A GENERALIZATION OF MILNOR'S \( \mu \)-INVARIANTS TO HIGHER-DIMENSIONAL LINK MAPS

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In this paper we generalize Milnor's \( \mu \)-invariants (which were originally defined for "almost trivial" classical links in \( \mathbb{R}^3 \)) to a (corresponding large class of) link maps in arbitrary higher dimensions. The resulting invariants play a central role in link homotopy classification theory. They turn out to be often even compatible with singular link concordances. Moreover, we compare them to linking coefficients of embedded links and to related invariants of Turaev and Nezhinskij. Along the way we also study certain auxiliary but important "Hopf homomorphisms".

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1. INTRODUCTION

Given \( r \geq 2 \) and arbitrary dimensions \( p_1, \ldots, p_r \geq 0 \) and \( m \geq 3 \), we want to study (spherical) link maps

\[
f = f_1 \sqcup \cdots \sqcup f_r : S^{p_1} \sqcup \cdots \sqcup S^{p_r} \to \mathbb{R}^m
\]

(i.e. the continuous component maps \( f_i \) have pairwise disjoint images) up to link homotopy, i.e. up to continuous deformations through such link maps. In other words, two link maps \( f \) and \( f' \) are called link homotopic if there is a continuous map

\[
F = F_1 \sqcup \cdots \sqcup F_r : \bigl( S^{p_1} \times I \bigr) \sqcup \cdots \sqcup \bigl( S^{p_r} \times I \bigr) \to \mathbb{R}^m \times I
\]

such that

(i) \( F_i \) and \( F_j \) have disjoint images for \( 1 \leq i < j \leq r \);
(ii) \( F(x, 0) = (f(x), 0) \) and \( F(x, 1) = (f'(x), 1) \) for all \( x \in \bigsqcup S^{p_i} \); and
(iii) \( F \) preserves \( I \)-levels.

If we drop the last condition, we say \( F \) is a singular link concordance and \( f, f' \) are link map concordant (cf. e.g. [1]).

The concept of link homotopy was introduced in 1954 by Milnor [2] in the case \( p_1 = \cdots = p_r = 1, m = 3 \) in order to get a first rough understanding of the overwhelming multitude of classical links. Moreover, starting from the fundamental group of a link complement, Milnor also defined \( \mathbb{Z} \)-valued link homotopy invariants \( \mu_\gamma \) (indexed by the permutations \( \gamma \in \Sigma_{r-2} \) of \( r-2 \) elements); they are sharp enough to decide whether or not a classical link homotopy class is trivial.

In an attempt to capture higher-order linking phenomena also in higher dimensions, one approach first suggests itself: just like the classical linking number is determined by

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overcrossings, we can define an invariant $h_j(f)$ which measures that part of the overcrossing locus of $f$ (w.r. to some decomposition $\mathbb{R}^m = \mathbb{R}^{m-1} \times \mathbb{R}$) where the component maps $f_i$ are "stacked on top of one another" according to the order given by the permutation $\gamma$. Unfortunately, when $r > 2$ this "Hopf invariant" $h_j$ turns out to vanish on all spherical link maps $f$ in codimensions $\geq 2$ (see Proposition 3.2). On the other hand, $h_j$ can be a highly nontrivial invariant for nonspherical link maps (where the spheres $S^{p_i}$ are replaced by arbitrary stably parallelized closed manifolds; see Theorem 3.1).

Therefore, we are led to adopt the following composite approach. Given a suitable ("$\kappa$-Brunnian", cf. Definition 4.1) spherical link map $f$ with $r$ components, we construct an (in general nonspherical) embedded framed link $\tilde{k}(f)$ with $r - 1$ components which is well-determined up to framed link bordism (as defined e.g. in [1]), and then we apply $h_j$ to $\tilde{k}(f)$.

This transition from $f$ to $\tilde{k}(f)$ involves the following three steps.

Step I (very natural). Form the product map $f = f_1 \times \cdots \times f_r$ into the configuration space

$$\mathcal{C}_r(\mathbb{R}^m) = \{(y_1, \ldots, y_r) \in (\mathbb{R}^m)^r | y_i \neq y_j \text{ for } 1 \leq i < j \leq r\}. \quad (3)$$

Step II (much more mysterious). Use the fact that the wedge $V = \bigvee^{r-1} S^{m-1}$ occurs as a subspace (and, up to deformation, as a fiber) in $\mathcal{C}_r(\mathbb{R}^m)$ and, starting from $\tilde{f}$, extract a homotopy class which lies in $\pi_*(V)$.

Step III (classical). Identify elements of $\pi_*(V)$ with bordism classes of links via the Pontrjagin–Thom procedure (i.e. by taking inverse images of regular values $z_1, \ldots, z_{r-1}$ in the different wedge factors $S^{m-1}$).

The resulting composite invariants

$$\mu_j(f) = h_j(\tilde{k}(f)) \in \pi_{p_1 + \cdots + p_r - (r-1)(m-2)-1} \gamma \in \Sigma_{r-2}, \quad (4)$$

were introduced in 1984 (cf. [3]) and have since then played a central role in the link homotopy theory of spherical link maps with more than two components (often leading to complete classification results, see e.g. [4]). However, the complicated homotopy theoretical nature of Step II made it hard until now really to understand these invariants.

In this paper we develop a new technique (embodied in the projectability Theorem 5.2) which often allows us to interpret our $\mu$-invariants directly in terms of the geometry of the original link map $f$. For example, if the first $r - 1$ components of $f$ are smooth embeddings and if the last component $f_r$ maps into a wedge of meridians, we can conclude in many cases that the "linking coefficient" (or "input")

$$\lambda(f) = [f_r] \in \pi_{p_1 + \cdots + p_r - (r-1)(m-2)-1} \gamma \in \Sigma_{r-2}$$

and our auxiliary link class (or "output")

$$\tilde{k}(f) \in \pi_{p_1 + \cdots + p_r} \gamma \in \Sigma_{r-2}$$

have $\pm$ equal values under the Hopf homomorphisms $h_j$ (which, however, are evaluated in two entirely different dimension settings). In particular, in the classical case of 1-dimensional links in $\mathbb{R}^3$ the scenario switches from the algebra of the free nonabelian group $\pi_1(\vee S^1)$ (where the values of $h_j$ in $\pi_0 = \mathbb{Z}$ turn out to be Magnus exponents) to the
geometry of the codimension-2 link $\tilde{r}(f)$ in $\mathbb{R}^r$. This is the bridge leading to the proof that Milnor's group theoretical $\mu$-invariants are but a special case of our geometric ones. As another consequence of our input--output analysis we can prove in codimensions greater than 2 that certain $\mu$-invariants of Turaev (which are defined for embedded spherical links, cf. [5]) suspend to our $\mu$-invariants (which are defined for all spherical link maps here; see the end of Section 4 and the appendix).

A second application of the projectability Theorem 5.2 concerns compatibilities with Nezhinskij's link suspension $E_i$, $1 \leq i \leq r$, which is based on rotations and which increases all dimensions $m$ and $p_j$, $j \neq i$, by 1 while leaving $p_i$ unchanged. In Theorem 7.2 we prove that the $\mu$-invariants of $f$ and of $E_i f$ agree up to a fixed sign whenever $f$ is $\kappa$-Brunnian. As a consequence, very often $\mu_i(f)$ turns out to be even invariant under singular link concordances (see Theorem 8.1 and Corollary 8.2). This may seem a little surprising since singular link concordances involve the transition from $\mathbb{R}^m$ to $\mathbb{R}^{m+1}$, but the corresponding inclusion of $\tilde{C}_i(\mathbb{R}^m)$ into $\tilde{C}_i(\mathbb{R}^{m+1})$ is obviously homotopy trivial. Anyway, this result allows us to find new cases where classification up to link homotopy and up to singular link concordance coincide and are performed by $\mu$-invariants (compare Theorem 8.4(c) of this paper to Theorems A and 5.1 in [1]). Actually, I am not aware of any case where these two equivalence relations are known to differ, and one might wonder whether there is a generally valid singular analogon of the Haefliger--Smale theorem ("embedded concordance implies isotopy in codimensions greater than 2", cf. [6, 2.2]).

Let us recall at this point that in the special case $r = 3$ the concordance invariance of $\mu$ was already obtained in [1]; it was deduced from a very different geometric interpretation of $\mu$ in terms of intersections.

A possible third application of the projectability theorem deals with link maps $f$ into the standard torus $S^1 \times D^{m-1} \subset \mathbb{R}^m$ and with the $\mu$-invariants of the "augmented" link maps in $\mathbb{R}^m$ consisting of $f$ and of a finite number of 2-codimensional meridians $\{z_i\} \times \partial D^{m-1}$, $z_i \in S^1$. For example in view of Hacon's classification theorem (cf. [7]) invariants of this type decide in the metastable dimension range whether a higher-dimensional knot in the torus is isotopic to the unknot. Details will be given elsewhere.

This paper is organized as follows. In Section 2 we present facts from homotopy theory which will be needed for the construction of $\tilde{r}(f)$ and in the proofs of the projectability Theorem 5.2 and of the suspension invariance Theorem 7.2. Section 3 contains a rather detailed geometric study of the omnipresent higher Hopf invariants $h$, both in terms of overcrossings and of intersections. After defining the $\mu$-invariants in Section 4 we can greatly increase their domain with the help of a few formal properties. For example, if $1 \leq p_1, \ldots, p_r \leq m - 3$ then $\mu_p$ becomes an additive link concordance invariant canonically defined for all link maps (see Corollary 8.2).

Sections 5 and 7 contain the two central technical Theorems 5.2 and 7.2 of the paper. These are exploited in the remaining sections and in the appendix in order to compare our $\mu$-invariants to linking coefficients and to link invariants of Milnor, Turaev and Nezhinskij, and also to deduce invariance results concerning singular link concordance.

Notations and Conventions. Throughout this paper we assume $r \geq 2$ and $m \geq 3$. Given an $s$-tuple $(p) = (p_1, \ldots, p_s)$ of nonnegative integers, we write $|p| := p_1 + \cdots + p_s$.

Spaces (and in particular spheres) are endowed with a base point $\ast$. Hence spheres are also endowed with a $CW$-decomposition $S^p = \ast \cup (S^p - \ast)$.

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1Note added in proof: P. Teichner has announced recently that singular link concordance implies link homotopy in codimensions at least 2.
the resulting product base point (\( \ast_1, \ldots, \ast_s \)) and the resulting product CW-structure. The configurations space \( \tilde{C}_s(\mathbb{R}^m) \) of ordered \( r \)-tuples of pairwise distinct points in \( \mathbb{R}^m \) inherits its topology from \( (\mathbb{R}^m)^r \).

All occurring manifolds are assumed to be smooth; framed means stably parallelized. An immersion is called framed if its normal bundle is trivialized.

2. A LITTLE HOMOTOPY THEORY

In this (and the next) section we will discuss the commuting diagram

\[
\begin{array}{ccc}
\pi_1[p(\tilde{C}_r(\mathbb{R}^m))] & \xrightarrow{\text{quot}} & [S^{(p)}, \tilde{C}_r(\mathbb{R}^m)] \\
\uparrow \text{incl.} & & \uparrow \text{incl.} \\
\pi_1[p((\mathbb{R}^m)^{r-1})] & \xrightarrow{\text{quot}^*} & [S^{(p)}/X, (\mathbb{R}^m)^{r-1}] & \xrightarrow{\text{quot}^*} & [S^{(p)}, (\mathbb{R}^m)^{r-1}] \\
\hline
\Omega^r & \cong & \pi_* & \cong & \pi_* \\
\end{array}
\]

Here we assume that \( p = (p_1, \ldots, p_s) \) is a given \( s \)-tuple of nonnegative integers and \( s \geq 1, r \geq 2 \).

We denote by

\[ S^{(p)} := S^{p_1} \times \cdots \times S^{p_s} \]

the resulting product (cell complex) of spheres which has dimension

\[ |p| := p_1 + \cdots + p_s. \]

In the next section we will assume \( X \) to equal

\[ X_i := S^{p_1} \times \cdots \times S^{p_{i-1}} \times S^{p_i + 1} \times \cdots \times S^{p_s}, \]

for some \( 1 \leq i \leq s \) such that \( p_i \geq 1 \). However, in Section 2 \( X \) can be any subcomplex of

\[ Y := \bigcup_{i=1}^{s} X_i. \]

quot\(_{(i)}\) denotes the obvious quotient map; e.g. \( X \subset Y \) induces the map

\[ \text{quot}^i : S^{(p)}/X \rightarrow S^{(p)}/Y = S^{(p)}. \]

Also recall that the wedge \( (\mathbb{R}^m)^{r-1} \) occurs as a deformation retract of the fiber \( \mathbb{R}^m - \{ y_1^{(0)}, \ldots, y_{r-1}^{(0)} \} \) of the fibration

\[ p : \tilde{C}_r(\mathbb{R}^m) \rightarrow \tilde{C}_{r-1}(\mathbb{R}^m) \]

\[ (y_1, \ldots, y_r) \rightarrow (y_1, \ldots, y_{r-1}), \]

\( r \geq 2 \).
we denote the resulting (fiber) inclusion by incl. Since p allows a section, e.g.
\[(y_1, \ldots, y_{r-1}) \to (y_1, \ldots, y_{r-1}; (\Sigma y_i + 1, 0, \ldots, 0)),\]
incl induces monomorphisms between homotopy groups (cf. [8]).

Due to our general assumption \(m \geq 3, r S^{m-1}\) and \(\mathcal{C}_r(\mathbb{R}^m)\) are simply connected, and hence base point free and base point preserving homotopy sets (denoted by square brackets) coincide here.

**Proposition 2.1 (Puppe).** The top horizontal map \(\text{quot}^*\) in diagram (5) is injective.

This holds (even if we replace \(\mathcal{C}_r(\mathbb{R}^m)\) by any topological space with base point) by [9, Satz 12, Satz 20 and the example following Satz 13].

**Proposition 2.2.** Let \(u \in \left[S^{(p)}, \mathcal{C}_r(\mathbb{R}^m)\right]\) satisfy

(i) \(u\) is trivial when restricted to each "\((s - 1)\)-fold product" \(X_i, i = 1, \ldots, s\) (cf. (6)), and
(ii) \(u\) becomes trivial when composed with the fiber projection \(p : \mathcal{C}_r(\mathbb{R}^m) \to \mathcal{C}_{r-1}(\mathbb{R}^m)\) (cf. (8)).

Then there exists a unique element
\[v \in \pi_{pj\left(r^{-1}S^{m-1}\right)}\]
such that \(\text{quot}^* \circ \text{incl}_*(v) = u\).

Moreover, for any element
\[w \in \left[S^{(p)}/X, r^{-1}S^{m-1}\right]\]
such that \(\text{incl}_* \circ \text{quot}^\sharp(w) = u\) we have
\[w = \text{quot}^\sharp(v)\].

**Proof.** Represent \(\text{quot}^\sharp(w)\) by a map \(g : S^{(p)} \to rS^{m-1}\) such that \(g|X = \ast\) (as wedge point). By induction over the dimension of the cells in \(Y - X\), we can deform \(g\) rel \(X\) until all of \(Y\) gets mapped to \(\ast\). Indeed, the closure \(\bar{c}\) of such a cell \(c\) in \(S^{(p)}\) is the product of some factor spheres \(S^{p_i}\) and some (at least one) base points. If, by induction, \(g\) is already constant on \(\bar{c} - c\), the obstruction to making \(g\) also constant on \(c\) lies in the kernel of
\[\pi_{pj\left(r^{-1}S^{m-1}\right)} \to \left[\bar{c}, \mathcal{C}_r(\mathbb{R}^m)\right]\]
(by condition (i)); but—as in (5) and Proposition 2.1—this kernel is trivial.

In the end \(g\) defines an element \(v\) as desired, which is the unique inverse image of \(w\) and hence also of \(u\).

Finally observe that \(u\) lies in the image of the right hand arrow \(\text{incl}_*\) (by condition (ii)). Thus, a suitable element \(w\) (for \(X = \ast\)), and hence \(v\), always exist. \(\square\)

### 3. HIGHER HOPF INVARIANTS OF (NONSPHERICAL) LINK MAPS AND THEIR GEOMETRY

In this section we study certain geometric invariants \(h_r\) and establish, in particular, the commuting triangle in diagram (5). Canonical desuspensions of \(h_r\) and their relations to the Hopf ladders of Boardman and Steer will be discussed briefly in the appendix.
Let $X$ be an $n$-dimensional framed manifold without boundary, let $k > 1$, $r > 2$ and $q_1, \ldots, q_{r-1}$ be integers and denote
\[ |q| := q_1 + q_2 + \cdots + q_{r-1}. \]
Given a permutation $\gamma \in \Sigma_{r-2}$ of $\{1, 2, \ldots, r-2\}$, we will construct an invariant
\[ h_\gamma(g) \in \pi_{n+k-l+q_{r-2}} \]
for every framed link map
\[ g = (g', g'') : N_1 \times \cdots \times N_{r-1} \to X \times \mathbb{R}^k \]
(i.e. the closed manifolds $N_i$ having the indicated dimensions are framed, $g$ is continuous and $g(N_i) \cap g(N_j) = \emptyset$ for $1 \leq i \neq j \leq r - 1$), as follows. After sufficiently small approximations we have:

(i) the product map
\[ \hat{g}' = g' \times \cdots \times g' : N_1 \times \cdots \times N_{r-1} \to X^{r-1} \]
is smooth and transverse to the $n$-dimensional diagonal $\Delta \subset X^{r-1}$; and

(ii) the map
\[ g'' : g^{-1}(\Delta) \to (S^{k-1})^{r-2} \]
obtained by putting
\[ g''_\gamma(x_1, \ldots, x_{r-1}) = \left(\frac{g''(x_{1(1)}) - g''(x_{2(1)})}{\|g''(x_{1(1)}) - g''(x_{2(1)})\|}, \ldots, \frac{g''(x_{1(3)}) - g''(x_{2(3)})}{\|g''(x_{1(3)}) - g''(x_{2(3)})\|}, \ldots, \frac{g''(x_{1(r-2)}) - g''(x_{2(r-2)})}{\|g''(x_{1(r-2)}) - g''(x_{2(r-2)})\|}\right) \]
is well-defined, smooth and transverse to some point $z$ in $(S^{k-1})^{r-2}$; if $k = 1$, $z := (+1, +1, \ldots, +1)$. Then we define $h_\gamma(g)$ to be the framed bordism class of $g''^{-1}(\{z\})$ or, equivalently, the corresponding stable homotopy class. This invariant is independent of our choices and depends on $g$ only up to bordism of framed link maps.

We can choose $z = (e_{k1}, e_{k2}, \ldots, e_k)$, where
\[ e_k := (0, \ldots, 0, 1) \in S^{k-1}, \]
and represent $h_\gamma(g)$ by that part of the “overcrossing locus” (with respect to the projection $\pi$ to $X \times \mathbb{R}^{k-1} \times \{0\}$) where we have
\[ g|N_{1(1)} > g|N_{2(2)} > \cdots > g|N_{r-2} > g|N_{r-1} \]
(with respect to the last coordinate in $\mathbb{R}^k$). This overcrossing interpretation is particularly interesting when all $q_i \geq 2$ and hence $\pi \circ g$ can be deformed into a nearby self-transverse framed immersion.

Another useful way of looking at $h_\gamma(g)$ is the intersection interpretation. Assume that $g|N_{r-1}$ allows a framed nullbordism
\[ G = (G', G'') : B \to X \times \mathbb{R}^k. \]
Then after suitable approximations the manifold
\[ \{(x, y) \in N_{i(r-2)} \times B | g'(x) = G'(y); g''(x) = G''(y) = \lambda e_k \text{ for some } \lambda \geq 0\} \]
is a bordism from the overcrossing locus $\mathcal{O}$ (where $g\mid N_{r-2} > g\mid N_{r-1}$) to the intersection $\mathcal{F}$ of $g\mid N_{r-2}$ with $G$ (where $\lambda = 0$). This gives rise to a framed bordism of link maps via the map $\tilde{g}$, $\tilde{g}(x, y) = g(x)$, whose values lie in $g(N_{r-2})$. Thus, in the original representation of $h_y(g)$ by the overcrossing locus where

$$g\mid N_{r(1)} > \cdots > g\mid N_{r-2} > \tilde{g}\mid \mathcal{O}$$

we can replace $\tilde{g}\mid \mathcal{O}$ by $\tilde{g}\mid \mathcal{F}$. If $\tilde{g}\mid \mathcal{F}$ is again nullbordant we can repeat the procedure. Often this method leads to a full description of $h_y(g)$ in terms of (iterated) intersections only.

Now let us assume $q_1, \ldots, q_{r-1} \geq 1$. Restricted to (embedded) links, our construction yields maps

$$h_y : (X \times \mathbb{R}^k) \cup \{\ast\} \rightarrow \pi_*^S,$$

Indeed, elements in the domain correspond—via the Pontryagin–Thom procedure—to bordism classes of framed links in $X \times \mathbb{R}^k$.

As a special case, this defines the maps $h_y$ in diagram (5). In order to obtain the left-hand homomorphism, put $X = \mathbb{R}^{r-1}$ and $k = 1$ when $|p| \geq 1$. As for the right-hand arrow $h_y$ in (5), we assume $X = X_i$ (cf. (6)) for some $1 \leq i \leq s$ such that $p_i \geq 1$; for a coherent sign convention we view $S^{(p)}/X \cong X \times \mathbb{R}^p \cup \{\ast\}$ as the one-point-compactification of $(X \times \mathbb{R}^{r-1}) \times \mathbb{R}$. These arrows commute up to a sign with the map $\text{quot}_y$ which is induced by the inclusion of the top cell.

Next we study the Hopf homomorphisms $h_y$ on the reduced homotopy groups

$$\tilde{\pi}_* \left( \bigvee_{i=1}^{r-1} S^{q_i} \right) \cong \bigcap_{i=1}^{r-1} \ker \left( \pi_* \left( \bigvee_{i=1}^{r-1} S^{q_i} \right) \rightarrow \pi_* \left( S^{q_1} \vee \cdots \vee S^{q_{r-1}} \right) \right)$$

where, in a way, linking phenomena of order $r-1$ are concentrated. (In the special case $* = q_1 = \cdots = q_{r-1} = 1$, the additivity of $h_y$ follows in the spirit of the proof of Corollary 6.2).

Consider the iterated Whitehead product

$$l_\gamma = \left[ [l_{\gamma(1)}], [l_{\gamma(2)}], \ldots, [l_{\gamma(r-2)}, l_{\gamma-1}] \ldots \right] \in \tilde{\pi}_{|q|-r+2} \left( \bigvee_{i=1}^{r-1} S^{q_i} \right),$$

$\gamma \in \Sigma_{r-2}$, where $l_{ij} \in \mathbb{N} \left( \bigvee_{i=1}^{r-1} S^{q_i} \right)$ is given by the obvious $j$th inclusion. These Whitehead products turn out to be in a sense dual to the homomorphisms $h_y$.

**Theorem 3.1.** If $q_1, \ldots, q_{r-1} \geq 1$, then the diagram of homomorphisms

\[ \begin{array}{ccc}
\tilde{\pi}_* \left( \bigvee_{i=1}^{r-1} S^{q_i} \right) & \xrightarrow{h_y} & \pi_* \left( S^{q_1} \cup \cdots \cup S^{q_{r-1}} \right) \\
\pi_* \left( S^{q_1} \cup \cdots \cup S^{q_{r-1}} \right) & \xrightarrow{\pm \delta_{\gamma, \gamma'} \cdot \text{E}^{oo}} & \pi_* \left( S^{q_1} \cup \cdots \cup S^{q_{r-1}} \right)
\end{array} \]

commutes for all permutations $\gamma, \gamma' \in \Sigma_{r-2}$.  


If in addition $q_i \geq \max\{2, p - |q| + r\}$ for $i = 1, \ldots, r - 1$, then

$$h := \bigoplus_{\gamma \in \Sigma_{r-2}} h_\gamma: \pi_2^p \left( \bigvee_{i=1}^{r-1} S^{n'_i} \right) \to \bigoplus_{\gamma \in \Sigma_{r-2}} \pi_2^p \left( \bigvee_{i=1}^{r-1} S^{n'_i - |q| + r - 2} \right)$$

is an isomorphism.

(Here $\delta_{\gamma'}$ is the usual Kronecker symbol, and $E^\infty$ denotes stable suspension.)

**Proof.** The construction of Whitehead products is based on the standard decomposition

$$S^{e_1'+e_2'-1} = B^{e_1'} \times S^{e_2'-1} \cup S^{e_1'-1} \times B^{e_2'}$$

of the unit sphere in $\mathbb{R}^{e_1'+e_2'} = \mathbb{R}^{e_1} \times \mathbb{R}^{e_2}$ and on the corresponding (projection) maps $t_j$ to the quotients $S^{e_j'}/\partial B^{e_j'}$ of the unit balls $B^{e_j'}$. Thus, we can describe a link corresponding to $[t_1, t_2] \in \pi_{e_1'+e_2'-1} \left( S^{e_1'} \times S^{e_2'} \right)$ by

$$\{0\} \times S^{e_2'-1} \cup S^{e_1'-1} \times \{z\} \subset S^{e_1'+e_2'-1} - \{\infty\} \cong \mathbb{R}^{e_1'+e_2'-1}$$

(16)

for any fixed $z$ in the interior of $B^{e_2'}$ (here we choose the base point $\{\infty\}$ to lie in $S^{e_1'-1} \times S^{e_2'-1}$).

The evaluation of our Hopf homomorphisms on such a link can be approached in the spirit of the intersection interpretation. When we apply our intersection procedure (discussed in (12)) to the two linked spheres above, we can describe $g|_S$ by the single point $z'$ where a suitable ball with boundary $S^{e_2'-1} \times \{z\}$ intersects $\{0\} \times S^{e_2'-1}$.

Similarly, a link in $\mathbb{R}^{e_2'+e_2'-2}$ corresponding to $[t_0, [t_1, t_2]]$ can take the form

$$\{0\} \times S^{e_2'+e_2'-2} \cup S^{e_2'-1} \times \{0\} \times S^{e_2'-1} \cup S^{e_2'-1} \times S^{e_2'-1} \times \{z\} ,$$

(17)

and we see again from the intersection approach (cf. (12)) that

$$h_{0,1,2}[t_0, [t_1, t_2]] = \pm h_{0,1}[t_0, [t_1, t_2]] = \pm \text{[point]}$$

(18)

where $t_0$ and $t_{1,2}$ are the canonical inclusions into $S^{e_2'} \times S^{e_1'+e_2'-1}$.

This procedure is compatible with forming further (iterated) Whitehead products since it works "fiberwise" when we replace a euclidian space $E$ of dimension $1 + \Sigma(e_l - 1)$ by any tubular neighborhood $M \times E$ in a higher-dimensional euclidean space.

Working our way successively from the innermost factors $t_{r-1}, t_{r-2}, \ldots$ of the iterated Whitehead product $t_\gamma$ towards the outer factors ... $t_{r(2)}, t_{r(1)}$ we get, on one hand,

$$h_i(t_{\gamma}) = \pm h_{i(1), i(2), \ldots, i(r-2)} \left( \gamma_{r(1)}, \gamma_{r(2)}, \ldots, \gamma_{r(r-2)} \right)$$

$$= \cdots = \pm \text{[point]}.$$

On the other hand, if $\gamma \neq \gamma'$ (and hence we have for some $1 \leq j \leq r - 2$ that $\gamma(i) = \gamma'(i)$, $i = j + 1, \ldots, r - 2$, but $\gamma(j) \neq \gamma'(j)$), then $h_i(t_{\gamma})$ has the form

$$h_{i(1), i(2), \ldots, i(r)} \left[ \gamma_{r(1)}, \gamma_{r(2)}, \ldots, \gamma_{r(r-2)} \right]$$

and a suitable nulbordism (constructed fiberwise) of the link component corresponding to $\tilde{\gamma}$ intersects only the $\gamma(j)$th component, not the $\gamma(j)$th component which lies "further outside"; therefore the iterated intersection which represents $h_i(t_{\gamma})$ is empty.

Now let $T = M \times \mathbb{R}^{\dim - r + 2} \subset \mathbb{R}^p$ be the tubular neighborhood of a closed smooth submanifold $M$ and let $g: S^p \to S^{\dim - r + 2}$ be given by the second projection on $T$ and
constant outside of $T$. Applying the previous discussion to every fiber of $T$, we obtain a link representing $(i*), [g]$; also we see that $h, (i*), [g]$ can be represented by the framed manifold $M_x + \{\text{point}\}$ whereas for $\gamma \neq \gamma', h, (i*), [g]$ can be represented by the empty manifold. This proves the first claim in our theorem.

Finally, assume that all $q_i \geq 2$. Then our epimorphism

$$h = \bigoplus h_\gamma : \pi_1(S^k - \{\text{point}\}) \to \mathbb{Z}^{(r-2)!}$$

is even bijective since its domain is freely generated by $(r - 2)!$ many “basic products” in the sense of Hilton (see [10, p. 155]).

More generally, the same basic products yield the full Hilton decomposition of $\pi_1(S^k)$ whenever $p < |q| + \min\{q_i\}$ and then the corresponding factor groups $\pi_i(S^k - \{\text{point}\})$ are also stable. Since the successive intersection description of $h_\gamma$ applies (fiberwise) to (composites of) these basic products, our homomorphism $h$, on one hand, and the Hilton isomorphism, on the other hand, are related by the invertible $(r - 2)! \times (r - 2)!$-matrix over $\mathbb{Z}$ which is formed by the values of $h$ on the basic products. The second claim in Theorem 3.1 follows.

The previous proof describes explicit links with nontrivial $h,\gamma$-values (see (16), (17) etc.). A striking common feature (e.g. of the standard links representing any Whitehead product of $1, \ldots, b - 1$) is: when the link has three or more nonempty components $N_i$, at most one of them is a sphere. This turns out to be very characteristic.

**Proposition 3.2.** Let

$$g = g_1 \prod \ldots \prod g_{r-1} : \bigcap_{i=1}^{r-1} N_i \to \mathbb{R}^k$$

be a framed link map such that $r - 1 \geq 3$ and $\dim N_i \leq k - 2$ for $i = 1, \ldots, r - 1$.

If for a given permutation $\gamma \in \Sigma_{r-2}$ the invariant $h_\gamma(g)$ is nontrivial, then none of the “intermediate” components $N_{\gamma(2)}, \ldots, N_{\gamma(r-2)}$ and at most one of the two “end” components $N_{\gamma(1)}$ and $N_{\gamma(r-1)}$ can possibly be a sphere.

In particular, if at least two of the components $N_i$ are spheres, then

$$h_\gamma(g) = 0 \quad \text{for all } \gamma \in \Sigma_{r-2}.$$

**Proof.** Fix $\gamma \in \Sigma_{r-2}$. Given $j$, $2 \leq j \leq r - 2$, observe that $h_\gamma(g)$ can be represented by the overcrossing locus where $g_{\gamma(1)} > g_{\gamma(2)} > \cdots > g_{\gamma(j-1)} > g_{\gamma(j)}$ (cf. (11) and (12)); here $g_{\gamma(j)}$ denotes the composite map

$$\tilde{g} : N = N_{\gamma(1)} \times \cdots \times N_{\gamma(2)} \times N_{\gamma(r-2)} \times N_{\gamma(r-1)} \to \mathbb{R}^k$$

and $N$ is the overcrossing locus where $g_{\gamma(1)} > \cdots > g_{\gamma(r-2)} > g_{\gamma(r-1)}$. Note that $N$ has strictly lower dimension than $N_{\gamma(j)}$. Hence if $N_{\gamma(j)}$ is a sphere then $\tilde{g}$ is homotopic—in $g(N_{\gamma(j)})$—to a constant map and $h_\gamma(g)$ must be trivial.

Assume now that both the “top component” $N_{\gamma(1)}$ and the “bottom component” $N_{\gamma(r-1)}$ are spheres. Since $k \geq 2$ and hence $(S^{k-1})^{r-2}$ is connected, we can base the construction of $h_\gamma$ on the regular value $z = (e_1, \ldots, e_{k-1})$. Thus, $h_\gamma(g)$ is represented by the overcrossing locus where simultaneously

$$g_{\gamma(1)} > g_{\gamma(2)} > \cdots > g_{\gamma(r-2)} \quad \text{and} \quad g_{\gamma(r-1)} > g_{\gamma(r-2)}.$$

Theorem 3.2.
This is obviously the union of the overcrossing manifolds corresponding to the sequences
\[(r - 1, \gamma(1), \ldots, \gamma(r - 2)); (\gamma(1), r - 1, \gamma(2), \ldots, \gamma(r - 2)); \ldots; (\gamma(1), \ldots, r - 1, \gamma(r - 2)).\]
But by the previous argument each of these manifolds is nullbordant since one of the spheres
\(N_{\gamma(1)} \text{ or } N_{r-1}\) occurs as an "intermediate" link component. Hence, \(h_\gamma(g) = 0.\)

Remark 3.3. If \(k \geq 2\) in our theory (see (10)), the last argument in the proof above shows also that we can safely neglect permutations \(\tilde{\gamma} \in \Sigma_{r-1}\) which do not fix the element \(r - 1.\) Indeed, the bordism class \(h_\gamma(g)\) of the overcrossing locus where
\[g|N_{\gamma(1)} > g|N_{\gamma(2)} > \cdots > g|N_{\gamma(r-1)}\]
allows a canonical decomposition into certain bordism classes of overcrossing manifolds of the form
\[g|N_{\gamma(1)} > \cdots > g|N_{\gamma(r-2)} > g|N_{\gamma(r-1)} \quad (\gamma \in \Sigma_{r-2}).\]
In other words, the invariant \(h_\gamma\) is a linear combination of the invariants \(h_\gamma \gamma \in \Sigma_{r-2},\) with coefficients \(\pm 1\) or 0.

4. THE \(\mu\)-INVARIANTS OF SPHERICAL LINK MAPS

Given a link map
\[f = f_1 \times \cdots \times f_r : S^{p_1} \times \cdots \times S^{p_r} \to \mathbb{R}^m\] (19)
such that \(r \geq 2, p_1, \ldots, p_r \geq 0\) and \(m \geq 3,\) the product map
\[\tilde{f} = f_1 \times \cdots \times f_r\] (20)
takes \(S^{(p)} = S^{p_1} \times \cdots \times S^{p_r}\) already into the configuration space \(\tilde{C}_r(\mathbb{R}^m) \subseteq (\mathbb{R}^m)^r\) (this restriction concerning the image of \(\tilde{f}\) is actually equivalent to the link map condition \(f_i(S^{p_i}) \cap f_j(S^{p_j}) = \emptyset\) for \(i \neq j).\) The resulting (ordinary) homotopy class
\[\kappa(f) = [\tilde{f}] \in [S^{(p)}, \tilde{C}_r(\mathbb{R}^m)]\] (21)
is our basic link homotopy invariant of \(f.\) Unfortunately it contains many redundancies and lies in a rather unwieldy homotopy set. We will therefore "break it up" into simpler parts and obtain our higher-dimensional \(\mu\)-invariants. For this purpose we will apply the discussion of Section 2 to \(u = r \in \mathbb{R}^m\) (and \(r = s).\)

Definition 4.1. We call the link map \(f \in \mathbb{R}^m\) Brunnian if
\[\tilde{f} | S^{p_1} \times \cdots \times S^{p_{r-1}} \times \{\ast\} \times S^{p_r} = \emptyset\]
is nullhomotopic (as a map into \(\tilde{C}_r(\mathbb{R}^m)\) for \(i = 1, \ldots, r).\)

Now let \(f \in \mathbb{R}^m\) be Brunnian. Due to the symmetry properties of \(\tilde{f}\) this implies in particular the homotopy triviality of the composite of \(\tilde{f}\) with the \(i\)th projection \(\tilde{C}_r(\mathbb{R}^m) \to \tilde{C}_{r-1}(\mathbb{R}^m)\) (which drops the \(i\)th component of a configuration) for \(i = 1, \ldots, r.\) It follows from Proposition 2.2 that there is a unique element
\[\tilde{\kappa}(f) \in \mathbb{R}^{p_1 + \cdots + p_r}, \left(\frac{r-1}{\sqrt{S^{m-1}}}\right)\] (22)
such that \( \text{quot}^* \circ \text{incl}_* (\tilde{\kappa}(f)) = \kappa(f) \) (for the definition of the reduced homotopy group above see (14)). Applying the Hopf homomorphisms of Section 3 we obtain the family of "\( r \)th order" link homotopy invariants

\[
\mu_r(f) := h_r(\tilde{\kappa}(f)) \in \pi_{p_1 + \cdots + p_r - (r-1)(m-2)-1}^S
\]

indexed by the permutations \( \gamma \in \Sigma_r \). If

\[
|p| := p_1 + p_2 + \cdots + p_r \leq r(m-2)
\]

then by Theorem 3.1 the resulting \((r-2)\)!-tuple of \( \mu \)-invariants of \( f \) contains precisely as much information as the original basic \( \kappa \)-invariant \( \kappa(f) = [f] \).

This applies in particular when all \( p_i \leq m-2 \). Then also \( f_i(*_i) \) can be joined to a faraway point outside the image of (a suitable approximation of) \( f_j, j \neq i \), and hence \( f \) is \( \kappa \)-Brunnian if and only if the \( \kappa \)-invariants of the sub-link maps \( f_1 \mid \cdots \mid f_{i-1} \mid f_{i+1} \mid \cdots \mid f_r \) are trivial for \( i = 1, \ldots, r \). We obtain by induction over \( r \):

**Theorem 4.2.** Let \( f \) be a link map with domain a disjoint union of spheres of dimensions not exceeding \( m-2 \). Then the link homotopy invariant \( \kappa(f) = [f] \) is trivial if and only if all (the consecutively defined) \( \mu \)-invariants of \( f \) and of all its sub-link maps vanish.

**Example 4.3.** \( r = 2 \), \( p_1, p_2 \leq m-2 \). Here the inclusion of \( \vee^{r-1} S^{m-1} = S^{m-1} \) into \( \tilde{C}_*(\mathbb{R}^m) \) is a homotopy equivalence (an inverse is given by the map which takes \((y_1, y_2)\) to \((y_2 - y_1)/\|y_2 - y_1\|)\); every link map \( f \) is \( \kappa \)-Brunnian, and the only \( \mu \)-invariant is just the generalized linking number

\[
\pm \alpha(f) = \pm h_{12}(f) \in \pi_{p_1 + p_2 - m+1}^S
\]

of \( f \) itself (cf. e.g. [11, Section 2]).

Next we assume \( p_1, \ldots, p_r \) to be strictly positive (but otherwise arbitrary) and we establish some compatibility results for the invariants \( \kappa \) and \( \mu \). Consider the link map

\[
\tilde{f} = f_1 \mid \cdots \mid f_{i-1} \mid f_i \circ \text{refl: } S^{p_1} \mid \cdots \mid S^{p_r} \to \mathbb{R}^m
\]

obtained by composing a given link map \( f \) in the indicated way with the reflection which flips the last coordinate. Also recall that the connected sum operation makes the set \( BLM^m_{\#} \) of base point preserving link homotopy classes of base point preserving link maps \( f \) (with the indicated dimensions) into a semigroup (we adopt the notations of [4, Section 1]).

**Proposition 4.4.** Assume \( p_1, \ldots, p_r \geq 1 \). Let \( \gamma \in \Sigma_{r-2} \).

(i) If a link map \( f \) is \( \kappa \)-Brunnian, then so is \( \tilde{f} \) and we have

\[
\tilde{\kappa}(\tilde{f}) = -\kappa(f) \quad \text{and} \quad \mu_r(\tilde{f}) = -\mu_r(f).
\]

(ii) If the link maps \( f^+, f^- \) are \( \kappa \)-Brunnian and base point preserving, then so is \( f^+ + f^- \), and we have

\[
\tilde{\kappa}(f^+ + f^-) = \tilde{\kappa}(f^+) + \tilde{\kappa}(f^-)
\]

and

\[
\mu_r(f^+ + f^-) = \mu_r(f^+) + \mu_r(f^-).
\]
Proof. Since (i) follows easily from standard techniques, we concentrate on claim (ii).
The $i$th component map of the connected sum $f = f^* + f^-$ factors through a wedge $S^i_+ \cup S^i_-$ of $p_i$-spheres. Hence, $\tilde{f}$ factors through the product
$$\prod_{i=1}^{r} (S^i_+ \cup S^i_-) = \bigcup S^i_+ \times \cdots \times S^i_-. $$

If a partial product $S$ to the right-hand side contains the "positive" factors $S^s_+, \ldots, S^r_+$ for some $1 \leq s < r$, then the corresponding as well as the remaining components of $\tilde{f}$ form two nullhomotopic maps into configuration spaces of complementary half-spaces of $\mathbb{R}^n$; hence $\tilde{f}|S$ is also nullhomotopic. Therefore, only $\tilde{f}|(S^s_+ \times \cdots \times S^r_-) \sim \tilde{f}^+$ and $\tilde{f}|(S^1_- \times \cdots \times S^s_-) \sim \tilde{f}^-$ contribute substantially to $\tilde{f}$. Using Puppe's result Proposition 2.1 in a cell-by-cell argument we see that $f$ is also $\kappa$-Brunnian and $\tilde{\kappa}(f) = \tilde{\kappa}(f^+) + \tilde{\kappa}(f^-)$ as claimed. \hfill \Box

If $p_1, \ldots, p_r \geq 1$, the compatibility result above can now be used to extend the invariants $\tilde{\kappa}$ and $\mu_\gamma$ to all base point preserving link maps whether they are $\kappa$-Brunnian or not. Given $2 \leq s \leq r - 1$, consider the map $\varphi_s$ from $BLM^r_{p_\gamma}$ to itself defined by
$$\varphi_s(f) = f + \sum_{1 < i_1 < \cdots < i_s < r} \tilde{f}_{i_1, \ldots, i_s},$$
where we choose some fixed (e.g., alphabetical) order of summation and where $\tilde{f}_{i_1, \ldots, i_s}$ denotes the link map consisting of $f_{i_1}, \ldots, f_{i_s}, f_{i_s} \circ \text{refl}$ as well as constant maps on the remaining spheres. Because of Proposition 4.4 we get: if every sub-link map of $f$ with $s - 1$ components has a trivial $\kappa$-invariant then so has every sub-link map of $\varphi_s(f)$ with $s$ components. Therefore, the composite $q = \varphi_{r-1} \circ \cdots \circ \varphi_2$ maps every $[f] \in BLM^r_{p_\gamma}$ to a $\kappa$-Brunnian element, and we define
$$\tilde{\kappa}(f) := \tilde{\kappa}(\varphi(f))$$
and
$$\mu_\gamma(f) := \mu_\gamma(\varphi(f)) \quad \text{for} \quad \gamma \in \Sigma_{r-2}.$$ 

Note:

(i) these definitions agree with the previous ones for every $\kappa$-Brunnian link map $f$; and
(ii) these invariants vanish on every link map $f$ such that at least one component $f_i$ is constant.

If we assume even that $p_1, \ldots, p_r \leq m - 3$ or that $m - 2, p_1, \ldots, p_r \geq 2$, then $BLM^r_{p_\gamma}$ is abelian (cf. e.g. [4, 1.4]) and therefore $\varphi$ and hence $\tilde{\kappa}$ and $\mu_\gamma$ are additive homomorphisms canonically defined on all of $BLM^r_{p_\gamma}$ (i.e. independent of choices). Moreover, if all spheres have dimensions strictly less than $m - 2$, base point free and base point preserving link homotopy coincides (cf. e.g. [4, 1.7]).

5. PROJECTABLE MAPS

The most important step in the construction of $\mu$-invariants is to associate the (usually nonspherical) link class $\tilde{\kappa}(f)$ to a suitable spherical link map $f$. This transition is very homotopy theoretical and abstract and at first sight not easy to understand geometrically.

In this section we will discuss a direct way which allows us often to find an element $w$ as in Section 2 for $u - \kappa(f)$ and, in particular, to read off $\mu_\gamma(f) = h_\gamma(w)$. 
Given a line \( l \subseteq \mathbb{R}^n \), let \( \pi : \mathbb{R}^n \to l^1 \) denote the projection onto its orthogonal complement and let \( \pi^{-1}(\bar{C}_r(l^1)) \) be the subset of \( \bar{C}_r(\mathbb{R}^m) \) consisting of those configurations which project already to configurations in \( l^1 \).

**Definition 5.1.** A map

\[
g : S^{(p)} = S^{p_1} \times \cdots \times S^{p_r} \to \bar{C}_r(\mathbb{R}^m)
\]

is called projectable if there exists a line \( l \subseteq \mathbb{R}^n \) such that

(i) \( p \circ g \) maps already into \( \pi^{-1}(\bar{C}_r_1(l^1)) \) and is nulhomotopic as a map into \( \pi^{-1}(\bar{C}_r_1(l^1)) \) (cf. (8)); and

(ii) \( g(S^{p_1} \times \cdots \times S^{p_{r-1}} \times \{\ast\}) \) lies already in \( \pi^{-1}(\bar{C}_r(l^1)) \).

Then we can choose an orientation (i.e. a "positive" half-line \( l_+ \)) of \( l \), and after a suitable approximation we obtain a framed link

\[
N := N_1 \cup \cdots \cup N_{r-1} \subseteq S^{p_1} \times \cdots \times S^{p_{r-1}} \times \mathbb{R}^{p_r}
\]

by putting

\[
N_i = \{ z \in S^{(p)} | g_i(z) - g_i(z) \in l_+, \quad i = 1, \ldots, r - 1 \}.
\]

In other words, \( N_i \) is the locus of points \( z \) where the \( r \)th component of the configuration \( g(z) \) "lies over" the \( i \)th component w.r. to the projection along \( l \). These submanifolds of \( S^{(p)} \) are disjoint by Definition 5.1(i) and do not meet \( S^{p_1} \times \cdots \times S^{p_{r-1}} \times \{\ast\} \) by Definition 5.1(ii).

**Theorem 5.2.** Assume \( p_r \geq 1 \). If \( g \) is projectable as above and if \( [g] \in \text{quot}^* (\pi_{|p}|(\bar{C}_r(\mathbb{R}^m))) \) (cf. (5)), then \( [g] = \text{quot}^* \circ \text{incl}_+(v) \) for a unique \( v \in \pi_{|p}(\vee_{r}^{m-1} S^{m-1}) \) and we have

\[
h_\gamma(v) = h_\gamma(N) \quad \text{for all } \gamma \in \Sigma_{r-2}.
\]

(cf. (28); the Hopf homomorphism to the right is defined via the projection along the \( x_{p_r} \)-axis in \( \mathbb{R}^{p_r} \)).

**Proof.** A nulhomotopy of \( p \circ g \) in \( \pi^{-1}(\bar{C}_r_1(l^1)) \subseteq \bar{C}_r_1(\mathbb{R}^m) \) lifts to a homotopy \( G \) (in \( \bar{C}_r(\mathbb{R}^m) \)) from \( g \) to a map \( g' \) which goes into a fiber \( p^{-1}(\{\ast\}) \sim \vee_{r-1}^{m-1} S^{m-1} \). On the other hand, as in the construction of \( N \) above, \( G \) defines also a framed link bordism from \( N \) to a link \( N' \) in \( S^{(p)} \). But under the Pontryagin-Thom construction \( N' \) corresponds to \( g' \), and hence \( N \) corresponds to a homotopy class \( w \) as in Proposition 2.2 (for \( X = X_r \), cf. (6)) such that \( \text{incl}_+ \circ \text{quot}(w) = [g] =: u \). Our claim follows now from Proposition 2.2 and (13).

6. COMPARISON TO MILNOR'S \( \mu \)-INVARIANTS AND TO LINKING COEFFICIENTS

For a first application of the "projectability theorem" (Theorem 5.2), we assume that

\[
g_i := m - p_i - 1 \geq 1 \quad \text{for } i = 1, \ldots, r - 1.
\]

In order to carry out the "input–output analysis" announced in [3] we will compare a map \( f_* : S^{p_r} \to \vee_{i=1}^{r} S^{(p)} \) ("input") to the resulting link map

\[
e_* (f_*) = e_1 \| \cdots \| e_{r-1} \| f_* : \prod_{i=1}^{r} S^{p_i} \to \mathbb{R}^m
\]
and its invariants ("output"). Here \( e_1, \ldots, e_{r-1} \) denote standard embeddings into parallel disjoint hyperplanes of \( \mathbb{R}^m \), and we interpret \( V := \sqrt{S^q} \) as a wedge of corresponding small meridians \( S^q \) such that the ball spanned by \( e_i(S^p) \) intersects \( V \) transversely and precisely in one point \( z_i \in S^q \) (see Fig. 1).

**Theorem 6.1.** Let \( e_*(f_i) \) be \( \kappa \)-Brunnian (this holds e.g. if \( \left[ f_i \right] \) lies in the reduced homotopy group \( \tilde{\pi}_p(\sqrt{S^q}) \), cf. (14)). Then there is a sign \( \varepsilon = \pm 1 \) such that

\[
\mu_\gamma(e_*(f_i)) = \varepsilon h_\gamma([f_i]) \quad \text{for all } \gamma \in \Sigma_{r-2}.
\]

**Note:** Both terms in this equation are defined by values of Hopf maps \( h_\gamma \), but in entirely different (dimension) settings: in \( \pi_{p1}([\sqrt{S^m-1}]) \) and in \( \pi_p(\sqrt{S^q}) \), resp.

**Proof:** We may assume \( p_r \geq 1 \) since otherwise the equation above holds trivially.

Clearly, the product map \( e_*(f_i) \) (cf. (20)) is projectable (cf. Definition 5.1): just choose \( l \) to be a suitable line in a hyperplane parallels of which contain the sets \( e_i(S^p) \) (which therefore project to disjoint balls). The components of the resulting link \( N \) (cf. (28)) are of the form

\[
N_i = S^{p1} \times \cdots \times S^{p_{i-1}} \times \{y_1\} \times S^{p_{i+1}} \times \cdots \times S^{p_{r-1}} \times f_r^{-1}\{z_i\}, \quad \text{for } i = 1, \ldots, r-1.
\]

Hence, \( h_\gamma(N) \) can be represented by that part of the overcrossing locus over \( \{y_1\} \times \cdots \times \{y_{r-1}\} \times \mathbb{R}^{p_{r-1}} \) where \( f_r^{-1}\{z_i\} \) lies over \( f_r^{-1}\{z_{j(i)}\} \) etc. w.r. to the \( x_{p_r} \)-coordinate (cf. (11)). On one hand, this locus defines \( \pm h_\gamma(f_i) \), but by Theorem 5.2 it also defines \( \mu_\gamma(e_*(f)) = h_\gamma(N) \).

**Corollary 6.2.** In the classical dimension setting \( m = 3, p_1 = \cdots = p_r = 1 \), a link is \( \kappa \)-Brunnian (cf. Definition 4.1) if and only if it is almost trivial in the sense of Milnor (cf. [2, Section 5]), and then its \( (\pi^S_0 = \mathbb{Z}) \)-valued invariant \( \mu_\gamma \) (as defined in (23)) coincides with Milnor's invariant \( \mu(\gamma(1), \ldots, \gamma(r-2), r-1, r) \) for every \( \gamma \in \Sigma_{r-2} \) (up to a fixed sign which depends only on \( r \)).

**Proof:** If \( f \) is almost trivial, i.e. if every proper sublink is homotopy trivial, then after a link homotopy we have \( f = e_*(f_i) \), where

\[
[f_i] = \prod_{\gamma \in \Sigma_{r-2}} \prod_{w = 1} w \cdot 1_{r-1}(\gamma(1), \ldots, \gamma(r-2), r-1, r) \cdot w^{-1}.
\]

Fig. 1. The construction of \( e_*(f) \).
(see [2, top of p. 190]); here \(i_1, \ldots, i_{r-1}\) denote the canonical generators of the free group \(\pi_1(\bigvee^{r-1} S^1)\) and \(w\) runs through all possible subproducts of \(t_i := t_{i(1)} \cdots t_{i(r-2)}\).

The calculation of the Hopf map \(h_r\) on the "input" \([f_r]\) amounts to counting the \((r - 1)\)-tuples ("subwords")

\[
i_{r-1}^\pm < i_{r-2}^\pm < \cdots < i_2^\pm < i_1^\pm < z_{y}^{(1)}< z_{y}^{(2)} < \cdots < z_{y}^{(r-2)} < z_{y}^{(r-1)}
\]

(with the appropriate sign = product of the exponents \(\pm 1\)) which occur in the word \([f_r]\). We count these \((r - 1)\)-tuples by first specifying which of the factors \(i_{z_{y}^{(i)}}\) we pick for the start. When counting the choices of the remaining sequence \(i_{z_{y}^{(r-2)}} < \cdots < i_{z_{y}^{(1)}}\) we can then drop all occurrences of \(t_{r-1}\) in our word, so only one factor \(w^{-1}\) remains. Hence, \(h_r\) counts just the occurrence \(i_{z_{y}^{(r-1)}} ((-1)^\mu \text{ times})\) if the subsequent word \(w^{-1}\) has full length \(r - 2\) and the corresponding permutation is \(\gamma\). But by Theorem 6.1, \(h_r[f_r] = (-1)^\mu \mu(\gamma(1), \ldots, \gamma(r-2), r-1, r)\) also coincides with \(\gamma f_r\).

It follows, in particular, from Milnor's standard form that \(f\) is link homotopy trivial if all its \(\mu\)-invariants vanish. Corollary 6.2 follows now from Theorem 4.2 by induction over \(r\).

The previous proof, based on a deep result of Milnor, shows that a classical link \(f\) is homotopy trivial if and only if \(\kappa(f)\) vanishes.

**Question 6.3.** Is the link homotopy class of every classical link \(f\) uniquely determined by \(\kappa(f) \in \pi_1([S^1]^r; C_r(\mathbb{R}^3)]\)?

A positive answer is valid for low \(r\). If true in general, it would mean a considerable conceptual simplification as compared to the homotopy classification of classical links by Habegger and Lin [12].

Next assume \(1 \leq p_1, \ldots, p_r \leq m - 3\) and consider any link map \(f\) such that \(f_1 \cdots f_{r-1}\) is already a link (i.e. a smooth embedding). Since the inclusion of a wedge of small meridians into the link complement induces isomorphisms on the homotopy groups of dimensions below \(m - 2\), \(f_r\) determines a unique homotopy class

\[
\lambda(f) \in \pi_* \left( \bigvee_{i=1}^{r-1} S^{q_i} \right)
\]

known as linking coefficient of \(f\) (cf. [6, 1.4]).

**Corollary 6.4.** There is a fixed sign \(\varepsilon = \pm 1\) such that

\[
\mu_\gamma(f) = \varepsilon h_\gamma(\lambda(f)) \quad \text{for all } \gamma \in \Sigma_{r-2}.
\]

**Proof.** Connected sums of links are again links if we modify the construction a little near the base points in a rather obvious way, and then linking coefficients are additive. In particular, the first \(r - 1\) components of \(\phi(f)\) (cf. (26)) form again a link \(l\), and the linking coefficient of \(\phi(f)\) is the image of \(\lambda(f)\) under the canonical retraction

\[
\text{retr} : \pi_* \left( \bigvee_{i=1}^{r-1} S^{q_i} \right) \to \pi_* \left( \bigvee_{i=1}^{r-1} S^{q_i} \right)
\]
(which kills all Hilton factors corresponding to basic Whitehead products not involving all meridians $t_1, \ldots, t_{r-1}$). It follows that both $\mu_i$ and $h_i \circ \lambda$ remain unchanged if we replace $f$ by $g(f) = e(f) = e\left(\text{retr}(\lambda(f))\right) + I$ (cf. (27) and (30)). Our claim follows now from (27)(ii) and Theorem 6.1.

7. COMPATIBILITY WITH NEZHINSKIJ SUSPENSIONS

In this section we describe a base point preserving version of a suspension construction introduced by Vladimir Nezhinskij (cf. [13, 1; 14, 2.2; 15]) and we study its compatibility with $\mu$-invariants. The central idea is to apply the projectability Theorem 5.2 to a map $G$ (cf. (42)) which is defined on $(S^1)^{r-1} \times S^p$ and which seems well-suited to capture the rotations involved in Nezhinskij's construction.

Throughout this section assume that $p_1, \ldots, p_r \geq 1$.

Fix an integer $1 \leq i \leq r$ and consider a base point preserving link map

$$f = f_1 \cup \ldots \cup f_r : S^{p_1} \cup \ldots \cup S^{p_r} \to \mathbb{R}^m_+$$

as in [4, 1.1] (i.e.

$$f_j(-1,0,\ldots,0) = *_j := (j,0,\ldots,0), \quad j = 1, \ldots, r, \quad (33)$$

and the half-space $\mathbb{R}^m_+$ is defined by the inequality $x_2 \geq 0$). Put

$$B_j = \{(x_1,\ldots,x_{p_{j+1}}) \in S^{p_j} | x_1 \geq 0\}.$$  

(34)

After a fairly obvious canonical deformation we may assume that

(i) $f_j$ maps actually the whole complement of the ball $B_j$ to $*_j, j \neq i$, and

(ii) the image of all of $f$ lies in the "quarter space" $\{(x_1,\ldots,x_{p_{j+1}}) \in S^{p_j} | x_1 \geq 0, x_3 \geq 0\}$ which we denote by $\mathbb{R}^m_+$ (cf. also $T_+$ in [4, 1.4]).

Then a new base point preserving link map, the Nezhinski suspension (Fig. 2)

$$E_if : S^{p_i+1} \cup \ldots \cup S^{p_{i-r+1}} \cup S^{p_i} \cup S^{p_{i+1}} \cup \ldots \cup S^{p_r+1} \to \mathbb{R}^m_+,$$

(35)

Fig. 2. Nezhinskij's suspension $E_i(f)$.  


is obtained by "rotating" all but the \( i \)th component of \( f \) along the \((x_3, x_{m+1})\) plane and by using standard homeomorphisms \( S^{p_i+1} \approx S^1 \times B_j/\sim \) and \( \mathbb{R}^m_{p_i+1} \approx S^1 \times \mathbb{R}^m_{p_i+1}/\sim \) (where \((z; x) \sim (z'; x)\) whenever \( z, z' \in S^1 \) and \( x \) lies in the boundary of the ball \( B_j \) or in the (half) hyperplane \( x_3 = 0 \), resp.).

In the same way a base point preserving link homotopy (cf. [4, 1.1]) from \( f \) to \( f' \), say, gives rise to a base point preserving link homotopy from \( E_i f \) to \( E_i f' \).

If \( f_1 \prod_i f_{i-1} \prod f_{i+1} \prod \cdots \prod f_r \), allows a base point preserving link nulhomotopy \( H \), there is another natural version of Nezhinskij's suspension. Indeed, the "track" map

\[
\bigg( \bigg[ \prod_{j \neq i} S^{p_j} \bigg] \times [-1, 1] \bigg] \rightarrow \mathbb{R}^{m+1} = \mathbb{R}^m \times \mathbb{R}, \quad (x, t) \rightarrow (H(x, It), t),
\]

together with \( f_i \times \{0\} \), also gives rise to a base point preserving link map

\[
E_i f : S^{p_i+1} \prod \cdots \prod S^{p_{i-1}+1} \prod S^{p_i} \prod S^{p_{i+1}+1} \prod \cdots \prod S^{p_r+1} \rightarrow \mathbb{R}^{m+1}. \tag{36}
\]

It is easy to find a base point preserving link homotopy from \( E_i f \) to \( E_i f \) (use a deformation involving \( H \) in the negative \( x_3 \)-range).

**Example 7.1.** If \( r = 2 \) each of the two components of \( f \), when taken alone, is of course nulhomotopic, and the constructions above yield well-known suspensions (cf. e.g. [16, p. 99]).

**Theorem 7.2.** Assume \( p_1, \ldots, p_r > 1 \), fix \( 1 \leq i \leq r \) and let the base point preserving link map

\[
f = f_1 \prod \cdots \prod f_r : S^{p_1} \prod \cdots \prod S^{p_r} \rightarrow \mathbb{R}^m
\]

be \( \kappa \)-Brunnian. Then:

(i) (Habegger) \( E_i f \) is also \( \kappa \)-Brunnian; and

(ii) there is a fixed sign \( \varepsilon = \pm 1 \) such that for all \( \gamma \in \Sigma_{r-2} \)

\[
\mu_{\gamma}(E_i f) = \varepsilon \cdot \mu_{\gamma}(f).
\]

**Proof.** Consider the homeomorphism \( h : \mathbb{R}^m \cong H := \{ x = (x_1, \ldots, x_m) \in \mathbb{R}^m | x_3 > 0 \} \) defined by \( h(x) = (x_1, x_2, \exp x_3, x_4, \ldots, x_m) \) and the resulting inclusion

\[
id \times h : S^1 \times \mathbb{R}^m \cong S^1 \times H \subset \mathbb{R}^{m+1} \tag{37}
\]

onto the complement of the \((x_1, x_2, x_4, \ldots, x_m)\)-subspace; here we identify the product of \( S^1 \) and of the positive \( x_3 \)-axis in the obvious way with the punctured \((x_3, x_{m+1})\)-plane.

We want to compare \( E_i f \) to the nonspherical link map

\[
F : S^1 \times S^{p_1} \prod \cdots \prod S^1 \times S^{p_{i-1}+1} \prod S^{p_i} \prod S^{p_{i+1}+1} \prod \cdots \prod S^{p_r} \rightarrow \mathbb{R}^{m+1}
\]

defined by \((id \times h) \circ (id \times f_j), j \neq i, \) and by \( h \circ f_i \). For \( j \neq i \) there are the quotient maps

\[
S^{p_j+1} \rightrightarrows S^{p_j+1} / \partial B_j = (S^1 \times S^{p_j}) / S^1 \times \{ * \} \rightrightarrows S^1 \times S^{p_j} \tag{38}
\]

(cf. (34) and (35)) and, after suitable link homotopies, \( E_i f \) and \( F \) factor through the same map

\[
F' : \left( \bigg[ \prod_{j \neq i} S^{p_j+1} / \partial B_j \bigg] \right) \times S^{p_i} \rightarrow \mathbb{R}^{m+1}. \tag{39}
\]
We obtain the commuting diagram:

\[
\begin{array}{c}
\pi_{[p]+r-1} (\mathcal{C}_r) \\
\otimes \\
\otimes \\
\otimes
\end{array}
\begin{array}{c}
\left( \prod_{j \neq i} S^{p_j + 1} \times S^{p_i}, \mathcal{C}_r \right) \ni \left[ \bar{E}_i f \right] \\
\uparrow Q^* \\
\left( \prod_{j \neq i} S^{p_j + 1} / \partial B_j \times S^{p_i}, \mathcal{C}_r \right) \ni \left[ \bar{F}^* \right] \\
\uparrow Q^{**} \\
\left( S^{1} \right)^{r-1} \times S^{[p]}, \mathcal{C}_r \right) \ni \left[ \bar{F} \right] \\
\uparrow \\
\left( S^{1} \right)^{r-1} \times S^{[p]}, \mathcal{C}_r \right) \ni \left[ G \right]
\end{array}
\]

where $\mathcal{C}_r := \mathcal{C}_r (\mathbb{R}^{m+1})$ and all arrows are induced by the obvious (products of) quotient maps (see also Proposition 2.1 and (38)). Moreover, recall from (22) that $\bar{f}$ is homotopic in $\mathcal{C}_r (\mathbb{R}^m)$ to a map of the form $g \circ \text{quot}$, where $g = (g_1, \ldots, g_r)$ maps $S^{[p]}$ into the subspace ("fiber") $\setminus r^{-1}S^{m-1}$ of $\mathcal{C}_r (\mathbb{R}^m)$ (in other words, $g_1 \equiv *_1, \ldots, g_{r-1} \equiv *_{r-1}$ are constant and $g_r$ represents $\bar{r}(f)$; here we choose the base point $(*_1, \ldots, *_r)$ of $\mathcal{C}_r (\mathbb{R}^m)$ to be defined by the convention in (33)). Hence,

\[
[\bar{F}] = (\text{id} \times \text{quot})^* [G] \tag{41}
\]

where

\[
G = (G_1, \ldots, G_r) : (S^1)^{r-1} \times S^{[p]} \to \mathcal{C}_r (S^1 \times \mathbb{R}^m) \subset \mathcal{C}_r (\mathbb{R}^{m+1}) \tag{42}
\]

is defined as follows: the $j$th component of the configuration $G(z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_r, x)$ is $(\text{id} \times h_j) (z_j, g_j (x))$ for $j \neq i$ while its $i$th component equals $h(g_i (x))$ (whenever $z_j \in S^1$ for all $j \neq i$ and $x \in S^{[p]}$; see also (37)).

Now observe that $G$ is projectable in the sense of Definition 5.1: indeed, the images of $G_1, \ldots, G_{r-1}$ are circles (or a point) which project along the (positive) $x_{m+1}$-axis $l$ to pairwise disjoint intervals. In order to apply Theorem 5.2, it suffices (by Proposition 2.2) to check that $G$ is nulhomotopic when restricted to

\[
\{(x_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_r, x) \in (S^1)^{r-1} \times S^{[p]} \mid z_j = 1\}
\]

for $j \neq i$. If $i = r$ (or $j = r$), rotate the constant $G_j$ (or $G_i$, resp.) by the angle $\pi$ in $\mathbb{R}^{m+1}$ (cf. (37)) and combine a nulhomotopy of $G_r = g_r \circ \text{proj}$ in $\mathbb{R}^m \setminus \{*_k \mid k \neq i, j\}$ with a contraction of the remaining components of $G$. If $i, j \neq r$ (and hence $G_r$ is obtained by rotating $g_r$), rotate the constants $G_i$ and $G_j$ to $z_i = 1$ and $z_j = -1$ and contract $G_r$ near $z_r = 1$ as above; after moving $G_i$ to a faraway point it is now easy to find the desired nulhomotopy of $G_i$.

It follows that $G$ and $\bar{F}$ are nulhomotopic on every strict subproduct in their domains. But since $S^{p_j + 1} / \partial B_j = (S^1 \times S^{p_j}) / S^1 \times \{\ast\}$, the domain of $\bar{F}$ (cf. (40)) inherits a cell decomposition from the domain of $\bar{F}$, and we can combine obstruction theory with Puppe's result Proposition 2.1 to show inductively that also $\bar{F}$ is nulhomotopic outside the top cell. We conclude from diagram (40) that $E_i f$ is $\kappa$-Brunnian and that $[\bar{E}_i f], [\bar{F}^*], [\bar{F}]$ and...
all lie in the image of $\text{quot}^*$ and have the same unique inverse image $v$ in $\pi_{1,p+r-1}(\sqrt{r-1} S^m)$, namely $v = \xi(E_\nu f)$.

In particular, the projectability Theorem 5.2 applies to $G$ and we obtain

$$\mu_\gamma(E_\nu f) = h_\nu(v) = \pm h_\nu(N), \quad \gamma \in \Sigma_{r-2},$$

(43)

where the link $N = N_1 \sqcup \cdots \sqcup N_{r-1} \subset (S^1)^{r-1} \times \mathbb{R}^{|p|}$ is defined by

$$N_k = \{ z \in (S^1)^{r-1} \times S^{|p|} | G_\nu(z) - G_\nu(kz) \in \ell_+ \}.$$

(44)

It remains to compare $h_\nu(N)$ to $h_\nu(g) = \mu(f)$. In the special case $i = r$ both $G$ and $N$ may also be obtained from the link map $e_\nu(g)$ in $\mathbb{R}^{m+1}$ which consists of $r - 1$ standard circles (in disjoint planes parallel to the $(x_3, x_{m+1})$-plane) and of the map $g_r$ into a wedge of meridians (cf. (30)). Indeed, clearly $G = e_\nu(g)$ and

$$h_\nu(N) = h_\nu(e_\nu(g)) = \pm h_\nu(g).$$

as in (the proof of) Theorem 6.1. Together with (43) this implies that

$$\mu_\gamma(E_\nu f) = \pm h_\nu(g) = \pm \mu(f).$$

(45)

Next we consider the case $i \neq r$. Then $G_i \equiv h^*(i) = (i, 0, 1, 0, \ldots, 0)$ (cf. (33) and (37)). More generally, for $1 \leq j \leq r - 1$ $G_j$ maps into the unit circle of the plane $l_j \times l$ which is parallel to the $(x_3, x_{m+1})$-plane; here we define

$$l_j := \{(j, 0, x_3, 0, \ldots, 0) \in \mathbb{R}^m | x_3 \in \mathbb{R}\}.$$

Fig. 3. Determining the overcrossing locus $N_j$ of $G$ (when $i, j, r$ are distinct).
In order also to study $G_r$, we may assume that

(i) the line $l_j$ intersects the wedge $h(\sqrt{r^{-1}} S^{m-1})$ transversely in $\mathbb{R}^m$, and precisely in two points of the form

$$y^*_j = (j, 0, 1 \pm \varepsilon, 0, \ldots, 0) \quad (\text{where } 0 < \varepsilon < 1)$$

which lie in the $j$th component sphere of the wedge; and

(ii) $y^*_j$ are regular values of $h \circ g_r$.

Let $M^\pm_j = (h \circ g_r)^{-1}(y^*_j) \subset \mathbb{R}^{[p]}$ be the corresponding inverse image manifolds. Then only the points of $(S^1)^{-1} \times (M^+_j \cup M^-_j)$ lie in $G_r^{-1}(l_j \oplus l)$ and hence can contribute to the overcrossing locus $N_j$ (cf. (44)). If $j \neq i$, the corresponding parts $N^\pm_j$ of $N_j$ have the form

$$N^+_j = \{(z_1, \ldots, \hat{z}_i, \ldots, z_r; x) \in (S^1)^{-1} \times \mathbb{R}^{[p]} \mid z_i = \sigma^+(z_j); x \in M^+_j \}$$

and

$$N^-_j = \{(z_1, \ldots, \hat{z}_i, \ldots, z_r; x) \in (S^1)^{-1} \times \mathbb{R}^{[p]} \mid z_j = \sigma^-(z_r); x \in M^-_j \}$$

where $\sigma^\pm$ assigns to each element of an “inner” circle the only element of the outer circle lying “above” it w.r. to the $\pm x_{r+1}$-coordinate (see Fig. 3). Since these two maps are nullhomotopic, we may replace them by constant maps $c_r, j$ and $c_i, j$ resp., without changing the link homotopy type of $N$. If $j = i$, $N^+_j$ has the desired product form $(S^1)^{-2} \times \{c_r, j \times M^+_j \}$ right away, but $N^-_j = \emptyset$.

For the calculation of $h_\nu(N)$ we may assume that the constants $c_{r, j}, 1 \leq j \leq r - 1$ are pairwise distinct. Then only the sublink

$$N' = N_1^{-1} \bigsqcup \cdots \bigsqcup N_{i-1}^{-1} \bigsqcup N_i^+ \bigsqcup N_{i+1}^+ \bigsqcup \cdots \bigsqcup N_{r-1}^{-1}$$

can contribute to the overcrossing locus of $N$ over $(S^1)^{-1}$ (which must lie entirely over the single point $(c_1, \ldots, c_{r-1}, c_{r+1}, \ldots, c_{r-1}, c_{r, i})$). As in the proof of Theorem 6.1, we conclude that

$$\pm \mu_\nu(E_i, f) = h_\nu(N) = h_\nu(N') \quad (\text{cf. (43)})$$

and

$$\pm h_\nu(M_1 \bigsqcup \cdots \bigsqcup M_{r-1}^+ \bigsqcup M^+_i \bigsqcup M_{i+1}^+ \bigsqcup \cdots \bigsqcup M^-_r \subset \mathbb{R}^{[p]}) = \mu_\nu(f)$$

agree up to a fixed sign.

\section*{8. CONCORDANCE INVARIANCE}

In this section we show that in many cases our $\mu$ invariants and certain components of linking coefficients remain unchanged by (singular) link concordances. As a consequence we can specify situations where classification up to link homotopy and up to singular link concordance coincides and is completely achieved by our $\mu$-invariants.

\textbf{Theorem 8.1.} Assume that $p_1, \ldots, p_r \geq 1$ and that the base point preserving link maps

$$f, f' : S^{p_1} \bigsqcup \cdots \bigsqcup S^{p_r} \to \mathbb{R}^m$$
are related by a base point preserving singular link concordance (i.e. \( F \) maps into \( \mathbb{R}_+^r \times I \), and
\[
F_j((0, \ldots, 0), t) = (j, t)
\]
for all \( t \in I, j = 1, \ldots, r; \)
cf. (2).

Then for all \( \gamma \in \Sigma_{r - 2} \)
\[
\mu_\gamma(f) = \mu_\gamma(f').
\]

**Proof.** Such concordances are compatible with addition and with \( \Phi \) (cf. (26)) as well as with Nezhinskij suspensions; thus we may assume \( f \) and \( f' \) to be \( \kappa \)-Brunnian (cf. (27)).

On the other hand, Nezhinskij's construction is also compatible with smooth embeddings. Indeed, if a component map \( f_j \) of \( f \) is already a base point preserving embedding into \( \mathbb{R}_+^n \), we may deform it until \( f_j \mid \partial B_j \) is a standard inclusion into a small sphere in \( \partial \mathbb{R}_+^n \) near \( *_j \) (see (33), (34)); the corresponding "rotated" component of \( E \) will again be an embedding. The same applies to concordances.

Now, if we apply an iterated Nezhinskij suspension \( E = E_1^{\kappa} \circ \cdots \circ E_r^{\kappa} \) to the link maps \( f \) and \( f' \), their \( \mu \)-invariants remain unchanged except for a fixed \( \pm \) sign (see Theorem 7.2). Choose \( N_j \) big enough so that the \( r \)-th component map \( S^p \to \mathbb{R}_+^n \) of \( E_r \) can be approximated by an embedding, and similarly for \( f \) and \( f' \). This procedure can be iterated until \( E_f \) and \( E_{f'} \) are embedded links which are joined by the embedded link concordance \( EF \). But according to a theorem of Haefliger and Smale (cf. [6, 2.21]) then \( E_f \) and \( E_{f'} \) are isotopic and hence link homotopic, and their \( \mu \)-invariants must agree. \( \square \)

Assume \( p_1, \ldots, p_r \leq m - 3 \) for the remainder of this section. Then the base point preserving and base point free singular link concordance classifications coincide.

**COROLLARY 8.2.** If \( 1 \leq p_1, \ldots, p_r \leq m - 3 \) then for every \( \gamma \in \Sigma_{r - 2} \) \( \mu_\gamma \) defines a homomorphism from the full group of singular link concordance classes of link maps \( f: [\Sigma_r = 1 \to \Sigma_r \to \mathbb{R}_+^n \) to \( \pi_1 \mathbb{R}_+^n - (r - 1)(m - 2) - 1 \).

In view of Corollary 6.4 this implies immediately

**COROLLARY 8.3.** Let \( f \) be an embedded link in codimensions greater than 2. Then for every \( \gamma \in \Sigma_{r - 2} \), the invariant \( h(\lambda(f)) \) depends only on the singular link concordance class of \( f \).

(Here \( \lambda(f) \) denotes the \( r \)-th linking coefficient of \( f \) as in (31)).

This result contains the main Theorem 8.1 of [17] (link homotopy invariance of \( \lambda^\infty(\lambda_3(f)) = \pm h(\lambda(f)) = \pm \mu(f) \) when \( r = 3 \)) as a special case. \( \square \)

Finally, recall that higher-dimensional \( \mu \)-invariants play a decisive role in the link homotopy classification results in [4]. Here is another illustration of their strength. (For the definition of the canonical retraction \( \text{retr} \) which ignores lower order linking phenomena see (32).)

**PROPOSITION 8.4.** Assume \( p_1, \ldots, p_r \leq r(m - 2) - |p| \) and let \( f \) be an embedded link in dimensions \( 1 \leq p_1, \ldots, p_r \leq m - 3 \). Then:

(a) The "highest-order part" \( \lambda(f) := \text{retr}(\lambda(f)) \) of its linking coefficient is precisely as strong as the total \( \mu \)-invariant
\[
\mu(f) := \{ \mu_\gamma(f) \}_{\gamma \in \Sigma_{r - 1}} \in \bigoplus_{|p| = (r - 2)! \times (r - 1)(m - 2) - 1} \pi_1 \mathbb{R}_+^n
\]
and hence depends only on the singular link concordance class of \( f \).
The link homotopy class (or the singular link concordance class, resp.) of $f$ is entirely determined by $\mu(f)$ and by the link homotopy classes (or the singular link concordance classes, resp.) of all strict sublinks of $f$.

(c) In particular, for two homotopy Brunnian links $f$ and $f'$ (i.e. all strict sublinks are homotopically trivial) the following are equivalent:

(i) $f$ and $f'$ are link homotopic;
(ii) $f$ and $f'$ are link map concordant; and
(iii) $\mu(f) = \mu(f')$.

Proof. (a) According to Theorem 3.1 the homomorphism $h$ which maps $\lambda(f)$ to $\mu(f)$ is bijective in our dimension setting.
(b) Since $1 \leq p_{1}, \ldots, p_{r} \leq m - 3$ link homotopy classes of embedded links form a full group (construct additive inverses by repeatedly using reflected components and thus subtracting linking coefficients). Therefore, when all strict sublinks of $f$ are known up to link homotopy, the link homotopy classes of $f$, $\varphi(f)$ and $e_{*}(\lambda(f))$ determine each other (cf. the proof of Corollary 6.4). The analogous statement holds also for singular link concordance.
(c) The link homotopy class $[f] = \varphi([f]) = e_{*}(\lambda(f))$ is determined by $\mu(f) = h(\lambda(f))$.

Remark 8.5. There is considerable overlap of our results Corollary 8.3 and Proposition 8.4 with deep and interesting work of Nezhinskij on singular link concordance (see e.g. [18]).

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APPENDIX. COMPARISON TO THE LINK INVARIANTS OF TURAEV AND NEZHINSKIJ

Assume $1 \leq p_{1}, \ldots, p_{r} \leq m - 3$, $\gamma \in \Sigma_{r-2}$ and let $f : [\gamma_{1}]_{*}^{r} S^{p_{r}} \to \mathbb{R}^{m}$ be a smoothly embedded link. Starting from the $(r)$th linking coefficient $\lambda(f)$ (cf. (31)) Turaev [5] obtained and studied in particular the invariant

$$\mu(\gamma(1), \ldots, \gamma(r - 2), r - 1, r)(f) := b_{*} a_{r-1}(\lambda(f)).$$

This definition is made clear by the following diagram:

$$\lambda(f) \in \pi_{*_{r}} \left( \bigvee_{i=1}^{r-1} S^{p_{i}} \right) = \left[ ES^{p_{r}-1} \vee S^{p_{i}} \right] \xrightarrow{a_{r-1}} \left[ E^{r-1} S^{p_{r}-1} \vee \cdots \vee S^{p_{1}} \right]$$

Here $E$ denotes the standard suspension of homotopy theory; $a_{r-1}$ is the $(r - 1)$st operation in the Hopf ladder of Boardman and Steer ([19], cf. Theorem 6.1) and $b_{j}$ is the collapsing map from the wedge $\vee S^{p_{j}}$ to the $j$th factor sphere $S^{p_{j}}$, $j = 1, \ldots, r - 1$. 
THEOREM A.3. Turaev's invariant, defined for embedded links, suspends to the invariant \( \pm \mu(f) \) which is defined for all link maps (cf. Section 4).

**Proof.** In view of Corollary 6.4 we need only to show that \( b_{r-1} a_{r-1} \) suspends to the geometric Hopf invariant \( \mu_{r-1} \), discussed in Section 3.

It follows from James [20, 5.6] that any element of \( \pi_*(\mathcal{C}v\mathbb{S}^q) \) corresponds, via the Pontryagin–Thom procedure, to the bordism class of a framed embedding

\[
g: N = \bigcup_{i=1}^{r-1} \mathbb{S}^q_i \hookrightarrow \mathbb{R}^{p-1} = \mathbb{R}^{p-1} \times \mathbb{R}
\]

which projects already to a self-transverse framed immersion into \( \mathbb{R}^{p-1} \) (see also [21]). Now recall that, by Definition 5.4 in [19], \( a_{r-1}(g) = \mu_{r-1}(\tilde{r} \circ g) \) where the pinch map \( \tilde{r} \) has the effect of stacking \( r - 1 \) “parallel” copies of \( N \) on top of one another, i.e. we replace \( g \) by the disjoint embeddings \( g + (0, \ldots, 0, i\epsilon), i = 0, 1, \ldots, r - 2 \), for a sufficiently small \( \epsilon > 0 \). From these data the “external” operation \( \mu_{r-1} \) then extracts the (bordism class of the) total \((r - 1)\)-fold overcrossing locus which is embedded into \( \mathbb{R}^{p-1} + (r-1) \) in a canonical fashion (cf. [19, 6.6 and 6.7]). Finally, \( b_{r-1} \) selects that part of the corresponding \((r - 1)\)-tuple point locus in \( \mathbb{R}^{p-1} \) over which the parallel copies (or equivalently, the components) of \( N \) are stacked on top of one another according to the (reverse) order described by \( \gamma \). Since the stable suspension \( E^\infty \) forgets embeddings, we get

\[
E^\infty \circ b_{r-1} \circ a_{r-1} = \pm \mu_{r-1}.
\]

Nezhinskij has proposed to extend Turaev's invariants to all link maps \( f \) in codimensions > 2 as follows. First apply a suitable composed Nezhinskij suspension \( E = E_1^N \circ \cdots \circ E_{r-1}^N \), so that \( Ef \) can automatically be approximated by an embedding (as in the proof of Theorem 8.1) and then apply Turaev's construction (with values in the stable group \( \pi_\infty^N \)).

**Corollary A.4.** For every link map \( f \) in codimensions > 2 Nezhinskij’s invariant

\[
\mu(\gamma(1), \ldots, \gamma(r - 2), r - 1, r)(Ef)
\]

agrees with \( \pm \mu(f) \).

**Proof.** Base point preserving and base point free link homotopy coincide here; therefore the Nezhinskij suspensions can be defined by rotating in the \((x_2, x_{m+1})\)-planes and are seen to commute with addition and hence with \( \rho \) (cf. (26) and (27)). Our claim follows from Theorem 7.2 even if \( f \) is not \( K \)-Brunnian.

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