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NOTE

ON k-STACKED POLYTOPES

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It is proved that equality in the Generalized Simplicial Lower Bound Conjecture can always be obtained by k-stacked polytopes.

Let P be a simplicial convex d-polytope with f_i faces of dimension i. The vector $f(P)=(f_0,\ldots,f_{d-1})$ is called the *f-vector* of P. The complete characterization of all f-vectors, known as McMullen's g-conjecture [3], has been obtained by Billera and Lee in [1] and by Stanley in [8]. Billera and Lee proved the sufficiency and Stanley the necessity of McMullen's conditions for a vector in $\mathbb{Z}^d$ to be the f-vector of some simplicial d-polytope. These conditions are formulated in terms of the h-vector of a polytope rather than in terms of the f-vector.

The vector $h(P)=(h_0,h_1,\ldots,h_d)$ is called the *h-vector* of P, where

$$h_i=\sum_{j=0}^{i}\binom{d-j}{d-i}(-1)^{i-j}f_{j-1}\qquad (f_{-1}:=1).$$

Then the g-conjecture (or rather the g-Theorem) may be formulated as follows:

A vector $h=(h_0,\ldots,h_d)$ in $\mathbb{Z}^{d+1}$ is the h-vector of some simplicial d-polytope if and only if the following conditions hold:

(i) $h_i=h_{d-i}$, $0\leq i\leq n:=[\frac{1}{2}d]$,
(ii) $h_i\geq h_{i-1}$, $1\leq i\leq n$,
(iii) $h_0=1$ *and* $h_{i+1}-h_i\leq(h_i-h_{i-1})^{\langle i\rangle}$, $1\leq i\leq n-1$.

(For the definition of the functional $x^{\langle i\rangle}$ see [1], [3] or [8].)

The inequality (ii) together with the following condition for equality is known as the "Generalized Simplicial Lower Bound Conjecture" first formulated by McMullen and Walkup [4]:

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(∗) If $d \geq 4$, then equality holds in (ii) for a d-polytope P if and only if P is an $(i-1)$-stacked polytope (a polytope P is called a k-stacked polytope if P (not the boundary-complex of P!) admits a subdivision into a simplicial complex, every $(d-k-1)$-face of which is a face of P).

The 'only if' part of condition (∗) is still open, and the purpose of this paper is to prove a related result which gives some support to the validity of the conjecture.

In [4], McMullen and Walkup proved the following: If d, k and v are integers satisfying $2 \leq 2k \leq d < v$, then there exists a k-neighbourly d-polytope with v vertices which is k-stacked.

This can be viewed as a special case of our following main result:

Theorem. *Let $h = (h_0, \ldots, h_d)$ be a vector satisfying the conditions* (i)–(iii) *of the* g*-conjecture and let $h_k = h_{k-1}$ for some k with $1 \leq k \leq n$, then there exists a $(k-1)$-stacked d-polytope P with $h(P) = h$.*

Of course, our theorem does not exclude the existence of a non-stacked polytope having the same h-vector as a k-stacked polytope, but at least it proves that equality in (ii) implies the *existence* of a stacked polytope with the right h-vector.

The proof of our theorem is based on the construction introduced by Billera and Lee in [1], and so we use their teminology.

Let h be the vector of the theorem. Then, according to [1], there exists a shellable subcomplex Δ of the boundary-complex of $C(h_1 + d, d+1)$($C(n, d)$ is the cyclic d-polytope with n vertices) such that $|\Delta|$ is a d-ball and $h(\partial\Delta) = h$.

Furthermore, it is shown that $\partial\Delta$ is a 'sharp shadow-boundary' of $C(h_1 + d, d+1)$, i.e. there is a point $z \in \mathbb{R}^{d+1}$ from which exactly those facets of $C(h_1 + d, d+1)$ are 'visible' which are in Δ.

Let H be a hyperplane in $\mathbb{R}^{d+1}$ which strictly separates z from $C(h_1 + d, d+1)$. We project $|\Delta|$ on H by central projection with center z. The image of Δ in H is a complex Δ' isomorphic to Δ and $|\Delta'|$ is a d-polytope P with $h(P) = h$.

It remains to prove that P is $(k-1)$-stacked. This follows from the fact that every cell of Δ' whose dimension is smaller than $d-k+1$ is a face of P. To show this, we use the following (compare [1, § 6]):

$$h_i(\Delta') = h_i(P) - h_{i-1}(P) \quad \text{for } 1 \leq i \leq n \quad \text{and} \quad h_0(\Delta') = 1.$$

So we may conclude that $h_i(\Delta') = 0$ for $i \geq k$.

We have remarked that Δ' is a shellable d-ball, i.e. there is an ordering of the d-cells $F_1, F_2, \ldots, F_m$ of Δ' such that for $2 \leq j \leq m$, $F_j \cap \bigcup_{i=1}^{j-1} F_i$ is the set of all faces of F_j which contain a certain face G_j of F_j, and it is easy to verify that a cell of Δ' is in the interior of Δ' (i.e. not a face of P) if and only if it contains such a face G_j. It follows from a well-known interpretation of the h-vector (compare [2] or [3]) that for a fixed shelling order $h_i(\Delta')$ is exactly the number of G_j's which have dimension $d-i$.

As we have $h_i(\Delta')=0$ for $i \geq k$, it follows that the dimension of every G_j and hence of every interior cell of Δ' is at least $d-k+1$.

This completes the proof of the theorem.

We should like to mention another interesting feature of k-stacked polytopes (where $1 \leq k \leq n$): For these polytopes the proof of the necessity of McMullen's conditions for f-vectors is much easier (Stanley mentions this in another context in [7]).

The crucial ω in Stanley's proof [8] of the general theorem can be found without the use of the hard Lefschetz-theorem as follows: Let Δ be the triangulation of P without interior faces of 'small' dimension and let $\theta_1, \ldots, \theta_{d+1}$ be a suitable system of parameters in the Stanley–Reisner-ring A_Δ (compare [2] and [6]). Let J be the ideal of A_Δ spanned by $\theta_1, \ldots, \theta_d$ and the interior faces of Δ and define $A := A_\Delta / J$. Taking ω as the image of θ_{d+1} in the homomorphism mapping A_Δ on A one gets all properties of ω required in [8] to solve the g-conjecture.

It is possible that another access to the characterization of equality in the Generalized Simplicial Lower Bound Conjecture could be provided by the method of bistellar operations (see [5]). One can easily verify that the equality $h_i(P) = h_{i-1}(P)$ $(1 \leq i \leq n)$ implies that no geometric $(i-1)$-operation (in the sense of [5]) can be performed in the boundary-complex of P.

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