$I(u) = \int_\Omega \left[\frac{1}{2}\left(\sum_{i,j=1}^{n} a_{ij}(x)\, u_{x_i} u_{x_j} + c(x)\, u^2\right) - P(x, u(x))\right] dx, \tag{0.2}$$

where P is the primitive of p and I is defined on an appropriate Hilbert space ($E = W_0^{1,2}(\Omega)$).

Only conditions governing the asymptotic behavior of $p(x, z)$ at $z = 0$ and $z = \infty$ need be imposed to verify that I satisfies the conditions at 0 and ∞ mentioned above. In particular if $p(x, z) = o(|\, z\,|)$ at $z = 0$, $p(x, z)\, z^{-1} \to \infty$ as $z \to \infty$, and some additional technical conditions are satisfied, then (0.1) possesses infinitely many distinct pairs of solutions and I has an unbounded sequence of critical values. For the nonodd case, new existence results for positive solutions of (0.1) are given. If $p(x, z)$ is replaced by $a(x)z + p(x, z)$ with $a > 0$ in $\bar{\Omega}$ the corresponding I is no longer positive near 0 making this case more subtle but the same conclusions as above obtain. Another interesting situation treated is (0.1) with p replaced by λp where $\lambda \in \mathbf{R}$, now $p(x, z)\, z^{-1} \to \infty$ as $z \to \infty$, and $zp(x, z)$ is positive near 0. Then in general there will only be finitely many solutions of (0.1), say say $\pm\bar{u}_j$, $\pm\underline{u}_j$ with $I(\bar{u}_j) > 0 > I(\underline{u}_j)$, $1 \leqslant j \leqslant k$, the number k increasing as λ increases. Note that for this case the solutions occur in pairs aside from oddness.

Preliminary technical material is given in Section 1, the abstract results are contained in Section 2, and the applications to partial differential equations are in Section 3. A Galerkin argument is used in Section 4 to give another development of some the results of Section 3. Finally, in Section 5 applications of the theory of Section 2 are made to obtain new existence theorems for nonlinear integral equation of the form

$$v(x) = \int_\Omega g(x, y)\, q(y, v(y))\, dy, \tag{0.3}$$

where again it is the behavior of $q(x, z)$ at $z = 0$ and ∞ that is of importance.

There is a sizable literature on the study of critical points of real valued even functionals I on Hilbert or Banach manifolds which are oddly diffeomorphic to a sphere. See e.g. [1–6] and the bibliography in [6]. The basic ideas go back to the work of Ljusternik and Schnirelman [1] and are also used in our work here. Since we are dealing with I on a Banach space, no Lagrange multiplier occurs when the gradient of I is equated to 0 at critical points of I, unlike the case arising when dealing with spherelike manifolds. Clark [7] has recently studied the existence of critical points of I on E under a rather different set of hypotheses. Use is made of one of his results in Section 2. We do not know of any other investigations which work with classes of sets such as ours or which have such dual characterizations for critical values.

Earlier existence theorems for (0.1) of the same nature as our work but generally under more restriction hypotheses have been obtained by several people [8–15]. Those closest to the results presented here are in [15] where a Galerkin argument dual to that of Section 4 was employed. The method of [15] is such that unlike our work one gets neither a variational characterization of solutions nor a multiplicity statement for degenerate critical values. We also give a much more complete treatment of case II^- of [15] and a simpler proof of some of the results of [14] on the existence of pairs of positive solutions of (0.1) with λ present.

Some of our applications to (0.3) improve the results of [16], [8]–[9] while the remainder are related to results for (0.1) which seem to have no analog in the integral equations literature.

1. Preliminaries

Some of the background material needed for Section 2 will be presented here. In particular the notion of genus and some of its properties are introduced and an appropriate "deformation" lemma related to flow homotopies is given.

$C(X, Y)$ and $C^1(X, Y)$ will denote respectively the spaces of continuous and continuously Fréchet differentiable maps from X into Y. The Fréchet derivative of I at the point u will be denoted by $I'(u)$.

Let E be a real Banach space and let $\Sigma(E)$ denote the class of closed (in E) subsets of $E - \{0\}$ symmetric with respect to the origin.

Definition 1.1. $A \in \Sigma(E)$ has *genus* n (denoted by $\gamma(A) = n$) if n is the smallest integer for which there exists $\phi \in C(A, \mathbf{R}^n - \{0\})$. $\gamma(A) = \infty$ if there exists no finite such n and $\gamma(\varnothing) = 0$.

The following properties of genus are required in Section 2. The proofs of (1)–(6) can be found in [8] or [17].

Lemma 1.2. *Let* $A, B \in \Sigma(E)$.

(1) *If there exists an odd* $\phi \in C(A, B)$, *then* $\gamma(A) \leqslant \gamma(B)$;

(2) *If* $A \subset B$, $\gamma(A) \leqslant \gamma(B)$:

(3) *If there exists an odd homeomorphism* $h \in C(A, B)$, *then*

$$\gamma(A) = \gamma(B) = \gamma(h(A));$$

(4) *If* $\gamma(B) < \infty$, $\gamma(\overline{A - B}) \geqslant \gamma(A) - \gamma(B)$;

(5) *If* A *is compact* , $\gamma(A) < \infty$ *and there exists a uniform neighborhood* $N_\delta(A)$ (*all points within* δ *of* A) *of* A *such that* $\gamma(N_\delta(A)) = \gamma(A)$;

(6) *If* A *is homeomorphic by an odd homeomorphism to the boundary of a symmetric bounded open neighborhood of* 0 *in* $\mathbf{R}^m$, $\gamma(A) = m$;

(7) *Let* $A \in \Sigma(E)$, V *be a* k *dimensional subspace of* E, *and* $V^\perp$ *an algebraically and topologically complementary subspace. If* $\gamma(A) > k$. *then* $A \cap V^\perp \neq \varnothing$.

Proof of 7. Let P denote the projection of E onto V along $V^{\perp}$. If $A \cap V^{\perp} = \varnothing$, then $P \in C(A, V - \{0\})$ since $Pu = 0$ for $u \in A$ implies $u \in V^{\perp}$. But then by (1) and the definition of genus, $\gamma(A) \leqslant k$, a contradiction. Q.E.D.

Next for $I \in C^1(E, \mathbf{R})$, let $A_c = \{u \in E \mid I(u) \leqslant c\}$, $\hat{A}_c = \{u \in E \mid I(u) \geqslant c\}$, and $K_c = \{u \in E \mid I(u) = c, I'(u) = 0\}$. The following result is quite similar to theorems to be found in the literature (e.g. [4, 7]) but does not seem to be stated in this generality or form.

LEMMA 1.3. *Suppose $I \in C^1(E, \mathbf{R})$ satisfies $(P - S)$: Any sequence (u_m) for which $|I(u_m)|$ is bounded and $I'(u_m) \to 0$ possesses a convergent subsequence. Let $c \in \mathbf{R}$ and $\mathbf{N}$ be any neighborhood of K_c. Then there exists $\eta_t(t, x) \equiv \eta_t(x) \in [C(0, 1] \times E, E)$ and constants $d_1 > e > 0$ such that:*

(1) $\eta_0(x) = x$ *for all* $x \in E$;

(2) $\eta_t(x) = x$ *for* $x \notin I^{-1}[c - d_1, c + d_1]$ *and all* $t \in [0, 1]$;

(3) η_t *is a homeomorphism of* E *onto* E *for all* $t \in [0, 1]$;

(4) $I(\eta_t(x)) \leqslant I(x)$ *for all* $x \in E$, $t \in [0, 1]$;

(5) $\eta_1(A_{c+d} - \mathbf{N}) \subset A_{c-d}$;

(6) *If* $K_c = \varnothing$, $\eta_1(A_{c+d}) \subset A_{c-d}$;

(7) *If* I *is even,* η_t *is odd in* x.

Proof. Although not all of the above is stated there explicitly, Lemma 1.3 is essentially contained in Theorem 4 of [7]. (Clark actually takes I to be even, $c < 0$, and an appropriately weaker version of $(P - S)$, but given $(P - S)$ the proof is unaffected). Therefore, in view of its length, we only indicate the small modifications of the proof of [7] necessary to get the lemma. Letting U_ϵ be as in [7], define $\bar{g}(x) \equiv 0$ for $x \in U_{\epsilon/8}$, $\bar{g}(x) \equiv 1$ for $x \in E - U_{\epsilon/4}$ and $\bar{g}$ Lipschitz continuous on E with $0 \leqslant \bar{g} \leqslant 1$. Replace V of [7] by $\bar{V} = \bar{g}V$. Then $\bar{V}$ is a bounded locally Lipschitz continuous vector field on E. Therefore, the flow $\eta_t(x)$ generated by $\bar{V}$ exists for all $t \in \mathbf{R}$ [4]. The proof of Theorem 4 of [7] then gives all but (3) above. But (3) follows from the semigroup property of η_t. Q.E.D.

Remark 1.4. If $c > 0$ $(c < 0)$, $(P - S)$ is only needed for $\{u \in E \mid I(u) > 0\}$ $(\{u \in E \mid I(u) < 0\})$ as in [7]. Then d_1 can be chosen so that $d_1 < c$ $(d_1 < -c)$. If $\bar{V}$ is replaced by $-\bar{V}$, the conclusions of Lemma 1.3 obtain with a sign reversal in (4), d replaced by $-d$ in (5), (6), and A_c replaced by $\hat{A}_c$.

2. The Abstract Theory

Let E be an infinite dimensional Banach space over $\mathbf{R}$. The finite dimensional case is also contained in what follows with some obvious changes in the statement of results. Let $B_r = \{u \in E \mid \| u \| < r\}$ and $S_r = \partial B_r$; B_1 and S_1 will be denoted by B and S, respectively. Let $I \in C^1(E, \mathbf{R})$. If I satisfies $I(0) = 0$ and

(I_1) there exists a $\rho > 0$ such that $I > 0$ in $B_\rho - \{0\}$ and

$$I \geqslant \alpha > 0 \quad \text{on} \quad S_\rho ,$$

then $u = 0$ is a local minimum for I. We will develop additional conditions under which I possesses more critical points. Further assume

(I_2) there exists $e \in E$, $e \neq 0$ with $I(e) = 0$;

(I_3) *If* $(u_m) \subset E$ with $0 < I(u_m)$, $I(u_m)$ bounded above, and $I'(u_m) \to 0$, then (u_m) possesses a convergent subsequence.

(I_3) is a slightly weakened version of $(P - S)$ [18]. The condition that $0 < I(u_m)$ in (I_3) can be replaced by $\beta < I(u_m)$ for any $\beta < \alpha$. (I_1)–(I_3) imply I possesses a second critical in $\hat{A}_\alpha$ (using the notation of Section 1).

Let

$$\Gamma = \{g \in C([0, 1], E) \mid g(0) = 0, g(1) = e\}.$$

Clearly $\Gamma \neq \varnothing$.

Theorem 2.1. *Suppose I satisfies (I_1)–(I_3). Then*

$$b = \inf_{g \in \Gamma} \max_{y \in [0,1]} I(g(y)) \tag{2.2}$$

is a critical value of I with $0 < \alpha \leqslant b < +\infty$.

Proof. (I_1)–(I_2) imply S_ρ separates 0 and e. Thus, for any $g \in \Gamma$ the connectedness of $g([0, 1])$ implies $g([0, 1]) \cap S_\rho \neq \varnothing$. Hence,

$$\max_{y \in [0,1]} I(g(y)) \geqslant \alpha$$

and, therefore, $b \geqslant \alpha$. That b is a critical value of I follows in a standard fashion. Arguing indirectly, if b is not a critical value of I, by (6) of

Lemma 1.3 and Remark 1.4, there exists $\epsilon \in (0, \alpha)$ and $\eta_1 \in C(E, E)$ such that $\eta_1(A_{b+\epsilon}) \subset A_{b-\epsilon}$. Choose $g \in \Gamma$ such that

$$\max_{y\in[0,1]} I(g(y)) \leqslant b + \epsilon.$$

Since $\eta_1(g(0)) = 0$, $\eta_1(g(1)) = e$ by (2) of Lemma 1.3, $\eta_1 \circ g \in \Gamma$. But

$$\max_{y\in[0,1]} I(\eta_1(g(y))) \leqslant b - \epsilon$$

contradicting the definition of b. Q.E.D.

Remark 2.3. We suspect that results such as Theorems 2.1 and 2.4 below exist in the literature although probably not in the generality given here. Note that I need not be bounded either from above or below. Indeed, thus will be the case in some of our applications such as Theorems 3.10 and 3.13. If I is also bounded from below and $(P - S)$, then $0 \geqslant \inf_E I$ is also a critical value of I with a corresponding non zero critical point, for if $\inf_E I = 0$, then e is a local minimum for I while if the infimum is negative, the result follows essentially from Theorem 4 of [7].

Next we give a dual version of Theorem 2.1. Let $\Gamma_* = \{h \in C(E, E) \mid h(0) = 0,\ h$ is a homeomorphism of E onto E, and $h(B) \subset \hat{A}_0\}$ and $\Gamma_*^e = \{h \in \Gamma_* \mid h(s)$ separates 0 and $e\}$. (I_1) implies $\Gamma_* \neq \varnothing$.

THEOREM 2.4. *Let I satisfy (I_1)–(I_3). Then*

$$c \equiv \sup_{h\in\Gamma_*^e} \inf_{u\in S} I(h(u)) \tag{2.5}$$

is a critical value of I with $0 < \alpha \leqslant c \leqslant b < \infty$.

Proof. $h(u) = \rho u \in \Gamma_*^e$ and, therefore, $\alpha \leqslant c$. Moreover, since $h(S)$ separates 0 and e for any $h \in \Gamma_*^e$ and $g([0, 1])$ is connected for any $g \in \Gamma$, there exists $w \in g([0, 1]) \cap h(S)$. Therefore,

$$\inf_{u\in S} I(h(u)) \leqslant I(w) \leqslant \max_{y\in[0,1]} I(g(y))$$

and, consequently, $c \leqslant b$. Finally if c is not a critical value of I, by (3) and (6) of Lemma 1.3 and Remark 1.4, there exists $\epsilon \in (0, \alpha)$ and a homeomorphism η_1 from E onto E such that $\eta_1(\hat{A}_{c-\epsilon}) \subset \hat{A}_{c+\epsilon}$. Note that $\eta_1(0) = 0$ by (2) of Lemma 1.3 and $\eta_1 \circ h$ is a homeomorphism of E onto E. By (4) of Lemma 1.3, $\eta : \hat{A}_0 \to \hat{A}_0$ and hence $\eta_1 \circ h: B \to \hat{A}_0$. To show that $\eta_1 \circ h \in \Gamma_*^e$, it suffices to show that

$\eta_1 \circ h(x) = e$ implies $x \notin \bar{B}$. But $\eta_1(e) = e$ so $h(x) = e$ and $x \notin \bar{B}$. Thus, $\eta_1 \circ h \in \Gamma_*^e$. But

$$\inf_{u \in S} I(\eta_1(h(u))) \geqslant c + \epsilon,$$

a contradiction. Q.E.D.

Remark 2.6. Simple examples in $E = \mathbf{R}^1$ show $c < b$ is possible. In infinite dimensional situations such as in Section 3 this would be a very difficult question to decide.

If I is even and negative at ∞ in an appropriate fashion, more can be said about the number of critical points of I. This situation will be explored next. Suppose that:

(I_4) $I(u) = I(-u)$ for all $\in E$;

(I_5) For any finite dimensional $\tilde{E} \subset E$, $\tilde{E} \cap \hat{A}_0$ is bounded.

Note that (I_4) implies nonzero critical points of I occur in antipodal pairs. (I_5) need not be required for all such $\tilde{E}$ but only for a nested sequence $E_1 \subset E_2 \subset \cdots$ of increasing dimension. Let

$$\Gamma^* = \{h \in \Gamma_* \mid h \text{ is odd}\}$$

and let

$$\Gamma_m = \{K \subset E \mid K \text{ is compact, symmetric with respect to the origin and for all } h \in \Gamma^*, \gamma(K \cap h(S)) \geqslant m\}.$$

Since $h(S) \subset E - \{0\}$ is closed and symmetric, $\gamma(K \cap h(S))$ is defined.

LEMMA 2.7. *Let I satisfy* (I_1), (I_5). *Then*

(1) $\Gamma_m \neq \varnothing$;

(2) $\Gamma_{m+1} \subset \Gamma_m$;

(3) $K \in \Gamma_m$ *and* $Y \in \Sigma(E)$ *with* $\gamma(Y) \leqslant r < m$ *implies* $\overline{K - Y} \in \Gamma_{m-r}$;

(4) *If ϕ is an odd homeomorphism of E onto E and $\phi^{-1}(\hat{A}_0) \subset \hat{A}_0$, then $\phi(K) \in \Gamma_m$ whenever $K \in \Gamma_m$.*

Proof. (1) Let $\tilde{E}$ be an m dimensional subspace of E and let $K_R = \tilde{E} \cap \bar{B}_R$. Then K_R is compact and symmetric. For R sufficiently large, by (I_5), $K_R \supset \tilde{E} \cap \hat{A}_0$; moreover, for any $h \in \Gamma^*$, $\tilde{E} \cap \hat{A}_0 \supset \tilde{E} \cap h(B)$ and then $K_R \supset \tilde{E} \cap h(B)$. Therefore, $K_R \cap h(S) = \tilde{E} \cap h(S)$. Since h is an homeomorphism of E onto E and $h(0) = 0$,

$h(B)$ is a neighborhood of 0 in E. Therefore, $\tilde{E} \cap h(B)$ is a neighborhood of 0 in E with boundary contained in $\tilde{E} \cap h(S)$. The interior (in $\tilde{E}$) of $\tilde{E} \cap h(B)$ is a symmetric bounded open neighborhood of 0 in $\tilde{E}$. Since $\tilde{E}$ is isomorphic to $\mathbf{R}^m$, it follows from (2) and (6) of Lemma 1.2 and Definition 1.1 that $\gamma(\tilde{E} \cap h(S)) = \gamma(K_R \cap h(S)) = m$. Hence, $K_R \in \Gamma_m$.

(2) is obvious.

(3) $\overline{K - Y}$ is compact and symmetric. Moreover, for $h \in \Gamma^*$, $\overline{K - Y} \cap h(S) = \overline{(K \cap h(S)) - Y}$ so by (4) of Lemma 1.2,

$$\gamma(\overline{K - Y} \cap h(S)) = \gamma(\overline{(K \cap h(S)) - Y}) \geqslant \gamma(K \cap h(S) - \gamma(Y) \geqslant m - r$$

(4) $\phi(K)$ is compact and symmetric. If $h \in \Gamma^*$, $\gamma(\phi(K) \cap h(S)) = \gamma(K \cap \phi^{-1}(h(S)))$ by (3) of Lemma 1.2. Since $\phi^{-1}(\hat{A}_0) \subset \hat{A}_0$ and $\phi^{-1} \circ h$ is odd, $\phi^{-1} \circ h \in \Gamma^*$. Hence, the result. Q.E.D.

THEOREM 2.8. *Let I satisfy (I_1), (I_3)–(I_5). For each $m \in \mathbf{N}$, let*

$$b_m = \inf_{K \in \Gamma_m} \max_{u \in K} I(u) \tag{2.9}$$

Then $0 < \alpha \leqslant b_m \leqslant b_{m+1}$ and b_m *is a critical value of I. Moreover, if $b_{m+1} = \cdots = b_{m+r} = b$, then $\gamma(K_b) \geqslant r$.*

Proof. Since $h(u) = \rho u \in \Gamma^*$, $K \cap B_\rho \neq \varnothing$ for each $K \in \Gamma_m$. Therefore, $b_m \geqslant \alpha$. (2) of Lemma 2.7 shows $b_{m+1} \geqslant b_m$. To prove b_m is a critical value of I, it suffices to prove the sharper multiplicity statement. If $\gamma(K_b) < r$, by (5) of Lemma 1.2, there is a neighborhood $N_\delta(K_b)$ with $\gamma(N_\delta(K_b)) < r$. By (3), (6), (7) of Lemma 1.3 and Remark 1.4, there is an $\epsilon \in (0, \alpha)$ and an odd homeomorphism η_1 of E onto E such that $\eta_1(A_{b+\epsilon} - N_\delta(K_b)) \subset A_{b-\epsilon}$. Let $K \in \Gamma_{m+r}$ such that

$$\max_{u \in K} I(u) \leqslant b + \epsilon.$$

Therefore, $\overline{K - N_\delta(K_b)} = K - \operatorname{int} N_\delta(K_b) \equiv Q \in \Gamma_{m+1}$ by (2) of Lemma 2.7. (int X denotes the interior of X). By (4) of Lemma 1.3, $\eta_1^{-1}(\hat{A}_0) \subset \hat{A}_0$. Hence, (4) of Lemma 2.7 shows $\eta_1(Q) \in \Gamma_{m+1}$. But

$$b \leqslant \max_{u \in \eta_1(Q)} I(u) \leqslant b - \epsilon$$

a contradiction and the proof is complete. Q.E.D.

An immediate consequence of Theorem 2.8 is the following corollary.

COROLLARY 2.9. *Under the hypotheses of Theorem 2.8, I possesses infinitely many distinct pair of critical points.*

We do not know if I must possess infinitely many distinct critical values.

The set Γ_m are somewhat cumbersome to define. Next a simpler class of sets $\tilde{\Gamma}_m$ is introduced which can also be used to furnish critical values of I. Let B^m denote the unit ball in $\mathbf{R}^m$ and $S^{m-1} = \partial B^m$. Let $\tilde{\Gamma}_m = \{g \in C(B^m, E) | \, g \text{ is odd }, 1 - 1, \text{ and } g(S^{m-1}) \subset A_0\}$.

The set $\tilde{\Gamma}_m$ possess similar properties to Γ_m.

LEMMA 2.10. *If I satisfies (I_1), (I_5), for all $m \in \mathbf{N}$,*

(1) $\tilde{\Gamma}_m \neq \varnothing$;

(2) $\tilde{\Gamma}_{m+1} \subset \tilde{\Gamma}_m$;

(3) *If ϕ is an odd homeomorphism of E onto E, with $\phi(A_0) \subset A_0$, then $\phi \circ g \in \tilde{\Gamma}_m$ whenever $g \in \tilde{\Gamma}_m$.*

Proof. (2) and (3) are obvious. To verify (1), let $\tilde{E}$ be an m dimensional subspace of E. By (I_5), there exists $R > 0$ such that $\hat{A}_0 \cap \tilde{E} \subset B_R$. Identifying $\tilde{E}$ with $\mathbf{R}^m$, an odd $1 - 1$ function g can be defined with $g(S^{m-1}) = S_R \cap \tilde{E}$. Hence $g \in \tilde{\Gamma}_m$. Q.E.D.

THEOREM 2.11. *Let I satisfy (I_1), (I_3)–(I_5). For each $m \in N$, let*

$$\tilde{b}_m = \inf_{g \in \tilde{\Gamma}_m} \max_{y \in B^m} I(g(y)).$$

Then $0 < \alpha \leqslant \tilde{b}_m \leqslant \tilde{b}_{m+1}$ and $\tilde{b}_m$ is a critical value of I.

Proof. As in Theorem 1.1, $g(B^m) \cap S_\rho \neq \varnothing$, so $\tilde{b}_m \geqslant \alpha$ and $\tilde{b}_m \leqslant \tilde{b}_{m+1}$ by (2) of Lemma 2.10. Finally the argument of Theorem 2.6 shows $\tilde{b}_m$ is a critical value of I. Note only that if $\tilde{b}_m$ is not a critical value, by (2) of Lemma 1.3, $\eta_1(u) = u$ on A_0 and, therefore, $\phi = \eta_1$ satisfies (3) of Lemma 2.10. Q.E.D.

Remark 2.12. The sets $\tilde{\Gamma}_m$ have the advantage of being simple to define. However we are unable to prove a multiplicity result for the corresponding critical values as in Theorem 2.8. We suspect that a "weak multiplicity lemma" obtains for the $\tilde{b}_m$, i.e., if $\tilde{b}_m = \tilde{b}_{m+1} = b$, then K_b contains infinitely many distinct points. In any event for some of the applications of Section 3 independent arguments show $\tilde{b}_m \to \infty$ as $m \to \infty$, and, therefore, there are infinitely many distinct critical values $\tilde{b}_m$ of I.

Next we obtain analogs of Theorem 2.4 for Theorem 2.8. For each

$m \in \mathbf{N}$, let E_m be any m-dimensional subspace of E and $E_m{}^\perp$ an algebraically and topologically complementary subspace.

THEOREM 2.13. *Let I satisfy (I_1), (I_3)–(I_5). For each $m \in \mathbf{N}$, let*

$$c_m = \sup_{h \in \Gamma^*} \inf_{u \in S \cup E_{m-1}^\perp} I(h(u)). \tag{2.14}$$

Then $0 < \alpha \leqslant c_m \leqslant b_m < \infty$, $c_m \leqslant c_{m+1}$, and c_m is a critical value of I.

Proof. That $c_m \geqslant \alpha$ follows as in Theorem 2.4. Since we can identify $S \cap E_m{}^\perp$ with a subset of $S \cap E_{m-1}^\perp$, $c_m \leqslant c_{m+1}$. To see that $c_m \leqslant b_m$, and, therefore, $c_m < \infty$, let $K \in \Gamma_m$ and $h \in \Gamma^*$. If $K \cap h(S \cap E_{m-1}^\perp) \neq \varnothing$ for all such K, h, it follows as in Theorem 2.4 that $c_m \leqslant b_m$. Since $h(S)$ is the boundary of a neighborhood of 0 in E, for all ϵ sufficiently small, $K \cap h(S) = (K - B_\epsilon) \cap h(S)$. Therefore, $\gamma(K - B_\epsilon) \geqslant \gamma((K - B_\epsilon) \cap h(S)) \geqslant m$ since $K \in \Gamma_m$. By (3) of Lemma 1.2, $\gamma(h^{-1}(K - B_\epsilon) \cap S) \geqslant m$. Consequently, (7) of Lemma 1.2 implies $h^{-1}(K - B_\epsilon) \cap S \cap E_{m-1}^\perp \neq \varnothing$. Equivalently, $(K - B_\epsilon) \cap h(S \cap E_{m-1}^\perp) \neq \varnothing$. Finally the proof of Theorem 2.4 shows that c_m is a critical value of I. Q.E.D.

Remark 2.15. As in Remark 2.12, we suspect that a weak multiplicitiy lemma obtains for the critical values c_m. Indeed, there is considerable amount of flexibility in defining the sets used to obtain dual critical values and a class of sets can probably be found which gives a full multiplicity lemma here. Observe also that c_m's can be defined as in (2.14) with Γ_* replacing Γ^* when I is not even. However, in general, $c_m = \infty$ for $m > 1$, for this case.

COROLLARY 2.16. $c_m \leqslant \tilde{b}_m$.

Proof. If $g \in \tilde{\Gamma}_m$ and $h \in \Gamma^*$, then $\gamma(\bar{B}^m - B_\epsilon{}^m) = m$ by Definition 1.1 and (2), (3), (6) of Lemma 1.2. (Here $B_\epsilon{}^m$ denotes a ball in $\mathbf{R}^n$ of radius ϵ about 0). Therefore, $h(S \cap E_{m-1}^\perp) \cap g(B^m) \neq \varnothing$ via (7) of Lemma 1.2 as in Theorem 2.13. Q.E.D.

Remark 2.17. The above two results remain unchanged if E_m and $E_m{}^\perp$ are permitted to vary over all such subsets of E. If there exists $h_m \in \Gamma^*$ such that

$$\inf_{u \in E_m{}^\perp \cap S} I(h_m(u)) \to \infty$$

as $m \to \infty$, then c_m and a fortiori b_m, $\tilde{b}_m \to \infty$ as $m \to \infty$. In this event, I possesses infinitely many distinct critical values. This situation arises in some of the applications of Section 3.

Next we explore the effect of weakening (I_1) so that I need not be positive in a deleted neighborhood of 0. (I_1) will be replaced by

(I_6) There exists an l dimensional subspace $\tilde{E}$ of E with algebraically and topologically complementary subspace $\tilde{E}^{\perp}$ such that $I > 0$ on $(B_\rho - \{0\}) \cap \tilde{E}^{\perp}$ and $I \geqslant \alpha > 0$ on $S_\rho \cap \tilde{E}^{\perp}$.

Once analogs of Γ^* and Γ_m are introduced, results generalizing our earlier ones obtain. Let

$$\Lambda^* = \{h \in C(E, E) \mid h \text{ is an odd homeomorphism of } E \text{ onto } E \text{ with } h(B) \subset \hat{A}_0 \cup \bar{B}_\rho\}.$$

Again $h(u) = \rho u \in \Lambda^*$ so $\Lambda^* \neq \varnothing$. Let

$$\Lambda_m = \{K \subset E \mid K \text{ is compact, symmetric, and } \gamma(K \cap h(S)) \geqslant m \text{ for all } h \in \Lambda^*\}.$$

LEMMA 2.18. *Let I satisfy (I_5)–(I_6). Then for all $m \in \mathbf{N}$,*

(1) $\Lambda_m \neq \varnothing$;

(2) $\Lambda_{m+1} \subset \Lambda_m$;

(3) $K \in \Lambda_m$ *and* $Y \in \Sigma(E)$ *with* $\gamma(Y) \leqslant r < m$ *implies* $\overline{K - Y} \in \Lambda_{m-r}$;

(4) *If ϕ is an odd homeomorphism of E onto E with $\phi(u) = u$ when $I(u) < 0$ and $\phi^{-1}(\hat{A}_0) \subset \hat{A}_0$, then $\phi(K) \in \Lambda_m$ whenever $K \in \Lambda_m$.*

Proof. (1) Let $\hat{E}$ be an m dimensional subspace of E with $m \geqslant l$ and $\hat{E} \supset \tilde{E}$. Let $K_R = \bar{B}_R \cap \hat{E}$. Then for R sufficiently large and any $h \in \Lambda^*$, $K_R \supset (\hat{A}_0 \cup B_\rho) \cap \hat{E} \supset h(B) \cap \hat{E}$ by (I_5). The proof for this case continues and concludes as in Lemma 2.7. (2) which is trivial then implies (1) for $m < l$.

(3) is proved as earlier.

(4) As in (4) of Lemma 2.7, this reduces to showing $\phi^{-1} \circ h \in \Lambda^*$ for all $h \in \Lambda^*$. Clearly $\phi^{-1} \circ h$ is an odd homeomorphsim of E onto E. Since $h(B) \subset \hat{A}_0 \cup \bar{B}_\rho$, $\phi^{-1}(\hat{A}_0) \subset \hat{A}_0$, and ϕ^{-1} is the identity on the set where $I < 0$, $\phi^{-1} \circ h \in \Lambda^*$. Q.E.D.

THEOREM 2.19. *Let I satisfy (I_3)–(I_6). For each $m > l$, let*

$$b_m = \inf_{K \in \Lambda_m} \max_{u \in K} I(u). \tag{2.20}$$

Then $0 < \alpha \leqslant b_m \leqslant b_{m+1}$ *and* b_m *is a critical value of* I. *Moreover, if* $b_{m+1} = \cdots = b_{m+r} = b$, *then* $\gamma(K_b) \geqslant r$.

Proof. (2) of Lemma 2.18 shows $b_m \leqslant b_{m+1}$. Let $K \in \Lambda_m$ and $h(u) = \rho u \in \Lambda^*$. Therefore, $\gamma(K \cap h(S)) = \gamma(K \cap S_\rho) \geqslant m$. By (7) of Lemma 1.2, $K \cap S_\rho \cap \tilde{E}^\perp \neq \varnothing$. Hence, by (I_6), $\max_{u \in K} I(u) \geqslant \alpha$ and $b_m \geqslant \alpha$. To obtain the remaining assertions of the theorem we argue as in Theorem 1.8 observing only that $\eta_1(u) = u$ on A_0 and $\eta_1^{-1}(\hat{A}) \subset \hat{A}_0$ via (2) and (4) of Lemma 1.3. Hence (4) of Lemma 2.18 is applicable with $\phi = \eta_1$. Q.E.D.

A dual result is also valid here.

THEOREM 2.21. *Let* I *satisfy* (I_3)–(I_6). *For each* $m > l$, *let*

$$c_m = \sup_{h \in \Lambda^*} \inf_{u \in S \cap E_{m-1}^\perp} I(h(u)). \tag{2.22}$$

Then $0 < \alpha \leqslant c_m \leqslant b_m < \infty$, $c_m \leqslant c_{m+1}$, *and* c_m *is a critical value of* I.

Proof. The proof is essentially the same as that of Theorem 2.13 and will be omitted. Q.E.D.

We conclude this section with generalizations of Theorem 2.8 and 2.13 which involve replacing (I_5) by

(I_7) There exists a j dimensional subspace $\hat{E} \subset E$ and a compact set $A \subset \hat{E}$ with $I < 0$ on A such that 0 lies in a bounded component (in $\hat{E}$) of $\hat{E} - A$.

It is clear that (I_5) implies (I_7).

THEOREM 2.23. *Let* I *satisfy* (I_1), (I_3)–(I_4), *and* (I_7). *Then the conclusions of Theorem* 2.8 *and* 2.13 *obtain for* b_m *and* c_m *as defined by* (2.9) *and* (2.14), $1 \leqslant m \leqslant j$.

Proof. The only role played by (I_5) in the earlier theorem is in showing that $\Gamma_m \neq \varnothing$. Thus we only need verify this here for $1 \leqslant m \leqslant j$. Let $\hat{E}$ be an m dimensional subspace of $\hat{E}$ and $K_R = \hat{E} \cap \bar{B}_R$. We prove that $K_R \in \Gamma_m$. Indeed for R sufficiently large, (I_7) implies $K_R \supset A \cap \hat{E}$, and, therefore, the component Q of $\hat{A}_0 \cap \hat{E}$ containing 0 lies in K_R. Thus, for $h \in \Gamma^*$, $Q \cap h(S)$ contains the boundary of a symmetric bounded neighborhood of 0 in $\hat{E}$ and by (2) and (6) of Lemma 1.2, $\gamma(K_R \cap h(S)) \geqslant (Q \cap h(S)) \geqslant m$. Q.E.D.

COROLLARY 2.24. *If I is also bounded from below and satisfies $(P - S)$, then*

$$-\infty < \inf_{\substack{X \in \Sigma(E) \\ \gamma(X) \geqslant m}} \max_{u \in X} I(u) \equiv d_m < 0, \qquad 1 \leqslant m \leqslant j, \tag{2.25}$$

and d_m is a critical value of I. If $d_{m+1} = \cdots = d_{m+r} = d$, $\gamma(K_d) \geqslant r$.

Proof. $\gamma(A) = j$ by (2) and (6) of Lemma 1.3. Therefore, $d_m < 0$, $1 \leqslant m \leqslant j$ since $X = A$ is admissible for the calculation of d_m. The remaining assertions of the corollary follows from Theorem 7 of [7]. Q.E.D.

Thus, under the hypotheses of Corollary 2.24, I has at least m distinct pairs of critical points in each of the sets int A_0, int $\hat{A}_0$. It is possible to combine the situations of (I_6)–(I_7) but we shall not do so.

3. APPLICATIONS TO ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS

This section is devoted to applications of the results of Section 2 to second order uniformly elliptic partial differential equations. Consider

$$\begin{aligned} Lu &\equiv -\sum_{i,j=1}^{n} (a_{ij}(x)\, u_{x_i})_{x_j} + c(x)u = p(x, u), \qquad x \in \Omega, \\ u &= 0, \qquad x \in \partial\Omega, \end{aligned} \tag{3.1}$$

where Ω is a smooth bounded domain in $\mathbf{R}^n$, L is uniformly elliptic in Ω, the a_{ij} are continuously differentiable in Ω with Hölder continuous first derivatives, and $c(x)$ is Hölder continuous in Ω with $c \geqslant 0$.

The results given here for (3.1) improve those of [8]–[10], [12], [14]–[15]. Properties of the nonlinearity $p(x, z)$ at $z = 0$ and $|z| = \infty$ will be used to verify the hypotheses imposed on I in Section 2. We begin by treating (3.1) under the following set of conditions for p.

(p_1) p is locally Hölder continuous in $\bar{\Omega} \times \mathbf{R}$ and $p(x, 0) = 0$.

(p_2) $|p(x, z)| \leqslant a_1 + a_2 |z|^s$ for $n > 2$ where $1 < s < \dfrac{n+2}{n-2}$
$\leqslant a_3 \exp a(z)$ for $n = 2$ where $a(z)z^{-2} \to 0$ as $z \to \infty$.

(p_3) $p(x, z) = o(|z|)$ at $z = 0$ uniformly in $x \in \bar{\Omega}$.

(p_4) $p(x, z)\, z^{-1} \to \infty$ as $z \to \infty$ or as $z \to -\infty$ uniformly in $x \in \bar{\Omega}$.

(p_5) If $P(x, z) = \int_0^z p(x, t)\, dt$, then there exists an $a_4 > 0$ such that for $|z| \geqslant a_4$, $P(x, z) \leqslant \theta p(x, z) z$ where $\theta \in [0, \frac{1}{2})$.

(p_6) p is odd in z.

In (p_1)–(p_6) and the sequel, a_1, a_2, etc. repeatedly denote positive constants. (p_5) is satisfied in particular if p is odd in z and $p(x, z) = z^s +$ a lower order term at $z = \infty$. The case $n = 1$ can also be included here and then (p_2) is unnecessary. However, except for Theorem 3.39, stronger results have been obtained by other means for this case. See e.g. [17, 19–20].

Formally, critical points of the functional

$$I(u) = \frac{1}{2} \int_\Omega \left(\sum_{i,j=1}^{n} a_{ij}(x)\, u_{x_i} u_{x_j} + c(x)\, u^2 \right) dx - f(u), \tag{3.2}$$

where

$$f(u) = \int_\Omega P(x, u(x))\, dx$$

satisfy (3.1).

Let $E = W_0^{1,2}(\Omega)$ where the usual Sobolev space notation is being employed. It follows from the Poincaré inequality that

$$\| u \| = \left(\int_\Omega \left(\sum_{i,j=1}^{n} a_{ij}(x)\, u_{x_i} u_{x_j} + c(x)\, u^2 \right) dx \right)^{1/2}$$

can be taken as a norm for E. (3.1) is equivalent to $u - \mathscr{P}u = 0$ where $\mathscr{P}(u)$ is defined by

$$\mathscr{P}(u)v = \int_\Omega p(x, u(x))\, v(x)\, dx$$

for all $v \in E$, i.e., $\mathscr{P}(u)$ is the potential operator on E corresponding to $f(u)$. Standard arguments using (p_2) show $\mathscr{P}$ is compact and f is weakly continuous. See e.g. [10, 11] for more details. It is clear that $I \in C^1(E, \mathbf{R})$ and $I(0) = 0$. It can also be shown that I need not be bounded from above or below [10]. We will use (p_1)–(p_6) to verify that the theory of Section 2 is applicable here.

LEMMA 3.3. *If p satisfies (p_2)–(p_3), I satisfies (I_1).*

Proof. (for $n > 2$). It suffices to show $f(u) = o(\| u \|^2)$ at $u = 0$. By (p_3) for all $\epsilon > 0$, there exists a $\delta > 0$ such that $|P(x, z)| \leqslant \epsilon z^2$

if $|z| \leqslant \delta$. Furthermore, (p_2) implies $|P(x, z)| \leqslant a_5 |z|^{s+1}$ for $|z| \geqslant \delta$. Therefore, by the Poincaré and Sobolev inequalities

$$|f(u)| \leqslant \int_\Omega (\epsilon u^2 + a_5 |u|^{s+1})\, dx \leqslant a_6(\epsilon \|u\|^2 + \|u\|^{s+1}),$$

where $s > 1$. Hence, $f(u) = o(\|u\|^2)$ at $u = 0$. The proof for $n = 2$ is similar. Q.E.D.

LEMMA 3.4. *If p satisfies (p_4), I satisfies (I_2).*

Proof. Suppose $p(x, z)\, z^{-1} \to \infty$ as $z \to \infty$. Let $u \in E$ with $\|u\| = 1$, $u > 0$ in Ω, and $\int_\Omega u^2\, dx = a_5$. Then

$$I(Ru) = \tfrac{1}{2}R^2 - f(Ru).$$

By (p_4), $p(x, z) \geqslant Kz$ for $z \geqslant M(K)$. Choosing $K = 4a_5^{-1}$ leads to

$$f(Ru) \geqslant \int_{\{x \in \Omega \mid u(x) > M/R\}} K(R^2/2)\, u^2\, dx - a_6\,,$$

where a_6 is a constant depending on p and a_5. For R sufficiently large,

$$\int_{\{x \in \Omega \mid u(x) > M/R\}} u^2\, dx \geqslant \tfrac{1}{2} a_5\,.$$

Therefore, $f(Ru) \geqslant R^2$ and $I(Ru) \leqslant a_6 - \frac{1}{2}R^2$ for R sufficiently large. Hence, I satisfies (I_2). The argument for $p(x, z)\, z^{-1} \to \infty$ as $z \to -\infty$ is essentially the same. Q.E.D.

COROLLARY 3.5. *If p satisfies (p_4) and (p_6), I satisfies (I_5).*

Proof. This is precisely Corollary 2.3 of [15]. The argument is essentially as in Lemma 3.4. Q.E.D.

LEMMA 3.6. *If p satisfies (p_2) and (p_5), I satisfies (I_3).*

Proof. In fact we prove a slightly stronger result, namely if $(u_m) \subset E$ with $I(u_m) \leqslant d$ and $I'(u_m) \to 0$, then u_m possesses a convergent subsequence. From (3.2),

$$\begin{aligned} d &\geqslant \tfrac{1}{2} \|u_m\|^2 - \left(\int_{\{x \in \Omega \mid |u_m(x)| \leqslant a_4\}} P(x, u_m)\, dx + \int_{\{x \in \Omega \mid |u_m(x)| > a_4\}} P(x, u_m)\, dx \right) \\ &\geqslant \tfrac{1}{2} \|u_m\|^2 - a_5 - \theta \int_{\{x \in \Omega \mid |u_m(x)| > a_4\}} p(x, u_m)\, u_m\, dx \\ &\geqslant \tfrac{1}{2} \|u_m\|^2 - a_6 - \theta \int_\Omega p(x, u_m)\, u_m\, dx. \end{aligned} \tag{3.7}$$

Since $I'(u_m) \to 0$, for any $\epsilon > 0$, there is an $M = M(\epsilon)$ such that for all $m \geqslant M$,

$$| I'(u_m)v | = \left| \int_\Omega \left[\sum_{i,j=1}^{n} a_{ij}(x)\, u_{x_i} v_{x_j} + c(x)\, uv - p(x, u_m)v \right] dx \right| \leqslant \epsilon \| v \| \tag{3.8}$$

for all $v \in E$. Choosing $\epsilon = 1$, $v = u_m$, and combining (3.7)–(3.8) yields

$$d \geqslant (\tfrac{1}{2} - \theta) \| u_m \|^2 - a_6 - \theta \| u_m \|. \tag{3.9}$$

Hence (u_m) is bounded in E. Since $\mathscr{P}$ is compact, $\mathscr{P}(u_m)$ possesses a convergent subsequence. But then $I'(u_m) = u_m - \mathscr{P}(u_m) \to 0$ implies (u_m) possesses a convergent subsequence. Q.E.D.

With the aid of these technical preliminaries we have the following theorem

THEOREM 3.10. *If p satisfies (p_1)–(p_5), then b and c as defined by* (2.2), (2.5) *are critical values of I with $0 < c \leqslant b < \infty$. The corresponding critical points are classical solutions of* (3.1).

Proof. The first statement is immediate from Lemmas 3.3, 3.4, 3.6 and Theorems 2.1 and 2.4. The second follows from standard regularity results [21]. Q.E.D.

COROLLARY 3.11. *Under the hypotheses of Theorem* 3.10, *if $p(x, z)\, z^{-1} \to \infty$ as $z \to \infty$ $(z \to -\infty)$,* (3.1) *possesses a solution $\bar{u} > 0$ $(\underline{u} < 0)$ in Ω.*

Proof. Let $\bar{p}(x, z) = p(x, z)$ for $z \geqslant 0$ and $\bar{p} \equiv 0$ for $z < 0$. Then $\bar{p}$ satisfies (p_1)–(p_5) so by Theorem 3.10, the corresponding functional I has a critical point $\bar{u} \not\equiv 0$ in Ω satisfying

$$\begin{aligned} L\bar{u} &= \bar{p}(x, \bar{u}), \qquad x \in \Omega; \\ \bar{u} &= 0, \qquad x \in \partial\Omega. \end{aligned} \tag{3.12}$$

Let $A = \{x \in \Omega \mid \bar{u}(x) < 0\}$. Then $L\bar{u} = 0$ in A and $\bar{u} = 0$ on ∂A. The maximum principle implies $\bar{u} \equiv 0$ in A and, therefore, $A = \varnothing$. Hence, $\bar{u} \geqslant 0$ in Ω. That $\bar{u} > 0$ in Ω follows, e.g. from arguments in [14]. Q.E.D.

If p also satisfies (p_6), the assertions of Theorem 3.10 can be considerably strengthened. In the context of Theorem 2.13 we take

(e_m) to be an orthonormal basis for E, $E_m = \text{span}\{e_1, \ldots, e_m\}$, and $E_m{}^{\perp}$ the orthogonal complement of E_m.

THEOREM 3.13. *If p satisfies (p_1)–(p_6), then for each $m \in \mathbf{N}$, b_m and c_m as defined by* (2.9) *and* (2.14) *are critical values of I with $0 < c_m \leqslant b_m < \infty$. If $b_{m+1} = \cdots = b_{m+r} \equiv b$, $\gamma(K_b) \geqslant r$. Therefore,* (3.1) *possesses infinitely many distinct pairs of solutions.*

Proof. Immediate from Lemmas 3.3, 3.6, Corollary 3.5, and Theorems 2.8 and 2.13. Q.E.D.

Although Theorem 3.13 guarantees the existence of infinitely distinct pairs of critical points for I, a priori there may only be finitely many b_m's or they may be bounded. The following results shows this is not possible.

THEOREM 3.14. *c_m (and, therefore, $b_{m'}$) $\to \infty$ as $m \to \infty$.*

Proof. As was noted in Lemma 3.3,

$$f(u) \leqslant \epsilon \int_\Omega u^2\, dx + a_7(s+1)^{-1} \int_\Omega |u|^{s+1}\, dx.$$

Choose ϵ so that $\| u \|^2 \geqslant 4^{-1}\epsilon \int_\Omega u^2\, dx$ for all $u \in E$. Let

$$J(u) = \tfrac{1}{4} \| u \|^2 - a_7(s+1)^{-1} \int_\Omega |u|^{s+1}\, dx.$$

Then $I(u) \geqslant J(u)$ for all $u \in E$. Hence,

$$c_m = \sup_{h \in \Gamma^*} \inf_{u \in S \cap E_{m-1}^{\perp}} I(h(u)) \geqslant \sup_{h \in \Gamma^*} \inf_{u \in S \cap E_{m-1}^{\perp}} J(h(u)) \geqslant \inf_{u \in S \cap E_{m-1}^{\perp}} J(h(u)) \tag{3.15}$$

for any $h \in \Gamma^*$. Let

$$T = \left\{ u \in E - \{0\} \,\middle|\, \tfrac{1}{2} \| u \|^2 = a_7 \int_\Omega |u|^{s+1}\, dx \right\},$$

and let $d_m = \inf\{\| u \| \mid u \in T \cap E_m{}^{\perp}\}$. Then $d_m \to \infty$ as $m \to \infty$ for otherwise there exists $u_m \in T \cap E_m{}^{\perp}$ with $d \geqslant \| u_m \|$ for all $m \in \mathbf{N}$. Since $u_m \in E_m{}^{\perp}$, u_m tends to 0 weakly in E and strongly in $L^{s+1}(\Omega)$. However, by [15, Lemma 2.7] there is an $r > 0$ such that $(\int_\Omega |u|^{s+1}\, dx)^{1/s+1} \geqslant r > 0$ for all $u \in T$. Hence, u_m must be bounded away from 0 in $L^{s+1}(\Omega)$ and $d_m \to \infty$ as $m \to \infty$.

Define $h_m(u) = M^{-1} d_m u$ for $u \in E_m{}^{\perp}$ where $M > 1$ satisfies

$$M > [4(s+1)^{-1}]^{1/s-1}. \tag{3.16}$$

For $u \neq 0$, choose $\beta(u) > 0$ such that $\beta(u)u \in T$. It is easily seen that there exists a unique such $\beta(u)$. (Indeed $\beta(u)$ is continuous [10]) Moreover, for each $0 \neq u \in E$, $J(tu)$ is a monotone increasing function for $t \in [0, \beta(u)]$ with a maximum at $t = \beta(u)$. Since $M^{-1} d_m \leqslant d_m \leqslant \beta(u)$ for $u \in (E_m{}^\perp \cap B) - \{0\}$ and $h_m(0) = 0$,

$$h_m(E_m{}^\perp \cap B) \subset \{u \in E \mid J(u) \geqslant 0\} \subset \{u \in E \mid I(u) \geqslant 0\}.$$

Suppose for the moment that h_m can be extended to $h_m \in \Gamma^*$. We will show

$$\inf_{u \in S \cap E_m{}^\perp} J(h_m(u)) \to \infty \qquad \text{as} \quad m \to \infty \tag{3.17}$$

and, hence, from (3.15), $c_m \to \infty$ as $m \to \infty$.

For each $u \in S \cap E_m{}^\perp$,

$$J(h_m(u)) = \tfrac{1}{4}(M^{-1}d_m)^2 - (a_7/(s+1)) \int_\Omega | M^{-1}d_m u |^{s+1} \, dx.$$

Since $\frac{1}{2}\beta(u)^2 = a_7 \int_\Omega | \beta(u)u |^{s+1} \, dx$,

$$\begin{aligned} J(h_m(u)) &= (M^{-1}d_m)^2 \left[\frac{1}{4} - \frac{M^{1-s}}{2(s+1)} \left(\frac{d_m}{\beta(u)}\right)^{s-1}\right] \\ &\geqslant (M^{-1}d_m)^2 \left[\frac{1}{4} - \frac{M^{1-s}}{2(s+1)}\right] \geqslant \frac{1}{8}(M^{-1}d_m)^2 \end{aligned} \tag{3.18}$$

by (3.16). Hence, (3.18) shows (3.17) is valid.

To complete the proof, we show the component

$$Q = \left\{\tfrac{1}{2} \| u \|^2 > a_7 \int_\Omega | u |^{s+1} \, dx\right\} \cup \{u = 0\}$$

of $E - T$ to which 0 belongs (i.e., the "inside" of the spherelike set T) contains $Z_\epsilon \equiv \{d_m M^{-1}(E_m{}^\perp \cap B)\} \times \{\epsilon(E_m \cap B)\}$ for some $\epsilon > 0$. For if not, there exists a sequence $\epsilon_j \to 0$ and $u_j \notin Q$ such that $u_j \in Z_{\epsilon_j}$. Therefore, (u_j) is bounded so a subsequence converges weakly in E and strongly in $L^{s+1}(\Omega)$ to $\bar{u}$ with $\int_\Omega | \bar{u} |^{s+1} > 0$ by above remarks. Since $\| \cdot \|$ is weakly lower semicontinuous and $\int_\Omega | \cdot |^{s+1} \, dx$ is weakly continuous, $\frac{1}{2} \| \bar{u} \|^2 \leqslant a_7 \int_\Omega | \bar{u} |^{s+1} \, dx$. Therefore $\bar{u} \in E - Q$. On the other hand, $\bar{u} \in d_m M^{-1}(E_m{}^\perp \cap B) \subset Q$, a contradiction. Thus, there exists ϵ as above.

Now on defining $\bar{h}_m(e_j) = \epsilon e_j$ for $1 \leqslant j \leqslant m$, it is easily seen that $\bar{h}_m : Z_\epsilon \to Z_\epsilon$ and $\bar{h}_m \in \Gamma^*$. So the proof is complete. Q.E.D.

Remark 3.19. Theorem 3.14 together with Corollary 2.16 imply

$\tilde{b}_m \to \infty$ as $m \to \infty$, and therefore, there are infinitely many distinct critical points $\tilde{b}_m$ for I.

Next the effect of adding a linear term to the right side of (3.1) is studied. This is a more subtle case and corresponds to that treated in Theorems 2.19 and 2.21. Consider

$$\begin{aligned} Lu &= a(x)u + p(x, u), \qquad x \in \Omega; \\ u &= 0, \qquad x \in \partial\Omega, \end{aligned} \tag{3.20}$$

where a is Hölder continuous and positive in $\bar{\Omega}$. The linear eigenvalue problem

$$\begin{aligned} Lv &= \lambda a v, \qquad x \in \Omega; \\ v &= 0, \qquad x \in \partial\Omega \end{aligned} \tag{3.21}$$

possesses an unbounded sequence of eigenvalues $0 < \lambda_1 < \lambda_2 \leqslant \cdots$ with each eigenvalue appearing according to its multiplicity. Each eigenvalue is of finite multiplicity and λ_1 is simple. Let the (e_j) of Theorem 3.13 now be the eigenfunctions of (3.21) with e_m corresponding to λ_m.

Let

$$J(u) = I(u) - \tfrac{1}{2} \int_\Omega a u^2 \, dx.$$

Then $I(u) \geqslant J(u)$ so if p satisfies (p_4), Lemma 3.4 implies J satisfies (I_2) and by Corollary 3.5, (p_4) and (p_6) imply (I_5) for J.

LEMMA 3.22. *If* $1 < \lambda_{l+1} \neq \lambda_l$, *and* p *satisfies* (p_2)–(p_3), *then* I *satisfies* (I_6) *with* $\hat{E} = E_l$.

Proof. For $u \in E_l^\perp$, $\| u \|^2 \geqslant \lambda_{l+1} \int_\Omega a u^2 \, dx$, and, therefore,

$$J(u) \geqslant \tfrac{1}{2}(1 - \lambda_{l+1}^{-1}) \| u \|^2 - f(u). \tag{3.23}$$

Since $f(u) = o(\| u \|^2)$ by Lemma 3.3, the result follows. Q.E.D.

Remark 3.24. If $l = 0$, J satisfies (I_1).

THEOREM 3.25. *If* $\lambda_1 > 1$ *and* p *satisfies* (p_1)–(p_5) *with* $p(x, z)\, z^{-1} \to \infty$ *as* $z \to \infty$ $(z \to -\infty)$, (3.20) *possesses a solution* $\bar{u} > 0$ $(\underline{u} < 0)$ in Ω.

Proof. Remark 3.24 shows J satisfies (I_1). Once (I_3) is verified for J, the theorem follows by arguing as in Theorem 3.10 and Corollary

3.11. The proof of Lemma 3.6 shows $\| u_m \|^2 - \int_\Omega a u_m^2 \, dx$ is bounded above in E. This together with $\lambda_1 > 1$ yields the boundedness of (u_m). Hence, (I_3) as earlier . Q.E.D.

Remark 3.26. If $\lambda_1 \leqslant 1$, (3.20) need not possess any positive solutions. For example, consider

$$-\Delta u = u + u^3, \quad x \in \Omega; \qquad u = 0, \quad x \in \partial\Omega, \tag{3.27}$$

where $\Omega \subset \mathbf{R}^3$ is such that $\lambda_1 \leqslant 1$. Comparison with the linear eigenvalues problem

$$-\Delta v = \lambda v, \quad x \in \Omega; \qquad v = 0, \quad x \in \partial\Omega$$

shows a necessary condition for $u > 0$ to be a solution of (3.27) is $\lambda_1 > 1$.

Once again stronger results obtain for the odd case.

LEMMA 3.28. *If p satisfies (p_2), (p_4)–(p_6), then J satisfies (I_3).*

Proof. By (p_4) and (p_6), $| p(x, z)| \geqslant M(t) \mid z \mid$ for $\mid z \mid \geqslant t$, where $M(t) \to \infty$ as $t \to \infty$. Therefore, $\frac{1}{2} a(x) z^2 \leqslant M(t)^{-1} a_1 P(x, z)$ for $\mid z \mid \geqslant t$ and t sufficiently large. Choose $t = \bar{t}$ so large that $\overline{M} = a_1 M(\bar{t})^{-1}$ satisfies $\theta(1 + \overline{M}) < \frac{1}{2}$. Suppose $d \geqslant J(u_m)$ and $J'(u_m) \to 0$. As in (3.7),

$$d + a_2 \geqslant \tfrac{1}{2} \| u_m \|^2 - (1 + \overline{M})\theta \int_\Omega p(x, u_m) \, u_m \, dx. \tag{3.29}$$

Since $J'(u_m) \to 0$, for m sufficiently large

$$\| u_m \| \geqslant \left| \| u_m \|^2 - \int_\Omega (a u_m^2 + p(x, u_m) \, u_m) \, dx \right|,$$

and therefore,

$$\int_\Omega u_m p(x, u_m) \, dx \leqslant \| u_m \| + \| u_m \|^2 - \int_\Omega a u_m^2 \, dx \leqslant \| u_m \| + \| u_m \|^2. \tag{3.30}$$

Combining (3.29)–(3.30) shows

$$a_3 \geqslant (\tfrac{1}{2} - (1 + \overline{M})\theta) \| u_m \|^2 - (1 + \overline{M})\theta \| u_m \| \tag{3.31}$$

so (u_m) is bounded and (I_3) follows as earlier. Q.E.D.

THEOREM 3.32. *If p satisfies (p_1)–(p_6) and $\lambda_l \leqslant 1 < \lambda_{l+1}$, then for all $m > l$, b_m and c_m as defined by (2.20) and (2.22) (with I replaced by J)*

are critical values of J *with* $0 < c_m \leqslant b_m$. *If* $b_{m+1} = \cdots = b_{m+r} \equiv b$, $\gamma(K_b) \geqslant r$. *Equation* (3.20) *possesses infinitely many distinct pairs of classical solutions.*

Proof. The proof follows from Lemmas 3.22, and 3.28, and Theorems 2.19 and 2.21. Q.E.D.

Slight modifications of the proof of Theorem 3.14 show the following corollary.

COROLLARY 3.33. c_m *(and, therefore,* b_m*)* $\to \infty$ *as* $m \to \infty$.

Equations (3.1) and (3.20) were treated in [15] with an additional parameter which plays no role in the context of the above theorems. However, its presence is essential for the next two applications. Consider

$$\begin{aligned} Lu &= \lambda p(x, u), \qquad x \in \Omega; \\ u &= 0, \qquad x \in \partial\Omega, \end{aligned} \tag{3.34}$$

where $\lambda \in \mathbf{R}$. Let

$$I(\lambda, u) = \tfrac{1}{2} \| u \|^2 - \lambda f(u)$$

Somewhat different conditions are imposed on p than earlier.

(p_7) $p(x, z) > 0$ for $z > 0$ in a deleted neighborhood of $z = 0$.

(p_8) There exists $\bar{z} > 0$ such that $p(x, \bar{z}) < 0$.

THEOREM 3.35. *Suppose p satisfies* (p_1), (p_3), (p_7)–(p_8). *Then there exists* $\underline{\lambda} > 0$ *such that for all* $\lambda > \underline{\lambda}$, (3.34) *possesses at least two distinct solutions* $\bar{u}(\lambda)$, $\underline{u}(\lambda) > 0$ *in* Ω *with* $I(\lambda, \bar{u}(\lambda)) > 0 > I(\lambda, \underline{u}(\lambda))$.

Proof. Let $\bar{p}(x, z) = 0$ for $z < 0$; $= p(x, z)$ for $0 \leqslant z \leqslant \bar{z}$; $= p(x, \bar{z})$ for $z > \bar{z}$. Then $\bar{p}$ satisfies (p_1)–(p_3). If u is a solution of

$$\begin{aligned} Lu &= \lambda \bar{p}(x, u), \qquad x \in \Omega; \\ u &= 0, \qquad x \in \partial\Omega, \end{aligned} \tag{3.36}$$

the argument of Corollary 3.11 shows $u > 0$ in Ω. Moreover, if $M = \max_\Omega u(x) = u(\bar{x})$, after a rotation of coordinates if necessary, (3.36) implies

$$0 \leqslant Lu(\bar{x}) = \lambda \bar{p}(\bar{x}, u(\bar{x})). \tag{3.37}$$

Hence, $M < \bar{z}$ by (p_8). Consequently, any nontrivial solution of (3.36) is positive in Ω and satisfies (3.34). Therefore, it suffices to prove (3.36) possesses two positive solutions as above.

Let

$$\bar{P}(x, z) = \int_0^z \bar{p}(x, t)\, dt$$

and

$$J(\lambda, u) = \tfrac{1}{2} \| u \|^2 - \lambda \int_\Omega \bar{P}(x, u(x))\, dx.$$

Note that $I(\lambda, u) = J(\lambda, u)$ if $0 \leqslant u(x) \leqslant \bar{z}$ in Ω. Since $\bar{p}$ satisfies (p_2)–(p_3), Lemma 3.3 shows $J(\lambda, \cdot)$ satisfies (I_1). If $w \in E$ is positive in Ω and pointwise small, (p_7) shows there exists $\lambda = \lambda(w) > 0$ such that $J(\lambda, w) = 0$. Let $\underline{\lambda} = \inf\{\lambda \in \mathbf{R}^+ \mid$ there exists $0 \not\equiv w \in E$ with $J(\lambda, w) = 0\}$. That $\underline{\lambda} > 0$ has been shown in [14]. Thus $J(\lambda, \cdot)$ satisfies (I_2) for all $\lambda > \underline{\lambda}$. Finally since $| \bar{P}(x, z)| \leqslant a_1 | z |$, $J(\lambda, u) \leqslant d$ implies $\| u \| \leqslant a_2$ where a_2 depends only on λ and d. It then easily follows that $J(\lambda, \cdot)$ satisfies (I_3). Hence, by Theorem 1.1, for all $\lambda > \underline{\lambda}$, there exists $\bar{u}(\lambda) \in E$ which is a critical point of $J(\lambda, \cdot)$ with $J(\lambda, \bar{u}(\lambda)) > 0$ and $\bar{u}(\lambda)$ is a classical solution of (3.36). Our above remarks imply $\bar{u} > 0$ in Ω and $I(\lambda, \bar{u}(\lambda)) = J(\lambda, \bar{u}(\lambda))$.

The existence of a solution $\underline{u}(\lambda) > 0$ in Ω with $I(\lambda, \underline{u}(\lambda)) < 0$ is proved in a straightforward fashion by minimizing $J(\lambda, \cdot)$ on E and has been carried out in [14]. Q.E.D.

Remark 3.38. A more general result than Theorem 3.35 was proved in [14] by a more complicated argument using a combination of variational and topological arguments. It was also shown in [14] that (3.36) has no nontrivial solutions for small positive λ so the requirement $\underline{\lambda} > 0$ is essential.

Next Theorem 3.35 will be improved for odd p. This is essentially case II^- of [15] but we get a much stronger result.

THEOREM 3.39. *Suppose p satisfies (p_1), (p_3), (p_6)–(p_8). Then for any $m \in \mathbf{N}$, there exists $\underline{\lambda}_m > 0$ such that for all $\lambda > \underline{\lambda}_m$, (3.34) possesses at least $2m$ distinct pairs of solutions $\pm\bar{u}_j(\lambda)$, $\pm\underline{u}_j(\lambda)$, $1 \leqslant j \leqslant m$ with $I(\lambda, \bar{u}_j(\lambda)) > 0 > I(\lambda, \underline{u}_j(\lambda))$.*

Proof. Let $\bar{p}$ be as in Theorem 3.35, $\hat{p} \equiv \bar{p}$ for $z \geqslant 0$, and $\hat{p}$ odd in z. As in Theorem 3.35, it suffices to prove the assertions for

$$\begin{aligned} Lu &= \lambda \hat{p}(x, u), \qquad x \in \Omega; \\ u &= 0, \qquad x \in \partial\Omega. \end{aligned} \tag{3.40}$$

Let $\hat{P}(x, z) = \int_0^z \hat{p}(x, t)\, dt$ and $\hat{I}(\lambda, u) = \frac{1}{2}\| u \|^2 - \lambda \int_\Omega \hat{P}(x, u(x))\, dx$. Let (e_j) be as in Theorem 3.13 and $A_r = E_m \cap S_r$. Then for r sufficiently small and $u \in A_r$, (p_7) implies $\int_\Omega \hat{P}(x, u(x))\, dx > 0$. By choosing $\underline{\lambda}_m$ sufficiently large and $\lambda > \underline{\lambda}_m$, $\hat{I}(\lambda, \cdot) < 0$ on A_r. Therefore, $\hat{I}(\lambda, \cdot)$ satisfies (I_7) with $\tilde{E} = E_m$ and $A = A_r$. It also satisfies (I_4). The proof of Theorem 3.35 shows $\hat{I}(\lambda, \cdot)$ also is bounded from below and satisfies (I_1) and $(P - S)$. Theorem 2.23 now provides the existence fo $\pm\bar{u}_j(\lambda)$ and Corollary 2.24 yields $\pm\underline{u}_j(\lambda)$, $1 \leqslant j \leqslant m$. · Q.E.D.

To conclude this section we remark that with small modifications, analogs of the results of this section for (3.1) and (3.20) can be obtained if the requirement that the coefficient c of L is nonnegative is dropped. Then (I_6) rather than (I_1) is used.

The results of Section 2 can also be applied to obtain new existence theorem for higher order elliptic equations. However, we preferred to stay in the technically simpler case treated here to illustrate the ideas. Since no constraints are imposed on the functionals handled here, Lagrange multipliers will not occur in the Euler equations as in other work on such questions, e.g. [5, 22, 23].

4. A Galerkin Method

A Galerkin argument was used in [15] to treat (3.1) and (3.20). In this section we will briefly show how a dual such method can be employed to solve (3.1). A disadvantage of the argument used in [15] is that one has a variational characterization of the approximate solutions of (3.1) but not of their limits. Such characterizations are desirable for several reasons such as comparison and approximation purposes. We get a variational characterization for the limit solutions obtained here although we do not get a multiplicity statement, as e.g. in Theorem 3.13.

Let E_m be an orthonormal basis for $E = W_0^{1,2}(\Omega)$, $E_m = \operatorname{span}\{e_1, ..., e_m\}$, and $F_m = I\,|_{E_m}$, where I is defined in (3.2) and p satisfies (p_1)–(p_6). It was shown in [15] that

$$c_{k,m} = \sup_{A \in \Sigma(E_m);\, \gamma(A) \geqslant m-k+1} \min_{u \in A} F_m(u)$$

is a critical value of F_m, $1 \leqslant k \leqslant m$ with $\gamma(K_c) \geqslant r$ if $c_{k+1,m} = \cdots = c_{k+r,m} = c$ (where by K_c we now mean $\{u \in E_m \mid F_m(u) = c, F_m'(u) = 0\}$). Moreover, c_{km} is a nonincreasing sequence in m converging to $\bar{c}_k > 0$. Letting $u_{k,m}$ be a critical point of F_m corresponding to $c_{k,m}$, a subsequence of $u_{k,m}$ converges to u_k where $I(u_k) = \bar{c}_k$ and u_k is a classical solution of (3.1). Finally it was shown that $\bar{c}_k \to \infty$ as $k \to \infty$, and, therefore, (3.1), possesses infinitely many distinct solutions.

To obtain the dual results let, $\hat{A}_m = A_0 \cap E_m$ and $\Gamma_k = \{K \subset E_m \mid K$ is compact, symmetric, and $K \cap A \neq \varnothing$ for all $A \in \Sigma(E_m)$ such that $A \subset \hat{A}_m$ with $\gamma(A) \geqslant m - k + 1\}$. As in Lemma 2.7 we have the following.

LEMMA 4.1. (1) *$\Gamma_{k,m} \neq \varnothing$, $1 \leqslant k \leqslant m$.*

(2) *$\Gamma_{k+1,m} \subset \Gamma_{k,m}$.*

(3) *$K \in \Gamma_{k,m}$ and $Y \in \Sigma(E_m)$ with $\gamma(Y) \leqslant r < k$ implies*

$$\overline{K - Y} \in \Gamma_{k-r,m}.$$

(4) *If ϕ is an odd homeomorphism of E_m to E_m and $\phi^{-1}(\hat{A}_m) \subset \hat{A}_m$, then $\phi(K) \in \Gamma_{k,m}$ whenever $K \in \Gamma_{k,m}$.*

Proof. (1) Take $K = B_R \cap E_k$ where R is chosen so that $K \supset \hat{A}_k$. Let $A \subset \hat{A}_m$ with $\gamma(A) \geqslant m - k + 1$. Therefore, by (7) of Lemma 1.2, $A \cap E_k \neq \varnothing$. Since $A \cap E_k \subset K$, the result follows.

(2)–(4) are proved essentially as earlier. Q.E.D.

LEMMA 4.2. *Let*

$$b_{k,m} = \inf_{K \in \Gamma_{k,m}} \max_{u \in K} F_m(u) \qquad 1 \leqslant k \leqslant m.$$

Then $0 < c_{k,m} \leqslant b_{k,m} \leqslant b_{k+1,m}$ and $b_{k,m}$ is a critical value of F_m. If $b_{k+1,m} = \cdots = b_{k+r,m} = b$, $\gamma(K_b) \geqslant r$.

Proof. Since I on E satisfies (I_1), (I_4)–(I_5), so does F_m on E_m. Moreover, F_m satisfies (I_3) since (I_5) implies $\hat{A}_m$ is bounded and we only need (I_3) for this set. Hence, $b_{k,m}$ is a critical value of F_m and the multiplicity statement obtains. To show $c_{k,m} \leqslant b_{k,m}$, let $K \in \Gamma_{k,m}$ and $A \subset \hat{A}_m$ with $\gamma(A) \geqslant m - k + 1$. Let $w \in K \cap A$. Then

$$\min_{u \in A} F_m(u) \leqslant F_m(w) \leqslant \max_{u \in K} F_m(u)$$

so $c_{k,m} \leqslant b_{k,m}$. Q.E.D.

Let $v_{k,m}$ be a critical point of $F_{k,m}$ corresponding to $b_{k,m}$.

THEOREM 4.3. *For each* $k \in \mathbf{N}$, $b_{k,m}$ *is a monotone nonincreasing sequence (in* m*) converging to a critical value* $\bar{b}_k$ *of* I *with* $\bar{b}_k \geqslant \bar{c}_k$. *A subsequence of* $v_{k,m}$ *converges to a corresponding critical point* v_k.

Proof. Since $\Gamma_{k,m} \subset \Gamma_{k,m+1}$ (as in the proof of Lemma 2.17 of [15]), $b_{k,m} \geqslant b_{k,m+1} \geqslant c_{k,m+1} \to \bar{c}_k$. Hence, $b_{k,m} \to \bar{b}_k \geqslant \bar{c}_k$. Lemmas 2.7, 2.12, 2.13 and the proof of Theorem 2.2 of [15] show $v_{k,m}$ is bounded away from 0 and ∞ in E and a subsequence of $v_{k,m}$ converges to v_k satisfying (3.1) and $I(v_k) = \bar{b}_k$. Q.E.D.

Remark 4.4. Since $\bar{c}_k \to \infty$ as $k \to \infty$, $\bar{b}_k \to \infty$ as $k \to \infty$. Hence, there exist infinitely many distinct critical values $\bar{b}_k$.

COROLLARY 4.5. *Let* $X_k = \bigcup_{m \in \mathbf{N}} \Gamma_{k,m}$. *Then*

$$\bar{b}_k = \inf_{K \in X_k} \max_{u \in K} I(u). \tag{4.6}$$

Proof. Let b denote the right hand side of (4.6). Clearly $b \leqslant b_{k,m}$ for all $m \in \mathbf{N}$ and, therefore, $b \leqslant \bar{b}_k$. On the other hand, if $K \in X_k$, $K \in \Gamma_{k,m}$ for some $m \in \mathbf{N}$ and, therefore,

$$\max_{u \in K} I(u) \geqslant b_{k,m} \geqslant \bar{b}_k.$$

Hence, $b = \bar{b}_k$. Q.E.D.

Remark. A similar Galerkin argument can be used to treat (3.20).

5. APPLICATIONS TO INTEGRAL EQUATIONS

In this final section, the theory developed earlier will be applied to nonlinear integral equations. Some of our results generalize work of Nehari [16] and Coffman [8] who studied similar equations making more restrictive assumptions for their nonlinearities As in Section 3 we show it is only the asymptotic behavior of the nonlinearity near 0 and ∞ that is of importance. Indeed the arguments and results given here closely resemble those of Section 3 and because of this similarity we will often be sketchy here. Some of the results of Section 3 can be obtained by converting the relevant partial differential equations to an integral equation as in [9], however, we preferred to handle the partial differential equations case first since sharper results obtain there.

Consider the integral equations

$$v(x) = \int_\Omega g(x, y)\, q(y, v(y))\, dy, \tag{5.1}$$

where $\Omega \subset \mathbf{R}^n$ is the closure of a bounded domain. In this section it is always assumed that g is real valued symmetric, and measurable on $\Omega \times \Omega$ and

$$G\phi(x) = \int_\Omega g(x, y)\,\phi(y)\,dy$$

is a compact linear map from $L^\sigma(\Omega) \to L^\tau(\Omega)$ where $\sigma^{-1} + \tau^{-1} = 1$ and $1 < \sigma < 2 < \tau$. G satisfies the latter condition if [2]

$$\int_{\Omega\times\Omega} |\, g(x, y)|^\tau\, dx\, dy < \infty. \tag{5.2}$$

It is further assumed that G is positive definite, i.e.,

$$\int_{\Omega\times\Omega} g(x, y)\,\phi(x)\,\phi(y)\,dx\,dy > 0 \qquad \text{for} \quad \phi \in L^\sigma(\Omega) - \{0\}. \tag{5.3}$$

The conditions imposed on q are

(q_1) q is continuous on $\Omega \times \mathbf{R}$;
(q_2) $|\, q(x, z)| \leqslant a_1 + a_2 \,|\, z\,|^s$, where $1 < s \leqslant \tau - 1$;
(q_3) $q(x, z)\, z^{-1} \to \infty$ as $|\, z\,| \to \infty$ uniformly in $x \in \Omega$

as well as (p_3), (p_5)–(p_6) of Section 3. (q_1) can be weakened to the Caratheodory conditions [2]

Under the above hypotheses on g, G admits a splitting [2], i.e., $G = HH^*$ where H: $L^2(\Omega) \to L^\tau(\Omega)$ is the positive self-adjoint square root of $G\,|_{L^2(\Omega)}$ and is compact, and H^*: $L^\tau(\Omega) \to L^2(\Omega)$ and is adjoint to H, i.e.,

$$\int_\Omega (H\phi(x))\,\psi(x)\,dx = \int_\Omega \phi(x)(H^*\psi(x))\,dx,$$

for all $\phi \in L^2(\Omega)$, $\psi \in L^\sigma(\Omega)$. In operator form, (5.1) becomes

$$v = HH^*q(v),$$

where we are abusing notation by suppressing the dependence of q on the independent variable. Since H is positive, it is a one-to-one map. Therefore, solving (5.1) is equivalent to solving

$$u = H^*q(Hu), \tag{5.4}$$

where $u \in E \equiv L^2(\Omega)$ and $v = Hu$.

The norm in $L^t(\Omega)$ will be denoted by $|\cdot|_t$. It is easily verified [2] that solutions of (5.4) are critical points of the functional

$$I(u) = \tfrac{1}{2}\,|u|_2^2 - \int_\Omega Q(y, Hu(y))\,dy \equiv \tfrac{1}{2}\,|u|_2^2 - f(u)$$

where $Q(x, z) = \int_0^z q(x, t)\,dt$ and $I \in C^1(E, \mathbf{R})$ with $I(0) = 0$.

LEMMA 5.5. *If q satisfies (q_1)–(q_3), (p_3), (p_5), I satisfies (I_1)–(I_3). If q also satisfies (p_6), I satisfies (I_4)–(I_5).*

Proof. (I_4) is trivial. To verify (I_1), it suffices to show $f(u) = o(|u|_2^2)$ at $u = 0$. As in Lemma 3.3,

$$\int_\Omega Q(x, v(x))\,dx \leqslant \epsilon\,|v|_2^2 + a_2\,|v|_\tau^\tau, \tag{5.6}$$

where $v = Hu$. By the Hölder inequality and the continuity of H, $|v|_2 \leqslant a_3\,|v|_\tau \leqslant a_4\,|u|_2$ so (5.6) implies (I_1).

(I_2) or (I_5) are proved essentially as in Lemma 3.4 where now (q_3) and $v = Hu$ replace, respectively, (p_4) and u in the estimates of the f term, and $|v|_2^2 = a_5$.

Finally, to verify (I_3), suppose $I(u_m) \leqslant d$ and $I'(u_m) \to 0$ as $m \to \infty$. Observe that

$$\begin{aligned} I'(u_m)\,u_m &= \int_\Omega (u_m - H^*q(x, Hu_m))\,u_m\,dx \\ &= \int_\Omega ({u_m}^2 - q(x, v_m)\,v_m)\,dx. \end{aligned} \tag{5.7}$$

As in Lemma 3.6, (5.7) and $I(u_m) \leqslant d$ yield

$$d + a_1 \geqslant \tfrac{1}{2}\,|u_m|_2^2 - \theta \int_\Omega q(x, v_m)\,v_m\,dx \geqslant (\tfrac{1}{2} - \theta)\,|u_m|_2^2 - \theta\,|u_m|_2$$

and the convergence of a subsequence of (u_m). Q.E.D.

Lemma 5.5 and Theorems 2.1 and 2.4 now immediately imply the following theorem.

THEOREM 5.8. *If q satisfies (q_1)–(q_3), (p_3), (p_5), then b and c as defined by* (2.2), (2.5) *are critical values of I with $0 < c \leqslant b$. The corresponding critical points provide nontrivial solutions of* (5.4) *and* (5.1).

In a similar fashion Lemma 5.5 and Theorems 1.8 and 1.9 imply the next theorem.

THEOREM 5.9. *If q satisfies* (q_1)–(q_3), (p_3)–(p_6), *then* (5.4) *and* (5.1) *possess infinitely many distinct nontrivial solutions.*

Remark 5.10. (1) The multiplicity statements of Theorem 2.8 are valid for (5.1) using the facts that H is continuous and one-to-one on $L^2(\Omega)$ and (1) of Lemma 1.2.

(2) By slightly modifying the above arguments, analogs of Theorems 3.25 (without positive assertions) and (3.32) can be obtained for

$$v(x) = \int_\Omega g(x, y)(a(y)\, v(y) + q(y, v(y))\, dy. \tag{5.11}$$

We will not carry out the details.

Next a version of Theorems 3.35 for integral equations will be given. Consider

$$v(x) = \lambda \int_\Omega g(x, y)\, q(y, v(y))\, dy. \tag{5.12}$$

Since we are now dealing with global integral rather than local differential operators, the truncation arguments used in Theorem 3.35 do not work here and (q_2) and (p_8) require strengthening. They will be replaced by

(q_4) $\quad | q(x, z)| \leqslant a_1 + a_2 \mid z \mid$.

(q_5) $\quad$ There exists $\bar{z} > 0$ such that $zq(x, z) \leqslant 0$ for $\mid z \mid \geqslant \bar{z}$.

The conditions (q_4)–(q_5) are fairly restrictive. However, they are satisfied in the context of Theorem 3.35 after p has been modified to $\bar{p}$. Since (q_4) is being assumed, it is natural to weaken the requirements on G so that $\sigma = 2 = \tau$.

Hypotheses like (q_5) have been used by Amann [24] to prove very general existence theorems for nonlinear integral equations. His results yield no useful information for (5.12) since $v \equiv 0$ is a solution.

THEOREM 5.13. *Suppose q satisfies* (q_1), (q_4)–(q_5), (p_3), *and* (p_7). *Then there exists* $\underline{\lambda} > 0$ *such that for all* $\lambda > \underline{\lambda}$, (5.12) *possesses at least two distinct nontrivial solutions* $\bar{v}(\lambda)$, $\underline{v}(\lambda)$.

Proof. It suffices to prove the result for

$$u = \lambda H^* q(Hu). \tag{5.14}$$

The solutions of (5.14) are critical points of

$$I(\lambda, u) = \tfrac{1}{2} | u |_2^2 - \lambda \int_\Omega Q(x, v(x))\, dx$$

with $v = Hu$. Lemma 5.5 shows $I(\lambda, \cdot)$ satisfies (I_1). Arguing as in Theorem 3.35, $I(\lambda, \cdot)$ satisfies (I_2) for all λ sufficiently large provided that there exists $u \in E$ such that $\int_\Omega Q(x, Hu(x))\, dx > 0$. To show this, observe first that since H is one-to-one and self-adjoint on E, $H = H^*$ has dense range. By (p_7), $Q(x, \eta) > 0$ for $x \in \Omega$ and any small constant $\eta > 0$. Let $\eta_1 = \int_\Omega Q(x, \eta)\, dx$. Choose $u \in E$ such that $| Hu - \eta |_2 \leqslant \delta < \eta$ where $\delta < [2a_1 \text{ meas } \Omega + 2\eta a_2(1 + 2 \text{ meas } \Omega)]^{-1} \eta_1$. Then using (q_4),

$$\begin{aligned} \int_\Omega Q(x, Hu(x)\, dx &\geqslant \eta_1 - \left| \int_\Omega (Q(x, Hu(x)) - Q(x, \eta))\, dx \right| \\ &\geqslant \eta_1 - \left| \int_\Omega \left(\int_\eta^{Hu(x)} q(x, z)\, dz \right) dz \right| \\ &\geqslant \eta_1 - | Hu - \eta |_2 \,| a_1 + a_2(| Hu(x)| + \eta)| \geqslant \eta_1/2. \end{aligned} \tag{5.15}$$

To verify (I_3), we need only to show $A_{\lambda,d} = \{u \in E \mid I(\lambda, u) \leqslant d\}$ is bounded for each $d \in \mathbf{R}$, $\lambda > 0$. For $u \in A_{\lambda,d}$, (q_5) shows

$$d + \lambda a_1 \geqslant \tfrac{1}{4} | u |_2^2, \tag{5.16}$$

where $a_1 = \bar{z}(\text{meas } \Omega) \max_{x \in \Omega; |z| \leqslant \bar{z}} | q(x, z)|$. Thus (I_3), follows as earlier and Theorem 1.1 gives a solution $\bar{u}(\lambda)$ of (5.14) with $I(\lambda, \bar{u}(\lambda)) > 0$ for all λ sufficiently large. A second solution $\underline{u}(\lambda)$ of (5.14) with $I(\lambda, \underline{u}(\lambda)) < 0$ is obtained by minimizing $I(\lambda, \cdot)$. Q E.D.

Remark 5.17. (1) (q_4) was required only to verify (I_2) and, therefore, can be eliminated if this condition can be verified by other means as e.g. in Corollary 5.20 below.

(2) As usual a stronger result analogous to Theorem 3.39 obtains here if q is odd. The proof differs from that of Theorem 3.39 only in the construction of the set A. An additional approximation argument using the density of the range of H in E as in Theorem 5.13 must be employed.

(3) If (5.1) is replaced by

$$v(x) = \int_\Omega g(x, y)(a(y)\, v(y) + q(y, v(y)))\, dy, \tag{5.18}$$

where $a > 0$ in Ω and is continuous, then the linear eigenvalue problem

$$w(x) = \mu \int_\Omega g(x, y)\, a(y)\, w(y)\, dy \tag{5.18}$$

has an unbounded sequence of eigenvalues $0 < \mu_1 \leqslant \cdots$. If q satisfies the hypotheses of theorem 5.13, as well as (p_6), then an application of [7, Theorem 7] shows (5.18) possesses at least k distinct pairs of solutions if $\mu_k < 1$.

It was observed above that the truncation devices of Section 2 do not seem to work in general for (5.12) (or (5.18)), and, therefore, q was required to satisfy (q_4)–(q_5). The next corollary shows (q_4) can sometimes be relaxed.

COROLLARY 5.20. *Suppose g is continuous on $\Omega \times \Omega$, and q satisfies (q_1), (q_5), (p_3), (p_7). Then the conclusions of Theorem 5.13 obtain.*

Proof. Since g is continuous, its eigenfunctions are continuous. Let e be an function of G. Then e is also an eigenfunction for H and therefore using (p_7), (I_2) can be satisfied here by choosing any sufficiently small multiple of e. Note that we satisfy (I_2) independently of the values of q for $|z| > \bar{z}$, i.e., $\underline{\lambda}$ depends only on q for $|z| \leqslant \bar{z}$. If v is a nontrivial solution of (5.1), then v is continuous since g is continuous and $v \in L^2(\Omega)$. Moreover, v is a priori bounded in $C(\Omega)$. This follows since first from (5.1) and (5.3),

$$\begin{aligned} 0 &< \lambda \int_{\Omega\times\Omega} g(x, y)\, q(y, v(y))\, q(x, v(x))\, dx\, dy \\ &= \int_\Omega v(x)\, q(x, v(x))\, dx \leqslant \int_{\{|v(x)|\leqslant \bar{z}\}} v(x)\, q(x, v(x))\, dx \qquad (5.21) \\ &\leqslant \bar{z}(\operatorname{meas} \Omega) \max_{x\in\Omega;|z|\leqslant\bar{z}} |q(x, z)| \equiv a_1 . \end{aligned}$$

Next let $t > 0$, $\phi_t(z) = 0$ if $|z| < t$, $\phi_t(z) = 1$ if $z > t$, and $\phi_t(z) = -1$ if $z < -t$. Then (5.1), the Schwarz inequality and (5.21) yield

$$\begin{aligned} t \int_\Omega |\phi_t(v(x))|\, dx &\leqslant \int_\Omega v(x)\, \phi_t(v(x))\, dx \\ &= \lambda \int_{\Omega\times\Omega} g(x, y)\, q(y, v(y))\, \phi_t(v(x))\, dx\, dy \\ &\leqslant \lambda \left(\int_{\Omega\times\Omega} g(x, y)\, q(x, v(x))\, q(y, v(y))\, dx\, dy\right)^{1/2} \\ &\quad \times \left(\int_{\Omega\times\Omega} g(x, y)\, \phi_t(v(x)) \cdot \phi_t(v(y))\, dx\, dy\right)^{1/2} \\ &\leqslant (\lambda a_1)^{1/2} \max_{\Omega\times\Omega} |g(x, y)| \int_\Omega |\phi_t(v(x))|\, dx. \qquad (5.22) \end{aligned}$$

Thus, $t \leqslant (\lambda a_1)^{1/2} \max_{\Omega\times\Omega} |g(x, y)|$ for any $t < \max |v(x)|$. Consequently,

$$\max_{\Omega} |v(x)| \leqslant (\lambda a_1)^{1/2} \max_{\Omega\times\Omega} |g(x, y)|. \tag{5.23}$$

Note that the a priori bound (5.23) is independent of how q is defined for $|z| > \bar{z}$. Let $M(\lambda) = 1 + \max((\lambda a_1)^{1/2}, \bar{z})$. Then q can be redefined for $|z| \geqslant M(\lambda)$ such that the new function $\bar{q}$ satisfies the hypotheses of Theorem 5.13 and λ is independent of the extension q while all solutions satisfy the bound (5.23). Hence, the conclusions of Theorem 5.13 obtain here. Q.E.D.

Remark. (5.21) and (5.22) with $\phi_i(v)$ replaced by v yields a bound for solutions of (5.12) valid under the hypotheses of Theorem 5.13.

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