Iterative Approximations of Fixed Points and Solutions for Strongly Accretive and Strongly Pseudo-Contractive Mappings in Banach Spaces*

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In this article, we prove some new convergence theorems of the Ishikawa and Mann iteration sequences for strongly accretive and strongly pseudo-contractive mappings by using the new inequality and the new approximation methods. Our main results improve and extend the corresponding results of [J. Math. Anal.

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1. INTRODUCTION

Throughout this article, we always assume that $X$ is a real Banach space, $X^*$ is the duality space of $X$, and $(\cdot, \cdot)$ is the pairing between $X$ and $X^*$. For $1 < p < \infty$, the mapping $J_p: X \to 2^{X^*}$ defined by

$$J_p(x) = \{f \in X^*: (x, f) = \|f\| \cdot \|x\|, \|f\| = \|x\|^{p-1}\}$$

is called the duality mapping with gauge function $\varphi(t) = t^{p-1}$. In particular, for $p = 2$, the duality mapping $J_2$ with gauge function $\varphi(t) = t$ is called the normalized duality mapping.

The following proposition gives some basic properties of duality mappings:

**Proposition 1.1.** Let $X$ be a real Banach space. For $1 < p < \infty$, the duality mapping $J_p: X \to 2^{X^*}$ has the following basic properties:

1. $J_p(x) \neq \emptyset$ for all $x \in X$ and $D(J_p)$ (the domain of $J_p$) = $X$,
2. $J_p(x) = \|x\|^{p-2} \cdot J_2(x)$ for all $x \in X$ ($x \neq 0$),
3. $J_p(\alpha x) = \alpha^{p-1} \cdot J_p(x)$ for all $\alpha \in [0, \infty)$,
4. $J_p(-x) = -J_p(x)$,
5. $J_p$ is bounded, i.e., for any bounded subset $A \subset X$, $J_p(A)$ is a bounded subset in $X^*$,
6. $J_p$ can be equivalently defined as the subdifferential of the functional $\psi(x) = \|x\|^{p-1} \cdot \|x\|^{p}$ (Asplund [1]), i.e.,

$$J_p(x) = \partial \psi(x) = \{f \in X^*: \psi(y) - \psi(x) \geq (f, y - x) \text{ for all } y \in X\},$$

7. $X$ is a uniformly smooth Banach space (equivalently, $X^*$ is a uniformly convex Banach space) if and only if $J_p$ is single-valued and uniformly continuous on any bounded subset of $X$ (Xu and Roach [21]).

**Definition 1.1.** Let $X$ be a real normed space and let $K$ be a nonempty subset of $X$. Let $T: K \to 2^X$ be a multivalued mapping.
(1) $T$ is said to be accretive if for any $x, y \in K$, $u \in Tx$, and $v \in Ty$, there exists $j_2 \in J_2(x - y)$ such that
\[(u - v, j_2) \geq 0,\]
or, equivalently, there exists $j_p \in J_p(x - y)$, $1 < p < \infty$, such that
\[(u - v, j_p) \geq 0.\]

(2) $T$ is said to be strongly accretive if for any $x, y \in K$, $u \in Tx$, and $v \in Ty$, there exists $j_2 \in J_2(x - y)$ such that
\[(u - v, j_2) \geq k \cdot \|x - y\|^2,\]
or, equivalently, there exists $j_p \in J_p(x - y)$, $1 < p < \infty$, such that
\[(u - v, j_p) \geq k \cdot \|x - y\|^p,\]
for some constant $k > 0$. Without loss of generality, we can assume that $k \in (0, 1)$ and such a number $k$ is called the strong accretive constant of $T$.

(3) $T$ is said to be (strongly) pseudo-contractive if $I - T$ (where $I$ denotes the identity mapping) is a (strongly) accretive mapping.

The concept of a single-valued accretive mapping was introduced independently by Browder [2] and Kato [13] in 1967. An early fundamental result in the theory of accretive mappings which is due to Browder states that the following initial value problem,
\[
\frac{du(t)}{dt} + Tu(t) = 0, \quad u(0) = u_0, \quad (1.1)
\]
is solvable if $T$ is locally Lipschitzian and accretive on $X$.

**Definition 1.2.** Let $X$ be a real Banach space, let $K$ be a nonempty convex subset of $X$ and let $T: K \to 2^K$ be a multivalued mapping. Given $x_0 \in K$, the sequence $(x_n)$ defined by
\[
\begin{cases}
x_{n+1} \in (1 - \alpha_n)x_n + \alpha_n Ty_n, \\
y_n \in (1 - \beta_n)x_n + \beta_n Tx_n,
\end{cases} \quad (1.2)
\]
for all $n = 0, 1, 2, \ldots$ is called the Ishikawa iteration sequence of $T$, where $(\alpha_n)$ and $(\beta_n)$ are two real sequences in $[0, 1]$ satisfying some conditions. Especially, if $\beta_n = 0$ for all $n = 0, 1, 2, \ldots$, then $(x_n)$ is called the Mann iteration sequence.
The convergence problems of Ishikawa and Mann iteration sequences were studied extensively by many authors (Chidume [4–6], Tan and Xu [18], Reich [16], Ishikawa [11, 12], Mann [14], Deng [8–10], Morales [15], Rhoades [17], Xu [20], and Zhou and Jia [22]).

In this article, by using the new inequality and new approximation methods, we study the convergence problem of the Ishikawa and Mann iteration sequences for strongly accretive mappings and strongly pseudo-contractive mappings, respectively. The main results in this article improve and extend the corresponding results in Chidume [4–6], Deng [8–10], Tan and Xu [18] and Zhou and Jia [22].

2. LEMMAS AND INEQUALITIES

**Lemma 2.1.** Let $X$ be a real Banach space and let $J_p: X \to 2^{x^*}$, $1 < p < \infty$, be a duality mapping. Then, for any given $x, y \in X$, we have
\[
\|x + y\|^p \leq \|x\|^p + p \cdot (y, j_p),
\]
for all $j_p \in J_p(x + y)$.

**Proof.** From Proposition 1.1(6), it follows that $J_p(x) = \partial \psi(x)$ (subdifferential of $\psi$), where $\psi(x) = p^{-1} \cdot \|x\|^p$. Also it follows from the definition of subdifferential of $\psi$ that
\[
\psi(x) - \psi(x + y) \geq (x - (x + y), j_p),
\]
for all $j_p \in J_p(x + y)$. Substituting $\psi(x)$ by $p^{-1} \cdot \|x\|^p$, we have
\[
\|x + y\|^p \leq \|x\|^p + p \cdot (y, j_p),
\]
for all $j_p \in J_p(x + y)$. This completes the proof.

**Remark 1.** In [16], Reich proved the following result: If $X$ is a uniformly smooth Banach space, then there exists a continuous and a nondecreasing function $b: [0, \infty) \to [0, \infty)$ such that $b(0) = 0$ and $b(ct) \leq cb(t)$ for all $c \geq 1$ and the following inequality holds
\[
\|x + y\|^2 \leq \|x\|^2 + 2(y, J_2(x)) + \max\{\|x\|, 1\} \cdot \|y\| \cdot b(\|y\|),
\]
for all $x, y \in X$. Because $X$ is a uniformly smooth Banach space, it follows from Proposition 1.1(7) that $J_p, 1 < p < \infty$, is a single-valued mapping. Taking $p = 2$, from (2.1) it follows that
\[
\|x + y\|^2 \leq \|x\|^2 + 2 \cdot (y, J_2(x + y)),
\]
for all $x, y \in X$. Comparing (2.3) with (2.2), we know that (2.1) is more succinct and more convenient than Reich’s inequality (2.2).
(2) The inequality (2.3) was proved by Zhou and Jia [22] and Chang [3] by using other methods, respectively.

**Lemma 2.2 [19].** Let \((\gamma_n)\) be a nonnegative real sequence and let \((\lambda_n)\) be a real sequence in \([0, 1]\) such that \(\sum_{n=0}^{\infty} \lambda_n = \infty\).

1. For any given \(\epsilon > 0\), if there exists a positive integer \(n_0\) such that
   \[\gamma_{n+1} \leq (1 - \lambda_n) \gamma_n + \epsilon \lambda_n,\]
   \hspace{1cm} (2.4)
   for all \(n \geq n_0\), then we have \(0 \leq \limsup_{n \to \infty} \gamma_n \leq \epsilon\).

2. If there exists a positive integer \(n_1\) such that
   \[\gamma_{n+1} \leq (1 - \lambda_n) \gamma_n + \lambda_n \cdot \sigma_n,\]
   \hspace{1cm} (2.5)
   for all \(n \geq n_1\), where \(\sigma_n \geq 0\) for all \(n = 0, 1, 2, \ldots\) and \(\sigma_n \to 0\) as \(n \to \infty\), then we have \(\lim_{n \to \infty} \gamma_n = 0\).

**Proof.** (1) By induction, from condition (2.4), we can prove that
\[\gamma_{n+i} \leq \exp \left( \frac{1}{n_0+i} \sum_{j=n_0}^{n_0+i-1} \lambda_j \right) \cdot \gamma_{n_0} + \epsilon,\]
for \(i = 1, 2, \ldots\). Because \(\sum_{n=0}^{\infty} \lambda_n = \infty\), letting \(i \to \infty\) and taking superior limits, we have
\[0 \leq \limsup_{i \to \infty} \gamma_i = \limsup_{i \to \infty} \gamma_{n_0+i} \leq \epsilon.\]

(2) Because \(\sigma_n \to 0\) as \(n \to \infty\), for any given \(\epsilon > 0\), there exists a positive integer \(n_2 \geq n_1\) such that \(\sigma_n < \epsilon\) for all \(n \geq n_2\). Therefore, from (2.5), it follows that
\[\gamma_{n+1} \leq (1 - \lambda_n) \cdot \gamma_n + \lambda_n \cdot \epsilon,\]
for all \(n \geq n_2\). By the conclusion (1), we know that \(0 \leq \limsup_{n \to \infty} \gamma_n \leq \epsilon\) and so, in view of the arbitrariness of \(\epsilon > 0\), we have \(\lim_{n \to \infty} \gamma_n = 0\). This completes the proof.

**3. Convergence of Ishikawa Iteration Sequences for Strongly Pseudo-Contractive Mappings**

**Lemma 3.1.** Let \(X\) be a Banach space and let \(T : X \to 2^X\) be a multivalued strongly pseudo-contractive mapping. Then, for any given \(x, y \in X\), \(u \in Tx\), and \(v \in Ty\), there exists \(\tilde{f}_p \in J_p(x - y)\), \(1 < p < \infty\), such that
\[(u - v, \tilde{f}_p) \leq (1 - k) \cdot \|x - y\|^p,\]
where \(k \in (0, 1)\) is the strongly accretive constant of \(I - T\).
THEOREM 3.2. Let $X$ be a uniformly smooth real Banach space, let $K$ be a nonempty closed convex subset of $X$ (it need not be bounded) and let $T: K \rightarrow K$ be a single-valued Lipschitzian strongly pseudo-contractive mapping. Let $L \geq 1$ be the Lipschitz constant of $T$ and let $k \in (0, 1)$ be the strongly accretive constant of $I - T$. Let $\{\alpha_n\}, \{\beta_n\}$ be two real sequences satisfying the following conditions:

(i) $0 \leq \alpha_n \leq 1$ for all $n = 0, 1, 2, \ldots$, 
(ii) $0 \leq \beta_n \leq k(1 - k)(L + L^2)^{-1}$ for all $n = 0, 1, 2, \ldots$, 
(iii) $\sum_{n=0}^{\infty} \alpha_n = \infty$ and $\alpha_n \rightarrow 0$ as $n \rightarrow \infty$.

If $F(T) \neq \emptyset$ (i.e., the set of all fixed points of $T$ in $K$), then, for any given $x_q \in K$, the Ishikawa iteration sequence $\{x_n\}$ defined by

\[
\begin{align*}
x_{n+1} &= (1 - \alpha_n)x_n + \alpha_n Ty_n, \\
y_n &= (1 - \beta_n)x_n + \beta_n Tx_n,
\end{align*}
\]

for all $n = 0, 1, 2, \ldots$ converges strongly to the unique fixed point of $T$ in $K$.

Proof. Take $q \in F(T)$ and hence $q = Tq$. If there exists a positive integer $n_0$ such that $x_{n_0} = q$, then we have

\[
\|y_{n_0} - q\| = \|(1 + \beta_{n_0})(x_{n_0} - q) + \beta_{n_0}(Tx_{n_0} - q)\| = \beta_{n_0}\|Tx_{n_0} - q\| \leq \beta_{n_0} \cdot L \cdot \|x_{n_0} - q\| = 0,
\]

i.e., $x_{n_0 + 1} = q$. By induction, we can prove that $x_{n_0 + i} = q$ for all $i \geq 1$. This implies that $x_n \rightarrow q$ as $n \rightarrow \infty$. Consequently, without loss of generality, we can assume that $x_n \neq q$ for all $n \geq 0$, i.e., $\|x_n - q\| > 0$ for all $n = 0, 1, 2, \ldots$. Because $X$ is uniformly smooth, by Proposition 1.1(7), $J_2$ is single-valued and uniformly continuous on any bounded subset of $X$. It follows from (3.1) and Lemma 2.1 that

\[
\|x_{n+1} - q\|^2 = \|(1 - \alpha_n)(x_n - q) + \alpha_n(Ty_n - q)\|^2 \\
\leq (1 - \alpha_n)^2\|x_n - q\|^2 + 2\alpha_n \cdot (Ty_n - q, J_2(x_{n+1} - q)) \\
= (1 - \alpha_n)^2\|x_n - q\|^2 + 2\alpha_n(Ty_n - q, J_2(x_n - q)) \\
+ 2\alpha_n \cdot b_n \cdot \|x_n - q\|^2,
\]

where

\[
b_n = \left(\frac{Ty_n - q}{\|x_n - q\|}, J_2\left(\frac{x_{n+1} - q}{\|x_n - q\|}ight) - J_2\left(\frac{x_n - q}{\|x_n - q\|}\right)\right).
\]
(I) First we consider the second term on the right side of (3.3). From Lemma 3.1 and the condition (ii), it follows that

$$\langle Tx_n - q, J_2(x_n - q) \rangle \leq (1 - k) \cdot \|x_n - q\|^2,$$

(3.4)

and

$$\|Ty_n - Tx_n\| \leq L \cdot \|y_n - x_n\| = L \cdot \beta_n \cdot \|Tx_n - x_n\|$$
$$\leq L \cdot \beta_n \cdot (\|Tx_n - q\| + \|x_n - q\|)$$
$$\leq L \cdot \beta_n \cdot (1 + L) \cdot \|x_n - q\|$$
$$\leq k \cdot (1 - k) \cdot \|x_n - q\|. \quad (3.5)$$

Thus, in view of (3.4) and (3.5), we have

$$\langle Ty_n - q, J_2(x_n - q) \rangle$$
$$= \langle Ty_n - Tx_n, J_2(x_n - q) \rangle + \langle Tx_n - q, J_2(x_n - q) \rangle$$
$$\leq k \cdot (1 - k) \cdot \|x_n - q\|^2 + (1 - k) \cdot \|x_n - q\|^2$$
$$= (1 - k^2)\|x_n - q\|^2. \quad (3.6)$$

(II) Next we consider the third term on the right side of (3.3). We prove that $b_n \to 0$ as $n \to \infty$. In fact, we have

$$\|y_n - q\| \geq \|(\beta_n - 1) \cdot (x_n - q) + \beta_n \cdot (Tx_n - q)\|$$
$$\leq (1 - \beta_n) \cdot \|x_n - q\| + \beta_n \cdot L \cdot \|x_n - q\|$$
$$\leq L \cdot \|x_n - q\|, \quad (3.7)$$

and so, from (3.7), it follows that

$$\frac{\|Ty_n - q\|}{\|x_n - q\|} \leq L \cdot \frac{\|y_n - q\|}{\|x_n - q\|} \leq L^2. \quad (3.8)$$

By the assumption $\alpha_n \to 0$ as $n \to \infty$, from (3.8) we have

$$\frac{x_{n+1} - q}{\|x_{n+1} - q\|} - \frac{x_n - q}{\|x_n - q\|} = \frac{x_{n+1} - x_n}{\|x_n - q\|} = \alpha_n \cdot \|Ty_n - x_n\|$$
$$\leq \frac{\alpha_n}{\|x_n - q\|} \cdot (\|Ty_n - q\| + \|x_n - q\|)$$
$$\leq \alpha_n (L^2 + 1) \to 0, \quad (3.9)$$
which implies that, as $n \to \infty$,
\[
J_2\left(\frac{x_{n+1} - q}{\|x_n - q\|}\right) - J_2\left(\frac{x_n - q}{\|x_n - q\|}\right) \to 0.
\]
Besides, from (3.8) it follows that $((Ty_n - q)/\|x_n - q\|)_{n \geq 0}$ is a bounded sequence in $X$. Therefore we have, as $n \to \infty$,
\[
b_n \to 0.
\]
(3.10)
Substituting (3.6) into (3.3), we have
\[
\|x_{n+1} - q\|^2 \leq \left[(1 - \alpha_n)^2 + 2\alpha_n(1 - k^2) + 2\alpha_n \cdot b_n\right] \cdot \|x_n - q\|^2
\]
\[
= \left[1 - k^2\alpha_n + \alpha_n(\alpha_n - k^2 + 2b_n)\right] \cdot \|x_n - q\|^2.
\]
Because $\alpha_n \to 0$ and $b_n \to 0$ as $n \to \infty$, there exists a positive integer $n_1$ such that $\alpha_n - k^2 + 2b_n \leq 0$ for all $n \geq n_1$. Therefore, we have
\[
\|x_{n+1} - q\|^2 \leq (1 - k^2\alpha_n) \cdot \|x_n - q\|^2,
\]
(3.11)
for all $n \geq n_1$. Letting $\|x_n - q\|^2 = \gamma_n$, $\lambda_n = k^2\alpha_n$, and $\sigma_n = 0$, it follows from Lemma 2.2(2) that $x_n \to q$ as $n \to \infty$.

(III) Finally, we prove that $q$ is the unique fixed point of $T$ in $K$. If $q_1$ is also a fixed point of $T$ in $K$, by Lemma 3.1, we have
\[
\|q - q_1\|^2 = (q - q_1, J_2(q - q_1)) \leq (1 - k) \cdot \|q - q_1\|^2.
\]
Because $k \in (0, 1)$, we have $q = q_1$. This completes the proof.

Remark 2. Theorem 3.2 improves and extends the results of Chidume [5, Theorem 2], Chidume [6, Theorem 4], Deng and Ding [10, Theorem 1], Deng [8, Theorem 2], Deng [9, Theorem 4], and Tan and Xu [18, Theorem 4.2].

Theorem 3.3. Let $X$ be a real uniformly smooth Banach space, let $K$ be a bounded closed convex subset of $X$ and let $T : K \to K$ be a strongly pseudocontractive mapping. Let $(\alpha_n), (\beta_n)$ be two real sequences satisfying the following conditions:

(i) $0 \leq \alpha_n, \beta_n \leq 1$ for all $n = 0, 1, 2, \ldots$,
(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\alpha_n \to 0$, and $\beta_n \to 0$ as $n \to \infty$.

If $F(T) \neq \emptyset$, then, for any given $x_0 \in K$, the Ishikawa iteration sequence $(x_n)$ defined by
\[
\begin{align*}
x_{n+1} & = (1 - \alpha_n)x_n + \alpha_n Ty_n, \\
y_n & = (1 - \beta_n)x_n + \beta_n Tx_n,
\end{align*}
\]
(3.12)
for all $n = 0, 1, 2 \ldots$ converges strongly to the unique fixed point of $T$ in $K$. 
**Proof.** Take $q \in F(T)$ and so $q = Tq$. By (3.12) and Lemma 2.1, we have, for $1 < p < \infty$,

\[
\|x_{n+1} - q\|^p = \|(1 - \alpha_n)(x_n - q) + \alpha_n(Ty_n - q)\|^p
\]

\[
\leq (1 - \alpha_n)^p \|x_n - q\|^p + p\alpha_n \cdot (Ty_n - q, J_p(x_{n+1} - q))
\]

\[
= (1 - \alpha_n)^p \|x_n - q\|^p + p \cdot \alpha_n \cdot (Ty_n - q, J_p(y_n - q))
\]

\[
+ p \cdot \alpha_n \cdot c_n,
\]

where

\[
c_n = (Ty_n - q, J_p(x_{n+1} - q) - J_p(y_n - q)).
\]

(1) First, from Lemma 3.1, it follows that

\[
(Ty_n - q, J_p(y_n - q)) \leq (1 - k) \cdot \|y_n - q\|^p.
\]

(11) Next we prove that $c_n \to 0$ as $n \to \infty$.

In fact, because $K$ is a bounded set in $X$ and $x_n, Tx_n, Ty_n, q \in K$, then $(Ty_n - q), (Tx_n), (Ty_n)$, and $(x_n)$ all are bounded sequences in $X$. It follows from the conditions (i) and (ii) that, as $n \to \infty$,

\[
x_{n+1} - q - (y_n - q) = (\beta_n - \alpha_n) x_n + \alpha_n Ty_n - \beta_n Tx_n \to 0.
\]

In view of the uniform continuity of $J_p$ on any bounded subset of $X$, we have

\[
J_p(x_{n+1} - q) - J_p(y_n - q) \to 0,
\]

and so $c_n \to 0$ as $n \to \infty$.

(111) Now we estimate $\|y_n - q\|^p$.

From (3.12) and Lemma 2.1,

\[
\|y_n - q\|^p = \|(1 - \beta_n)(x_n - q) + \beta_n(Tx_n - q)\|^p
\]

\[
\leq (1 - \beta_n)^p \|x_n - q\|^p + p \cdot \beta_n \cdot (Tx_n - q, J_p(y_n - q))
\]

\[
\leq (1 - \beta_n)^p \|x_n - q\|^p + \beta_n \cdot M,
\]

where $M = \max \{\sup_{n \geq 0} \|Tx_n - q\| \|y_n - q\|^{p-1}, \sup_{n \geq 0} \|x_n - q\|^p\} < \infty$.

Substituting (3.16) into (3.15) and (3.15) into (3.13), we have

\[
\|x_{n+1} - q\|^p \leq \left[\left(1 - \alpha_n\right)^p + p \alpha_n (1 - k)(1 - \beta_n)^p\right]
\]

\[
\cdot \|x_n - q\|^p + \alpha_n \cdot e_n,
\]

(3.17)
where \( e_n = p \cdot [c_n + p(1 - k)\beta_n \cdot M] \). Because we have

\[
0 \leq (1 - \alpha_n)^p + p\alpha_n(1 - k)(1 - \beta_n)^p \leq (1 - \alpha_n)^p + p\alpha_n(1 - k)
\]

\[
= 1 - p\alpha_n + \frac{p(p - 1)}{2!} \alpha_n^2 - \frac{p(p - 1)(p - 2)}{3!} \alpha_n^3
\]

\[
+ \cdots + (-\alpha_n)^p + p\alpha_n(1 - k)
\]

\[
\leq 1 - k\alpha_n + \alpha_n h_n,
\]

where \( h_n = \left\{ \frac{p(p - 1)}{2!} \alpha_n - \left[ \frac{p(p - 1)(p - 2)}{3!} \right] \alpha_n^2 + \cdots + (-\alpha_n)^p - 1 \right\} \). (3.17) can be written as follows,

\[
\|x_{n+1} - q\|^p \leq (1 - k\alpha_n)\|x_n - q\|^p + \alpha_n h_n\|x_n - q\|^p + \alpha_n c_n
\]

\[
\leq (1 - k\alpha_n)\|x_n - q\|^p + \alpha_n (h_n M + e_n).
\]

Taking \( r_n = \|x_n - q\|^p \), \( \lambda_n = k\alpha_n \), and \( \sigma_n = (h_n M + e_n)/k \), we have

\[
r_{n+1} \leq (1 - \lambda_n) r_n + \lambda_n \sigma_n,
\]

for \( n = 0, 1, 2, \ldots \). From Lemma 2.2 it follows that \( x_n \to q \) as \( n \to \infty \).

The uniqueness of the fixed point \( q \) can be proved as in Theorem 3.2. This completes the proof.

**Remark 3.** (1) Theorem 3.3 generalized Chidume [5, Theorem 2] in several aspects and contains it as a special case.

(2) Theorem 3.3 improves and extends the results of Chidume [6, Theorem 4], Tan and Xu [18, Theorem 4.2] and Zhou and Jia [22, Theorem 2.1].

**Theorem 3.4.** Let \( X \) be a real Banach space, let \( K \) be a nonempty bounded closed convex subset of \( X \) and let \( T: K \to K \) be a uniformly continuous strongly pseudo-contractive mapping. Let \( \{\alpha_n\}, \{\beta_n\} \) be two real sequences satisfying the following conditions:

(i) \( 0 \leq \alpha_n, \beta_n < 1 \), and \( \alpha_n \to 0, \beta_n \to 0 \) as \( n \to \infty \),

(ii) \( \sum_{n=0}^{\infty} \alpha_n = \infty \).

If \( F(T) \neq \emptyset \), then, for any given \( x_0 \in K \), the Ishikawa iteration sequence \( \{x_n\} \) defined by

\[
\begin{align*}
x_{n+1} &= (1 - \alpha_n)x_n + \alpha_n T y_n, \\
y_n &= (1 - \beta_n)x_n + \beta_n T x_n
\end{align*}
\]

for all \( n = 0, 1, 2, \ldots \) converges strongly to the unique fixed point of \( T \) in \( K \).
Proof. Take \( q \in F(T) \) and so \( q = Tq \). From (3.18) and Lemma 2.1, it follows that
\[
\|x_{n+1} - q\|^2 \leq (1 - \alpha_n)^2\|x_n - q\|^2 + 2\alpha_n \cdot (T_{y_n} - q, j_2) \\
= (1 - \alpha_n)^2\|x_n - q\|^2 + 2\alpha_n(T_{y_n} - T_{x_{n+1}}, j_2) \\
+ 2\alpha_n(T_{x_{n+1}} - q, j_2),
\]
for all \( j_2 \in J_2(x_{n+1} - q) \).

(1) First we consider the third term on the right side of (3.19).

From Lemma 3.1, it follows that there exists \( j_2(x_{n+1} - q) \in J_2(x_{n+1} - q) \) such that
\[
(T_{x_{n+1}} - q, j_2(x_{n+1} - q)) \leq (1 - k) \cdot \|x_{n+1} - q\|^2. \tag{3.20}
\]
Substituting (3.20) into (3.19), we have
\[
\|x_{n+1} - q\|^2 \leq (1 - \alpha_n)^2\|x_n - q\|^2 + 2\alpha_n(T_{y_n} - T_{x_{n-1}}, j_2(x_{n+1} - q)) \\
+ 2\alpha_n(1 - k) \cdot \|x_{n+1} - q\|^2, \tag{3.21}
\]
for all \( n = 0, 1, 2, \ldots \).

(11) Letting \( d_n = (T_{y_n} - T_{x_{n+1}}, j_2(x_{n+1} - q)) \), we prove that \( d_n \to 0 \) as \( n \to \infty \). In fact, because \( (x_n), (T_{y_n}), (T_{y_n}) \) all are bounded sequences in \( K \),
\[
y_n - x_{n+1} = (\alpha_n - \beta_n) x_n + \beta_n T_{x_n} - \alpha_n T_{y_n} \to 0,
\]
as \( n \to \infty \). By virtue of the uniform continuity of \( T \), we have, as \( n \to \infty \),
\[
\|T_{y_n} - T_{x_{n+1}}\| \to 0. \tag{3.22}
\]
Again because \( (x_n - q) \) is a bounded sequence in \( X \), by Proposition 1.1(5) we know that \( J_2((x_n - q)) \) is a bounded subset in \( X^* \). Because \( J_2((x_{n+1} - q)) \) is also a bounded sequence in \( X^* \). From (3.22), it follows that \( d_n \to 0 \) as \( n \to \infty \). Because \( \alpha_n \to 0 \) as \( n \to \infty \), there exists a positive integer \( n_3 \) such that, for all \( n \geq n_3, 1 - 2\alpha_n \cdot (1 - k) > 0 \). Therefore, for \( n \geq n_3 \), (3.21) can be written as follows,
\[
\|x_{n+1} - q\|^2 \leq \frac{(1 - \alpha_n)^2}{1 - 2\alpha_n(1 - k)}\|x_n - q\|^2 + \frac{2\alpha_n d_n}{1 - 2\alpha_n(1 - k)}. \tag{3.23}
\]
Take $\gamma_1, \gamma_2 \in (0, \infty)$ such that

$$0 < \gamma_1 < \min \left\{ 1, 2k, \frac{1}{2(1-k)} \right\},$$

$$\gamma_2 = \frac{(2k - \gamma_1)(1 - 2(1-k)\gamma_1)^{-1}}{1 - 2\gamma_2(1-k)}.$$

Because $\alpha_n \to 0$ as $n \to \infty$, there exists a positive integer $n_4 \geq n_3$ such that $\alpha_n < \gamma_2$ for all $n \geq n_4$. It is easy to prove that

$$\frac{(1 - \alpha_n)^2}{1 - 2\alpha_n(1-k)} \leq (1 - \gamma_1 \cdot \alpha_n),$$

(3.24)

for all $n \geq n_4$. Therefore, (3.23) can be written as the following form: For all $n \geq n_4$,

$$\|x_{n+1} - q\|^2 \leq (1 - \gamma_1 \alpha_n)\|x_n - q\|^2 + \gamma_1 \alpha_n \cdot \sigma_n,$$

(3.25)

where $\sigma_n = 2d_n / [\gamma_1 (1 - 2\alpha_n(1-k))]$. Thus by Lemma 2.2(2), we have $x_n \to q$ as $n \to \infty$.

The uniqueness of the fixed point $q$ can be proved as in Theorem 3.2. This completes the proof.

Remark 4. Theorem 3.4 also improves and extends the results of Chidume [4], Chidume [5, Theorem 2], Chidume [6, Theorem 4], Tan and Xu [18, Theorem 4.2], and Deng and Ding [10, Theorem 1].

4. CONVERGENCE OF MANN ITERATION SEQUENCES FOR STRONGLY PSEUDO-CONTRACTION MAPPINGS

Theorem 4.1. Let $X$ be a real uniformly smooth Banach space, let $K$ be a nonempty bounded closed convex subset of $X$ and let $T: K \to K$ be a strongly pseudo-contractive mapping. Let $(\alpha_n)$ be a real sequence in $[0, 1]$ satisfying $\sum_{n=0}^{\infty} \alpha_n = \infty$ and $\alpha_n \to 0$ as $n \to \infty$. If $F(T) \neq \emptyset$, then, for any given $x_0 \in K$, the Mann iteration sequence $(x_n)$ defined by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_nTx_n,$$

for $n = 0, 1, 2, \ldots$ converges strongly to the unique fixed point of $T$ in $K$.

Proof. Taking $\beta_n = 0$ for all $n = 0, 1, 2, \ldots$ in Theorem 3.3, then the conclusion of Theorem 4.1 can be obtained from Theorem 3.3 immediately.
Remark 5. (1) Because $L_p, 1 < p < \infty$, is a special uniformly smooth Banach space, Theorem 4.1 improves and extends the results of Chidume [4, Theorem 2].

(2) Theorem 4.1 also improves and extends the results of Chidume [5, Theorem 1], Chidume [6, Theorem 3], and Tan and Xu [18, Theorem 3.2].

Theorem 4.2. Let $X$ be a real Banach space, let $K$ be a nonempty bounded closed convex subset of $X$, let $T: K \to K$ be a uniformly continuous strongly pseudo-contractive mapping. Let $(\alpha_n)$ be a real sequence satisfying the following conditions:

(i) $0 \leq \alpha_n < 1$ for all $n = 0, 1, 2, \ldots$ and $\alpha_n \to 0$ as $n \to \infty$,

(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$.

If $F(T) \neq \emptyset$, then, for any given $x_0 \in K$, the Mann iteration sequence $(x_n)$ defined by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Tx_n,$$

for all $n = 0, 1, 2, \ldots$ converges strongly to the unique fixed point of $T$ in $K$.

Proof. Taking $\beta_n = 0$ for all $n = 0, 1, 2, \ldots$ in Theorem 3.4, the conclusion of Theorem 4.2 can be obtained immediately.

5. CONVERGENCE OF ISHIKAWA ITERATION SEQUENCES FOR STRONGLY ACCRETIVE MAPPINGS

Lemma 5.1. Let $X$ be a real Banach space and let $T: X \to 2^X$ be a multivalued strongly accretive mapping with a strongly accretive constant $k \in (0, 1)$. For any given $f \in X$, define a mapping $S: X \to 2^X$ by

$$Sx = f - Tx + x,$$

for all $x \in X$. Then for any given $x, y \in X$, $u \in Sx$, and $v \in Sy$, there exists $\tilde{j}_p \in J_p(x - y), 1 < p < \infty$, such that

$$\langle u - v, \tilde{j}_p \rangle \leq (1 - k)\|x - y\|^p.$$

Because $T$ is strongly accretive if and only if $(I - T)$ is strongly pseudo-contractive, the following theorem is obtained from Theorem 3.2.

Theorem 5.2. Let $X$ be a real uniformly smooth Banach space and let $T: X \to X$ be a Lipschitzian strongly accretive mapping. Let $k \in (0, 1)$ and
\( L > 1 \) be the strongly accretive constant and Lipschitz constant of \( T \), respectively. For any given \( f \in X \), define a mapping \( S : X \rightarrow X \) by

\[
Sx = f - Tx + x,
\]

for all \( x \in X \). Let \( \{\alpha_n\}, \{\beta_n\} \) be two real sequences satisfying the following conditions:

(i) \( 0 \leq \alpha_n \leq 1 \) for all \( n = 0, 1, 2, \ldots \),

(ii) \( 0 \leq \beta_n \leq k \cdot [2(1 + L)(2 + L)]^{-1} \) for all \( n = 0, 1, 2, \ldots \),

(iii) \( \sum_{n=0}^{\infty} \alpha_n = \infty \) and \( \alpha_n \to 0 \) as \( n \to \infty \).

If \( ST \neq \emptyset \) (the set of solutions of the equation \( f = Tx \) in \( X \)), then, for any given \( x_0 \in X \), the Ishikawa iteration sequence \( \{x_n\} \) defined by

\[
\begin{align*}
x_{n+1} &= (1 - \alpha_n)x_n + \alpha_nSy_n, \\
y_n &= (1 - \beta_n)x_n + \beta_nSx_n,
\end{align*}
\]

for \( n = 0, 1, 2, \ldots \) converges strongly to the unique solution of the equation \( f = Tx \) in \( X \).

Proof. Taking \( q \in ST \), we have \( f = Tq \) and so \( q = Sq \). If there exists a positive integer \( n_0 \) such that \( x_{n_0} = q \), then, by the same way as stated in Theorem 3.2, we can prove that \( x_{n+n} = q \) for all \( n = 1, 2, \ldots \) and so \( x_n \to q \) as \( n \to \infty \). Therefore, without loss of generality, we can assume that \( x_n \neq q \) for all \( n = 0, 1, 2, \ldots \) and so \( \|x_n - q\| > 0 \) for all \( n = 0, 1, 2, \ldots \). Because \( X \) is uniformly smooth, it follows from Proposition 1.1(7) that \( J_2 \) is single-valued and uniformly continuous on any bounded subset of \( X \). From (5.1) and Lemma 2.1, it follows that

\[
\begin{align*}
\|x_{n+1} - q\|^2 &\leq (1 - \alpha_n)^2\|x_n - q\|^2 + 2\alpha_n(Sx_n - q, J_2(x_{n+1} - q)) \\
&\leq (1 - \alpha_n)^2\|x_n - q\|^2 + 2\alpha_n(Sy_n - q, J_2(x_n - q)) \\
&\quad + 2\alpha_n \cdot \beta_n \cdot \|x_n - q\|^2,
\end{align*}
\]

where

\[
g_n = \left( \frac{Sy_n - q}{\|x_n - q\|}, J_2 \left( \frac{x_{n+1} - q}{\|x_n - q\|} \right) - J_2 \left( \frac{x_n - q}{\|x_n - q\|} \right) \right).
\]

(1) First we consider the second term on the right side of (5.2). We have

\[
(Sy_n - q, J_2(x_n - q)) = (Sy_n - Sx_n, J_2(x_n - q)) \\
+ (Sx_n - q, J_2(x_n - q)).
\]
By Lemma 5.1, we have
\[(S x_n - q, J_2(x_n - q)) \leq (1 - k) \cdot \|x_n - q\|^2,\]  \hfill (5.4)
\[\|S y_n - S x_n\| \leq (1 + L) \cdot \|y_n - x_n\|.\]  \hfill (5.5)

Hence we have, by the condition (ii),
\[(S y_n - S x_n, J_2(x_n - q)) \leq (1 + L) \|y_n - x_n\| \cdot \|x_n - q\| \]
\[= (1 + L) \cdot \beta_n \cdot ||S x_n - x_n|| \cdot \|x_n - q\| \]
\[\leq (1 + L) \cdot \beta_n \|S x_n - q\| + \|x_n - q\| \cdot \|x_n - q\| \]
\[\leq (1 + L) \cdot \beta_n (2 + L) \cdot \|x_n - q\|^2 \]
\[\leq \frac{1}{2} \cdot k \cdot \|x_n - q\|^2.\]  \hfill (5.6)

Substituting (5.4) and (5.6) into (5.3), we have
\[(S y_n - q, J_2(x_n - q)) \leq (1 - \frac{1}{2}k) \cdot \|x_n - q\|^2.\]  \hfill (5.7)

(11) Next we prove that \(g_n \to 0, n \to \infty\). In fact, by the same way as in the proof of (3.7)–(3.9), we can prove that
\[\|y_n - q\| \leq (1 + L) \|x_n - q\|,\]  \hfill (5.8)
\[\frac{\|S y_n - q\|}{\|x_n - q\|} \leq \frac{(1 + L) \|y_n - q\|}{\|x_n - q\|} \leq (1 + L)^2,\]  \hfill (5.9)
\[\frac{x_{n+1} - q}{\|x_n - q\|} - \frac{x_n - q}{\|x_n - q\|} \leq \alpha_n \left( (1 + L)^2 + 1 \right) \to 0, \quad (n \to \infty).\]  \hfill (5.10)

By the uniform continuity of \(J_2\), from (5.9) and (5.10), it follows that \(g_n \to 0\) as \(n \to \infty\). Substituting (5.7) into (5.2), we obtain
\[\|x_{n+1} - q\|^2 \leq \left[ (1 - \alpha_n)^2 + 2 \alpha_n \left( 1 - \frac{k}{2} \right) + 2 \alpha_n \cdot g_n \right] \cdot \|x_n - q\|^2 \]
\[= \left[ 1 - \frac{k}{2} \alpha_n + \alpha_n \left( \alpha_n - \frac{k}{2} + 2 g_n \right) \right] \cdot \|x_n - q\|^2.\]  \hfill (5.11)

Because \(\alpha_n \to 0\) and \(g_n \to 0\) as \(n \to \infty\), there exists a positive integer \(n_6\) such that, for all \(n \geq n_6\), \(\alpha_n + 2 g_n - k/2 < 0\). Therefore, from (5.11), we have, for all \(n \geq n_6\),
\[\|x_{n+1} - q\|^2 \leq \left( 1 - \frac{k}{2} \alpha_n \right) \cdot \|x_n - q\|^2.\]

It follows from Lemma 2.2(2) that \(x_n \to q\) as \(n \to \infty\).
In addition, it is easy to prove that $q$ is the unique solution of the equation $f = Tx$ in $X$. This completes the proof.

Remark 6. Theorem 5.2 improves and extends the results of Deng and Ding [10, Theorem 2], Chidume [6, Theorem 2], Tan and Xu [18, Theorem 4.1], and Deng [8, Theorem 1].

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