Non-convex self-dual Lagrangians: New variational principles of symmetric boundary value problems

Abbas Moameni a, b

a Department of Mathematical Sciences, Sharif University of Technology, Tehran, Iran
b Department of Mathematics and Statistics, Queen’s University, Kingston, ON, K7L 3N6 Canada

Received 31 March 2010; accepted 15 January 2011
Available online 28 January 2011
Communicated by L.C. Evans

Abstract

We study the concept and the calculus of Non-convex self-dual (Nc-SD) Lagrangians and their derived vector fields which are associated to many partial differential equations and evolution systems. They indeed provide new representations and formulations for the superposition of convex functions and symmetric operators. They yield new variational resolutions for large class of Hamiltonian partial differential equations with variety of linear and nonlinear boundary conditions including many of the standard ones. This approach seems to offer several useful advantages: It associates to a boundary value problem several potential functions which can often be used with relative ease compared to other methods such as the use of Euler–Lagrange functions. These potential functions are quite flexible, and can be adapted to easily deal with both nonlinear and homogeneous boundary value problems. Additionally, in most cases the solutions generated using this new method have greater regularity than the solutions obtained using the standard Euler–Lagrange function. Perhaps most remarkable, however, are the permanence properties of Nc-SD Lagrangians; their calculus is relatively manageable, and their applications are quite broad.

© 2011 Elsevier Inc. All rights reserved.

Keywords: Non-convex duality; Variational principles; Variational methods; Partial differential equations

Contents

1. Introduction .................................................................................................................. 2675
  1.1. Homogeneous boundary conditions ........................................................................ 2678

E-mail address: momeni@mast.queensu.ca.

0022-1236/$ – see front matter © 2011 Elsevier Inc. All rights reserved.
doi:10.1016/j.jfa.2011.01.010
1. Introduction

The aim of this paper is to develop the concept of self-duality for non-convex functions well adapted to the study of certain partial differential equations that the standard Euler–Lagrange functions may not be quite manageable. Starting with an equation of the form

\[ \Lambda u \in \partial \varphi(u), \]  

it is well known that it can be formulated – and sometimes solved – whenever \( \Lambda : \text{Dom}(\Lambda) \subset V \to V^* \) is a linear self-adjoint operator. Indeed, in this case it can be reduced to the inclusion

\[ 0 \in \partial F(u) \]  

where

\[ F(u) = \frac{1}{2} \langle \Lambda u, u \rangle - \varphi(u). \]  

In the case where \( \varphi \) is convex and the linear operator \( \Lambda \) is positive, the functional \( F \) can be written as difference of two convex functions,

\[ F(u) = \psi(\Lambda u) - \varphi(u) \]

where the quadratic convex function \( \psi \) on \( V^* \) is defined by \( \psi(p) = \frac{1}{2} \langle A^{-1} p, p \rangle \). Such problems where the objective is the difference of two convex functions has received a lot of attentions in the literature starting the works of J.F. Toland [29] and I. Singer [26]. Indeed, Toland introduced the notion of critical points of \( \psi \circ \Lambda - \varphi \) that generalizes the classical definition in the case where \( \varphi \circ \Lambda \) and \( \varphi \) are not necessarily differentiable functions. He also established an interesting one-to-one correspondence between the critical points of \( \psi \circ \Lambda - \varphi \) and \( \varphi^* \circ \Lambda - \psi^* \) on \( V^* \), where \( \varphi^* \) and \( \psi^* \) are Fenchel–Legendre dual of \( \varphi \) and \( \psi \) respectively. There was also established by F. Clarke and I. Ekeland [9,8,11] an interesting dual variational formulation for the case where
the operator \( \Lambda \) is not necessarily positive and may have an infinite sequence of eigenvalues going from \(-\infty\) to \(\infty\). In fact, similar to Toland duality they established a one-to-one correspondence between critical points of the functional \( F \) and the functional

\[
F_{CE}(u) = \frac{1}{2} \langle \Lambda u, u \rangle - \varphi^*(\Lambda u).
\]

Note that even though the Clarke–Ekeland least action principle may have a somewhat related idea with Toland duality, it cannot be deduced directly from Toland’s dual principle. As seen, one can associate to an inclusion of the form (1) at least two functionals in such a way that their critical points may generate solutions for the corresponding inclusion. The question of whether these functions are all possible choices associated to a given inclusion (1) and also finding a unified source for all functions with such a property are addressed in this work. In fact, while analyzing these principles we were led to an abstract scheme that provides a unified way to obtain many more of such functions. To explain this scheme, let us start with Toland’s principle [28–30], with a minor modification, for a class of optimization problems. Indeed, let \( V \) and \( V^* \) be two Banach spaces in duality and with \( \langle ., . \rangle : V \times V^* \to \mathbb{R} \) the corresponding bilinear form compatible with the topologies on \( V \) and \( V^* \). Denote by \((P)\) the problem of evaluating

\[
\inf_{u \in V} J(u),
\]

where \( J : V \to \mathbb{R} \) is possibly a non-convex function. By embedding this problem in a family of perturbed problems a dual problem was established. In fact, by considering the perturbation \( \Phi : V \times V^* \to \mathbb{R} \) for which \( p \to \Phi(u, p) \) is convex and lower semi-continuous for each \( u \in V \) and

\[
\Phi(u, 0) = -J(u),
\]

one can generate a dual problem as follows: Let \( LF_2(\Phi) \) be the Fenchel–Legendre dual of \( \Phi \) with respect to the second variable, that is a function on \( V \times V \) given by:

\[
LF_2(\Phi)(u, v) = \sup_{p \in V^*} \left\{ \langle p, v \rangle - \Phi(u, p) \right\}.
\]

Denote by \( \Phi^\# \) the Fenchel–Legendre dual of \( LF_2(\Phi)(., v) \) with respect to the first variable. Therefore \( \Phi^\# \) is a function on phase space \( V \times V^* \) given by

\[
\Phi^\#(v, q) = \sup_{u \in V} \left\{ \langle q, u \rangle - LF_2(\Phi)(u, v) \right\}.
\]

It was established that the problem

\[
\inf_{v \in V} \Phi^\#(v, 0)
\]

is a dual problem for \((P)\) in such a way that \( \inf_{u \in V} J(u) = \inf_{v \in V} \Phi^\#(v, 0) \) provided \( p \to \Phi(u, p) \) is bounded in a neighborhood of \( p = 0 \). There is also a one-to-one relation between minimizers of \((P)\) and \((P^\#)\).

Following this idea of obtaining a dual problem, we are led to the following notion.
Definition 1.1. Say that the Lagrangian $\Phi$ on $V \times V^*$ is Non-convex self-dual if the following property hold.

$$\Phi^\#(u, p) = \Phi(u, p) \quad \text{for all } (u, p) \in V \times V^*.$$ 

Some basic examples of $\text{Nc-SD}$ Lagrangians are of the form:

1. $\Phi_0(u, p) = \phi^\ast(p) - \phi(u)$,
2. $\Phi_1(u, p) = 2\phi^\ast(p) - \langle p, u \rangle$,
3. $\Phi_2(u, p) = \langle p, u \rangle - 2\phi(u)$,

where $\phi : V \to \mathbb{R}$ is convex and lower semi-continuous and $\phi^\ast$ its Fenchel–Legendre dual defined on $V^*$. The class of Non-convex self-dual Lagrangians is much richer though and goes well beyond saddle functions stated above, since they are naturally compatible with symmetric operators. Indeed, if $\Lambda : V \to V^*$ is self-adjoint and $\Phi$ is any $\text{Nc-SD}$ Lagrangian on $V \times V^*$ then the Lagrangian

$$\Psi(u, p) = \Phi(u, \Lambda u + p)$$

is also Non-convex self-dual. There are also situations where the operator $\Lambda$ is not purely self-adjoint provided one takes into account certain boundary terms. In fact, the operator $\Lambda$ modulo the boundary operator $B := (\beta_1, \beta_2) : V \to Y \times Y^*$ (for some Banach spaces $Y$ and $Y^*$ that are in duality) corresponds to the “Green formula”

$$\langle \Lambda u, v \rangle_{V \times V^*} = \langle u, \Lambda v \rangle_{V \times V^*} + \langle \beta_1 u, \beta_2 v \rangle_{Y \times Y^*} - \langle \beta_1 v, \beta_2 u \rangle_{Y \times Y^*}.$$

In this case if $\Phi : V \times V^* \to \mathbb{R}$ and $\ell : Y \times Y^* \to \mathbb{R}$ are Non-convex self-dual Lagrangians then the Lagrangian $\Psi : V \times (V^* \times Y^*) \to \mathbb{R}$ defined by

$$\Psi(u, (p, e)) = \Phi(u, \Lambda u + p) + \ell(\beta_1 u, \beta_2 u + e)$$

is also a Non-convex self-dual Lagrangian.

To connect this notion to the solutions of inclusion (1), note that $u$ is a solution of inclusion (1) if and only if the pair $(\Lambda u, u)$ is a solution of one of the following inclusions on the phase space $V \times V^*$:

1. $(-\Lambda u, u) \in (-\partial \phi(u), \partial \phi^\ast(\Lambda u))$,
2. $(-\Lambda u, u) \in (-\Lambda u, \partial \phi^\ast(\Lambda u))$,
3. $(-\Lambda u, u) \in (-\partial \phi(u), u)$.

Now taking into account Non-convex self-dual Lagrangians $\Phi_1$, $\Phi_2$ and $\Phi_3$ the above inclusions can be rewritten as follows,

1. $(-\Lambda u, u) \in \partial \Phi_1(u, \Lambda u)$,
2. $(-\Lambda u, u) \in \partial \Phi_2(u, \Lambda u)$,
3. $(-\Lambda u, u) \in \partial \Phi_3(u, \Lambda u)$,
respectively. Note that \( \Phi_i, i = 1, 2, 3 \) are saddle functions and \( \partial \Phi_i \) stands for subdifferential of saddle functions introduced by Rockafellar. As it turns out there is a close correspondence between solutions of inclusions of type (1) and critical points of Non-convex self-dual Lagrangians generated by the pair \((\Lambda, \varphi)\). We shall state and summarize some particular cases of our main results in two cases: Homogeneous boundary conditions and nonlinear boundary conditions.

### 1.1. Homogeneous boundary conditions

In this case we assume the linear operator \( \Lambda \) is purely self-adjoint. Here is a useful result of the variational principle we establish for homogeneous boundary conditions in Section 4.

**Theorem 1.2.** Suppose \( \Phi : V \times V^* \to \mathbb{R} \cup \{\infty\} \) is a saddle Nc-SD Lagrangian and \( \Lambda : \text{Dom}(\Lambda) \subset V \to V^* \) is a self-adjoint linear operator that is also onto. Suppose one of the following conditions holds:

1. The operator \( \Lambda \) is non-negative.
2. For each \( p \in V^* \), the function \( u \to \Phi(u, p) \) is Gâteaux differentiable and \( \nabla_1 \Phi(u, p) = -p \).

Then for every critical point \( u \) of \( \Phi(u, \Lambda u) \) there exists \( v \in V \) with \( \Lambda u = \Lambda v \) and

\[
(-\Lambda v, v) \in \partial \Phi(u, \Lambda u).
\]

As a straightforward application of the above theorem the functionals \( \Phi_1(u, \Lambda u) := \varphi^*(\Lambda u) - \varphi(u) \) and \( \Phi_2(u, \Lambda u) := 2\varphi^*(\Lambda u) - \langle \Lambda u, u \rangle \) can be seen as new potentials for the inclusion (1) as follows.

**Corollary 1.3.** Let \( \Lambda : \text{Dom}(\Lambda) \subset V \to V^* \) be a non-negative self-adjoint operator that is also onto. Let \( \varphi : V \to \mathbb{R} \) be convex, lower-semi continuous and also continuous. Then every critical point of

\[
I(u) = \varphi^*(\Lambda u) - \varphi(u)
\]

is a solution of the equation

\[
\Lambda u \in \partial \varphi(u).
\]

This corollary was first established in [24] by the author via a direct computation (see also [22, 21] for more applications). It was then understood that all variational principles of this type fall under a unified principle as discussed in this paper. We shall use this corollary to provide an existence result for system of super-linear transport equations with a small parameter \( \epsilon \),

\[
\begin{align*}
\epsilon \alpha \nabla u &= \Delta v + |v|^{p-2} v, & x \in \Omega, \\
-\epsilon \alpha \nabla v &= \Delta u + |u|^{q-2}, & x \in \Omega, \\
u = v = 0, & x \in \partial \Omega,
\end{align*}
\]

by finding critical points of
\[ I(u, v) = \frac{1}{p'} \int_{\Omega} |\epsilon a. \nabla u - \Delta v|^p \, dx + \frac{1}{q'} \int_{\Omega} |\epsilon a. \nabla v + \Delta u|^q \, dx - \frac{1}{p} \int_{\Omega} |v|^p \, dx - \frac{1}{q} \int_{\Omega} |u|^q \, dx \]

on \( W^{2,q'}(\Omega) \times W^{2,p'}(\Omega) \) where \( p' = \frac{p}{p-1} \) and \( q' = \frac{q}{q-1} \).

Taking into account the Lagrangian \( \Phi_2 \), here is another application of Theorem 1.2.

**Corollary 1.4.** Let \( \Lambda : \text{Dom}(\Lambda) \subset V \rightarrow V^* \) be a surjective self-adjoint operator and \( \varphi : V \rightarrow \mathbb{R} \) be convex and lower-semi continuous. If \( u \) is a critical point of

\[ I(w) = 2\varphi^*(\Lambda w) - \langle \Lambda w, w \rangle \]

then there exists \( v \in V \) such that \( \frac{v+u}{2} \) is a solution of

\[ \Lambda w \in \partial \varphi(w). \]

Note that the above corollary is nothing but the well-known Clarke–Ekeland least action principle. It is also remarkable that \( \Phi_1 \) and \( \Phi_2 \) are just two typical examples of \( Nc-SD \) Lagrangians that have already provided two different variational principles for the inclusion (1). By characterizing the class of \( Nc-SD \) Lagrangians in Section 3, we shall see one can actually obtain many more principles that fit within this theory.

### 1.2. Nonlinear boundary conditions

Here is another useful result of the variational principle we establish for nonlinear boundary conditions in Section 4.

**Theorem 1.5.** Let \( \Lambda : \text{Dom}(\Lambda) \subset V \rightarrow V^* \) be an operator correspond to the above “Green formula” modulo the boundary operator \( B := (\beta_1, \beta_2) : \text{Dom}(\Lambda) \rightarrow Y \times Y^* \) such that \( (\Lambda, \beta_2) : \text{Dom}(\Lambda) \subset V \rightarrow V^* \times Y^* \) and \( \beta_1 : \text{Dom}(\Lambda) \subset V \rightarrow Y \) are onto. Let \( \Phi : V \times V^* \rightarrow \mathbb{R} \) and \( \ell : Y \times Y^* \rightarrow \mathbb{R} \) be saddle Non-convex self-dual Lagrangians that are Gâteaux differentiable with respect to their first variables. We also assume that \( \text{Dom}(\Lambda) \cap \text{Ker}(\beta_1) \) is dense in \( V \).

Suppose one of the following conditions holds:

(i) For each \( u \in \text{Dom}(\Lambda) \), \( \langle u, Au \rangle_{V \times V^*} + \langle \beta_1 u, \beta_2 u \rangle_{Y \times Y^*} \geq 0 \).

(ii) For each \( (p, e) \in V^* \times Y^* \) the functions \( u \rightarrow \Phi(u, p) \) and \( l \rightarrow \ell(l, e) \) are Gâteaux differentiable and, \( \nabla_1 \Phi(u, p) = -p \) and \( \nabla_1 \ell(l, e) = -e. \)

Suppose \( u \) is a critical point of \( I(u) = \Psi(u, 0) \) where \( \Psi \) is defined on \( \text{Dom}(\Lambda) \times (V^* \times Y^*) \) by

\[ \Psi(u, (p, e)) = \Phi(u, Au + p) + \ell(\beta_1 u, \beta_2 u + e). \]

Set \( v \in \partial_2 \Phi(u, Au) \). Then \( \Lambda u = \Lambda v \), \( \beta_2 u = \beta_2 v \) and the pair \( (u, v) \) is a solution of the system

\[
\begin{cases}
(-\Lambda v, v) \in \partial \Phi(u, Au), \\
(-\beta_2 v, \beta_1 v) \in \partial \ell(\beta_1 u, \beta_2 u).
\end{cases}
\]
To get a better understanding and see concrete applications of this theorem we shall discuss some particular cases.

**Corollary 1.6.** Let $\Lambda : \text{Dom}(\Lambda) \subset V \to V^*$ and $\mathcal{B} := (\beta_1, \beta_2) : \text{Dom}(\Lambda) \to Y \times Y^*$ satisfy part (i) of Theorem 1.5. Let $\varphi : V \to \mathbb{R}$ and $\psi : Y \to \mathbb{R}$ be convex, lower-semi continuous and also Gâteaux differentiable. Then every critical point of

$$I(u) = \varphi^*(\Lambda u) - \varphi(u) + \psi^*(\beta_2 u) - \psi(\beta_1 u)$$

is a solution of the inclusion

$$\begin{cases}
\Lambda u = \nabla \varphi(u), \\
\beta_2 u = \nabla \psi(\beta_1 u).
\end{cases} \quad (2)$$

The following result can be seen as a generalization of Clarke–Ekeland duality when the operator $\Lambda$ is not purely self-adjoint and one deals with boundary terms as well.

**Corollary 1.7.** Let $\Lambda : \text{Dom}(\Lambda) \subset V \to V^*$ and $\mathcal{B} := (\beta_1, \beta_2) : \text{Dom}(\Lambda) \to Y \times Y^*$ be as in Theorem 1.5. Let $\varphi : V \to \mathbb{R}$ and $\psi : Y \to \mathbb{R}$ be convex, lower-semi continuous and also Gâteaux differentiable. If $u$ is a critical point of

$$I(w) = 2\varphi^*(\Lambda w) - \langle \Lambda w, w \rangle_{V \times V^*} + 2\psi^*(\beta_2 w) - \langle \beta_2 w, \beta_1 w \rangle_{Y \times Y^*}$$

then there exists $v \in V$ such that $\frac{v + \beta_1}{2}$ is a solution of (2).

As an application of this corollary we provide a new variational principle for convex Hamiltonian systems with nonlinear boundary conditions of the form:

$$\begin{cases}
J \dot{t}(t) = \nabla \varphi(t, u(t)), \\
\frac{u(T) + u(0)}{2} = \nabla \psi(J u(T) - J u(0)).
\end{cases}$$

The above results are actually particular cases of a much more general *Non-convex self-dual* variational principle that will be stated and established in full generality in the following sections. As applications, we shall also provide many more concrete examples of this principle throughout the paper.

The interested reader is referred to [10,6,7,5,2,3,25,20] for more applications of the related results to PDE’s and monotone operators. We also refer to [16–18,15] for results in convex self-duality.

The paper is organized as follows. We start by reviewing in Section 2, some important definitions and results in Convex Analysis, theory of saddle functions and symmetric linear operators. In Section 3, we start by establishing some basic permanence properties of *Non-convex self-dual* Lagrangians and their calculus and we conclude this section by a characterization of *Nc-SD* Lagrangians. In Section 4, we first establish a variational principle for homogeneous boundary value problems then we deal with boundary value problems where compatible boundary Lagrangians are appropriately added to the “interior Lagrangian”, in order to solve problems with prescribed nonlinear boundary terms. In Section 5, by making use of a minimax principle for
lower semi-continuous functionals we proceed with the proof of existence theorems stated in previous sections.

2. Preliminaries

In this section we recall some important definitions and results in Convex Analysis, theory of saddle functions and linear symmetric operators used in this work. We also introduce the terminology used consistently throughout the paper for the convenience of the reader. For the proof of these results the interested reader is referred to [14,13,15,29].

2.1. Separating duality and convex analysis

Let $V$ and $V^*$ be two real Banach spaces and let $\langle \cdot , \cdot \rangle$ be a bilinear form on the phase space $V \times V^*$. The following definition is due to J.F. Toland [29].

Definition 2.1. We say that the bilinear form puts $V$ and $V^*$ in duality. This duality is said to be separating if,

(1) for $0 \neq u \in V$, there exists an element $p \in V^*$ such that $\langle u, p \rangle \neq 0$,
(2) for $0 \neq p \in V^*$, there exists an element $u \in V$ such that $\langle u, p \rangle \neq 0$.

The weak topology on $V$ induced by $\langle \cdot , \cdot \rangle$ is denoted by $\sigma(V,V^*)$ and analogously $\sigma(V^*,V)$ is the weak topology on $V^*$. It is known that $\sigma(V, V^*)$ and $\sigma(V^*, V)$ are Hausdorff topologies if and only if the duality between $V$ and $V^*$ is separating. Throughout this paper we shall assume the spaces $V$ and $V^*$ are in separating duality. A function $\Phi : V \rightarrow \mathbb{R}$ is said to be lower semi-continuous if

$$\Phi(u) \leq \liminf_{n \rightarrow \infty} \Phi(u_n),$$

for each $u \in V$ and any sequence $u_n$ approaching $u$ in the weak topology $\sigma(V, V^*)$. Let $\Phi : V \rightarrow \mathbb{R} \cup \{\infty\}$ be a proper convex function. The subdifferential $\partial \Phi$ of $\Phi$ is defined to be the following set-valued operator: if $u \in Dom(\Phi)$, set

$$\partial \Phi(u) = \{ p \in V^*; \langle p, v - u \rangle + \Phi(u) \leq \Phi(v) \text{ for all } v \in V \}$$

and if $u \notin Dom(\Phi)$, set $\partial \Phi(u) = \emptyset$. If $\Phi$ is Gâteaux differentiable at $u$ then $\partial \Phi(u) = \{ \nabla \Phi(u) \}$.

The Fenchel–Legendre dual of an arbitrary function $\Phi$ is denoted by $\Phi^*$ that is a function on $V^*$ and is defined by

$$\Phi^*(p) = \sup \{ \langle p, u \rangle - \Phi(u); \ u \in V \}.$$ 

Clearly $\Phi^* : V^* \rightarrow \mathbb{R} \cup \{\infty\}$ is convex and lower semi-continuous. Consequently $\Phi^{**} : V \rightarrow \mathbb{R} \cup \{\infty\}$ is always convex and lower semi-continuous. The following observation is crucial in the subsequent analysis.
Proposition 2.1. Let $\Phi : V \to \mathbb{R} \cup \{ \infty \}$ be an arbitrary function. The following statements hold:

1. $\Phi^{**}(u) \leq \Phi(u)$ for all $u \in V$.
2. $\Phi(u) + \Phi^*(p) \geq \langle p, u \rangle$ for all $u \in V$ and $p \in V^*$.
3. If $\Phi$ is convex and lower-semi continuous then $\Phi^{**} = \Phi$ and the following are equivalent

$$\Phi(u) + \Phi^*(p) = \langle u, p \rangle \iff p \in \partial \Phi(u) \iff u \in \partial \Phi^*(p).$$

The following is a crucial property of convex functions.

Proposition 2.2. Let $V$ and $V^*$ be in separating duality and $\Phi : V \to \mathbb{R} \cup \{ \infty \}$ be a proper convex function. Suppose $\Phi$ is sub-differentiable at $u, v \in V$.

If there exist $p \in \partial \Phi(u)$ and $q \in \partial \Phi(v)$ with

$$\langle p-q, u-v \rangle = 0$$

then $p, q \in \partial \Phi(u) \cap \partial \Phi(v)$.

**Proof.** It follows from $p \in \partial \Phi(u)$ and $q \in \partial \Phi(v)$ that

$$\Phi(u) + \Phi^*(p) = \langle p, u \rangle \quad \& \quad \Phi(v) + \Phi^*(q) = \langle q, v \rangle.$$

Adding up this equalities, we obtain $\langle p, u \rangle + \langle q, v \rangle = \Phi(u) + \Phi^*(p) + \Phi(v) + \Phi^*(q)$. It also follows from (3) that $\langle p, u \rangle + \langle q, v \rangle = \langle p, v \rangle + \langle q, u \rangle$, which together with the above equation imply that

$$\langle p, v \rangle + \langle q, u \rangle = \Phi(u) + \Phi^*(p) + \Phi(v) + \Phi^*(q)$$

and therefore $\Phi(v) + \Phi^*(p) - \langle p, v \rangle + \Phi(u) + \Phi^*(q) - \langle q, u \rangle = 0$. This together with the fact that

$$\Phi(v) + \Phi^*(p) - \langle p, v \rangle \geq 0, \quad \Phi(u) + \Phi^*(q) - \langle q, u \rangle \geq 0$$

imply that both terms are indeed zero,

$$\Phi(v) + \Phi^*(p) - \langle p, v \rangle = 0,$$

$$\Phi(u) + \Phi^*(q) - \langle q, u \rangle = 0,$$

from which we have $p \in \partial \Phi(v)$ and $q \in \partial \Phi(u)$. \qed

As an important and straightforward consequence of the above proposition we have the following.

Theorem 2.2. Let $V$ and $V^*$ be in separating duality and $\Phi : V \to \mathbb{R} \cup \{ \infty \}$ be a proper convex function. Suppose $\Phi$ is Gâteaux differentiable at $u, v \in X$. Then
\[ \langle \nabla \Phi(u) - \nabla \Phi(v), u - v \rangle = 0 \]

if and only if \( \nabla \Phi(u) = \nabla \Phi(v) \).

**Proof.** Since \( \Phi \) is Gâteaux differentiable at \( u, v \in X \), we have \( \partial \Phi(u) = \{ \nabla \Phi(u) \} \) and \( \partial \Phi(v) = \{ \nabla \Phi(v) \} \). Set \( p = \nabla \Phi(u) \) and \( q = \nabla \Phi(v) \). If \( \langle \nabla \Phi(u) - \nabla \Phi(v), u - v \rangle = 0 \), it follows from the above proposition that \( p, q \in \partial \Phi(u) \cap \partial \Phi(v) \). This implies \( \nabla \Phi(u) = \nabla \Phi(v) \). \( \square \)

### 2.2. Saddle functions on phase spaces

Here we summarize some of the results in the theory of saddle functions on the product space \( X \times Y \) for some Banach spaces \( X \) and \( Y \). We start with the definition of saddle functions:

**Definition 2.3.** We call a function \( H : X \times Y \to \mathbb{R} \) a saddle function if the following properties hold:

1. \( H(x,\cdot) \) is convex and lower semi-continuous for each \( x \in X \).
2. \( H(\cdot,y) \) is concave and upper semi-continuous for each \( y \in Y \).

Assuming the Banach spaces \( X \) and \( Y \) are in separating duality with \( X^* \) and \( Y^* \) respectively, it is easily seen that the bilinear form on \( (X \times Y) \times (X^* \times Y^*) \) defined by

\[ \langle (x,y), (p,q) \rangle_{(X \times Y) \times (X^* \times Y^*)} = \langle x, p \rangle_{X \times X^*} + \langle y, q \rangle_{Y \times Y^*} \]

puts \( X \times Y \) and \( X^* \times Y^* \) in separating duality. For a saddle function \( H : X \times Y \to \mathbb{R} \) the notion of subdifferential is introduced by Rockafellar as the multivalued mapping \( \partial H : X \times Y \to X^* \times Y^* \) defined by

\[ \partial H(x,y) = \{ (-p,q) ; \ p \text{ is a subdifferential of the convex function } -H(\cdot,y) \text{ at } x \text{ and } q \text{ is a subdifferential of the convex function } H(x,\cdot) \text{ at } y \}. \]

Thus, denoting the subdifferential with respect to the first variable by \( \partial_1 \) and subdifferential with respect to the second variable by \( \partial_2 \) we have \( \partial H(x,y) = \partial_1(-H(x,y)) \times \partial_2 H(x,y) \).

For a saddle function \( H \), the function on \( X^* \times Y \) obtained by taking the Fenchel–Legendre dual of \( -H(\cdot,y) \) when the second argument is fixed, i.e., \( F(\cdot,y) = (-H(\cdot,y))^* \) or

\[ F(p,y) = \sup_{x \in X} \{ (p,x)_{X \times X^*} + H(x,y) \} \]

is called the first convex parent of \( H \). The second convex parent of \( H \) is a function on \( X \times Y^* \) defined by \( G(x,\cdot) = (H(x,\cdot))^* \), or

\[ G(x,q) = \sup_{y \in Y} \{ (q,y)_{Y \times Y^*} - H(x,y) \}. \]

The following is rather standard.
**Proposition 2.3.** If $H$ is a saddle function on $X \times Y$ then the following hold:

(1) The first convex parent $F$ and the second convex parent $G$ are convex and lower semi continuous with respect to both variable and indeed

$$F^*(x, q) = G(x, q) \text{ for all } (x, q) \in X \times Y^*,$$

and

$$G^*(p, y) = F(p, y) \text{ for all } (p, y) \in X^* \times Y,$$

where $F^*$ and $G^*$ are Fenchel–Legendre dual of $F$ and $G$ with respect to both variables.

(2) The following are equivalent:

$$(-p, q) \in \partial H(x, y) \iff (x, q) \in \partial F(p, y) \iff (p, y) \in \partial G(x, q).$$

**2.3. Linear self-adjoint operators modulo boundary operators**

For the proof of the main theorem regarding nonlinear boundary conditions and also in various applications, we are often faced with an unbounded operator $\Lambda : Dom(\Lambda) \subset V \to V^*$ which may still satisfy various aspects of symmetry. For the convenience, we now recall some standard notions on this subject.

**Definition 2.4.** Let $V$ and $V^*$ be in separating duality. A linear operator $\Lambda : Dom(\Lambda) \subset V \to V^*$ is called symmetric if $Dom(\Lambda)$ is dense in $V$ and $\langle \Lambda u, v \rangle = \langle u, \Lambda v \rangle$ for all elements $u$ and $v$ in the domain of $\Lambda$. The operator $\Lambda$ is said to be non-negative if $\langle \Lambda u, u \rangle \geq 0$ for all $u \in Dom(\Lambda)$.

We shall also deal with situations where the operator $\Lambda$ is not purely symmetric provided one takes into account certain boundary terms. In fact, the operator $\Lambda$ modulo the boundary operator $B := (\beta_1, \beta_2) : V \to Y \times Y^*$ (for some Banach spaces $Y$ and $Y^*$ that are in duality) corresponds to the “Green formula”

$$\langle \Lambda u, v \rangle_{V \times V^*} = \langle u, Av \rangle_{V \times V^*} + \langle \beta_1 u, \beta_2 v \rangle_{Y \times Y^*} - \langle \beta_1 v, \beta_2 u \rangle_{Y \times Y^*}.$$

We introduce the following notion.

**Definition 2.5.** Suppose the spaces $V$ and $V^*$ and also $Y$ and $Y^*$ are in separating duality. We say that a linear operator $\Lambda : Dom(\Lambda) \subset V \to V^*$ is symmetric modulo the linear boundary operator $B := (\beta_1, \beta_2) : Dom(\Lambda) \to Y \times Y^*$ if the following properties are satisfied:

(1) The space $V_0 = Dom(\Lambda) \cap ker(\beta_1)$ is dense in $V$.

(2) The operator $\beta_1 : Dom(\Lambda) \subset V \to Y$ has a dense range.

(3) For every $u, v \in Dom(\Lambda)$ we have $\langle \Lambda u, v \rangle_{V \times V^*} = \langle u, Av \rangle_{V \times V^*} + \langle \beta_1 u, \beta_2 v \rangle_{Y \times Y^*} - \langle \beta_1 v, \beta_2 u \rangle_{Y \times Y^*}$.

Our definition of non-negative symmetric operators modulo the boundary operator $B := (\beta_1, \beta_2) : V \to Y \times Y^*$ will change accordingly. Indeed,
Definition 2.6. We say that an operator \( \Lambda \) is non-negative modulo the boundary operator \( \mathcal{B} = (\beta_1, \beta_2) \) if the following property is satisfied:

- For every \( u \in \text{Dom}(\Lambda) \) we have \( \langle \Lambda u, u \rangle_{V \times V^*} + \langle \beta_1 u, \beta_2 u \rangle_{Y \times Y^*} \geq 0 \).

Example 1. Let \( \Omega \) be a smooth domain in \( \mathbb{R}^N \), and \( \partial \Omega \) its boundary. Note that for \( p > 1 \) and \( p' = \frac{p}{p-1} \) Banach spaces \( L^p(\Omega) \) and \( L^{p'}(\Omega) \) are in separating duality with the bilinear form

\[
\langle u, v \rangle = \int_{\Omega} u(x)v(x) \, dx.
\]

Consider the Laplace operator \( -\Delta : \text{Dom}(-\Delta) \subset L^p(\Omega) \rightarrow L^{p'}(\Omega) \). Let \( H^{\frac{1}{2}}(\Omega) \) be the fractional Sobolev space of order \( \frac{1}{2} \) and \( H^{-\frac{1}{2}}(\Omega) \) its dual. Define the boundary operators \( \beta_1 : \text{Dom}(-\Delta) \subset L^p(\Omega) \rightarrow H^{\frac{1}{2}}(\partial\Omega) \) and \( \beta_2 : \text{Dom}(-\Delta) \subset L^p(\Omega) \rightarrow H^{-\frac{1}{2}}(\Omega) \) by \( \beta_1 u = u|_{\partial\Omega} \) and \( \beta_2 u = \frac{\partial u}{\partial \nu} \) where \( \frac{\partial u}{\partial \nu} \) is the normal derivative on \( \partial\Omega \). We shall show that \( -\Delta : \text{Dom}(-\Delta) \subset L^p(\Omega) \rightarrow L^{p'}(\Omega) \) is symmetric and non-negative modulo the boundary operator \( \mathcal{B} := (\beta_1, \beta_2) \) and also \( \text{Dom}(-\Delta) \subset L^p(\Omega) \rightarrow H^{\frac{1}{2}}(\Omega) \times H^{-\frac{1}{2}}(\Omega) \) is surjective by Theorem 8.3 in [23, Chapter 1]. Condition (3) of Definition 2.5 is nothing but the integration by parts in \( H^1(\Omega) \). In fact, for \( u, v \in \text{Dom}(-\Delta) \) we have

\[
\langle -\Delta u, v \rangle = -\int_{\Omega} \Delta u(x)v(x) \, dx
\]

\[
= \int_{\Omega} \nabla u(x) \cdot \nabla v(x) \, dx - \int_{\partial\Omega} \frac{\partial u}{\partial \nu} v \, d\sigma
\]

\[
= -\int_{\Omega} u(x) \Delta v(x) \, dx - \int_{\partial\Omega} \frac{\partial u}{\partial \nu} v \, d\sigma + \int_{\partial\Omega} \frac{\partial v}{\partial \nu} u \, d\sigma
\]

\[
= \langle -\Delta v, u \rangle + \langle \beta_1 u, \beta_2 v \rangle_{H^{\frac{1}{2}}(\partial\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)} - \langle \beta_1 v, \beta_2 u \rangle_{H^{\frac{1}{2}}(\partial\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)}.
\]

The above computation also shows that \( -\Delta \) is non-negative modulo the boundary operator \( \mathcal{B} := (\beta_1, \beta_2) \). Indeed, \( \langle -\Delta u, u \rangle + \langle \beta_1 u, \beta_2 u \rangle_{H^{\frac{1}{2}}(\partial\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)} = \int_{\Omega} |\nabla u(x)|^2 \, dx \geq 0 \).

We shall use the following result in Section 4.

Proposition 2.4. Suppose the spaces \( V \) and \( V^* \) and also \( Y \) and \( Y^* \) are in separating duality. The following hold:

1. Assume that the linear operator \( \Lambda : \text{Dom}(\Lambda) \subset X \rightarrow X^* \) is symmetric and non-negative. If \( \langle \Lambda u - \Lambda v, u - v \rangle = 0 \) for some \( u, v \in \text{Dom}(\Lambda) \) then \( \Lambda u = \Lambda v \).
(2) Assume that the linear operator \( \Lambda : \text{Dom}(\Lambda) \subset V \to V^* \) is symmetric and non-negative modulo the linear boundary operator \( B = (\beta_1, \beta_2) : V \to Y \times Y^* \). If \( \langle \Lambda u - \Lambda v, u - v \rangle_{V \times V^*} + \langle \beta_1 u - \beta_1 v, \beta_2 u - \beta_2 v \rangle_{Y \times Y^*} = 0 \) for some \( u, v \in \text{Dom}(\Lambda) \) then \( \Lambda u = \Lambda v \) and \( \beta_2 u = \beta_2 v \).

**Proof.** For the proof of part (1), note that since \( \Lambda \) is symmetric and non-negative, the function \( \Phi \) defined on \( V \) by
\[
\Phi(u) = \frac{1}{2} \langle \Lambda u, u \rangle
\]
is convex and Gâteaux differentiable on \( \text{Dom}(\Lambda) \). In fact
\[
\nabla \Phi(u) = \Lambda u
\]
for all \( u \in \text{Dom}(\Lambda) \).

Now if \( \langle \Lambda u - \Lambda v, u - v \rangle_{V \times V^*} = 0 \) we have
\[
\langle \nabla \Phi(u) - \nabla \Phi(v), u - v \rangle = 0,
\]
from which together with Theorem 2.2 one obtains \( \nabla \Phi(u) = \nabla \Phi(v) \).

We now prove part (2). As in part (1) since \( \Lambda \) is symmetric and non-negative modulo the linear boundary operator \( B = (\beta_1, \beta_2) \) the function \( \Phi \) defined on \( V \) by
\[
\Phi(u) = \frac{1}{2} \langle \Lambda u, u \rangle + \frac{1}{2} \langle \beta_1 u, \beta_2 u \rangle_{Y \times Y^*}
\]
is convex and Gâteaux differentiable on \( \text{Dom}(\Lambda) \). A straightforward computation shows that for \( \eta \in \text{Dom}(\Lambda) \) we have
\[
\langle \nabla \Phi(u), \eta \rangle = \frac{1}{2} \langle \Lambda u, \eta \rangle_{X \times X^*} + \frac{1}{2} \langle \Lambda \eta, u \rangle_{X \times X^*} + \frac{1}{2} \langle \beta_2 u, \beta_1 \eta \rangle_{Y \times Y^*} + \frac{1}{2} \langle \beta_2 \eta, \beta_1 u \rangle_{Y \times Y^*}.
\]

It now follows from part (3) of Definition 2.5 that
\[
\langle \Lambda \eta, u \rangle_{X \times X^*} + \langle \beta_2 \eta, \beta_1 u \rangle_{Y \times Y^*} = \langle \Lambda u, \eta \rangle_{X \times X^*} + \langle \beta_2 u, \beta_1 \eta \rangle_{Y \times Y^*}
\]
and therefore
\[
\langle \nabla \Phi(u), \eta \rangle = \langle \Lambda u, \eta \rangle_{X \times X^*} + \langle \beta_2 u, \beta_1 \eta \rangle_{Y \times Y^*}.
\]

By assumption we have
\[
\langle \Lambda u - \Lambda v, u - v \rangle_{V \times V^*} + \langle \beta_1 u - \beta_1 v, \beta_2 u - \beta_2 v \rangle_{Y \times Y^*} = 0.
\]
This together with (4) imply that
\[
\langle \nabla \Phi(u) - \nabla \Phi(v), u - v \rangle = 0.
\]
By Theorem 2.2 we obtain \( \nabla \Phi(u) = \nabla \Phi(v) \) from which together with (4) one has
\[
\langle \Lambda u - \Lambda v, \eta \rangle + \langle \beta_2 u - \beta_2 v, \beta_1 \eta \rangle_{Y \times Y^*} = 0 \quad \text{for all } \eta \in \text{Dom}(\Lambda).
\]

It follows from part (1) of Definition 2.5 that \( \text{Ker}(\beta_1) \) is dense in \( V \). This and the above equation yield that \( \langle \Lambda u - \Lambda v, \eta \rangle = 0 \) for all \( \eta \in \text{Ker}(\beta_1) \) and therefore \( \Lambda u = \Lambda v \). It then follows from (5) that
\[
\langle \beta_2 u - \beta_2 v, \beta_1 \eta \rangle_{Y \times Y^*} = 0 \quad \text{for all } \eta \in \text{Dom}(\Lambda),
\]
from which together with density of range \( \beta_1 \) in \( Y \), due to part (2) of Definition 2.5, we obtain \( \beta_2 u = \beta_2 v \). \( \square \)
3. Non-convex self-dual Lagrangians

Let $V$ be a Banach space that is in separating duality with the Banach space $V^*$. Functions $\Phi : V \times V^* \to \mathbb{R} \cup \{\infty\}$ on phase space $V \times V^*$ will be called Lagrangians. We shall consider the class of Lagrangians that are convex and lower semi-continuous on the second variable. The Fenchel–Legendre dual of $\Phi$ with respect to the second variable will be denoted by $LF_2(\Phi)$ and is a function on $V \times V$ given by:

$$LF_2(\Phi)(u, v) = \sup_{p \in V^*} \left\{ \langle p, v \rangle - \Phi(u, p) \right\}.$$

We define the Non-convex dual, $\Phi^\#$, of $\Phi$ by computing the Fenchel–Legendre dual of $LF_2(\Phi)(., v)$ with respect to the first variable. Therefore $\Phi^\#$ is a Lagrangian on the phase space $V \times V^*$ given by

$$\Phi^\#(v, q) = \sup_{u \in V} \left\{ \langle q, u \rangle - LF_2(\Phi)(u, v) \right\}.$$

**Definition 3.1.** Suppose $\Phi$ is a Lagrangian on phase space $V \times V^*$. Say that the Lagrangian $\Phi$ on $V \times V^*$ is Non-convex self-dual if the following property hold.

$$\Phi^\#(u, p) = \Phi(u, p) \quad \text{for all } (u, p) \in V \times V^*.$$

We now list some permanence properties of Nc-SD Lagrangians.

**Proposition 3.1.** Let $V$ be a Banach space that is in separating duality with the Banach space $V^*$. The following statements hold:

1. If $\varphi : V \to \mathbb{R}$ is convex and lower semi-continuous and $\varphi^*$ its Fenchel–Legendre dual defined on $V^*$, then the following Lagrangians are Non-convex self-dual:
   (i) $\Phi_1(u, p) := \varphi^*(p) - \varphi(u), \quad (u, p) \in V \times V^*$,
   (ii) $\Phi_2(u, p) := \varphi^*(p) - \langle p, u \rangle, \quad (u, p) \in V \times V^*$,
   (iii) $\Phi_3(u, p) := \langle p, u \rangle - \varphi(u), \quad (u, p) \in V \times V^*$.
2. If $A : V \to V^*$ is symmetric and $\Phi$ is any Nc-SD Lagrangian then the Lagrangian

$$\Psi(u, p) := \Phi(u,Au + p)$$

is also Non-convex self-dual.
3. If $\Phi$ is an Nc-SD Lagrangian and $\mu > 0$ then the Lagrangian $\mu.\Phi$ on $V \times V^*$ defined by

$$\mu.\Phi)(u, p) = \mu^{-2}\Phi(\mu u, \mu p)$$

is also Nc-SD.

**Proof.** Fix $(v, q) \in V \times V^*$. Note that $LF_2(\Phi_1)(u, v) = \varphi(u) + \varphi(v)$ from which we have

$$\Phi_1^\#(v, p) = \varphi^*(p) - \varphi(v) = \Phi_1(v, p).$$
Thus, $\Phi_1$ is an Nc-SD Lagrangian. For the Lagrangian $\Phi_2$ we have

$$LF_2(\Phi_2)(u, v) = \sup_{p \in V^*} \{ (p, v) - \Phi_2(u, p) \}$$

$$= \sup_{p \in V^*} \{ (p, v) - \varphi^*(p) + (p, u) \}$$

$$= \sup_{p \in V^*} \{ (p, v + u) - \varphi^*(p) \} = \varphi(u + v).$$

This implies that

$$\Phi_2^\#(v, q) = \sup_{u \in V} \{ (q, u) - \varphi(u + v) \}$$

$$= \sup_{u \in V} \{ (q, u + v) - \varphi(u + v) \} - \langle q, v \rangle$$

$$= \varphi^*(q) - \langle q, v \rangle = \Phi_2(v, q)$$

thereby giving that $\Phi_2$ is Nc-SD.

For the Lagrangian $\Phi_3$ we have

$$LF_2(\Phi_3)(u, v) = \sup_{p \in V^*} \{ (p, v) - \Phi_3(u, p) \}$$

$$= \sup_{p \in V^*} \{ (p, v) - (p, u) + \varphi(u) \}$$

$$= \sup_{p \in V^*} \{ (p, v - u) + \varphi(u) \},$$

from which we obtain

$$LF_2(\Phi_3)(u, v) = \begin{cases} \varphi(u), & u = v, \\ +\infty, & u \neq v, \end{cases}$$

and therefore

$$\Phi_3^\#(v, q) = \sup_{u \in V} \{ (q, u) - LF_2(\Phi_3)(u, v) \}$$

$$= \langle q, v \rangle - \varphi(v) = \Phi_3(v, q).$$

This completes the proof of part (1).

For the proof of part (2), we first compute $LF_2(\Psi)(u, v)$.

$$LF_2(\Psi)(u, v) = \sup_{p \in V^*} \{ (p, v) - \Phi(u, p + \Lambda u) \}$$

$$= \sup_{p \in V^*} \{ (p + \Lambda u, v) - \Phi(u, p + \Lambda u) \} - \langle \Lambda u, v \rangle$$

$$= LF_2(\Phi)(u, v) - \langle \Lambda u, v \rangle.$$
It follows that

\[ \Psi^*(v, q) = \sup_{u \in V} \{ \langle q, u \rangle - LF_2(\Psi)(u, v) \} \]

\[ = \sup_{u \in V} \{ \langle q, u \rangle - LF_2(\Phi)(u, v) + \langle \Lambda u, v \rangle \} \]

\[ = \sup_{u \in V} \{ \langle q + \Lambda v, u \rangle - LF_2(\Phi)(u, v) \} \]

\[ = \Phi^*(v, q + \Lambda v) = \Phi(v, q + \Lambda v) = \Psi(v, q). \]

This proves part (2).

For part (3), we have

\[ LF_2(\mu \Phi)(u, v) = \sup_{p \in V^*} \{ \langle p, v \rangle - (\mu \Phi)(u, p) \} \]

\[ = \sup_{p \in V^*} \{ \langle p, v \rangle - \mu^{-2} \Phi(\mu u, \mu p) \} \]

\[ = \mu^{-2} \sup_{p \in V^*} \{ \langle \mu p, \mu v \rangle - \Phi(\mu u, \mu p) \} \]

\[ = \mu^{-2} LF_2(\Phi)(\mu u, \mu v). \]

Thus

\[ (\mu \Phi)^*(v, q) = \sup_{u \in V} \{ \langle q, u \rangle - \mu^{-2} LF_2(\Phi)(\mu u, \mu v) \} \]

\[ = \mu^{-2} \sup_{u \in V} \{ \langle \mu q, \mu u \rangle - LF_2(\Phi)(\mu u, \mu v) \} \]

\[ = \mu^{-2} \Phi(\mu v, \mu q) = (\mu \Phi)(v, q). \]

This completes the proof of part (3). ⊓⊔

It follows from Proposition 3.1 that, if \( \Lambda : X \to X^* \) is a linear symmetric operator then the following Lagrangians are Nc-SD:

1. \( \Psi_1(u, p) := \Phi_1(u, p + \Lambda u) = \varphi^*(p + \Lambda u) - \varphi(u) \),
2. \( \Psi_2(u, p) := \Phi_2(u, p + \Lambda u) = \varphi^*(p + \Lambda u) - \langle p + \Lambda u, u \rangle \),
3. \( \Psi_3(u, p) := \Phi_3(u, p + \Lambda u) = \langle p + \Lambda u, u \rangle - \varphi(u) \).

They will be called basic Non-convex self-dual Lagrangians.
3.1. Unbounded operators

We shall see that in practice and in various applications, we are often faced with an unbounded symmetric operator \( \Lambda \). As seen, in Proposition 3.1 the iteration, in an appropriate way, of \( Nc-SD \) Lagrangians with bounded symmetric operators are still \( Nc-SD \). Here we extend this result to unbounded operators as well. Indeed, let \( \Lambda : Dom(\Lambda) \subset V \rightarrow V^* \) be a possibly unbounded symmetric operator. If \( \Lambda \) is closed, consider \( V_\Lambda \) to be the Banach space \( Dom(\Lambda) \) equipped with the norm:

\[
\|u\|_{V_\Lambda} = \|u\|_V + \|\Lambda u\|_{V^*}.
\]

Since \( V_\Lambda \) is dense in \( V \), it is easily seen that \( V_\Lambda \) and \( V^* \) are still in separating duality with the same bilinear form that puts \( V \) and \( V^* \) in separating duality.

**Proposition 3.2.** Let \( \Lambda : Dom(\Lambda) \subset V \rightarrow V^* \) be a closed symmetric operator and \( \Phi : V \times V^* \rightarrow \mathbb{R} \) be a Non-convex self-dual Lagrangian. If \( LF_2(\Phi) \) is continuous on \( V \times V \) then the Lagrangian \( \Psi : V_\Lambda \times V^* \rightarrow \mathbb{R} \) defined by

\[
\Psi(u, p) = \Phi(u, \Lambda u + p)
\]

is also a Non-convex self-dual Lagrangian.

**Proof.** Let us first compute \( LF_2(\Psi)(u, v) \), for \( u, v \in V_\Lambda \),

\[
LF_2(\Psi)(u, v) = \sup_{p \in V^*} \{ \langle p, v \rangle - \Phi(u, p + \Lambda u) \}
\]

\[
= \sup_{p \in V^*} \{ \langle p + \Lambda u, v \rangle - \Phi(u, p + \Lambda u) \} - \langle \Lambda u, v \rangle
\]

\[
= LF_2(\Phi)(u, v) - \langle \Lambda u, v \rangle.
\]

It follows that

\[
\Psi^\#(v, q) = \sup_{u \in V_\Lambda} \{ \langle q, u \rangle - LF_2(\Psi)(u, v) \}
\]

\[
= \sup_{u \in V_\Lambda} \{ \langle q, u \rangle - LF_2(\Phi)(u, v) + \langle \Lambda u, v \rangle \}
\]

\[
= \sup_{u \in V_\Lambda} \{ \langle q + \Lambda v, u \rangle - LF_2(\Phi)(u, v) \}.
\]

Since \( V_\Lambda \) is dense in \( V \) and \( LF_2(\Phi) \) is continuous on \( V \times V \) we have

\[
\sup_{u \in V_\Lambda} \{ \langle q + \Lambda v, u \rangle - LF_2(\Phi)(u, v) \} = \sup_{u \in V} \{ \langle q + \Lambda v, u \rangle - LF_2(\Phi)(u, v) \}.
\]
from which we have

$$\Psi^#(v, q) = \sup_{u \in V} \{ \langle q + \Lambda v, u \rangle - \Lambda F_2(\Phi)(u, v) \} = \Phi^#(v, q + \Lambda v) = \Phi(v, q + \Lambda v) = \Psi(v, q).$$

\[ \square \]

**Example 2.** Let $\Omega$ be a smooth domain in $\mathbb{R}^N$ and $\partial \Omega$ its boundary. Consider the Laplace operator with Dirichlet boundary condition $-\Delta : Dom(-\Delta) \subset L^p(\Omega) \to L^{p'}(\Omega)$, for $p > 1$ and $p' = \frac{p}{p-1}$. It follows that

$$Dom(\Delta) = \{ u \in L^p(\Omega); \ \Delta u \in L^{p'}(\Omega) \ \& \ u = 0 \text{ on } \partial \Omega \}$$

is a Banach space when equipped with the norm:

$$\|u\| = \|u\|_{L^p(\Omega)} + \|\Delta u\|_{L^{p'}(\Omega)}.$$

Consider also the convex function $\varphi : L^p(\Omega) \to \mathbb{R}$ defined by $\varphi(u) = \frac{1}{p} \int_{\Omega} |u(x)|^p \, dx + \int_{\Omega} u(x) f(x) \, dx$ where $f \in L^{p'}(\Omega)$. An easy computation shows that $\varphi^*(r) = \frac{1}{p} \int_{\Omega} |r(x) - f(x)|^{p'} \, dx$ for all $r \in L^{p'}(\Omega)$. We have that $\Phi(u, r) = \varphi^*(r) - \varphi(u)$ is an $Nc$-SD Lagrangian on $L^p(\Omega) \times L^{p'}(\Omega)$. Now since

$$LF_2(\Phi)(u, v) = \frac{1}{p} \int_{\Omega} \left[ |u(x)|^p + |v(x)|^p \right] \, dx + \int_{\Omega} (u(x) + v(x)) f(x) \, dx$$

is continuous on $L^p(\Omega) \times L^{p'}(\Omega)$ and $Dom(-\Delta)$ is dense in $L^p(\Omega)$ it follows from the above proposition that

$$\Psi(u, r) := \frac{1}{p'} \int_{\Omega} |r(x) - \Delta u(x) - f(x)|^{p'} \, dx - \frac{1}{p} \int_{\Omega} |u(x)|^p \, dx - \int_{\Omega} u(x) f(x) \, dx$$

is also an $Nc$-SD Lagrangian on $Dom(-\Delta) \times L^{p'}(\Omega)$.

3.2. Symmetric operators modulo boundary operators

For problems involving nonlinear boundary terms, we may start with an $Nc$-SD Lagrangian $\Phi$, but the operator $A : Dom(A) \subset V \to V^*$ may be symmetric modulo a term involving a boundary operator $B := (\beta_1, \beta_2) : V \to Y \times Y^*$ for some Banach spaces $Y$ and $Y^*$ that are in separating duality. We can then try to recover Non-convex self-duality by adding a correcting term via a boundary Lagrangian on $Y \times Y^*$, in such a way that a new Lagrangian

$$\Psi(u, (p, e)) := \Phi(u, Au + p) + \ell(\beta_1 u, \beta_2 u + e)$$

becomes $Nc$-SD on $V_A \times (V^* \times Y^*)$. Thus, we first need to define a bilinear form between $V_A$ and $V^* \times Y^*$ in such a way that it puts $V_A$ and $V^* \times Y^*$ in separating duality.
Lemma 3.2. Let $\Lambda : \text{Dom}(\Lambda) \subset V \to V^*$ be a symmetric operator modulo the boundary operator $B := (\beta_1, \beta_2) : V \to Y \times Y^*$. Then the bilinear form
\[
\langle u, (p, e) \rangle_{V^* \times (V^* \times Y^*)} := \langle u, p \rangle_{V \times V^*} + \langle \beta_1 u, e \rangle_{V \times Y^*}
\]
puts $V_\Lambda$ and $V^* \times Y^*$ in separating duality.

**Proof.** First assume $0 \neq u \in V_\Lambda$. Since $V$ and $V^*$ are in separating duality, there exists $p \in V^*$ such that $\langle u, p \rangle_{V \times V^*} \neq 0$ and therefore $\langle u, (p, 0) \rangle_{V^* \times (V^* \times Y^*)} \neq 0$. Now suppose $0 \neq (p, e) \in V^* \times Y^*$. If $p \neq 0$, since $V$ and $V^*$ are in separating duality, there exists $u \in V$ such that $\langle u, p \rangle_{V \times V^*} \neq 0$. It also follows from Definition 2.5 that $\text{Ker}(\beta_1)$ is dense in $V$. Thus, there exists a sequence $\{u_n\} \subset \text{Ker}(\beta_1)$ such that $u_n \to u$ in $V$. It follows that
\[
\langle u_n, (p, e) \rangle_{V^* \times (V^* \times Y^*)} = \langle u_n, p \rangle_{V \times V^*} \neq 0,
\]
for $n$ large enough.

If $p = 0$, then $e$ must be a non-zero element of $Y^*$ and the result follows from the fact that $\beta_1$ has a dense range in $Y$.

We now state our result.

**Proposition 3.3.** Let $\Lambda : \text{Dom}(\Lambda) \subset V \to V^*$ be a possibly unbounded symmetric operator modulo the boundary operator $B := (\beta_1, \beta_2) : \text{Dom}(\Lambda) \to Y \times Y^*$. Let $\Phi : V_\Lambda \times V^* \to \mathbb{R}$ and $\ell : Y \times Y^* \to \mathbb{R}$ be Non-convex self-dual Lagrangians. If $LF_2(\Phi)$ and $LF_2(\ell)$ are continuous on $V \times V$ and $Y \times Y$ respectively then the Lagrangian $\Psi : V_\Lambda \times (V^* \times Y^*) \to \mathbb{R}$ defined by
\[
\Psi(u, (p, e)) = \Phi(u, \Lambda u + p) + \ell(\beta_1 u, \beta_2 u + e)
\]
is also a Non-convex self-dual Lagrangian.

**Proof.** Let us first compute $LF_2(\Psi)(u, v)$, for $u, v \in V_\Lambda$,
\[
LF_2(\Psi)(u, v) = \sup_{(p, e) \in V^* \times Y^*} \left\{ \langle p, v \rangle_{V \times V^*} + \langle \beta_1 v, e \rangle_{V \times Y^*} - \Phi(u, \Lambda u + p) - \ell(\beta_1 u, \beta_2 u + e) \right\}
\]
\[
= \sup_{(p, e) \in V^* \times Y^*} \left\{ \langle p + \Lambda u, v \rangle_{V \times V^*} + \langle \beta_1 v, e + \beta_2 u \rangle_{V \times Y^*} \right.
\]
\[
- \Phi(u, p + \Lambda u) - \ell(\beta_1 u, \beta_2 u + e) \left. \right\} - \langle \Lambda u, v \rangle_{V \times V^*} - \langle \beta_1 v, \beta_2 u \rangle_{V \times Y^*}
\]
\[
= LF_2(\Phi)(u, v) + LF_2(\ell)(\beta_1 u, \beta_1 v) - \langle \Lambda u, v \rangle_{V \times V^*} - \langle \beta_1 v, \beta_2 u \rangle_{Y \times Y^*}.
\]

It follows that
\[
\Psi^#(v, q) = \sup_{u \in V_\Lambda} \left\{ \langle q, u \rangle_{V \times V^*} + \langle \beta_1 u, e \rangle_{Y \times Y^*} - LF_2(\Psi)(u, v) \right\}
\]
\[
= \sup_{u \in V_\Lambda} \left\{ \langle q, u \rangle_{V \times V^*} + \langle \beta_1 u, e \rangle_{Y \times Y^*} + \langle \Lambda u, v \rangle_{V \times V^*} + \langle \beta_1 v, \beta_2 u \rangle_{Y \times Y^*} \right\}
\]
\[
- LF_2(\Phi)(u, v) - LF_2(\ell)(\beta_1 u, \beta_1 v) \\
= \sup_{u \in V_A} \left\{ \langle q, u \rangle_{V \times V^*} + \langle \beta_1 u, e \rangle_{Y \times Y^*} + \langle u, \Lambda v \rangle_{V \times V^*} + \langle \beta_1 u, \beta_2 v \rangle_{Y \times Y^*} \\
- LF_2(\Phi)(u, v) - LF_2(\ell)(\beta_1 u, \beta_1 v) \right\} \\
= \sup_{u \in V_A} \left\{ \langle q + \Lambda v, u \rangle_{V \times V^*} + \langle \beta_1 u, e + \beta_2 v \rangle_{Y \times Y^*} \\
- LF_2(\Phi)(u, v) - LF_2(\ell)(\beta_1 u, \beta_1 v) \right\} \\
= \sup_{u \in V_A, u_0 \in V_0} \left\{ \langle q + \Lambda v, u \rangle_{V \times V^*} + \langle \beta_1 (u + u_0), e + \beta_2 v \rangle_{Y \times Y^*} \\
- LF_2(\Phi)(w - u_0, v) - LF_2(\ell)(\beta_1 (u + u_0), \beta_1 v) \right\}
\]

where \( V_0 = \text{Dom}(\Lambda) \cap \ker(\beta_1) \). Setting \( w = u + u_0 \) we have \( u = w - u_0 \) and therefore

\[
\Psi^#(v, q) = \sup_{w \in V_A, u_0 \in V_0} \left\{ \langle q + \Lambda v, w - u_0 \rangle_{V \times V^*} + \langle \beta_1 (w), e + \beta_2 v \rangle_{Y \times Y^*} \\
- LF_2(\Phi)(w - u_0, v) - LF_2(\ell)(\beta_1 (w), \beta_1 v) \right\}.
\]

Since \( V_0 \) is dense in \( V \) and \( LF_2(\Phi) \) is continuous on \( V \times V \) we have

\[
\sup_{u_0 \in V_0} \left\{ \langle q + \Lambda v, w - u_0 \rangle - LF_2(\Phi)(w - u_0, v) \right\} = \sup_{u \in V} \left\{ \langle q + \Lambda v, u \rangle - LF_2(\Phi)(u, v) \right\} = \Phi^#(v, q + \Lambda v)
\]

from which we have

\[
\Psi^#(v, q) = \sup_{w \in V_A} \left\{ \langle \beta_1 (w), e + \beta_2 v \rangle_{Y \times Y^*} - LF_2(\ell)(\beta_1 (w), \beta_1 v) \right\} + \Phi^#(v, q + \Lambda v).
\]

Also taking into account that \( \beta_1 : V_A \to Y \) has a dense range in \( Y \) and \( LF_2(\ell) \) is continuous on \( Y \times Y \) we have

\[
\sup_{w \in V_A} \left\{ \langle \beta_1 (w), e + \beta_2 v \rangle_{Y \times Y^*} - LF_2(\ell)(\beta_1 (w), \beta_1 v) \right\} = \ell^#(\beta_1 v, e + \beta_2 v).
\]

This implies that

\[
\Psi^#(v, q) = \Phi^#(v, q + \Lambda v) + \ell^#(\beta_1 v, e + \beta_2 v) \\
= \Phi(v, q + \Lambda v) + \ell(\beta_1 v, e + \beta_2 v) \\
= \Psi(v, q).
\]

**Example 3.** Let \( N > 4 \), \( 1 < p < \frac{2N}{N-4} \), \( p' = \frac{p}{p-1} \) and \( \Omega \) be a bounded smooth domain in \( \mathbb{R}^N \) and \( \partial \Omega \) its boundary. Consider the fourth-order operator \( \Lambda u : \text{Dom}(\Lambda) \subset L^p(\Omega) \to L^{p'}(\Omega) \), defined by \( \Lambda u = \Delta^2 u + u \). It follows from the Sobolev embedding
that $\text{Dom}(\Lambda) = W^{2,p'}(\Omega)$. By the same argument as in Example 1, one can easily deduce that $\Lambda$ is a symmetric operator modulo the boundary operators $\beta_1 : \text{Dom}(\Lambda) \to H^{1,2}(\partial\Omega) \times H^{3,2}(\partial\Omega)$ defined by $\beta_1 u = (\frac{\partial u}{\partial n}, u)_{\partial\Omega}$ and $\beta_2 : \text{Dom}(\Lambda) \to H^{-1,2}(\partial\Omega) \times H^{-3,2}(\partial\Omega)$ defined by $\beta_2 u = (-\Delta u, \frac{\partial u}{\partial n})_{\partial\Omega}$. If $\phi : L^p(\Omega) \to \mathbb{R}$ and $\psi : H^{1,2}(\partial\Omega) \times H^{3,2}(\partial\Omega) \to \mathbb{R}$ are two convex and continuous functions then it follows from Proposition 3.3 that the functional

$$\Phi : W^{2,p'}(\Omega) \times (L^{p'}(\Omega) \times (H^{-1,2}(\partial\Omega) \times H^{-3,2}(\partial\Omega))) \to \mathbb{R} \cup \{\infty\},$$

defined by

$$\Phi(u,(p,e_1,e_2)) = \phi^*\left(\Delta^2 u + u + p\right) - \phi(u) + \psi^*\left(-\Delta u + e_1, \frac{\partial u}{\partial n} + e_2\right) - \psi\left(\frac{\partial u}{\partial n}, u\right),$$

is an $Nc$-SD Lagrangian.

3.3. Characterization of non-convex self-dual Lagrangians

We first introduce the notion of symmetric Hamiltonians as follows.

**Definition 3.3.** Let $V$ be a real Banach space. Say that a function $F : V \times V \to \mathbb{R}$ is a symmetric Hamiltonian if it satisfies the following properties:

1. $F(u,.)$ is convex and lower semi-continuous for each $u \in V$.
2. $F(.,v)$ is convex and lower semi-continuous for each $v \in V$.
3. For all $u,v \in V$ we have $F(u,v) = F(v,u)$.

We shall establish a one-to-one correspondence between Non-convex self-dual Lagrangians on $V \times V^*$ and symmetric Hamiltonians on $V \times V$:

**Theorem 3.4.** Let $V$ and $V^*$ be in separating duality. If $\Phi : V \times V^* \to \mathbb{R}$ is a Non-convex self-dual Lagrangian then $LF^2(\Phi)$ is a symmetric Hamiltonian.

Conversely, if a function $F : V \times V \to \mathbb{R}$ is a symmetric Hamiltonian then the Lagrangian $\Phi$ on $V \times V^*$ obtained by computing the Fenchel–Legendre dual of $F$ with respect to the second variable, i.e.,

$$\Phi(u,p) = \sup \{\langle p,v \rangle - F(u,v) ; v \in V \},$$

is a Non-convex self-dual Lagrangian.

**Proof.** Let $\Phi : V \times V^* \to \mathbb{R}$ be a Non-convex self-dual Lagrangian. We prove $LF^2(\Phi)$ satisfies part (3) of Definition 3.3. Parts (1) and (2) are a direct consequence of part (3). Let $u, v \in V$. By the definition of $LF^2(\Phi)$ we have
\[ LF_2(\Phi)(u, v) = \sup_{p \in V^*} \{ \langle p, v \rangle - \Phi(u, p) \}. \]

It follows that \( LF_2(\Phi)(u, \cdot) \) is Fenchel–Legendre dual of \( \Phi(u, \cdot) \), i.e.,

\[ LF_2(\Phi)(u, \cdot) = [\Phi(u, \cdot)]^* \quad \text{on } V. \] (6)

On the other hand since \( \Phi \) is \( Nc\)-SD, we have

\[ \Phi(u, p) = \Phi^\#(u, p) = \sup_{w \in V} \{ \langle p, w \rangle - LF_2(\Phi)(w, u) \}. \] (7)

This implies that \( \Phi(u, \cdot) \) is Fenchel–Legendre dual of \( LF_2(\Phi)(\cdot, u) \). From Fenchel duality we have

\[ [LF_2(\Phi)(\cdot, u)]^* = \Phi(u, \cdot) \quad \text{on } V^*, \]

from which we have

\[ [LF_2(\Phi)(\cdot, u)]^{**} = [\Phi(u, \cdot)]^* \quad \text{on } V. \]

This together with (6) imply that

\[ [LF_2(\Phi)(\cdot, u)]^{**} = LF_2(\Phi)(u, \cdot) \quad \text{on } V, \]

thereby giving

\[ LF_2(\Phi)(\cdot, u) \leq LF_2(\Phi)(u, \cdot) \quad \text{on } V. \]

Since \( u \) is an arbitrary element in \( V \), the above inequality implies that

\[ LF_2(\Phi)(v, u) \leq LF_2(\Phi)(u, v) \quad \text{for all } u, v \in V, \]

and in fact the equality holds.

Converse, obviously the function \( \Phi \) obtained by computing the Fenchel–Legendre dual of \( F \) with respect to the second variable, \( \Phi(u, \cdot) = [F(u, \cdot)]^* \), is convex and lower semi-continuous with respect to the second variable. We shall show that \( LF_2(\Phi)(u, v) = F(u, v) \) for all \( u, v \in V \). It follows from \( \Phi(u, \cdot) = [F(u, \cdot)]^* \) together with \( F \) being convex and lower semi-continuous with respect to the second variable that

\[ LF_2(\Phi)(u, \cdot) = [\Phi(u, \cdot)]^* = [F(u, \cdot)]^{**} = F(u, \cdot). \]

It follows that
\[ \Phi^h(u, p) = \sup \{ \langle p, v \rangle - L F_2(\Phi)(v, u); \, v \in V \} \]
\[ = \sup \{ \langle p, v \rangle - F(v, u); \, v \in V \} \]
\[ = \sup \{ \langle p, v \rangle - F(u, v); \, v \in V \} \]
\[ = \Phi(u, p), \]

and therefore \( \Phi \) is a Non-convex self-dual Lagrangian. \( \square \)

4. Applications to calculus of variations

As mentioned in the Introduction, there is a large class of symmetric differential equations that can be written as
\[ (\Lambda u, u) \in \partial \Phi(u, \Lambda u), \]
where \( \Lambda : \text{Dom}(\Lambda) \subset V \rightarrow V^* \) is a symmetric linear operator and \( \Phi : V \times V^* \rightarrow \mathbb{R} \cup \{ \infty \} \) is a Non-convex self-dual Lagrangian. In this section, we shall state and establish in full generality the relationship between solutions of such inclusions with the corresponding Non-convex self-dual Lagrangians in both homogeneous and nonlinear boundary conditions.

4.1. Homogeneous boundary conditions

Here is our main result regarding homogeneous boundary conditions.

**Theorem 4.1.** Suppose \( \Phi : V \times V^* \rightarrow \mathbb{R} \cup \{ \infty \} \) is a saddle Nc-SD Lagrangian and \( \Lambda : \text{Dom}(\Lambda) \subset V \rightarrow V^* \) is a symmetric operator that is also onto. Suppose one of the following conditions holds:

(i) The operator \( \Lambda \) is non-negative.

(ii) For each \( p \in V^* \), the function \( u \rightarrow \Phi(u, p) \) is Gâteaux differentiable and \( \nabla_1 \Phi(u, p) = -p \).

(iii) For each \( u \in V \), the function \( p \rightarrow \Phi(u, p) \) is Gâteaux differentiable and \( \nabla_2 \Phi(u, p) = u \).

Then for every critical point \( u \) of \( \Phi(u, \Lambda u) \) there exists \( v \in V \) with \( \Lambda u = \Lambda v \) and
\[ (-\Lambda v, v) \in \partial \Phi(u, \Lambda u). \]

**Proof.** Suppose \( u \) is a critical point of \( \Phi(u, \Lambda u) \). It follows that there exists \( v \in \partial_2 \Phi(u, \Lambda u) \) such that \( \Lambda v \in \partial_1 (-\Phi(u, \Lambda u)) \). This implies that
\[ (-\Lambda v, v) \in \partial \Phi(u, \Lambda u). \]  

Now we show that if either of conditions (i), (ii) or (iii) is satisfied then \( \Lambda v = \Lambda u \).

**Proof with condition (i):** By part (2) of Proposition 2.3 we have that \( (\Lambda u, \Lambda v) \in \partial L F_2(\Phi)(u, v) \). It then follows from Theorem 3.4 that \( (\Lambda u, \Lambda v) \in \partial L F_2(\Phi)(v, u) \). Thus \( (u, v) \) is a solution of the following system
\[
\begin{align*}
(\Lambda u, \Lambda u) & \in \partial L F_2(\Phi)(u, v), \\
(\Lambda u, \Lambda v) & \in \partial L F_2(\Phi)(v, u).
\end{align*}
\]
Since $\Phi$ is a saddle function, we have $LF_2(\Phi)$ is convex in both variables by virtue of Proposition 2.3. However, subdifferential of convex functions are monotone and therefore

$$\langle \partial LF_2(\Phi)(v, u) - \partial LF_2(\Phi)(u, v), (v - u, u - v) \rangle_{(V^* \times V^*) \times (V \times V)} \geq 0.$$ 

By plugging $(\Lambda v, \Lambda u) \in \partial LF_2(\Phi)(u, v)$ and $(\Lambda u, \Lambda v) \in \partial LF_2(\Phi)(v, u)$ in the above inequality we have

$$0 \leq \langle (\Lambda u, \Lambda v) - (\Lambda v, \Lambda u), (v - u, u - v) \rangle_{(V^* \times V^*) \times (V \times V)}$$
$$= \langle (\Lambda u - \Lambda v, \Lambda v - \Lambda u), (v - u, u - v) \rangle_{(V^* \times V^*) \times (V \times V)}$$
$$= \langle \Lambda u - \Lambda v, v - u \rangle_{V \times V^*} + \langle \Lambda v - \Lambda u, u - v \rangle_{V \times V^*}$$
$$= -2 \langle \Lambda u - \Lambda v, u - v \rangle_{V \times V^*}.$$ 

On the other hand $\Lambda : Dom(\Lambda) \subset V \rightarrow V^*$ is a non-negative operator and therefore $(\Lambda u - \Lambda v, u - v)_{V \times V^*} \geq 0$ from which we have the latter is indeed zero, i.e.,

$$\langle \Lambda u - \Lambda v, u - v \rangle_{V \times V^*} = 0,$$

and therefore $\Lambda u = \Lambda v$ by virtue of Proposition 2.4.

Proof with condition (ii): Since the function $w \rightarrow \Phi(w, p)$ is Gâteaux differentiable and $\nabla_1 \Phi(w, p) = -p$, it follows from (8) that $-\Lambda v = -\Lambda u$.

Proof with condition (iii): Since the function $p \rightarrow \Phi(u, p)$ is Gâteaux differentiable and $\nabla_2 \Phi(u, p) = u$, it follows from (8) that $v = u$. 

Here is one useful corollary of Theorem 4.1 that provides a new variational principle for certain PDE’s.

**Corollary 4.2.** Let $\Lambda : Dom(\Lambda) \subset V \rightarrow V^*$ be a non-negative symmetric operator. If $\Lambda$ is onto and $\varphi : V \rightarrow \mathbb{R}$ is convex and lower-semi continuous, then every critical point of

$$I(u) = \varphi^*(\Lambda u) - \varphi(u)$$

is a solution of the equation

$$\Lambda u \in \partial \varphi(u).$$

**Proof.** Define the saddle function $\Phi : V \times V^* \rightarrow \mathbb{R}$ by $\Phi(u, p) = \varphi^*(p) - \varphi(u)$. It follows from part (1) of Proposition 3.1 that $\Phi$ is a non-convex self-dual Lagrangian on $V \times V^*$. By Theorem 4.1, if $u$ is a critical point of $I(u) = \varphi^*(\Lambda u) - \varphi(u)$ then there exists $v \in V$ with $\Lambda u = \Lambda v$ and

$$(-\Lambda v, v) \in \partial \Phi(u, \Lambda u).$$
Also note that \( \partial \Phi(u, p) = (-\partial \varphi(u), \partial \varphi^*(p)) \), from which we obtain
\[
(-\Lambda v, v) \in \left(-\partial \varphi(u), \partial \varphi^*(\Lambda u)\right).
\]
This implies \(-\Lambda v \in -\partial \varphi(u)\) for which together with the fact that \(\Lambda u = \Lambda v\) we have
\[
\Lambda u \in \partial \varphi(u).
\]

**Example 4** (System of transport equations). Let \(a : \Omega \to \mathbb{R}^N\) be a smooth function on a bounded domain \(\Omega\) of \(\mathbb{R}^N\). Consider the first-order operator \(Aw = a \cdot \nabla w = \sum_{i=1}^{N} a_i \frac{\partial w_i}{\partial x_i}\). Assume that the vector field \(\sum_{i=1}^{N} a_i \frac{\partial w_i}{\partial x_i}\) is actually the restriction of a smooth vector field \(\sum_{i=1}^{N} \bar{a}_i \frac{\partial w_i}{\partial x_i}\) defined on an open neighborhood of \(\bar{\Omega}\) and each \(\bar{a}_i\) is a \(C^1\) function on that neighborhood. Consider the system
\[
\begin{cases}
\epsilon a \cdot \nabla u = \Delta v + |v|^{p-2} v, & x \in \Omega, \\
-\epsilon a \cdot \nabla v = \Delta u + |u|^{q-2}, & x \in \Omega, \\
u = v = 0, & x \in \partial \Omega.
\end{cases}
\tag{9}
\]

We can use Corollary 4.2 to establish the following existence result.

**Theorem 4.3.** Assume \(\text{div}(a) = 0\) on \(\Omega\), \(2 < p, q < \frac{2N}{N-2}\) and \(|\frac{1}{p} - \frac{1}{q}| \leq \frac{1}{N}\). Then there exists \(\epsilon_0 > 0\) such that for \(0 < \epsilon < \epsilon_0\) the functional
\[
I(u, v) = \frac{1}{p} \int_{\Omega} |\epsilon a \cdot \nabla u - \Delta v|^p \, dx + \frac{1}{q'} \int_{\Omega} |\epsilon a \cdot \nabla v + \Delta u|^q \, dx - \frac{1}{p} \int_{\Omega} |v|^p \, dx - \frac{1}{q} \int_{\Omega} |u|^q \, dx
\]
has a critical point \((u, v) \in (W^{2,q'}(\Omega) \cap W_0^{1, p'}(\Omega)) \times (W^{2,q'}(\Omega) \cap W_0^{1, q'}(\Omega))\) that is indeed a solution of the system (9).

We shall prove this theorem in Section 5. Here is another application of Theorem 4.1

**Corollary 4.4.** Let \(\Lambda : \text{Dom}(\Lambda) \subset V \to V^*\) be a symmetric operator and \(\varphi : V \to \mathbb{R}\) be convex, lower-semi continuous. If \(\Lambda\) is onto and \(u\) is a critical point of
\[
I(u) = 2\varphi^*(\Lambda u) - \langle \Lambda u, w \rangle
\]
then there exists \(v \in V\) with \(\Lambda u = \Lambda v\) such that \(\frac{v+u}{2}\) is a solution of
\[
\Lambda w \in \partial \varphi(w).
\]

**Proof.** Define the function \(\Phi : V \times V^* \to \mathbb{R}\) by \(\Phi(u, p) = 2\varphi^*(p) - \langle u, p \rangle\), which is a Non-convex self-dual Lagrangian on \(V \times V^*\) by view of part (1) of Proposition 3.1. By Theorem 4.1, if \(u\) is a critical point of \(I(u) = 2\varphi^*(\Lambda u) - \langle \Lambda u, u \rangle\) then there exists \(v \in V\) with \(\Lambda u = \Lambda v\) and
\[
(-\Lambda v, v) \in \partial \Phi(u, \Lambda u).
\]
We have \( \partial \Phi(u, p) = (-p, 2\partial \varphi^*(p) - u) \), from which we obtain
\[
(-\Lambda v, v) \in (-\Lambda u, 2\partial \varphi^*(\Lambda u) - u).
\]

This implies \( v \in -2\partial \varphi^*(\Lambda u) - u \) and therefore \( \frac{v + u}{2} \in \partial \varphi^*(\Lambda u) \), from which we have \( \Lambda u = \Lambda v \). Taking into account that \( \Lambda u = \Lambda v \) we get
\[
\Lambda \left( \frac{v + u}{2} \right) \in \partial \varphi \left( \frac{v + u}{2} \right). \quad \square
\]

**Remark 4.5.** Note that the above corollary is indeed the well-known Clarke–Ekeland duality. In fact, Clarke and Ekeland introduced an interesting dual variational formulation for Hamiltonian systems associated with a convex Hamiltonian (see [9,8,11,12]). Such a duality principle has turned out to be extremely useful for various purposes such as existence of periodic solutions and solutions with minimum period. Their duality principle in abstract links the critical points of the functionals \( F(u) = \varphi(u) - \frac{1}{2} \langle \Lambda u, u \rangle \) and \( \tilde{F}(u) = \varphi^*(\Lambda u) - \frac{1}{2} \langle \Lambda u, u \rangle \) in such a way that if \( \tilde{u} \) is a critical point of \( \tilde{F} \), then there exists \( u_0 \in \text{Ker}(\Lambda) \) such that \( \tilde{u} + u_0 \) is a critical point of \( F \).

**Corollary 4.6.** Let \( \Lambda : \text{Dom}(\Lambda) \subset V \to V^* \) be a symmetric operator and \( \varphi : V \to \mathbb{R} \) be convex, lower-semi continuous. If \( u \) is a critical point of
\[
I(w) = \langle \Lambda w, w \rangle - 2\varphi(w)
\]
then \( u \) is a solution of
\[
\Lambda w \in \partial \varphi(w).
\]

**Proof.** Define the function \( \Phi : V \times V^* \to \mathbb{R} \) by \( \Phi(u, p) = \langle u, p \rangle - 2\varphi(u) \), which is a Non-convex self-dual Lagrangian on \( V \times V^* \) by view of part (1) of Proposition 3.1. By Theorem 4.1, if \( u \) is a critical point of \( I(u) = \langle \Lambda u, u \rangle - 2\varphi(u) \) then there exists \( v \in V \) with \( \Lambda u = \Lambda v \) and
\[
(-\Lambda v, v) \in \partial \Phi(u, \Lambda u).
\]

We have \( \partial \Phi(u, p) = (p - 2\partial \varphi(u), u) \), from which we obtain
\[
(-\Lambda v, v) \in (\Lambda u - 2\partial \varphi(u), u).
\]

This implies \( v = u \) and \( \Lambda u \in \partial \varphi(u) \). \( \square \)

This is nothing but the classical Euler–Lagrange functional associated to the inclusion \( \Lambda u \in \partial \varphi(u) \). As seen, this theory allows us to have various functionals associated to certain inclusions that gives us the flexibility to choose the most appropriate one to study the corresponding inclusion.
4.2. Nonlinear boundary conditions

Here is our main result when one considers certain boundary terms.

**Theorem 4.7.** Let $\Lambda : \text{Dom}(\Lambda) \subset V \to V^*$ be a symmetric operator modulo the boundary operator $B := (\beta_1, \beta_2) : V \to Y \times Y^*$ such that $(\Lambda, \beta_2) : \text{Dom}(\Lambda) \subset V \to V^* \times Y^*$ is onto. Let $\Phi : V \times V^* \to \mathbb{R}$ and $\ell : Y \times Y^* \to \mathbb{R}$ be saddle Non-convex self-dual Lagrangians that are Gâteaux differentiable with respect to their first variables. Suppose one of the following conditions holds:

(i) The operator $\Lambda$ modulo the boundary operator $B := (\beta_1, \beta_2)$ is non-negative.
(ii) For each $(u, p) \in V \times V^*$ and each $(l, e) \in Y \times Y^*$ we have $\nabla_1 \Phi(u, p) = -p$ and $\nabla_1 \ell(l, e) = -e$.
(iii) For each $u \in V$ the function $p \to \Phi(u, p)$ is Gâteaux differentiable and $\nabla_2 \Phi(u, p) = u$.

Suppose $u$ is a critical point of $I(u) = \Psi(u, 0)$ where

$$\Psi(u, (p, e)) = \Phi(u, \Lambda u + p) + \ell(\beta_1 u, \beta_2 u + e).$$

Then there exists $v \in V$ with $\Lambda u = \Lambda v$ and $\beta_2 u = \beta_2 v$ such that $(u, v)$ is a solution of the system

$$\begin{cases}
-\Lambda v, v \\
-\beta_2 v, \beta_1 v
\end{cases} \in \partial \Phi(u, \Lambda u),$$

$$\begin{cases}
-\Lambda x, \eta \\
-\beta_2 x, \beta_1 \eta
\end{cases} \in \partial \ell(\beta_1 u, \beta_2 u).$$

**Proof.** Since $u$ is a critical point of $I$, there exist $v \in \partial_2 \Phi(u, \Lambda u)$ and $w \in \partial_2 \ell(\beta_1 u, \beta_2 u)$ such that

$$\langle \nabla_1 \Phi(u, \Lambda u), \eta \rangle_{V \times V^*} + \langle v, \Lambda \eta \rangle_{V \times V^*} + \langle \nabla_1 \ell(\beta_1 u, \beta_2 u), \beta_1 \eta \rangle_{Y \times Y^*} + \langle w, \beta_2 \eta \rangle_{Y \times Y^*} = 0,$$

for all $\eta \in \text{Dom}(\Lambda)$. Since $(\Lambda, \beta_2) : \text{Dom}(\Lambda) \subset V \to V^* \times Y^*$ is onto, there exists $x \in \text{Dom}(\Lambda)$ such that

$$\begin{cases}
-\Lambda x = \nabla_1 \Phi(u, \Lambda u), \\
-\beta_2 x = \nabla_1 \ell(\beta_1 u, \beta_2 u).
\end{cases}$$

This together with (10) imply that

$$\langle v, \Lambda \eta \rangle_{V \times V^*} - \langle \Lambda x, \eta \rangle_{V \times V^*} + \langle w, \beta_2 \eta \rangle_{Y \times Y^*} - \langle \beta_2 x, \beta_1 \eta \rangle_{Y \times Y^*} = 0,$$

for all $\eta \in \text{Dom}(\Lambda)$.

Now we show that $x = v$ and $w = \beta_1(v)$. Indeed, it follows from (12) and part (3) of Definition 2.5 that

$$\langle v, \Lambda \eta \rangle_{V \times V^*} - \langle \Lambda x, \eta \rangle_{V \times V^*} + \langle w, \beta_2 \eta \rangle_{Y \times Y^*} - \langle \beta_2 x, \beta_1 \eta \rangle_{Y \times Y^*} = 0,$$

for all $\eta \in \text{Dom}(\Lambda)$,
by giving
\[ \langle v - x, \Lambda \eta \rangle_{V \times V^*} + \langle \beta_2 \eta, \beta_1 x - w \rangle_{Y \times Y^*} = 0, \quad \text{for all } \eta \in \text{Dom}(\Lambda). \]

This together with the fact that \((\Lambda, \beta_2) : \text{Dom}(\Lambda) \subset V \to V^* \times Y^*\) is onto imply that \(x = v\) and \(w = \beta_1(x) = \beta_1(v)\). It then follows from (11), \(v \in \partial_2 \Phi(u, \Lambda u)\) and \(w \in \partial_2 \ell(\beta_1 u, \beta_2 u)\) that
\[
\begin{aligned}
\langle -\Lambda v, v \rangle &\in \partial \Phi(u, \Lambda u), \\
\langle -\beta_2 v, \beta_1 v \rangle &\in \partial \ell(\beta_1 u, \beta_2 u).
\end{aligned}
\tag{13}
\]

Now we show that if either of conditions (i), (ii) or (iii) is satisfied then \(\Lambda v = \Lambda u\) and \(\beta_2 v = \beta_2 u\).

**Proof with condition (i):** In this step we first show that the following inequality holds:
\[ \langle \beta_1 u, \beta_2 u \rangle_{Y \times Y^*} + \langle \beta_1 v, \beta_2 v \rangle_{Y \times Y^*} \leq \langle \beta_1 u, \beta_2 v \rangle_{Y \times Y^*} + \langle \beta_1 v, \beta_2 u \rangle_{Y \times Y^*}. \]

Indeed, it follows from (13) that \((-\beta_2 v, \beta_1 v) \in \partial \ell(\beta_1 u, \beta_2 u)\) from which together with part (2) of Proposition 2.3 we obtain \((\beta_2 v, \beta_2 u) \in \partial LF_2(\ell)(\beta_1 u, \beta_1 v)\). It then follows from Theorem 3.4 that \((\beta_2 u, \beta_2 v) \in \partial LF_2(\ell)(\beta_1 v, \beta_1 u)\).

Since \(\ell\) is a saddle function, by Proposition 2.3 we have that \(LF_2(\ell)\) is convex in both variables. It is also standard that the subdifferential of convex functions are monotone. It follows that
\[ \langle \partial LF_2(\ell)(\beta_1 u, \beta_1 v) - \partial LF_2(\ell)(\beta_1 v, \beta_1 u), (\beta_1 u - \beta_1 v, \beta_1 v - \beta_1 u) \rangle_{(Y^* \times Y^*) \times (Y \times Y)} \geq 0. \]

By plugging \((\beta_2 v, \beta_2 u) \in \partial LF_2(\ell)(\beta_1 u, \beta_1 v)\) and \((\beta_2 u, \beta_2 v) \in \partial LF_2(\ell)(\beta_1 v, \beta_1 u)\) in the above inequality we have
\[
0 \leq \langle (\beta_2 v, \beta_2 u) - (\beta_2 u, \beta_2 v), (\beta_1 u - \beta_1 v, \beta_1 v - \beta_1 u) \rangle_{(Y^* \times Y^*) \times (Y \times Y)}
= \langle (\beta_2 v - \beta_2 u, \beta_2 u - \beta_2 v), (\beta_1 u - \beta_1 v, \beta_1 v - \beta_1 u) \rangle_{(Y^* \times Y^*) \times (Y \times Y)}
= 2\langle \beta_2 v, \beta_1 u \rangle_{Y^* \times Y} + 2\langle \beta_2 u, \beta_1 v \rangle_{Y^* \times Y} - 2\langle \beta_2 v, \beta_1 v \rangle_{Y^* \times Y} - 2\langle \beta_2 u, \beta_1 u \rangle_{Y^* \times Y},
\]
from which we obtain
\[
\langle \beta_2 v, \beta_1 v \rangle_{Y^* \times Y} + \langle \beta_2 u, \beta_1 u \rangle_{Y^* \times Y} \leq \langle \beta_2 v, \beta_1 u \rangle_{Y^* \times Y} + \langle \beta_2 u, \beta_1 v \rangle_{Y^* \times Y}. \tag{14}
\]

This proves the desired claim. By the same argument, one can deduce from \((-\Lambda v, v) \in \partial \Phi(u, \Lambda u)\) that
\[
(\Lambda v, v)_{V \times V^*} + (\Lambda u, u)_{V \times V^*} \leq (\Lambda v, u)_{V \times V^*} + (\Lambda v, u)_{V \times V^*}. \tag{15}
\]

Taking the sum of inequalities (14) and (15), we have
\[
(\Lambda u, u)_{V \times V^*} + (\beta_1 u, \beta_2 u)_{Y \times Y^*} + (\Lambda v, v)_{V \times V^*} + (\beta_1 w, \beta_2 v)_{Y \times Y^*}
\leq (\Lambda u, v)_{V \times V^*} + (\beta_1 u, \beta_2 v)_{Y \times Y^*} + (\Lambda v, u)_{V \times V^*} + (\beta_1 u, \beta_2 v)_{Y \times Y^*}. \tag{16}
\]
This inequality is equivalent to
\[ \langle \Lambda v - \Lambda u, v - u \rangle + \beta_1 (v - u), \beta_2 (v - u) \rangle_{Y \times Y^*} \leq 0. \quad (17) \]

On the other hand the operator \( \Lambda \) is non-negative modulo the boundary operator \( B = (\beta_1, \beta_2) \), from which together with (17) we have that the latter is indeed zero and we have \( \Lambda v = \Lambda u \) and \( \beta_2 v = \beta_2 u \) due to Proposition 2.4.

**Proof with condition (ii):** Since \( \Phi(., p) \) and \( \ell(., e) \) are Gâteaux differentiable and \( \nabla_1 \Phi(u, p) = -p \) and \( \nabla_1 \ell(l, e) = -e \), it follows from (13) that \( -\Lambda v = -\Lambda u \) and \( \beta_2 v = \beta_2 u \).

**Proof with condition (iii):** Since \( \Phi(u, .) \) is Gâteaux differentiable and \( \nabla_2 \Phi(u, p) = u \), it follows from (13) that \( v = u \).

**Corollary 4.8.** Let \( \Lambda : \text{Dom}(\Lambda) \subset V \to V^* \) be a non-negative symmetric operator modulo the boundary operator \( B := (\beta_1, \beta_2) : V \to Y \times Y^* \) in such a way that \( \langle \Lambda, \beta_2 \rangle : \text{Dom}(\Lambda) \to V^* \times Y^* \) is onto. Let \( \varphi : V \to \mathbb{R} \) and \( \psi : Y \to \mathbb{R} \) be convex, lower-semi continuous and also Gâteaux differentiable. Then every critical point of
\[ I(u) = \varphi^*(\Lambda u) - \varphi(u) + \psi^*(\beta_2 u) - \psi(\beta_1 u) \]
is a solution of the equation
\[ \begin{cases} 
\Lambda u = \nabla \varphi(u), \\
\beta_2 u = \nabla \psi(\beta_1 u). 
\end{cases} \quad (18) \]

**Proof.** Define the saddle *Non-convex self-dual* Lagrangians \( \Phi : V \times V^* \to \mathbb{R} \cup \{+\infty\} \) and \( \ell : Y \times Y^* \to \mathbb{R} \cup \{+\infty\} \) by \( \Phi(u, p) = \varphi^*(p) - \varphi(u) \) and \( \ell(l, e) = \psi^*(e) - \psi(l) \) respectively. By Theorem 4.7, if \( u \) is a critical point of \( I(u) = \varphi^*(\Lambda u) - \varphi(u) + \psi^*(\beta_2 u) - \psi(\beta_1 u) \), there exists \( v \in V \) with \( \Lambda u = \Lambda v \) and \( \beta_2 u = \beta_2 v \) and the pair \((u, v)\) is a solution of the system
\[ \begin{cases} 
(-\Lambda v, v) \in \partial \Phi(u, \Lambda u), \\
(-\beta_2 v, \beta_1 v) \in \partial \ell(\beta_1 u, \beta_2 u). 
\end{cases} \]

It follows that
\[ \begin{cases} 
(-\Lambda v, v) \in \left(-\nabla \varphi(u), \partial \varphi^*(\Lambda u)\right), \\
(-\beta_2 v, \beta_1 v) \in \left(-\nabla \psi(\beta_1 u), \partial \psi^*(\beta_2 u)\right). 
\end{cases} \]

This implies \(-\Lambda v = -\nabla \varphi(u)\) and \(-\beta_2 v = -\nabla \psi(\beta_1 u)\) from which together with the fact that \( \Lambda u = \Lambda v \) and \( \beta_2 u = \beta_2 v \) we have
\[ \begin{cases} 
\Lambda u = \nabla \varphi(u), \\
\beta_2 u = \nabla \psi(\beta_1 u). 
\end{cases} \]

**Example 5 (A semi-linear bi-Laplace equation).** Let \( N > 4 \), \( 1 < p < \frac{2N}{N-4} \) and \( \Omega \) be a smooth domain in \( \mathbb{R}^N \) and \( \partial \Omega \) its boundary. Consider the fourth-order equation with nonlinear boundary
conditions
\[
\begin{cases}
\Delta^2 u + u = |u|^{p-2} u, & x \in \Omega, \\
\frac{\partial u}{\partial n} = \nabla \psi_1(u), & x \in \partial \Omega, \\
-\Delta u = \nabla \psi_2 \left( \frac{\partial u}{\partial n} \right), & x \in \partial \Omega.
\end{cases}
\]  

(19)

We have the following result.

**Theorem 4.9.** Suppose \( \psi_1 : H^\frac{3}{2}(\partial \Omega) \rightarrow \mathbb{R} \) and \( \psi_2 : H^\frac{1}{2}(\partial \Omega) \rightarrow \mathbb{R} \) are continuously differentiable and convex. Then every critical point of the functional

\[
I(u) = \int_\Omega \left[ \frac{1}{p'} \Delta^2 u + u |^{p'} - \frac{1}{p} |u|^p \right] dx + \psi_1^*(\frac{\partial u}{\partial n}) + \psi_2^*(-\Delta u) - \psi_1(u)
\]

is a solution of (19).

**Proof.** It follows from Example 3 that the operator \( \Lambda u = \Delta^2 u + u \) is a symmetric and non-negative operator modulo the boundary operators \( \beta_1 : \text{Dom}(\Lambda) \rightarrow H^\frac{1}{2}(\partial \Omega) \times H^\frac{3}{2}(\partial \Omega) \) defined by \( \beta_1 u = (\frac{\partial u}{\partial n}, u)_{\partial \Omega} \) and \( \beta_2 : \text{Dom}(\Lambda) \rightarrow H^{-\frac{1}{2}}(\partial \Omega) \times H^{-\frac{3}{2}}(\partial \Omega) \) defined by \( \beta_2 u = (-\Delta u, \frac{\partial u}{\partial n})_{\partial \Omega} \). It also follows from Example 3 that the functional \( \Phi : W^{2,p'}(\Omega) \times (L^p(\Omega) \times (H^{-\frac{1}{2}}(\partial \Omega) \times H^{-\frac{3}{2}}(\partial \Omega))) \) defined by

\[
\Phi(u, (p, e_1, e_2)) = \varphi^*(\Delta^2 u + u + p) - \varphi(u) + \psi^*_2(-\Delta u + e_1)
\]

\[
- \psi_2^* \left( \frac{\partial u}{\partial n} \right) + \psi^*_1 \left( \frac{\partial u}{\partial n} + e_2 \right) - \psi_1(u)
\]

is an Nc-SD Lagrangian. Therefore, taking into account that \( I(u) = \Phi(u, 0) \), the result follows from Corollary 4.8. \( \square \)

**Corollary 4.10.** Let \( \Lambda : \text{Dom}(\Lambda) \subset V \rightarrow V^* \) be a symmetric operator modulo the boundary operator \( B := (\beta_1, \beta_2) : V \rightarrow Y \times Y^* \) in such a way that \( \Lambda, \beta_2 : \text{Dom}(\Lambda) \rightarrow Y^* \times Y^* \) is onto. Let \( \varphi : V \rightarrow \mathbb{R} \) and \( \psi : Y \rightarrow \mathbb{R} \) be convex, lower-semi continuous and also Gâteaux differentiable. If \( u \) is a critical point of

\[
I(w) = 2\varphi^*(\Lambda w) - \langle \Lambda w, w \rangle + 2\psi^*(\beta_2 w) - \langle \beta_2 w, \beta_1 w \rangle
\]

then there exists \( v \in V \) such that \( \frac{v+u}{2} \) is a solution of (18).

**Proof.** Define the Non-convex self-dual Lagrangians \( \Phi : V \times V^* \rightarrow \mathbb{R} \) and \( \ell : Y \times Y^* \rightarrow \mathbb{R} \) by \( \Phi(u, p) = 2\varphi^*(p) - \langle u, p \rangle \) and \( \ell(l, e) = 2\psi^*(e) - \langle e, l \rangle \) respectively. It follows from Theorem 4.7 that if \( u \) is a critical point of \( I \), there exists \( v \in V \) with \( \Lambda u = \Lambda v \) and \( \beta_2 u = \beta_2 v \).
satisfying

\[
\begin{align*}
(\lambda v, v) &\in (\lambda u, 2\partial \varphi^*(\lambda u) - u), \\
(-\beta_2 v, \beta_1 v) &\in (-\beta_2 u, 2\partial \psi^*(\beta_2 u) - \beta_1 u).
\end{align*}
\]

This implies \( \frac{v + u}{2} \in \partial \varphi^*(\lambda u) \) and \( \beta_1 \left( \frac{v + u}{2} \right) \in \partial \psi^*(\beta_2 u) \) from which we have \( \lambda u = \lambda v \) and \( \beta_2 u = \beta_2 v \) we get

\[
\begin{align*}
\lambda \left( \frac{v + u}{2} \right) &= \nabla \varphi \left( \frac{v + u}{2} \right), \\
\beta_2 \left( \frac{v + u}{2} \right) &= \nabla \psi \left( \beta_1 \left( \frac{v + u}{2} \right) \right).
\end{align*}
\]

\[
\square
\]

**Remark 4.11.** The above corollary can be seen as a generalization of Clarke–Ekeland duality when the operator \( \Lambda \) is not purely symmetric and one deals with boundary terms as well.

**Example 6 (Finite dimensional Hamiltonian systems with nonlinear boundary conditions).** Let \( T > 0 \) and \( J : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N \times \mathbb{R}^N \) be the symplectic operator defined by \( J(x, y) = (-y, x) \) and consider the following finite dimensional Hamiltonian systems in \( \mathbb{R}^{2N} \),

\[
\begin{align*}
J \dot{u}(t) &= \nabla \varphi(t, u(t)), \\
\frac{1}{2} [u(T) + u(0)] &= \nabla \psi(Ju(T) - Ju(0)).
\end{align*}
\]

Hamiltonian systems with this type of boundary conditions are also treated in [4,16].

Here is an application of Corollary 4.10.

**Theorem 4.12.** Let \( \varphi : [0, T] \times (\mathbb{R}^N \times \mathbb{R}^N) \to \mathbb{R} \) be differentiable, convex and lower semi-continuous in the second variable. Also let \( \psi : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R} \) be differentiable and convex. Then every critical point of the functional

\[
I(u) = \int_0^T \varphi^*(t, J\dot{u}(t)) dt - \frac{1}{2} [J\dot{u}(t), u(t)]
\]

\[
+ \psi^* \left( \frac{u(T) + u(0)}{2} \right) - \frac{1}{2} \left( \frac{u(T) + u(0)}{2} , J(u(T) - u(0)) \right)
\]

is a solution of (20).

**Example 7 (A Hamiltonian system of PDE’s with nonlinear Neumann boundary conditions).** Let \( N > 2 \) and \( \Omega \) be a smooth domain in \( \mathbb{R}^N \) and \( \partial \Omega \) its boundary. Consider the following infinite
dimensional Hamiltonian system,
\[
\begin{align*}
-\Delta u + u &= |v|^{q-2}v, & x \in \Omega, \\
-\Delta v + v &= |u|^{p-2}u, & x \in \Omega, \\
\frac{\partial u}{\partial n} &= |v|^{q-2}v, & x \in \partial\Omega, \\
\frac{\partial v}{\partial n} &= |u|^{p-2}u, & x \in \partial\Omega.
\end{align*}
\]

(21)

We have the following result.

**Theorem 4.13.** Assume \( p, q > 2 \) and \( \min\{\frac{1}{p} + \frac{N-1}{Nq}, \frac{1}{q} + \frac{N-1}{Np}\} > \frac{N-1}{N} \). Let \( p' = \frac{p}{p-1} \) and \( q' = \frac{q}{q-1} \). Then the functional
\[
I(u, v) = \frac{1}{p'} \int_{\Omega} |-\Delta v + v|^{p'} \, dx + \frac{1}{q} \int_{\Omega} |-\Delta u + u|^{q} \, dx + \frac{1}{p'} \int_{\partial\Omega} \left| \frac{\partial v}{\partial n} \right|^{p'} \, dx
\]
\[
+ \frac{1}{q'} \int_{\partial\Omega} \left| \frac{\partial u}{\partial n} \right|^{q'} \, dx - 2 \int_{\Omega} \nabla u \cdot \nabla v \, dx - 2 \int_{\Omega} uv \, dx
\]

has a critical point in \( W^{2, q'}(\Omega) \times W^{2, p'}(\Omega) \) that is indeed a solution of the system (21).

We shall prove this theorem in Section 5.

As in Corollaries 4.8 and 4.10, by considering different combination of interior \( Nc-SD \) Lagrangians and boundary \( Nc-SD \) Lagrangians one can obtain different variational principles of Eq. (18). Here we state one more application of Theorem 4.7 and leave it to interested readers to generate more new principles by making use of Theorem 4.7.

**Corollary 4.14.** Let \( \Lambda : Dom(\Lambda) \subset V \to V^* \) be a symmetric operator modulo the boundary operator \( B := (\beta_1, \beta_2) : V \to Y \times Y^* \) in such a way that \( (\Lambda, \beta_2) : Dom(\Lambda) \to V^* \times Y^* \) is onto. Let \( \varphi : V \to \mathbb{R} \) and \( \psi : Y \to \mathbb{R} \) be convex lower semi-continuous and Gâteaux differentiable. If \( u \) is a critical point of
\[
I(w) = \langle \Lambda w, w \rangle - 2\varphi(w) + \psi^*(\beta_2 w) - \psi(\beta_1 w)
\]
then \( u \) is a solution of (18).

**5. Critical points of lower semi-continuous functionals**

In this section we shall provide a minimax principle for lower semi-continuous functionals applicable for the proof of existence theorems stated in previous sections. We first recall the following minimax principle for lower semi-continuous functionals due to Szulkin [27].

Let \( X \) be a real Banach space and \( I : X \to \mathbb{R} \cup \{+\infty\} \) a functional on \( X \) such that \( I = \Phi + \psi \) with \( \Phi \in C^1(X, \mathbb{R}) \) and \( \psi : X \to \mathbb{R} \cup \{+\infty\} \) convex and lower semi-continuous. A point \( u \in X \) is said to be a critical point of \( I \) if \( u \in Dom(\psi) \) and if it satisfies the inequality
\[ \langle \nabla \Phi(u), v - u \rangle + \psi(v) - \psi(u) \geq 0 \quad \text{for all } v \in X. \]

We shall say that \( I \) satisfies the compactness condition of Palais–Smale type provided:

\((PS)\) If \( u_n \) is a sequence such that \( I(u_n) \to c \in \mathbb{R} \) and

\[ \langle \nabla \Phi(u_n), v - u_n \rangle + \psi(v) - \psi(u_n) \geq -\epsilon_n \| v - u_n \| \quad \text{for all } v \in X, \]

where \( \epsilon_n \to 0 \) then \( u_n \) possesses a subsequent that converges strongly.

The following is established by Szulkin [27].

**Theorem 5.1.** Suppose that \( I : X \to \mathbb{R} \cup \{+\infty\} \) is as above satisfying \((PS)\) and the mountain pass geometry (in short MPG), i.e.,

(i) \( I(0) = 0 \) and there exist \( \alpha, \rho > 0 \) such that \( I(u) \geq \alpha \) when \( \| u \| = \rho \),

(ii) \( I(e) \leq 0 \) for some \( e \in X \) with \( \| e \| > \rho \).

Then \( I \) has a critical value \( c \geq \alpha \) which may be characterized by

\[ c = \inf_{\gamma \in \Gamma} \sup_{t \in [0,1]} I(\gamma(t)), \]

where \( \Gamma = \{ \gamma \in C([0,1], X) : \gamma(0) = 0, \gamma(1) = e \} \).

By some minor changes in the proof of the above theorem one can replace \((PS)\) condition by the following condition

\((PS^*)\) If \( u_n \) is a sequence such that \( I(u_n) \to c \in \mathbb{R} \) and

\[ \langle \nabla \Phi(u_n), v - u_n \rangle + \psi(v) - \psi(u_n) \geq -\epsilon_n \| v - u_n \| \quad \text{for all } v \in X, \]

where \( \epsilon_n \to 0 \) then there exists \( u \in X \) such that, up to a subsequence,

(i) \( u_n \) converges weakly to \( u \in X \),

(ii) \( \Phi(u_n) \to \Phi(u) \) and \( \langle \nabla \Phi(u_n), v - u_n \rangle \to \langle \nabla \Phi(u), v - u \rangle \) for all \( v \in X \),

(iii) \( I(u_n) \to I(u) \).

We shall use the above theorem with \((PS^*)\) to deal with the existence theorems stated in previous sections. To be more precise, let \( X \) and \( X^* \) be two real Banach spaces in separating duality and \( \Lambda : \text{Dom}(\Lambda) \subset X \to X^* \) be a closed linear operator. As in Section 3 set \( X_\Lambda = \{ u \in X ; \Lambda u \in X^* \} \) that is Banach space when equipped with the norm:

\[ \| u \|_{X_\Lambda} = \| u \|_V + \| \Lambda u \|_{V^*}. \]

We shall assume \( \| u \|_X \leq c \| \Lambda u \|_{X^*} \) for some constant \( c \) and all \( u \in X \). It then follows that \( \| u \| = \| \Lambda u \|_{X^*} \) is an equivalent norm for \( X_\Lambda \). We shall also assume that the embedding \( X_\Lambda \hookrightarrow X \) is compact. We establish the following result as a consequence of Theorem 5.1.
Theorem 5.2. Suppose the operator \( \Lambda : \text{Dom}(\Lambda) \subset X \to X^* \) is as above and \( I : X_\Lambda \to \mathbb{R} \) is a functional of the form \( I(u) = F(\Lambda u) - G(u) \) where:

1. \( F : X^* \to \mathbb{R} \) is convex lower semi-continuous and differentiable.
2. \( F \) is coercive, i.e.,
   \[
   \frac{F(p)}{\|p\|_{X^*}} \to +\infty \quad \text{as} \quad \|p\|_{X^*} \to +\infty,
   \]
   and we also have \( \langle p, \nabla F(p) \rangle \leq \beta F(p) \) for some \( 1 < \beta < 2 \) and all \( p \in X^* \).
3. \( G \in C^1(X_\Lambda, \mathbb{R}) \) and \( \langle u, \nabla G(u) \rangle \geq 2G(u) \) for all \( u \in X_\Lambda \).
4. The functional \( G : X_\Lambda \to \mathbb{R} \) is weakly continuous and the map \( \nabla G : X_\Lambda \to X^* \) is weak to weak continuous.

If \( I \) satisfies the mountain pass geometry then \( I \) has a critical value \( c \geq \alpha \) which may be characterized by

\[
c = \inf_{\gamma \in \Gamma} \sup_{t \in [0, 1]} I(\gamma(t)),
\]
where \( \Gamma = \{ \gamma \in C([0, 1], X) : \gamma(0) = 0, \gamma(1) = e \} \).

Proof. By virtue of Theorem 5.1 we just need to show that the functional \( I \) satisfies \((PS^*)\). To do this, suppose \( u_n \in X_\Lambda \) is a sequence such that \( I(u_n) \to c \in \mathbb{R} \) and

\[
\langle -\nabla G(u_n), v - u_n \rangle + F(\Lambda v) - F(\Lambda u_n) \geq -\epsilon_n \| v - u_n \| \quad \text{for all} \quad v \in X_\Lambda,
\]

(22)

where \( \epsilon_n \to 0 \). Note that since \( I \) is differentiable the above inequality simply means \( \nabla I(u_n) \to 0 \).

We first show that \( u_n \) is bounded in \( X_\Lambda \). It follows from conditions (2) and (3) that

\[
c + o(1) = I(u_n) - \frac{1}{2}\langle I'(u_n(t)), u_n(t) \rangle
\]

\[
= F(\Lambda u_n) - G(u_n) - \frac{1}{2}\langle \nabla F(\Lambda u_n), \Lambda u_n \rangle + \frac{1}{2}\langle \nabla G(u_n), u_n \rangle
\]

\[
\geq F(\Lambda u_n) - \frac{\beta}{2}\langle \nabla F(\Lambda u_n), \Lambda u_n \rangle + \frac{1}{2}(\langle \nabla G(u_n), u_n \rangle - 2G(u_n))
\]

\[
\geq \left( 2 - \frac{\beta}{2} \right) F(\Lambda u_n).
\]

Since \( 1 < \beta < 2 \), it follows from the coercivity of \( F \) that \( u_n \) is bounded in \( X_\Lambda \). Thus, up to a subsequence, there exists \( u \in X_\Lambda \) such that \( u_n \to u \) weakly in \( X_\Lambda \). Due to the compact embedding \( X_\Lambda \hookrightarrow X \), we have that \( u_n \to u \) strongly in \( X \). Fix \( v \in X_\Lambda \), it follows from condition (4) together with and convergence of \( u_n \) to \( u \) in a weak sense in \( X_\Lambda \) and in a strong sense in \( X \) that

\[
\langle \nabla G(u_n), v - u_n \rangle \to \langle \nabla G(u), v - u \rangle,
\]

for all \( v \in X_\Lambda \). This together with (22) and lower semi-continuity of \( F \) implies that
\[
\langle -\nabla G(u), v - u \rangle + F(\Lambda v) = \liminf_{n \to \infty} \langle -\nabla G(u_n), v - u_n \rangle + F(\Lambda v)
\geq \liminf_{n \to \infty} (F(\Lambda u_n) - \epsilon_n \|v - u_n\|_{X_A})
\geq F(\Lambda u).
\]

Therefore,
\[
\langle -\nabla G(u), v - u \rangle + F(\Lambda v) \geq F(\Lambda u),
\]
and since \(v \in X_A\) is arbitrary, it follows that \(u\) is a critical point of \(I\). Now we show that \(I(u_n) \to I(u)\). Since \(G : X_A \to \mathbb{R}\) is weakly continuous we just need to show that \(\lim_{n \to \infty} F(\Lambda u_n) = F(\Lambda u)\). Note first that since \(F\) is lower semi-continuous we have
\[
F(\Lambda u) \leq \liminf_{n \to \infty} F(\Lambda u_n).
\]
On the other hand by taking \(v = u\) in (22) we have
\[
\langle -\nabla G(u_n), u - u_n \rangle + F(\Lambda u) \geq F(\Lambda u_n) - \epsilon_n \|u - u_n\|.
\] (23)

By taking \(\limsup\) from both sides we get
\[
F(\Lambda u) = \limsup_{n \to \infty} \langle -\nabla G(u_n), u - u_n \rangle + F(\Lambda u)
\geq \limsup_{n \to \infty} (F(\Lambda u_n) - \epsilon_n \|v - u_n\|_{X_A})
= \limsup_{n \to \infty} F(\Lambda u_n).
\]
Thus \(\lim_{n \to \infty} F(\Lambda u_n) = F(\Lambda u)\) and therefore \(\lim_{n \to \infty} I(u_n) = I(u)\). \(\square\)

5.1. System of transport equations

We now proceed with the proof of Theorem 4.3. We shall make frequent use of the following theorem while proving our existence results (see [1] for the proof).

**Theorem 5.3.** Let \(\Omega\) be a smooth domain in \(\mathbb{R}^N\), \(\Omega_0\) a bounded subdomain of \(\Omega\), and \(\Omega_k^0\) the intersection of \(\Omega_0\) with a \(k\)-dimensional plane in \(\mathbb{R}^N\). Let \(j, m\) be integers, \(j \geq 0, m \geq 1\) and let \(1 \leq r < +\infty\). Then the following embeddings are continuous.

\[
W_j^{j+m,r}(\Omega) \hookrightarrow W_j^{j,s}(\Omega_k^0) \quad \text{if} \quad 0 < N - mr < k \leq N \quad \text{and} \quad 1 \leq s \leq \frac{kr}{N - mr},
\]

\[
W_j^{j+m,r}(\Omega) \hookrightarrow W_j^{j,s}(\Omega) \quad \text{if} \quad mr = N, \quad 1 \leq k \leq N \quad \text{and} \quad 1 \leq s < \infty.
\]

We need the following lemma to prove Theorem 4.3.

**Lemma 5.4.** Let \(\Omega\) be a smooth bounded domain of \(\mathbb{R}^N\). If \(p, q > 2\) and \(\left| \frac{1}{p} - \frac{1}{q} \right| \leq \frac{1}{N}\) then the following embeddings are continuous.
(1) \(W^{2,q}(\Omega) \hookrightarrow W^{1,p'}(\Omega)\).
(2) \(W^{2,p'}(\Omega) \hookrightarrow W^{1,q}(\Omega)\).

**Proof.** By Theorem 5.3 the embedding \(W^{2,q}(\Omega) \hookrightarrow W^{1,p'}(\Omega)\) is continuous provided \(p' \leq \frac{Nq'}{N-q}\) that is equivalent to \(\frac{1}{p} - \frac{1}{q} \leq \frac{1}{N}\). Similarly \(W^{2,p'}(\Omega) \hookrightarrow W^{1,q}(\Omega)\) is continuous provided \(\frac{1}{q} - \frac{1}{p} \leq \frac{1}{N}\). \(\square\)

**Proof of Theorem 4.3.** Set \(V = L^q(\Omega) \times L^p(\Omega)\) and \(V^* = L^{q'}(\Omega) \times L^{p'}(\Omega)\). Define \(G : V \rightarrow \mathbb{R}\) by \(G(u,v) = \frac{1}{q} \int_{\Omega} |u|^q \, dx + \frac{1}{p} \int_{\Omega} |v|^p \, dx\). Let \(\Lambda : Dom(\Lambda) \subset V \rightarrow V^*\) be the operator \(\Lambda(u,v) = (-\epsilon a. \nabla v - \Delta u, \epsilon a. \nabla u - \Delta v)\) with

\[
\text{Dom}(\Lambda) = \{(u,v) \in V; \; (u,v) \in V^* \text{ and } u = v = 0, \; x \in \partial \Omega\}.
\]

Note that \(\Lambda\) is a symmetric operator and for each \((u,v) \in \text{Dom}(\Lambda)\) we have

\[
\langle \Lambda(u,v), (u,v) \rangle = \int_{\Omega} |\nabla v|^2 \, dx + \int_{\Omega} |\nabla u|^2 \, dx + 2\epsilon \int_{\Omega} (a. \nabla u)v \, dx
\]

\[
\geq \int_{\Omega} |\nabla v|^2 \, dx + \int_{\Omega} |\nabla u|^2 \, dx - \frac{2\epsilon \|a\|_\infty}{\lambda_1} \|\nabla u\|_{L^2(\Omega;\mathbb{R}^N)} \|\nabla v\|_{L^2(\Omega;\mathbb{R}^N)},
\]

where \(\lambda_1\) is the first eigenvalue of \(-\Delta\) with Dirichlet boundary condition. The above estimate indeed shows that \(\Lambda\) is non-negative provided \(\epsilon \leq \frac{\lambda_1}{\|a\|_\infty}\). Since \(2 < p, q < \frac{2N}{N-2}\) and \(\left|\frac{1}{p} - \frac{1}{q}\right| \leq \frac{1}{N}\), it follows from Lemma 5.4 and Sobolev embeddings

\[
W^{2,q'}(\Omega) \hookrightarrow L^q(\Omega), \quad \text{for } 2 \leq q < \frac{2N}{N-2},
\]

\[
W^{2,p'}(\Omega) \hookrightarrow L^p(\Omega), \quad \text{for } 2 \leq p < \frac{2N}{N-2},
\]

that \(V_\Lambda = (W^{2,q'}(\Omega) \cap W_0^{1,p'}(\Omega)) \times (W^{2,p'}(\Omega) \cap W_0^{1,q'}(\Omega))\). Note also that

\[
\| (u,v) \| = \|\Delta u\|_{L^{p'}(\Omega)} + \|\Delta v\|_{L^{q'}(\Omega)}
\]

is an equivalent norm for \(V_\Lambda\) (Lemma 9.17 of [19]). The functional \(I\) can be rewritten as

\[
I(u,v) = G^*(\Lambda(u,v)) - G(u,v),
\]

and by Corollary 4.2 each critical point of \(I\) is a solution of \(\Lambda(u,v) = \nabla G(u,v)\) that is indeed the system (9). We shall make use of Theorem 5.2 in Section 6 to prove this functional has at least one non-trivial critical point. Set \(F(u,v) = G^*(u,v) = \frac{1}{p} \int_{\Omega} |v|^p \, dx + \frac{1}{q} \int_{\Omega} |u|^q \, dx\), and note that the functionals \(F\) and \(G\) and the operator \(\Lambda\) satisfy all assumptions in Theorem 5.2. Therefore, we just need to prove that the functional \(I : V_\Lambda \rightarrow \mathbb{R}\) satisfies the mountain pass geometry. Let us first recall the elementary inequality
\[ |a + b|^s \leq 2^{s-1}(|a|^s + |b|^s) \] for all \( a, b \in \mathbb{R} \) and \( s > 1 \),

from which we get the inequality

\[ 2^{1-s}|a|^s - |b|^s \leq |a - b|^s \] for all \( a, b \in \mathbb{R} \) and \( s > 1 \).

It follows from this inequality that

\[
I(u, v) \geq \frac{2^{1-p'}}{p'} \int_\Omega |\Delta v|^{p'} \, dx - \frac{1}{p'} \int_\Omega |\epsilon a.\nabla u|^{p'} \, dx - \frac{1}{p'} \int_\Omega |v|^{p} \, dx \\
+ \frac{2^{1-q'}}{q'} \int_\Omega |\Delta u|^{q'} \, dx - \frac{1}{q'} \int_\Omega |\epsilon a.\nabla v|^{q'} \, dx - \frac{1}{q'} \int_\Omega |u|^{q} \, dx
\]

\[
\geq \frac{2^{1-p'}}{p'} \int_\Omega |\Delta v|^{p'} \, dx - \frac{(\|a\|_\infty)^{p'}}{p'} \int_\Omega |\nabla u|^{p'} \, dx - \frac{1}{p'} \int_\Omega |v|^{p} \, dx \\
+ \frac{2^{1-q'}}{q'} \int_\Omega |\Delta u|^{q'} \, dx - \frac{(\|a\|_\infty)^{q'}}{q'} \int_\Omega |\nabla v|^{q'} \, dx - \frac{1}{q'} \int_\Omega |u|^{q} \, dx.
\]

It follows from embeddings (24), (25) and embeddings in Lemma 5.4 that there exist \( c_1, c_2, c_3 \) and \( c_4 \) such that

\[ \|u\|^q_{L^q(\Omega)} \leq c_1 \|\Delta u\|^q_{L^{p'}(\Omega)}, \quad \|v\|^p_{L^p(\Omega)} \leq c_2 \|\Delta v\|^p_{L^{p'}(\Omega)}, \]

\[ \|\nabla u\|^p_{L^{p'}(\Omega)} \leq c_3 \|\Delta u\|^p_{L^{p'}(\Omega)} \quad \text{and} \quad \|\nabla v\|^q_{L^q(\Omega)} \leq c_4 \|\Delta v\|^q_{L^{q'}(\Omega)} \] for all \((u, v) \in V_A\).

It then follows that

\[
I(u, v) \geq \frac{2^{1-p'}}{p'} \int_\Omega |\Delta v|^{p'} \, dx - c_3 \frac{(\|a\|_\infty)^{p'}}{p'} \left( \int_\Omega |\Delta u|^{q'} \, dx \right)^{\frac{p'}{q'}} - c_2 \frac{1}{p'} \left( \int_\Omega |\Delta v|^{p'} \, dx \right)^{\frac{p'}{q'}} \\
+ \frac{2^{1-q'}}{q'} \int_\Omega |\Delta u|^{q'} \, dx - c_4 \frac{(\|a\|_\infty)^{q'}}{q'} \left( \int_\Omega |\Delta v|^{p'} \, dx \right)^{\frac{q'}{p'}} - c_1 \frac{1}{q'} \left( \int_\Omega |\Delta u|^{q'} \, dx \right)^{\frac{q'}{p'}}
\]

\[
= \frac{2^{1-p'}}{p'} \|\Delta v\|^p_{L^{p'}(\Omega)} - c_3 \frac{(\|a\|_\infty)^{p'}}{p'} \|\Delta u\|^p_{L^{p'}(\Omega)} - c_2 \frac{1}{p'} \|\Delta v\|^p_{L^{p'}(\Omega)} \\
+ \frac{2^{1-q'}}{q'} \|\Delta u\|^q_{L^{q'}(\Omega)} - c_4 \frac{(\|a\|_\infty)^{q'}}{q'} \|\Delta v\|^q_{L^{q'}(\Omega)} - c_1 \frac{1}{q'} \|\Delta u\|^q_{L^{q'}(\Omega)}.
\]

Note that \( p > p' \) and \( q > q' \). Now take \( \rho > 0 \) small enough such that for \( \|(u, v)\| = \rho \) we have

\[ \frac{2^{1-p'}}{p'} \|\Delta v\|^p_{L^{p'}(\Omega)} - c_2 \frac{1}{p'} \|\Delta v\|^p_{L^{p'}(\Omega)} \geq \frac{1}{p'^2} \|\Delta v\|^p_{L^{p'}(\Omega)} \quad \text{and} \quad \frac{2^{1-q'}}{q'} \|\Delta u\|^q_{L^{q'}(\Omega)} - c_4 \frac{1}{q'} \|\Delta u\|^q_{L^{q'}(\Omega)} \geq \frac{1}{q'^2} \|\Delta u\|^q_{L^{q'}(\Omega)} \]

It then follows from \( \|(u, v)\| = \rho \) that
\[ I(u, v) \geq \frac{1}{p'2^p} \| \Delta v \|_{p'}^p - c_4 \left( \frac{\epsilon \| a \|_{\infty}}{q'} \right)^q \| \Delta v \|_{L^q(\Omega)}^q + \frac{1}{q'2^q} \| \Delta u \|_{L^q(\Omega)}^q \]

Claim. Let \( a, b, c, \rho > 0 \) be four constants, \( 0 \leq \rho_0 \leq \rho \) and \( r, s > 1 \). Let \( \delta_0 = \min \left\{ a \left( \frac{\rho}{\rho_0} \right)^r + c \left( \rho \frac{s}{\rho_0} \right)^s, b \left( \frac{\rho}{\rho_0} \right)^s + c \delta \left( \rho - \rho_0 \right)^s + \rho_0 \right\} \). Then for \( 0 < \delta < \delta_0 \) we have

\[ a \left( \rho - \rho_0 \right)^r + b \rho_0^s - c \delta \left( \rho - \rho_0 \right)^s + \rho_0 \delta > \delta. \]

The proof for this claim is elementary. Now by assuming \( a = \frac{1}{p'2^p}, b = \frac{1}{q'2^q}, \)
\( c = \max \{ c_4(\| a \|_{\infty})^{\frac{q'}{p}}, c_3(\| a \|_{\infty})^{\frac{q'}{p}} \} \), \( r = p', s = q' \), \( \| \Delta u \|_{L^p(\Omega)} = \rho_0 \), \( \| \Delta v \|_{L^q(\Omega)} = \rho - \rho_0 \) and \( \delta = \epsilon \rho_0^{s/q} \), it follows from the above claim that for some \( \epsilon_0 \) if \( 0 < \epsilon < \epsilon_0 \) and \( \| (u, v) \| = \rho \) then \( I(u, v) \geq \epsilon \). The second condition of the mountain pass geometry holds for any \((ru, rv) \in V_A\) where \((u, v) \neq (0, 0)\) and \( r \in \mathbb{R} \) is large enough. \( \square \)

5.2. Hamiltonian systems of PDE’s with Neumann boundary conditions

Here a proof to Theorem 4.13 is provided. The following lemma is a direct consequence of Theorem 5.3.

Lemma 5.5. Let \( \Omega \) be a smooth bounded domain in \( \mathbb{R}^N \) and \( \partial \Omega \) its boundary. Let \( p, q > 1 \) and \( p' = \frac{p}{p-1} \) and \( q' = \frac{q}{q-1} \). The following embeddings hold.

\[ W^{2,p'}(\Omega) \hookrightarrow L^q(\partial \Omega), \quad \text{if} \quad \frac{1}{p} + \frac{N-1}{Nq} > \frac{N-1}{N}, \]
\[ W^{2,q'}(\Omega) \hookrightarrow L^p(\partial \Omega), \quad \text{if} \quad \frac{1}{q} + \frac{N-1}{Np} > \frac{N-1}{N}, \]
\[ W^{2,p'}(\Omega) \hookrightarrow L^q(\Omega), \quad \text{if} \quad \frac{1}{q} + \frac{1}{p} > \frac{N-1}{N}, \]
\[ W^{2,q'}(\Omega) \hookrightarrow L^p(\Omega), \quad \text{if} \quad \frac{1}{q} + \frac{1}{p} > \frac{N-1}{N}. \]

Proof of Theorem 4.13. Note first that if \( \Omega \) is a smooth bounded domain in \( \mathbb{R}^N \) then for each \( r > 1, W^{2,r}(\Omega) \) has an equivalent norm of the form

\[ \| u \|_{W^{2,r}(\Omega)} := \left( \| -\Delta u + u \|_{L^r(\Omega)}^r + \| \frac{\partial u}{\partial n} \|_{L^r(\partial \Omega)}^r \right)^{\frac{1}{r}}. \]

Set \( V = L^p(\Omega) \times L^q(\Omega), V^* = L^{p'}(\Omega) \times L^{q'}(\Omega), Y = L^p(\partial \Omega) \times L^q(\partial \Omega) \) and \( Y^* = L^{p'}(\partial \Omega) \times L^{q'}(\partial \Omega) \). Define \( \Phi : V \to \mathbb{R} \) by \( \Phi(u, v) = \frac{1}{p} \int_{\Omega} |u|^p \, dx + \frac{1}{q} \int_{\Omega} |v|^q \, dx \) and
\( \Psi : Y \to \mathbb{R} \) by \( \Psi(u, v) = \frac{1}{p} \int_{\partial \Omega} |u|^p \, dx + \frac{1}{q} \int_{\partial \Omega} |v|^q \, dx \). It follows that \( \Phi^*(f, g) = \frac{1}{p} \int_{\partial \Omega} |f|^p \, dx + \frac{1}{q} \int_{\partial \Omega} |g|^q \, dx \) and \( \Psi^*(f_0, g_0) = \frac{1}{p} \int_{\partial \Omega} |f_0|^p \, dx + \frac{1}{q} \int_{\partial \Omega} |g_0|^q \, dx \).

Let \( \Lambda : \text{Dom}(\Lambda) \subset V \to V^* \) and \( \beta_1 : \text{Dom}(\Lambda) \to Y \) and \( \beta_2 : \text{Dom}(\Lambda) \to Y^* \) be the operators \( \Lambda(u, v) = (-\Delta u + v, -\Delta u + u) \), \( \beta_1(u, v) = (u|_{\partial \Omega}, v|_{\partial \Omega}) \) and \( \beta_2(u, v) = (\frac{\partial u}{\partial n}|_{\partial \Omega}, \frac{\partial v}{\partial n}|_{\partial \Omega}) \) respectively. An easy computation shows that \( \Lambda \) is symmetric, but not necessary non-negative, modulo the boundary operator \( (\beta_1, \beta_2) \). Note also that due to the inequality \( \frac{1}{p} + \frac{1}{q} > \frac{N-2}{N} \) we have \( \text{Dom}(\Lambda) = W^{2,q}(\Omega) \times W^{2,p}(\Omega) \). It follows from Corollary 4.10 that if \((u, v)\) is a critical point of

\[
F(u, v) := 2\Phi^*(\Lambda(u, v)) - \langle \Lambda(u, v), (u, v) \rangle_{V \times V^*} + \Psi^*(\beta_2(u, v)) - \langle \beta_2(u, v), \beta_1(u, v) \rangle_{Y \times Y^*},
\]

then there exists \((\tilde{u}, \tilde{v})\) with \( \Lambda(\tilde{u}, \tilde{v}) = \Lambda(u, v) \) and \( \beta_2(\tilde{u}, \tilde{v}) = \beta_2(u, v) \) such that \((u + \tilde{u}, v + \tilde{v})\) is a solution of (21). We shall show that \( u = \tilde{u} \) and \( v = \tilde{v} \) and therefore \((u, v)\) is indeed a solution of (21). In fact, if \( \Lambda(\tilde{u}, \tilde{v}) = \Lambda(u, v) \) and \( \beta_2(\tilde{u}, \tilde{v}) = \beta_2(u, v) \) we have \(-\Delta(u - \tilde{u}) + (u - \tilde{u}) = 0\) and \( \frac{\partial(u - \tilde{u})}{\partial n} = 0 \). Note that by regularity theory of Elliptic equations with Neumann boundary conditions [19], we have that \( u - \tilde{u} \in C^{1,\alpha}(\Omega) \) for some \( \alpha > 0 \). It then follows

\[
0 = \int_{\Omega} (-\Delta(u - \tilde{u} + (u - \tilde{u}))(u - \tilde{u}) \, dx = \int_{\Omega} |\nabla(u - \tilde{u})|^2 \, dx + \int_{\Omega} |u - \tilde{u}|^2 \, dx,
\]

from which we obtain \( u = \tilde{u} \). The same argument shows that \( v = \tilde{v} \). Note also that \( I(u, v) = \frac{1}{2} F(u, v) \). In fact

\[
F(u, v) = \frac{2}{q} \int_{\Omega} |\Delta u + u|^q \, dx + \frac{2}{p} \int_{\Omega} |\Delta v + v|^p \, dx + \frac{2}{q'} \int_{\partial \Omega} |\frac{\partial u}{\partial n}|^{q'} \, dx
\]

\[
+ \frac{2}{p'} \int_{\partial \Omega} |\frac{\partial v}{\partial n}|^{p'} \, dx - \int_{\Omega} (-\Delta u + u)v \, dx - \int_{\Omega} (-\Delta v + v)u \, dx
\]

\[
- \int_{\partial \Omega} \frac{\partial v}{\partial n} u \, dx - \int_{\partial \Omega} \frac{\partial u}{\partial n} v \, dx
\]

\[
= \frac{2}{q} \int_{\Omega} |\Delta u + u|^q \, dx + \frac{2}{p} \int_{\Omega} |\Delta v + v|^p \, dx + \frac{2}{q'} \int_{\partial \Omega} |\frac{\partial u}{\partial n}|^{q'} \, dx
\]

\[
+ \frac{2}{p'} \int_{\partial \Omega} |\frac{\partial v}{\partial n}|^{p'} \, dx - 2 \int_{\Omega} \nabla u \cdot \nabla v \, dx - 2 \int_{\Omega} uv \, dx
\]

\[
= 2I(u, v).
\]

Thus, \( F \) and \( I \) have the same family of critical points. So far we have proved that if \((u, v)\) is a critical point of \( I \) then \((u, v)\) is a solution of (21). Now we need to show that \( I \) has at least one non-trivial critical point.
Define $F : V^* \times Y^* \to \mathbb{R}$ and $G : \text{Dom}(A) = W^{2, q'}(\Omega) \times W^{2, p'}(\Omega) \to \mathbb{R}$ by $F((f, g), (f_0, g_0)) = \Phi^*(f, g) + \Psi^*(f_0, g_0)$ and $G(u, v) = \int_{\Omega} \nabla u \nabla v \, dx + \int_{\Omega} uv \, dx$. Define $\Lambda : W^{2, q'}(\Omega) \times W^{2, p'} \to V^* \times Y^*$ by $\Lambda(u, v) = (\Lambda(u, v), \beta_2(u, v))$. Note that $I(u, v) = F(\Lambda(u, v)) - G(u, v)$, and $F, G$ and $\Lambda$ satisfy all conditions of Theorem 5.2. Therefore, to show that $I$ has a non-trivial critical point, we just need to verify the mountain pass geometry. To do this, note that

$$|G(u, v)| = \left| \int_{\Omega} (-\Delta u + u) v \, dx + \int_{\Omega} (-\Delta v + v) u \, dx + \int_{\partial \Omega} \frac{\partial v}{\partial \eta} u \, d\eta + \int_{\partial \Omega} \frac{\partial u}{\partial \eta} v \, d\eta \right|$$

$$\leq \| -\Delta u + u \|_{L^{q'}(\Omega)} \| v \|_{L^q(\Omega)} + \| -\Delta v + v \|_{L^{p'}(\Omega)} \| u \|_{L^p(\Omega)}$$

$$+ \left\| \frac{\partial v}{\partial \eta} \right\|_{L^{p'}(\partial \Omega)} \| u \|_{L^p(\partial \Omega)} + \left\| \frac{\partial u}{\partial \eta} \right\|_{L^{q'}(\partial \Omega)} \| v \|_{L^q(\partial \Omega)}$$

$$\leq C (\| v \|^2_{W^{2, q'}(\Omega)} + \| u \|^2_{W^{2, p'}(\Omega)}) \quad \text{(by Lemma 5.5)}$$

for some constant $C$. We have

$$I(u, v) = \frac{1}{p'} \int_{\Omega} | -\Delta v + v |^{p'} \, dx + \frac{1}{q'} \int_{\Omega} | -\Delta u + u |^{q'} \, dx + \frac{1}{p'} \int_{\partial \Omega} \left| \frac{\partial v}{\partial \eta} \right|^{p'} \, d\eta$$

$$+ \frac{1}{q'} \int_{\partial \Omega} \left| \frac{\partial u}{\partial \eta} \right|^{q'} \, d\eta + G(u, v)$$

$$\geq \frac{1}{p'} \int_{\Omega} | -\Delta v + v |^{p'} \, dx + \frac{1}{q'} \int_{\Omega} | -\Delta u + u |^{q'} \, dx + \frac{1}{p'} \int_{\partial \Omega} \left| \frac{\partial v}{\partial \eta} \right|^{p'} \, d\eta$$

$$+ \frac{1}{q'} \int_{\partial \Omega} \left| \frac{\partial u}{\partial \eta} \right|^{q'} \, d\eta - C \left( \| v \|^2_{W^{2, q'}(\Omega)} + \| u \|^2_{W^{2, p'}(\Omega)} \right)$$

$$= \frac{1}{p'} \| v \|^2_{W^{2, p'}(\Omega)} + \frac{1}{q'} \| u \|^2_{W^{2, q'}(\Omega)} - C \left( \| v \|^2_{W^{2, q'}(\Omega)} + \| u \|^2_{W^{2, p'}(\Omega)} \right).$$

Since $p', q' < 2$, there exists $\alpha > 0$ such that for $\rho > 0$ small enough we have $I(u, v) > \alpha$ when $\| (u, v) \| = \rho$. For the second condition of the mountain pass geometry take $(u_0, v_0) \in W^{2, q'}(\Omega) \times W^{2, p'}(\Omega)$ with $G(u_0, v_0) > 0$. It then follows that $I(ru_0, rv_0) < 0$ for $r \in \mathbb{R}$ large enough. □

**Acknowledgments**

Some of the results on this work are related to my previous joint works with Professor Nassif Ghoussoub on self-dual Lagrangians for convex functions. I would like to express my gratitude to him for the numerous and fruitful discussions on projects in convex self-duality. I would also like to express my thanks and appreciation to Professor Ivar Ekeland for pointing me towards Toland’s duality that is indeed the main ingredient of this project.
References