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Linear Algebra and its Applications 402 (2005) 245–254
LINEAR ALGEBRA AND ITS APPLICATIONS
www.elsevier.com/locate/laa

Some reverses of the generalised triangle inequality in complex inner product spaces

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Received 25 May 2004; accepted 18 January 2005
Available online 5 March 2005
Submitted by V. Mehrmann

Abstract

Some reverses for the generalised triangle inequality in complex inner product spaces are given. They improve the classical Diaz–Metcalf inequalities. They are applied to obtain inequalities for complex numbers.

AMS classification: Primary 46C05; Secondary 26D15

Keywords: Triangle inequality; Diaz–Metcalf inequality; Reverse inequality; Complex inner product space

1. Introduction

The following reverse of the generalised triangle inequality

$$\cos\theta \sum_{k=1}^{n} |z_k| \leqslant \left|\sum_{k=1}^{n} z_k\right| \tag{1.1}$$

provided the complex numbers z_k, $k \in \{1, \ldots, n\}$ satisfy the assumption

$$a - \theta \leqslant \arg(z_k) \leqslant a + \theta, \quad \text{for any } k \in \{1, \ldots, n\}, \tag{1.2}$$

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doi:10.1016/j.laa.2005.01.015

where $a \in \mathbb{R}$ and $\theta \in \left(0, \frac{\pi}{2}\right)$ was first discovered by Petrovich in 1917, [5] (see [4, p. 492]) and subsequently was rediscovered by other authors, including Karamata [2, p. 300–301], Wilf [6], and in an equivalent form by Marden [3].

The first to consider the problem of obtaining reverses for the triangle inequality in the more general case of Hilbert and Banach spaces were Diaz and Metcalf [1] who showed that in an inner product space H over the real or complex number field, the following reverse of the triangle inequality holds

$$r \sum_{k=1}^{n} \|x_k\| \leqslant \left\| \sum_{k=1}^{n} x_k \right\| \tag{1.3}$$

provided

$$0 \leqslant r\|x_k\| \leqslant \operatorname{Re}\langle x_k, a\rangle \quad \text{for } k \in \{1, \ldots, n\},$$

where $a \in H$ is a unit vector, i.e. $\|a\| = 1$.

The case of equality holds in (1.3) if and only if

$$\sum_{k=1}^{n} x_k = r\left(\sum_{k=1}^{n} \|x_k\|\right) a. \tag{1.4}$$

The main purpose of this paper is to investigate the same problem of reversing the generalised triangle inequality in complex inner product spaces under additional assumptions for the imaginary part $\operatorname{Im}\langle x_k, a\rangle$. A refinement of the Diaz–Metcalf result is obtained. Applications for complex numbers are pointed out.

2. Main results

In [1], the authors have proved the following reverse of the generalised triangle inequality in terms of orthonormal vectors.

Theorem 1. *Let $e_1, \ldots, e_m$ be orthonormal vectors in $(H; \langle\cdot,\cdot\rangle)$, i.e., we recall that $\langle e_i, e_j\rangle = 0$ if $i \neq j$ and $\|e_i\| = 1$, $i, j \in \{1, \ldots, m\}$. Suppose that the vectors $x_1, \ldots, x_n \in H$ satisfy*

$$0 \leqslant r_k\|x_j\| \leqslant \operatorname{Re}\langle x_j, e_k\rangle, \quad j \in \{1, \ldots, n\},\ k \in \{1, \ldots, m\}, \tag{2.1}$$

where $r_k \geqslant 0$ for $k \in \{1, \ldots, m\}$. Then

$$\left(\sum_{k=1}^{m} r_k^2\right)^{\frac{1}{2}} \sum_{j=1}^{n} \|x_j\| \leqslant \left\| \sum_{j=1}^{n} x_j \right\|, \tag{2.2}$$

where equality holds if and only if

$$\sum_{j=1}^{n} x_j = \left(\sum_{j=1}^{n} \|x_j\|\right) \sum_{k=1}^{m} r_k e_k. \tag{2.3}$$

If the space $(H; \langle\cdot,\cdot\rangle)$ is complex and more information is available for the imaginary part, then the following result may be stated as well.

Theorem 2. *Let $e_1, \dots, e_m \in H$ be an orthonormal family of vectors in the complex inner product space H. If the vectors $x_1, \dots, x_n \in H$ satisfy the conditions*

$$0 \leqslant r_k \|x_j\| \leqslant \operatorname{Re}\langle x_j, e_k\rangle, \quad 0 \leqslant \rho_k \|x_j\| \leqslant \operatorname{Im}\langle x_j, e_k\rangle \tag{2.4}$$

for each $j \in \{1, \dots, n\}$ and $k \in \{1, \dots, m\}$, where $r_k, \rho_k \geqslant 0$ for $k \in \{1, \dots, m\}$, then we have the following reverse of the generalised triangle inequality:

$$\left[\sum_{k=1}^{m}\left(r_k^2 + \rho_k^2\right)\right]^{\frac{1}{2}} \sum_{j=1}^{n} \|x_j\| \leqslant \left\|\sum_{j=1}^{n} x_j\right\|. \tag{2.5}$$

The equality holds in (2.5) *if and only if*

$$\sum_{j=1}^{n} x_j = \left(\sum_{j=1}^{n} \|x_j\|\right) \sum_{k=1}^{m} (r_k + \mathrm{i}\rho_k) e_k. \tag{2.6}$$

Proof. Before we prove the theorem, let us recall that, if $x \in H$ and $e_1, \dots, e_m$ are orthogonal vectors, then the following identity holds true:

$$\left\| x - \sum_{k=1}^{m} \langle x, e_k\rangle e_k \right\|^2 = \|x\|^2 - \sum_{k=1}^{m} |\langle x, e_k\rangle|^2. \tag{2.7}$$

As a consequence of this identity, we note the *Bessel inequality*

$$\sum_{k=1}^{m} |\langle x, e_k\rangle|^2 \leqslant \|x\|^2, \quad x \in H. \tag{2.8}$$

The case of equality holds in (2.8) if and only if (see (2.7))

$$x = \sum_{k=1}^{m} \langle x, e_k\rangle e_k. \tag{2.9}$$

Applying Bessel's inequality for $x = \sum_{j=1}^{n} x_j$, we have

$$\begin{aligned} \left\|\sum_{j=1}^{n} x_j\right\|^2 &\geqslant \sum_{k=1}^{m} \left|\left\langle \sum_{j=1}^{n} x_j, e_k \right\rangle\right|^2 = \sum_{k=1}^{m} \left|\sum_{j=1}^{n} \langle x_j, e_k\rangle\right|^2 \\ &= \sum_{k=1}^{m} \left| \left(\sum_{j=1}^{n} \operatorname{Re}\left\langle x_j, e_k\right\rangle\right) + \mathrm{i}\left(\sum_{j=1}^{n} \operatorname{Im}\langle x_j, e_k\rangle\right)\right|^2 \\ &= \sum_{k=1}^{m} \left[\left(\sum_{j=1}^{n} \operatorname{Re}\langle x_j, e_k\rangle\right)^2 + \left(\sum_{j=1}^{n} \operatorname{Im}\langle x_j, e_k\rangle\right)^2\right]. \end{aligned} \tag{2.10}$$

Now, by the hypothesis (2.4) we have

$$\left(\sum_{j=1}^{n} \operatorname{Re}\langle x_j, e_k\rangle\right)^2 \geqslant r_k^2 \left(\sum_{j=1}^{n} \|x_j\|\right)^2 \tag{2.11}$$

and

$$\left(\sum_{j=1}^{n} \operatorname{Im}\langle x_j, e_k\rangle\right)^2 \geqslant \rho_k^2 \left(\sum_{j=1}^{n} \|x_j\|\right)^2. \tag{2.12}$$

Further, on making use of (2.10)–(2.12), we deduce

$$\begin{aligned}\left\|\sum_{j=1}^{n} x_j\right\|^2 &\geqslant \sum_{k=1}^{m}\left[r_k^2\left(\sum_{j=1}^{n}\|x_j\|\right)^2 + \rho_k^2\left(\sum_{j=1}^{n}\|x_j\|\right)^2\right] \\ &= \left(\sum_{j=1}^{n}\|x_j\|\right)^2 \sum_{k=1}^{m}\left(r_k^2+\rho_k^2\right),\end{aligned}$$

which is clearly equivalent to (2.5).

Now, if (2.6) holds, then the case of equality holds in (2.5).

Conversely, if the equality holds in (2.5), then it must hold in all the inequalities used to prove (2.5) and therefore we must have

$$\left\|\sum_{j=1}^{n} x_j\right\|^2 = \sum_{k=1}^{m}\left|\sum_{j=1}^{n}\langle x_j, e_k\rangle\right|^2 \tag{2.13}$$

and

$$r_k\|x_j\| = \operatorname{Re}\langle x_j, e_k\rangle, \quad \rho_k\|x_j\| = \operatorname{Im}\langle x_j, e_k\rangle \tag{2.14}$$

for each $j \in \{1, \ldots, n\}$ and $k \in \{1, \ldots, m\}$.

Using the identity (2.7), we deduce from (2.13) that

$$\sum_{j=1}^{n} x_j = \sum_{k=1}^{m}\left\langle \sum_{j=1}^{n} x_j, e_k\right\rangle e_k. \tag{2.15}$$

Multiplying the second equality in (2.14) with the imaginary unit i and summing the equality over j from 1 to n, we deduce

$$(r_k + \mathrm{i}\rho_k)\sum_{j=1}^{n}\|x_j\| = \left\langle \sum_{j=1}^{n} x_j, e_k\right\rangle \tag{2.16}$$

for each $k \in \{1, \ldots, n\}$.

Finally, utilising (2.15) and (2.16), we deduce (2.6) and the theorem is proved. □

The case of a unit vector, which improves the Diaz–Metcalf inequality (1.3) is useful for applications (see Propositions 1 and 2). It will be stated as a separate theorem for which some corollaries are also provided.

Theorem 3. *Let $(H; \langle\cdot,\cdot\rangle)$ be a complex inner product space. Suppose that the vectors $x_k \in H$, $k \in \{1, \ldots, n\}$ satisfy the condition*

$$0 \leqslant r_1\|x_k\| \leqslant \operatorname{Re}\langle x_k, e\rangle, \quad 0 \leqslant r_2\|x_k\| \leqslant \operatorname{Im}\langle x_k, e\rangle \tag{2.17}$$

for each $k \in \{1, \ldots, n\}$, where $e \in H$ is such that $\|e\| = 1$ and $r_1, r_2 \geqslant 0$. Then we have the inequality

$$\sqrt{r_1^2 + r_2^2}\sum_{k=1}^{n}\|x_k\| \leqslant \left\|\sum_{k=1}^{n} x_k\right\|, \tag{2.18}$$

where equality holds if and only if

$$\sum_{k=1}^{n} x_k = (r_1 + \mathrm{i}r_2)\left(\sum_{k=1}^{n}\|x_k\|\right)e. \tag{2.19}$$

The following corollaries of Theorem 3 are of interest for applications.

Corollary 1. *Let e a unit vector in the complex inner product space $(H; \langle\cdot,\cdot\rangle)$ and $\rho_1, \rho_2 \in (0, 1)$. If $x_k \in H$, $k \in \{1, \ldots, n\}$ are such that*

$$\|x_k - e\| \leqslant \rho_1, \quad \|x_k - \mathrm{i}e\| \leqslant \rho_2 \quad \text{for each } k \in \{1, \ldots, n\}, \tag{2.20}$$

then we have the inequality

$$\sqrt{2 - \rho_1^2 - \rho_2^2}\sum_{k=1}^{n}\|x_k\| \leqslant \left\|\sum_{k=1}^{n} x_k\right\|, \tag{2.21}$$

with equality if and only if

$$\sum_{k=1}^{n} x_k = \left(\sqrt{1-\rho_1^2} + \mathrm{i}\sqrt{1-\rho_2^2}\right)\left(\sum_{k=1}^{n}\|x_k\|\right)e. \tag{2.22}$$

Proof. From the first inequality in (2.20) we deduce, by taking the square, that

$$\|x_k\|^2 + 1 - \rho_1^2 \leqslant 2\operatorname{Re}\langle x_k, e\rangle,$$

implying

$$\frac{\|x_k\|^2}{\sqrt{1-\rho_1^2}} + \sqrt{1-\rho_1^2} \leqslant \frac{2\operatorname{Re}\langle x_k, e\rangle}{\sqrt{1-\rho_1^2}} \tag{2.23}$$

for each $k \in \{1, \ldots, n\}$.

Since, obviously

$$2\|x_k\| \leqslant \frac{\|x_k\|^2}{\sqrt{1-\rho_1^2}} + \sqrt{1-\rho_1^2}, \quad k \in \{1, \ldots, n\}, \tag{2.24}$$

hence, by (2.23) and (2.24),

$$0 \leqslant \sqrt{1-\rho_1^2}\|x_k\| \leqslant \operatorname{Re}\langle x_k, e\rangle \tag{2.25}$$

for each $k \in \{1, \ldots, n\}$.

From the second inequality in (2.20) we deduce

$$0 \leqslant \sqrt{1-\rho_2^2}\|x_k\| \leqslant \operatorname{Re}\langle x_k, \mathrm{i}e\rangle$$

for each $k \in \{1, \ldots, n\}$.

Since

$$\operatorname{Re}\langle x_k, \mathrm{i}e\rangle = \operatorname{Im}\langle x_k, e\rangle,$$

hence

$$0 \leqslant \sqrt{1-\rho_2^2}\|x_k\| \leqslant \operatorname{Im}\langle x_k, e\rangle \tag{2.26}$$

for each $k \in \{1, \ldots, n\}$.

Now, observe from (2.25) and (2.26), that the condition (2.17) of Theorem 3 is satisfied for $r_1 = \sqrt{1-\rho_1^2}$, $r_2 = \sqrt{1-\rho_2^2} \in (0, 1)$, and thus the corollary is proved. □

Corollary 2. *Let e be a unit vector in the complex inner product space $(H; \langle\cdot,\cdot\rangle)$ and $M_1 \geqslant m_1 > 0$, $M_2 \geqslant m_2 > 0$. If $x_k \in H$, $k \in \{1, \ldots, n\}$ are such that either*

$$\operatorname{Re}\langle M_1 e - x_k, x_k - m_1 e\rangle \geqslant 0, \quad \operatorname{Re}\langle M_2 \mathrm{i}e - x_k, x_k - m_2 \mathrm{i}e\rangle \geqslant 0 \tag{2.27}$$

or, equivalently,

$$\begin{aligned} \left\| x_k - \frac{M_1+m_1}{2} e \right\| &\leqslant \frac{1}{2}(M_1 - m_1), \\ \left\| x_k - \frac{M_2+m_2}{2} \mathrm{i}e \right\| &\leqslant \frac{1}{2}(M_2 - m_2) \end{aligned} \tag{2.28}$$

for each $k \in \{1, \ldots, n\}$, then we have the inequality

$$2\left[\frac{m_1 M_1}{(M_1+m_1)^2} + \frac{m_2 M_2}{(M_2+m_2)^2}\right]^{1/2} \sum_{k=1}^{n} \|x_k\| \leqslant \left\| \sum_{k=1}^{n} x_k \right\|. \tag{2.29}$$

The equality holds in (2.29) if and only if

$$\sum_{k=1}^{n} x_k = 2\left(\frac{\sqrt{m_1 M_1}}{M_1+m_1} + \mathrm{i}\frac{\sqrt{m_2 M_2}}{M_2+m_2}\right)\left(\sum_{k=1}^{n} \|x_k\|\right) e. \tag{2.30}$$

Proof. Firstly, remark that, for $x, z, Z \in H$, the following statements are equivalent:

(i) $\operatorname{Re}\langle Z - x, x - z\rangle \geqslant 0$ and
(ii) $\left\| x - \frac{Z+z}{2} \right\| \leqslant \frac{1}{2}\|Z - z\|$.

Using this fact, we may simply realize that (2.27) and (2.29) are equivalent.

Now, from the first inequality in (2.27), we get

$$\|x_k\|^2 + m_1 M_1 \leqslant (M_1 + m_1)\operatorname{Re}\langle x_k, e\rangle$$

implying

$$\frac{\|x_k\|^2}{\sqrt{m_1 M_1}} + \sqrt{m_1 M_1} \leqslant \frac{M_1 + m_1}{\sqrt{m_1 M_1}}\operatorname{Re}\langle x_k, e\rangle \tag{2.31}$$

for each $k \in \{1, \dots, n\}$.

Since, obviously

$$2\|x_k\| \leqslant \frac{\|x_k\|^2}{\sqrt{m_1 M_1}} + \sqrt{m_1 M_1}, \tag{2.32}$$

hence, by (2.31) and (2.32)

$$0 \leqslant \frac{2\sqrt{m_1 M_1}}{M_1 + m_1}\|x_k\| \leqslant \operatorname{Re}\langle x_k, e\rangle \tag{2.33}$$

for each $k \in \{1, \dots, n\}$.

Now, the proof follows the same path as the one of Corollary 1 and we omit the details. □

Finally, the following corollaries of the Theorem 2 may be stated as well.

Corollary 3. *Let $e_1, \dots, e_m$ be orthonormal vectors in the complex inner product space $(H; \langle\cdot,\cdot\rangle)$ and $\rho_k, \eta_k \in (0, 1)$, $k \in \{1, \dots, n\}$. If $x_1, \dots, x_n \in H$ are such that*

$$\left\|x_j - e_k\right\| \leqslant \rho_k, \qquad \left\|x_j - \mathrm{i}e_k\right\| \leqslant \eta_k$$

for each $j \in \{1, \dots, n\}$ and $k \in \{1, \dots, m\}$, then we have the inequality

$$\left[\sum_{k=1}^{m}\left(2 - \rho_k^2 - \eta_k^2\right)\right]^{\frac{1}{2}} \sum_{j=1}^{n}\|x_j\| \leqslant \left\|\sum_{j=1}^{n} x_j\right\|. \tag{2.34}$$

The case of equality holds in (2.34) if and only if

$$\sum_{j=1}^{n} x_j = \left(\sum_{j=1}^{n}\|x_j\|\right)\sum_{k=1}^{m}\left(\sqrt{1 - \rho_k^2} + \mathrm{i}\sqrt{1 - \eta_k^2}\right) e_k. \tag{2.35}$$

The proof employs Theorem 2 and is similar to the one from Corollary 1. We omit the details.

Corollary 4. *Let* $e_1, \dots, e_m$ *be as in Corollary* 3 *and* $M_k \geqslant m_k > 0$, $N_k \geqslant n_k > 0$, $k \in \{1, \dots, m\}$. *If* $x_1, \dots, x_n \in H$ *are such that either*

$$\operatorname{Re}\langle M_k e_k - x_j, x_j - m_k e_k\rangle \geqslant 0, \quad \operatorname{Re}\langle N_k \mathrm{i} e_k - x_j, x_j - n_k \mathrm{i} e_k\rangle \geqslant 0$$

or, equivalently,

$$\left\| x_j - \frac{M_k + m_k}{2} e_k \right\| \leqslant \frac{1}{2}(M_k - m_k),$$
$$\left\| x_j - \frac{N_k + n_k}{2} \mathrm{i} e_k \right\| \leqslant \frac{1}{2}(N_k - n_k)$$

for each $j \in \{1, \dots, n\}$ *and* $k \in \{1, \dots, m\}$, *then we have the inequality*

$$2\left\{\sum_{k=1}^{m}\left[\frac{m_k M_k}{(M_k + m_k)^2} + \frac{n_k N_k}{(N_k + n_k)^2}\right]\right\}^{\frac{1}{2}} \sum_{j=1}^{n} \|x_j\| \leqslant \left\| \sum_{j=1}^{n} x_j \right\|. \tag{2.36}$$

The case of equality holds in (2.36) *if and only if*

$$\sum_{j=1}^{n} x_j = 2\left(\sum_{j=1}^{n} \|x_j\|\right) \sum_{k=1}^{m}\left(\frac{\sqrt{m_k M_k}}{M_k + m_k} + \mathrm{i}\frac{\sqrt{n_k N_k}}{N_k + n_k}\right) e_k. \tag{2.37}$$

The proof employs Theorem 2 and is similar to the one in Corollary 2. We omit the details.

3. Applications for complex numbers

The following reverse of the generalised triangle inequality with a clear geometric meaning may be stated.

Proposition 1. *Let* $z_1, \dots, z_n$ *be complex numbers with the property that*

$$0 < \varphi_1 \leqslant \arg(z_k) \leqslant \varphi_2 < \frac{\pi}{2} \tag{3.1}$$

for each $k \in \{1, \dots, n\}$. *Then we have the inequality*

$$\sqrt{\sin^2 \varphi_1 + \cos^2 \varphi_2} \sum_{k=1}^{n} |z_k| \leqslant \left| \sum_{k=1}^{n} z_k \right|. \tag{3.2}$$

The equality holds in (3.2) *if and only if*

$$\sum_{k=1}^{n} z_k = (\cos \varphi_2 + \mathrm{i} \sin \varphi_1) \sum_{k=1}^{n} |z_k|. \tag{3.3}$$

Proof. Let $z_k = a_k + \mathrm{i}b_k$. We may assume that $b_k \geqslant 0$, $a_k > 0$, $k \in \{1, \dots, n\}$, since, by (3.1), $\frac{b_k}{a_k} = \tan[\arg(z_k)] \in [0, \infty)$, $k \in \{1, \dots, n\}$. By (3.1), we obviously have

$$0 \leqslant \tan^2 \varphi_1 \leqslant \frac{b_k^2}{a_k^2} \leqslant \tan^2 \varphi_2, \quad k \in \{1, \dots, n\}$$

from where we get

$$\frac{b_k^2 + a_k^2}{a_k^2} \leqslant \frac{1}{\cos^2 \varphi_2}, \quad k \in \{1, \dots, n\},\ \varphi_2 \in \left(0, \frac{\pi}{2}\right)$$

and

$$\frac{a_k^2 + b_k^2}{a_k^2} \leqslant \frac{1 + \tan^2 \varphi_1}{\tan^2 \varphi_1} = \frac{1}{\sin^2 \varphi_1}, \quad k \in \{1, \dots, n\},\ \varphi_1 \in \left(0, \frac{\pi}{2}\right)$$

giving the inequalities

$$|z_k| \cos \varphi_2 \leqslant \operatorname{Re}(z_k), \quad |z_k| \sin \varphi_1 \leqslant \operatorname{Im}(z_k)$$

for each $k \in \{1, \dots, n\}$.

Now, applying Theorem 3 for the complex inner product space $\mathbb{C}$ endowed with the inner product $\langle z, w \rangle = z \cdot \bar{w}$ for $x_k = z_k$, $r_1 = \cos \varphi_2$, $r_2 = \sin \varphi_1$ and $e = 1$, we deduce the desired inequality (3.2). The case of equality is also obvious by Theorem 3 and the proposition is proven. □

Another result that has an obvious geometrical interpretation is the following one.

Proposition 2. *Let $c \in \mathbb{C}$ with $|c| = 1$ and $\rho_1, \rho_2 \in (0, 1)$. If $z_k \in \mathbb{C}$, $k \in \{1, \dots, n\}$ are such that*

$$|z_k - c| \leqslant \rho_1, \quad |z_k - \mathrm{i}c| \leqslant \rho_2 \quad \text{for each } k \in \{1, \dots, n\}, \tag{3.4}$$

then we have the inequality

$$\sqrt{2 - \rho_1^2 - \rho_2^2} \sum_{k=1}^{n} |z_k| \leqslant \left| \sum_{k=1}^{n} z_k \right|, \tag{3.5}$$

with equality if and only if

$$\sum_{k=1}^{n} z_k = \left(\sqrt{1 - \rho_1^2} + \mathrm{i}\sqrt{1 - \rho_2^2} \right) \left(\sum_{k=1}^{n} |z_k| \right) c. \tag{3.6}$$

The proof is obvious by Corollary 1 applied for $H = \mathbb{C}$.

Remark 1. If we choose $c = 1$, and for $\rho_1, \rho_2 \in (0, 1)$ we define $\bar{D}(1, \rho_1) := \{z \in \mathbb{C} \| z - 1| \leqslant \rho_1\}$, $\bar{D}(\mathrm{i}, \rho_2) := \{z \in \mathbb{C} \| z - \mathrm{i}| \leqslant \rho_2\}$, then obviously the intersection

$$S_{\rho_1,\rho_2} := \bar{D}(1, \rho_1) \cap \bar{D}(\mathrm{i}, \rho_2)$$

is nonempty if and only if $\rho_1 + \rho_2 \geqslant \sqrt{2}$.

If $z_k \in S_{\rho_1,\rho_2}$ for $k \in \{1, \ldots, n\}$, then (3.5) holds true. The equality holds in (3.5) if and only if

$$\sum_{k=1}^{n} z_k = \left(\sqrt{1-\rho_1^2} + \mathrm{i}\sqrt{1-\rho_2^2}\right) \sum_{k=1}^{n} |z_k|.$$

Acknowledgement

The author would like to thank the anonymous referees for their valuable comments which have been implemented in the final version of this paper.

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