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# Parameter estimation in a spatial unilateral unit root autoregressive model

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ABSTRACT

Spatial unilateral autoregressive model $X_{k,\ell} = \alpha X_{k-1,\ell} + \beta X_{k,\ell-1} + \gamma X_{k-1,\ell-1} + \varepsilon_{k,\ell}$ is investigated in the unit root case, that is when the parameters are on the boundary of the domain of stability that forms a tetrahedron with vertices $(1, 1, -1)$, $(1, -1, 1)$, $(-1, 1, 1)$ and $(-1, -1, -1)$. It is shown that the limiting distribution of the least squares estimator of the parameters is normal and the rate of convergence is $n$ when the parameters are in the faces or on the edges of the tetrahedron, while on the vertices the rate is $n^{3/2}$.


## 1. Introduction

The analysis of spatial autoregressive models is of interest in many different fields of science such as geography, geology, biology and agriculture. A detailed discussion of these applications is given in [8] where the authors considered a special case of the so called unilateral autoregressive model having the form

$$X_{k,\ell} = \sum_{i=0}^{p_1}\sum_{j=0}^{p_2} \alpha_{i,j} X_{k-i,\ell-j} + \varepsilon_{k,\ell}, \qquad \alpha_{0,0} = 0. \tag{1.1}$$

A particular case of the above model is the so-called doubly geometric spatial autoregressive process

$$X_{k,\ell} = \alpha X_{k-1,\ell} + \beta X_{k,\ell-1} - \alpha\beta X_{k-1,\ell-1} + \varepsilon_{k,\ell},$$

introduced by Martin [22]. This was the first spatial autoregressive model for which unstability has been studied. It is, in fact, the simplest spatial model, since the product structure $\varphi(x, y) = xy - \alpha x - \beta y + \alpha\beta = (x - \alpha)(y - \beta)$ of its characteristic polynomial ensures that it can be considered as some kind of combination of two autoregressive processes on the line, and several properties can be derived by the analogy of one-dimensional autoregressive processes. This model has been used by Jain [21] in the study of image processing, by Martin [23], Cullis and Gleeson [16], Basu and Reinsel [9] in agricultural trials and by Tjøstheim [27] in digital filtering.

In the stable case when $|\alpha| < 1$ and $|\beta| < 1$, asymptotic normality of several estimators $(\widehat{\alpha}_{m,n}, \widehat{\beta}_{m,n})$ of $(\alpha, \beta)$ based on the observations $\{X_{k,\ell} : 1 \le k \le m \text{ and } 1 \le \ell \le n\}$ has been shown (e.g. [7,8,26,28]), namely,

$$\sqrt{mn}\begin{bmatrix}\widehat{\alpha}_{m,n} - \alpha \\ \widehat{\beta}_{m,n} - \beta\end{bmatrix} \xrightarrow{\mathcal{D}} \mathcal{N}(0, \Sigma_{\alpha,\beta})$$

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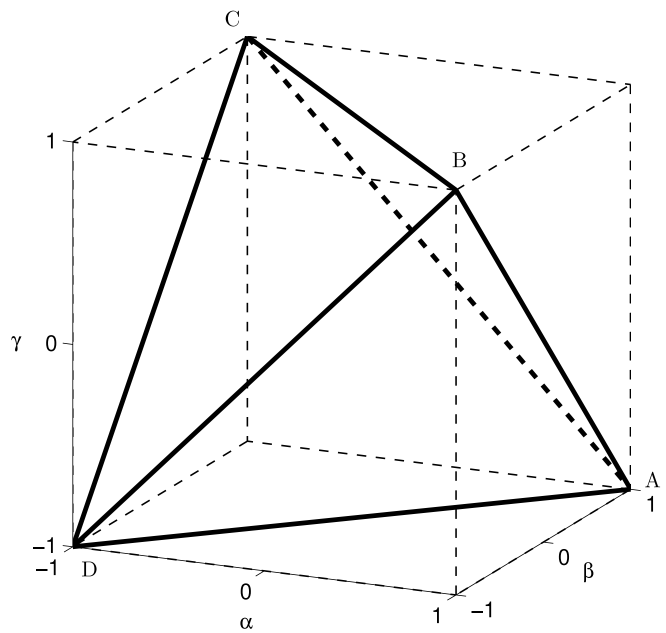


**Fig. 1.** The domain of stability of model (1.2).

as $m, n \to \infty$ with $m/n \to$ constant $> 0$ with some covariance matrix $\Sigma_{\alpha,\beta}$. Further, Davydov and Paulauskas [17] considered the $d$-dimensional case and under a less restrictive assumption on the increase of domain of observations (which for the above rectangular domain means $\min\{m, n\} \to \infty$) showed that the self-normalized least squares estimator is asymptotically normal and the limiting distribution is independent of the parameters.

In the unstable case when $\alpha = \beta = 1$, in contrast to the classical first order autoregressive time series model, where the appropriately normed least squares estimator (LSE) of the autoregressive parameter converges to a fraction of functionals of the standard Brownian motion (see e.g. [15] or [25]), the sequence of Gauss–Newton estimators $(\widehat{\alpha}_{n,n}, \widehat{\beta}_{n,n})$ of $(\alpha, \beta)$ has been shown to be asymptotically normal [11,12]. In the unstable case $\alpha = 1, |\beta| < 1$ the LSE turns out to be asymptotically normal again [11].

Baran et al. [2] discussed a special case of the model (1.1), namely, when $p_1 = p_2 = 1, \alpha_{0,1} = \alpha_{1,0} =: \alpha$ and $\alpha_{1,1} = 0$, which is the simplest spatial model, that cannot be reduced somehow to autoregressive models on the line. This model is stable in case $|\alpha| < 1/2$ (see e.g. [8,13,30]), and unstable if $|\alpha| = 1/2$. In Baran et al. [2] the asymptotic normality of the LSE of the unknown parameter $\alpha$ is proved both in stable and unstable cases. The case $p_1 = p_2 = 1, \alpha_{1,0} =: \alpha, \alpha_{0,1} =: \beta$ and $\alpha_{1,1} = 0$ was studied by Paulauskas [24] and Baran et al. [4]. This model is stable in case $|\alpha| + |\beta| < 1$ and unstable if $|\alpha| + |\beta| = 1$ [8]. Paulauskas [24] determined the exact asymptotic behaviour of the variances of the process, while Baran et al. [4] proved the asymptotic normality of the LSE of the parameters both in stable and unstable cases.

In the present paper we study the asymptotic properties of a more complicated special case of the unilateral model (1.1) with $p_1 = p_2 = 1, \alpha_{1,0} =: \alpha, \alpha_{0,1} =: \beta$ and $\alpha_{1,1} =: \gamma$. In a recent paper Genton and Koul [18] proved the asymptotic normality of minimum distance estimators in the case when model equation is valid on $\mathbb{Z}^2$. Here we deal with a model with boundary conditions, namely, we consider the spatial autoregressive process $\{X_{k,\ell} : k, \ell \in \mathbb{Z}, k, \ell \geq 0\}$ defined as

$$\begin{cases} X_{k,\ell} = \alpha X_{k-1,\ell} + \beta X_{k,\ell-1} + \gamma X_{k-1,\ell-1} + \varepsilon_{k,\ell}, & \text{for } k, \ell \geq 1, \\ X_{k,0} = X_{0,\ell} = 0, & \text{for } k, \ell \geq 0. \end{cases} \tag{1.2}$$

This process has already been examined in [1] where the asymptotic behaviour of the variances is clarified. The model is stable if $(\alpha, \beta, \gamma) \in \mathcal{S}$, where $\mathcal{S}$ is the open tetrahedron with vertices

$$\mathcal{V} := \{(1, 1, -1), (1, -1, 1), (-1, 1, 1), (-1, -1, -1)\}$$

(see Fig. 1, where $\{A, B, C, D\} := \mathcal{V}$). This result was proved by Basu and Reinsel [8] where the tetrahedron $\mathcal{S}$ was described by conditions $|\alpha| < 1, |\beta| < 1$ and $|\gamma| < 1, |1 + \alpha^2 - \beta^2 - \gamma^2| > 2|\alpha + \beta\gamma|$ and $1 - \beta^2 > |\alpha + \beta\gamma|$. Short calculation shows that condition of stability means that $|\alpha| < 1, |\beta| < 1$ and $|\gamma| < 1$, and inequalities

$$\alpha - \beta - \gamma < 1, \qquad -\alpha + \beta - \gamma < 1, \qquad -\alpha - \beta + \gamma < 1, \qquad \alpha + \beta + \gamma < 1$$

hold. Obviously, in case $\alpha\beta\gamma \geq 0$ the above set of conditions reduces to $|\alpha| + |\beta| + |\gamma| < 1$.

The model is unstable if $(\alpha, \beta, \gamma)$ lies on the boundary of $\mathcal{S}$, when one can distinguish three cases:
*Case* A. The parameters are in the interior of the faces of the boundary of $\mathcal{S}$, i.e. $(\alpha, \beta, \gamma) \in \mathcal{F}$, where $\mathcal{F} := \mathcal{F}_+ \cup \mathcal{F}_-$ with

$$\begin{aligned}\mathcal{F}_+ &:= \{(\alpha, \beta, \gamma) \in (-1, 1)^3 : \alpha\beta\gamma \ge 0, |\alpha| + |\beta| + |\gamma| = 1\}\\ &\quad \cup \{(\alpha, \beta, \gamma) \in (-1, 1)^3 : \alpha\beta\gamma < 0, |\alpha| + |\beta| - |\gamma| = 1\},\\ \mathcal{F}_- &:= \{(\alpha, \beta, \gamma) \in (-1, 1)^3 : \alpha\beta\gamma < 0, |\alpha| - |\beta| + |\gamma| = 1\}\\ &\quad \cup \{(\alpha, \beta, \gamma) \in (-1, 1)^3 : \alpha\beta\gamma < 0, -|\alpha| + |\beta| + |\gamma| = 1\}\end{aligned}$$

(see Fig. 1, where faces *ABC* and *DBC* form $\mathcal{F}_+$, while $\mathcal{F}_-$ contains faces *ACD* and *ABD*).
*Case* B. The parameters are in the interior of the edges of the boundary of $\mathcal{S}$, i.e. $(\alpha, \beta, \gamma) \in \mathcal{E}$, where $\mathcal{E} := \mathcal{E}_1 \cup \mathcal{E}_2 \cup \mathcal{E}_3$ with

$$\begin{aligned}\mathcal{E}_1 &:= \{(1, \beta, \gamma) : \beta \in (-1, 1), \gamma = -\beta\} \cup \{(-1, \beta, \gamma) : \beta \in (-1, 1), \gamma = \beta\},\\ \mathcal{E}_2 &:= \{(\alpha, 1, \gamma) : \alpha \in (-1, 1), \gamma = -\alpha\} \cup \{(\alpha, -1, \gamma) : \alpha \in (-1, 1), \gamma = \alpha\},\\ \mathcal{E}_3 &:= \{(\alpha, \beta, 1) : \alpha \in (-1, 1), \beta = -\alpha\} \cup \{(\alpha, \beta, -1) : \alpha \in (-1, 1), \beta = \alpha\}\end{aligned}$$

(see Fig. 1, where pairs of edges $\{AB, CD\}$, $\{AC, BD\}$ and $\{BC, AD\}$ form $\mathcal{E}_1$, $\mathcal{E}_2$ and $\mathcal{E}_3$, respectively). Observe that in each of the above three cases exactly two of the defining equations of set $\mathcal{F}$ are satisfied. In this way Case B can be considered as an extension of Case A to the situation when $\alpha\beta\gamma \le 0$ and one of the parameters equals $\pm 1$, while the other two parameters have absolute values less than one. Further, observe that in the first two cases $\gamma = -\alpha\beta$, so we obtain special cases of the doubly geometric model. If $(\alpha, \beta, \gamma) \in \mathcal{E}_1$ then for $k \in \mathbb{N}$ the difference $\Delta_{1,\alpha} X_{k,\ell} := X_{k,\ell} - \alpha X_{k-1,\ell}$ is a classical AR(1) process, i.e. $\Delta_{1,\alpha} X_{k,\ell} = \beta \Delta_{1,\alpha} X_{k,\ell-1} + \varepsilon_{k,\ell}$. Similarly, if $(\alpha, \beta, \gamma) \in \mathcal{E}_2$ then $\Delta_{2,\beta} X_{k,\ell} = \alpha \Delta_{2,\beta} X_{k-1,\ell} + \varepsilon_{k,\ell}$, where $\Delta_{2,\beta} X_{k,\ell} := X_{k,\ell} - \beta X_{k,\ell-1}$, $\ell \in \mathbb{N}$.
*Case* C. The parameters are in the vertices of the boundary of the domain of stability, i.e. $(\alpha, \beta, \gamma) \in \mathcal{V}$.

For a set $H \subset \{(k, \ell) \in \mathbb{Z}^2 : k, \ell \ge 1\}$, the least squares estimator $(\widehat{\alpha}_H, \widehat{\beta}_H, \widehat{\gamma}_H)$ of $(\alpha, \beta, \gamma)$ based on the observations $\{X_{k,\ell} : (k, \ell) \in H\}$ can be obtained by minimizing the sum of squares

$$\sum_{(k,\ell)\in H} \left(X_{k,\ell} - \alpha X_{k-1,\ell} - \beta X_{k,\ell-1} - \gamma X_{k-1,\ell-1}\right)^2$$

with respect to $\alpha$, $\beta$ and $\gamma$, and it has the form

$$\begin{bmatrix}\widehat{\alpha}_H\\ \widehat{\beta}_H\\ \widehat{\gamma}_H\end{bmatrix} = \left(\sum_{(k,\ell)\in H} \begin{bmatrix}X_{k-1,\ell}\\ X_{k,\ell-1}\\ X_{k-1,\ell-1}\end{bmatrix}\begin{bmatrix}X_{k-1,\ell}\\ X_{k,\ell-1}\\ X_{k-1,\ell-1}\end{bmatrix}^\top\right)^{-1} \sum_{(k,\ell)\in H} X_{k,\ell}\begin{bmatrix}X_{k-1,\ell}\\ X_{k,\ell-1}\\ X_{k-1,\ell-1}\end{bmatrix}.$$

For $n, m \in \mathbb{N}$ consider the rectangle

$$R_{n,m} := \{(i, j) \in \mathbb{N}^2 : 1 \le i \le n \text{ and } 1 \le j \le m\}.$$

For simplicity, we shall write $R_n := R_{n,n}$ for $n \in \mathbb{N}$. Further, let

$$\mathcal{H} := \begin{bmatrix}1 & 0 & -1\\ 0 & 1 & -1\end{bmatrix}, \qquad \Theta_{\alpha,\beta} := 2\begin{bmatrix}1 & 0 & -\beta\\ 0 & 1 & -\alpha\\ -\beta & -\alpha & 2\end{bmatrix}, \qquad |\alpha| = |\beta| = 1, \quad \text{and}$$

$$\mathcal{K}_{\alpha,\beta} := \begin{bmatrix}\kappa^{(1)}_{\beta,\alpha} & \kappa^{(2)}_{\alpha,\beta}\\ \kappa^{(2)}_{\alpha,\beta} & \kappa^{(1)}_{\alpha,\beta}\end{bmatrix}, \quad \alpha, \beta \in (-1, 1),$$

with

$$\kappa^{(1)}_{\alpha,\beta} := (1 - \beta^2)^{-1} + (1 - \alpha)^2 \varrho^{(1)}_{\alpha,\beta}, \qquad \kappa^{(2)}_{\alpha,\beta} := (1 - \alpha)(1 - \beta)\varrho^{(2)}_{\alpha,\beta},$$

and

$$\varrho^{(1)}_{\alpha,\beta} := \sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty}\left(\mathsf{P}\left(S^{(|\alpha|,1-|\beta|)}_{k,\ell} = k + 1\right) - \mathsf{P}\left(S^{(|\alpha|,1-|\beta|)}_{k,\ell} = k\right)\right)^2,$$

$$\varrho^{(2)}_{\alpha,\beta} := \sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty}\left(\mathsf{P}\left(S^{(|\alpha|,1-|\beta|)}_{k,\ell} = k + 1\right) - \mathsf{P}\left(S^{(|\alpha|,1-|\beta|)}_{k,\ell} = k\right)\right)\left(\mathsf{P}\left(S^{(|\beta|,1-|\alpha|)}_{\ell,k} = \ell + 1\right) - \mathsf{P}\left(S^{(|\beta|,1-|\alpha|)}_{\ell,k} = \ell\right)\right),$$

where $S^{(\nu,\mu)}_{k,\ell} := \xi^{(\nu)}_k + \eta^{(\mu)}_\ell$ and $\xi^{(\nu)}_k$ and $\eta^{(\mu)}_\ell$ are independent binomial random variables with parameters $(k, \nu)$ and $(\ell, \mu)$, respectively. Finally, for $|\alpha| = 1$, $\beta \in (-1, 1)$ or $|\beta| = 1$, $\alpha \in (-1, 1)$ let us introduce

$$\Sigma_{\alpha,\beta} := \begin{cases}\begin{bmatrix}1 & |\beta|\mathrm{sign}(\alpha) & |\beta|\\ |\beta|\mathrm{sign}(\alpha) & 1 & \mathrm{sign}(\alpha)\\ |\beta| & \mathrm{sign}(\alpha) & 1\end{bmatrix}, & \text{if } |\alpha| = 1,\\[2ex] \begin{bmatrix}1 & |\alpha|\mathrm{sign}(\beta) & \mathrm{sign}(\beta)\\ |\alpha|\mathrm{sign}(\beta) & 1 & |\alpha|\\ \mathrm{sign}(\beta) & |\alpha| & 1\end{bmatrix}, & \text{if } |\beta| = 1.\end{cases}$$

Using these notations we can formulate our main result.

**Theorem 1.1.** *Let* $\{\varepsilon_{k,\ell} : k, \ell \in \mathbb{N}\}$ *be independent random variables with* $\mathsf{E}\,\varepsilon_{k,\ell} = 0$, $\operatorname{Var}\varepsilon_{k,\ell} = 1$ *and* $\sup\{\mathsf{E}\,\varepsilon_{k,\ell}^8 : k, \ell \in \mathbb{N}\} < \infty$. *Assume that model (1.2) is satisfied.*

*If* $(\alpha, \beta, \gamma) \in \mathcal{F}_+$ *then*

$$(nm)^{1/2}\left(\widehat{\alpha}_{R_{n,m}} - \alpha, \widehat{\beta}_{R_{n,m}} - \beta, \widehat{\gamma}_{R_{n,m}} - \gamma\right)^\top \xrightarrow{\mathcal{D}} \mathcal{N}\left(0, \mathcal{H}^\top \mathcal{K}_{\alpha,\beta}^{-1}\mathcal{H}\right),$$

*if* $(\alpha, \beta, \gamma) \in \mathcal{E}_1 \cup \mathcal{E}_2$ *then*

$$(nm)^{1/2}\left(\widehat{\alpha}_{R_{n,m}} - \alpha, \widehat{\beta}_{R_{n,m}} - \beta, \widehat{\gamma}_{R_{n,m}} - \gamma\right)^\top \xrightarrow{\mathcal{D}} \mathcal{N}\left(0, \overline{\Sigma}_{\alpha,\beta}\right),$$

*while for* $(\alpha, \beta, \gamma) \in \mathcal{V}$ *we have*

$$(nm)^{3/4}\left(\widehat{\alpha}_{R_{n,m}} - \alpha, \widehat{\beta}_{R_{n,m}} - \beta, \widehat{\gamma}_{R_{n,m}} - \gamma\right)^\top \xrightarrow{\mathcal{D}} \mathcal{N}\left(0, \Theta_{\alpha,\beta}\right)$$

*as* $m, n \to \infty$ *with* $m/n \to$ *constant* $> 0$, *where* $\overline{A}$ *denotes the adjoint of a matrix A.*

We remark that results given Theorem 1.1 do not cover the cases $(\alpha, \beta, \gamma) \in \mathcal{F}_- \cup \mathcal{E}_3$. The main problem is that in these cases we could not handle the asymptotic behaviour of the covariance structure. A more detailed explanation and some results on the missing cases can be found in [1]. Another problem is that we were not able to find closed forms of $\varrho_{\alpha,\beta}^{(i)}$, $i = 1, 2$, and in this way we do not know how they depend on the parameters.

**Remark 1.2.** In Theorem 1.1 the results are not continuous with respect to the parameters $(\alpha, \beta, \gamma)$. However, the lack of continuity is a usual phenomenon in models of type (1.2). Even in the simplest case when $\alpha = \beta$ and $\gamma = 0$, from the results derived for the stable case ($|\alpha| < 1/2$) one cannot get a result for the unstable model ($|\alpha| = 1/2$) by taking the limit $\alpha \to 1/2$ (see e.g. [2]). However, it is possible to examine the nearly unstable situation, where one has a sequence of stable models with parameters converging to values corresponding to the unit root cases (see e.g. [3,5]).

**Remark 1.3.** In all cases considered in Theorem 1.1 the covariance matrices of the limiting distributions are singular with ranks being equal to 2, which is the consequence of the linear relation between the parameters $(\alpha, \beta, \gamma)$.

**Remark 1.4.** It is still an open question, whether one can get rid of condition $m/n \to$ constant $> 0$ and consider e.g. $\min\{m, n\} \to \infty$ instead (see e.g. [17]). Unfortunately, the method presented here cannot be used under such condition, because the Martingale Central Limit Theorem [20] applied in the proofs of Propositions 1.7 and 1.9 does not allow this generalization.

For the sake of simplicity, we carry out the proof of Theorem 1.1 only for $m = n$. The general case can be handled with slight modifications. We can write

$$\begin{bmatrix} \widehat{\alpha}_{R_n} - \alpha \\ \widehat{\beta}_{R_n} - \beta \\ \widehat{\gamma}_{R_n} - \gamma \end{bmatrix} = B_n^{-1} A_n,$$

with

$$A_n := \sum_{(k,\ell)\in R_n} \varepsilon_{k,\ell} \begin{bmatrix} X_{k-1,\ell} \\ X_{k,\ell-1} \\ X_{k-1,\ell-1} \end{bmatrix}, \qquad B_n := \sum_{(k,\ell)\in R_n} \begin{bmatrix} X_{k-1,\ell} \\ X_{k,\ell-1} \\ X_{k-1,\ell-1} \end{bmatrix} \begin{bmatrix} X_{k-1,\ell} \\ X_{k,\ell-1} \\ X_{k-1,\ell-1} \end{bmatrix}^\top.$$

Now, the idea of the proof is the following. First we show that in all considered cases $n^{-\tau}B_n$ has a limit, where $\tau$ is an appropriate rate, then we find the limiting distribution of $n^{-\tau/2}A_n$ (Propositions 1.6 and 1.7, respectively). However, as the limits in Proposition 1.6 are singular, the statements of Theorem 1.1 cannot be obtained directly from Propositions 1.6 and 1.7. Hence, one has to use the same idea as in [4] and consider $B_n^{-1} = \overline{B}_n/\det(B_n)$. Proposition 1.8 clarifies the asymptotic behaviour of $\det(B_n)$. We show that for $(\alpha, \beta, \gamma) \in \mathcal{F}_+$ and for $(\alpha, \beta, \gamma) \in \mathcal{E}_1 \cup \mathcal{E}_2$, random sequence $\det(B_n)$ normed by an appropriate power of $n$, has a positive, deterministic limit. Combining this with Proposition 1.10, where in the above mentioned two cases the asymptotic normality of the $\overline{B}_n A_n$ is proved, we immediately obtain the first two statements of Theorem 1.1. The situation is completely different for $(\alpha, \beta, \gamma) \in \mathcal{V}$, when all considered quantities converge to functionals of the standard Wiener sheet. However, if we take directly the limit of $n^{3/2}\overline{B}_n A_n/\det(B_n)$, we obtain asymptotic normality.

**Remark 1.5.** Observe that in Theorem 1.1 on the faces $\mathcal{F}_+$ and on the edges $\mathcal{E}_i$, $i = 1, 2$, the normalization is the same. A possible explanation is that the rates of convergence are connected to the rank of $\mathbb{I}_n = \mathsf{E}B_n$ which is the observed Fisher information matrix of $(\alpha, \beta, \gamma)$. For $(\alpha, \beta, \gamma) \in \mathcal{F}_+$ we have $\mathbb{I}_n \approx n^{5/2}\sigma_{\alpha,\beta}^2\Psi_{\alpha,\beta}$ and $\operatorname{rank}(\Psi_{\alpha,\beta}) = 1$, while for $(\alpha, \beta, \gamma) \in \mathcal{E}_1 \cup \mathcal{E}_2$ the corresponding result is $\mathbb{I}_n \approx n^3\left(2(1-\gamma^2)\right)^{-1}\Sigma_{\alpha,\beta}$ and $\operatorname{rank}(\Sigma_{\alpha,\beta}) = 2$. This difference in ranks determines the rate of convergence of $\det(B_n)$ and of $\overline{B}_n A_n$. A similar phenomenon can be observed in case $\gamma = 0$, where in the stable case $|\alpha| + |\beta| < 1$ and in the unstable case $|\alpha| + |\beta| = 1$, $0 < |\alpha| < 1$, the normalization is the same (see [4]).

**Proposition 1.6.** *If* $(\alpha, \beta, \gamma) \in \mathcal{F}_+$ *then*

$$n^{-5/2} B_n \xrightarrow{\mathsf{L}_2} \sigma^2_{\alpha,\beta} \Psi_{\alpha,\beta} \quad \text{as } n \to \infty,$$

*where*

$$\sigma^2_{\alpha,\beta} := \frac{2}{3}\left(\frac{1 - |\alpha| \vee |\beta|}{\pi(|\alpha| + |\beta|)}\right)^{1/2} \left(\frac{1}{(1-|\alpha|)(1-|\beta|)} - \frac{1}{5(1-|\alpha| \wedge |\beta|)^2}\right),$$

$$\Psi_{\alpha,\beta} := \begin{bmatrix} 1 \\ \operatorname{sign}(\alpha\beta) \\ \operatorname{sign}(\beta) \end{bmatrix} \begin{bmatrix} 1 \\ \operatorname{sign}(\alpha\beta) \\ \operatorname{sign}(\beta) \end{bmatrix}^\top .$$

*If* $(\alpha, \beta, \gamma) \in \mathcal{E}_1 \cup \mathcal{E}_2$ *then*

$$n^{-3} B_n \xrightarrow{\mathsf{L}_2} \left(2(1-\gamma^2)\right)^{-1} \Sigma_{\alpha,\beta} \quad \text{as } n \to \infty.$$

*If* $(\alpha, \beta, \gamma) \in \mathcal{V}$ *then*

$$n^{-4} B_n \xrightarrow{\mathcal{D}} \int_0^1 \int_0^1 \mathcal{W}^2(s,t)\,\mathrm{d}s\,\mathrm{d}t \begin{bmatrix} \alpha \\ \beta \\ \alpha\beta \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \alpha\beta \end{bmatrix}^\top \quad \text{as } n \to \infty,$$

*where* $\mathcal{W}(s, t)$ *is a standard Wiener sheet.*

**Proposition 1.7.** *If* $(\alpha, \beta, \gamma) \in \mathcal{F}_+$ *then*

$$n^{-5/4} A_n \xrightarrow{\mathcal{D}} \mathcal{N}\left(0, \sigma^2_{\alpha,\beta} \Psi_{\alpha,\beta}\right) \quad \text{as } n \to \infty.$$

*If* $(\alpha, \beta, \gamma) \in \mathcal{E}_1 \cup \mathcal{E}_2$ *then*

$$n^{-3/2} A_n \xrightarrow{\mathcal{D}} \mathcal{N}\left(0, \left(2(1-\gamma^2)\right)^{-1} \Sigma_{\alpha,\beta}\right) \quad \text{as } n \to \infty.$$

*If* $(\alpha, \beta, \gamma) \in \mathcal{V}$ *then*

$$n^{-2} A_n \xrightarrow{\mathcal{D}} \int_0^1 \int_0^1 \mathcal{W}(s,t)\mathcal{W}(\mathrm{d}s, \mathrm{d}t) \begin{bmatrix} \alpha \\ \beta \\ \alpha\beta \end{bmatrix} \quad \text{as } n \to \infty.$$

**Proposition 1.8.** *If* $(\alpha, \beta, \gamma) \in \mathcal{F}_+$ *then*

$$n^{-13/2} \det(B_n) \xrightarrow{\mathsf{P}} \sigma^2_{\alpha,\beta} \det\left(\mathcal{K}_{\alpha,\beta}\right) > 0 \quad \text{as } n \to \infty.$$

*If* $(\alpha, \beta, \gamma) \in \mathcal{E}_1 \cup \mathcal{E}_2$ *then*

$$n^{-8} \det(B_n) \xrightarrow{\mathsf{P}} \left(2(1-\gamma^2)\right)^{-2} \quad \text{as } n \to \infty.$$

*If* $(\alpha, \beta, \gamma) \in \mathcal{V}$ *then*

$$n^{-10} \det(B_n) \xrightarrow{\mathcal{D}} \frac{1}{4} \int_0^1 \int_0^1 \mathcal{W}^2(s,t)\,\mathrm{d}s\,\mathrm{d}t \quad \text{as } n \to \infty.$$

We remark that using higher moment conditions on the innovations $\varepsilon_{k,\ell}$, after tedious but straightforward calculations, instead of stochastic convergence one can also prove $\mathsf{L}_2$ convergence in the first two statements of Proposition 1.8.

Further, if we take appropriate linear transformations of $A_n$ we have asymptotic normality in all of the unstable cases considered. Let $C_n := \mathcal{H} A_n = \left(C_n^{(1)},\ C_n^{(2)}\right)^\top$, where

$$C_n^{(1)} := (1, 0, -1)\, A_n = \sum_{(k,\ell)\in R_n} \left(X_{k-1,\ell} - X_{k-1,\ell-1}\right) \varepsilon_{k,\ell}, \tag{1.3}$$

$$C_n^{(2)} := (0, 1, -1)\, A_n = \sum_{(k,\ell)\in R_n} \left(X_{k,\ell-1} - X_{k-1,\ell-1}\right) \varepsilon_{k,\ell}. \tag{1.4}$$

**Proposition 1.9.** *If* $(\alpha, \beta, \gamma) \in \mathcal{F}_+$ *then*

$$n^{-1} C_n \xrightarrow{\mathcal{D}} \mathcal{N}\left(0, \mathcal{K}_{\alpha,\beta}\right) \quad \text{as } n \to \infty.$$

*If* $(\alpha,\beta,\gamma)\in\mathcal{E}_1$ *then*

$$n^{-3/2}C_n^{(1)}\xrightarrow{\mathcal{D}}\mathcal{N}\left(0,(1-\gamma)^{-1}\right)\quad\text{and}\quad n^{-1}C_n^{(2)}\xrightarrow{\mathcal{D}}\mathcal{N}\left(0,(1-\gamma^2)^{-1}\right)\quad\text{as } n\to\infty.$$

*If* $(\alpha,\beta,\gamma)\in\mathcal{E}_2$ *then*

$$n^{-1}C_n^{(1)}\xrightarrow{\mathcal{D}}\mathcal{N}\left(0,(1-\gamma^2)^{-1}\right)\quad\text{and}\quad n^{-3/2}C_n^{(2)}\xrightarrow{\mathcal{D}}\mathcal{N}\left(0,(1-\gamma)^{-1}\right)\quad\text{as } n\to\infty.$$

*If* $(\alpha,\beta,\gamma)\in\mathcal{V}$ *then*

$$n^{-3/2}C_n\xrightarrow{\mathcal{D}}\mathcal{N}\left(0,\mathcal{I}_2/2\right)\quad\text{as } n\to\infty,$$

*where* $\mathcal{I}_2$ *denotes the two-by-two unit matrix.*

**Proposition 1.10.** *If* $(\alpha,\beta,\gamma)\in\mathcal{F}_+$ *then*

$$n^{-11/2}\overline{B}_nA_n\xrightarrow{\mathcal{D}}\mathcal{N}\left(0,\sigma^4_{\alpha,\beta}\det\left(\mathcal{K}_{\alpha,\beta}\right)\mathcal{H}^\top\overline{\mathcal{K}}_{\alpha,\beta}\mathcal{H}\right)\quad\text{as } n\to\infty.$$

*If* $(\alpha,\beta,\gamma)\in\mathcal{E}_1\cup\mathcal{E}_2$ *then*

$$n^{-7}\overline{B}_nA_n\xrightarrow{\mathcal{D}}\mathcal{N}\left(0,\left(2\left(1-\gamma^2\right)\right)^{-4}\overline{\Sigma}_{\alpha,\beta}\right)\quad\text{as } n\to\infty.$$

Note that direct calculations imply

$$X_{k,\ell}=\sum_{i=1}^{k}\sum_{j=1}^{\ell}G(k-i,\ell-j;\alpha,\beta,\gamma)\varepsilon_{i,j} \tag{1.5}$$

$$=\sum_{i=1}^{k}\sum_{j=1}^{\ell}\binom{k+\ell-i-j}{\ell-j}\alpha^{k-i}\beta^{\ell-j}F\left(i-k,j-\ell;i+j-k-\ell;-\frac{\gamma}{\alpha\beta}\right)\varepsilon_{i,j}, \tag{1.6}$$

$k,\ell\geq 1$, where (1.6) holds only for $\alpha\beta\neq 0$,

$$G(m,n;\alpha,\beta,\gamma):=\sum_{r=0}^{m\wedge n}\frac{(m+n-r)!}{(m-r)!(n-r)!r!}\alpha^{m-r}\beta^{n-r}\gamma^r,\quad m,n\in\mathbb{N}\cup\{0\},$$

and $F(-n,b;c;z)$ is the Gauss hypergeometric function defined by

$$F(-n,b;c;z):=\sum_{r=0}^{n}\frac{(-n)_r(b)_r}{(c)_r r!}z^r,\quad n\in\mathbb{N}, b,c,z\in\mathbb{C},$$

and $(a)_r:=a(a+1)\cdots(a+r-1)$ (for the definition in more general cases see e.g. [10]).

Observe that as for $m,n\in\mathbb{N}$ we have $F(-n,-m;-n-m;1)=\binom{m+n}{n}^{-1}$ and $F(-n,-m;-n-m;0)=1$, moving average representations of the doubly geometric model of Martin [22] and of the spatial models studied by Paulauskas [24] and Baran et al. [2,4], respectively, are special forms of (1.6).

Using representation (1.5) one can show that it suffices to prove Theorem 1.1 for $\alpha\geq 0$, $\beta\geq 0$ and $\gamma\geq 0$ if $\alpha\beta\gamma\geq 0$ and for $\alpha>0$, $\beta>0$ and $\gamma<0$ if $\alpha\beta\gamma<0$ (for more details, see [6]). In this way instead of $\mathcal{F}_+$, $\mathcal{V}$, $\mathcal{E}_1$ and $\mathcal{E}_2$ it suffices to use their subsets

$$\mathcal{F}_{++}:=\{(\alpha,\beta,\gamma):0\leq\alpha,\beta<1,|\gamma|<1,\alpha+\beta+\gamma=1\},\qquad\mathcal{V}_+:=\{(1,1,-1)\},$$
$$\mathcal{E}_{1+}:=\{(1,\beta,\gamma):0\leq\beta=-\gamma<1\},\qquad\mathcal{E}_{2+}:=\{(\alpha,1,\gamma):0\leq\alpha=-\gamma<1\},$$

respectively.

## 2. Covariance structure

By representations (1.5) and (1.6) we obtain that for $k_1,\ell_1,k_2,\ell_2\in\mathbb{N}$ and $\alpha,\beta,\gamma\in\mathbb{R}$ we have

$$\operatorname{Cov}\left(X_{k_1,\ell_1},X_{k_2,\ell_2}\right)=\sum_{i=1}^{k_1\wedge k_2}\sum_{j=1}^{\ell_1\wedge\ell_2}G(k_1-i,\ell_1-j;\alpha,\beta,\gamma)G(k_2-i,\ell_2-j;\alpha,\beta,\gamma) \tag{2.1}$$

$$=\sum_{i=1}^{k_1\wedge k_2}\sum_{j=1}^{\ell_1\wedge\ell_2}\binom{k_1+\ell_1-i-j}{\ell_1-j}\binom{k_2+\ell_2-i-j}{\ell_2-j}\alpha^{k_1+k_2-2i}\beta^{\ell_1+\ell_2-2j} \tag{2.2}$$
$$\times F\left(i-k_1,j-\ell_1;i+j-k_1-\ell_1;-\frac{\gamma}{\alpha\beta}\right)F\left(i-k_2,j-\ell_2;i+j-k_2-\ell_2;-\frac{\gamma}{\alpha\beta}\right),$$

where $x\wedge y:=\min\{x,y\}$, $x,y\in\mathbb{R}$, an empty sum is defined to be equal to 0, and (2.2) holds only for $\alpha\beta\neq 0$.

The following lemma [1, Corollary 2.2] helps us to find a more convenient form of the covariances.

**Lemma 2.1.** *If $0 \le \alpha, \beta < 1$ and $\alpha + \beta + \gamma = 1$, then*

$$G(m, n; \alpha, \beta, \gamma) = \mathsf{P}\left(S_{m,n}^{(\alpha,1-\beta)} = m\right) = \mathsf{P}\left(S_{n,m}^{(\beta,1-\alpha)} = n\right).$$

With the help of (2.1) and Lemma 2.1 one can find upper bounds for the covariances [1, Theorem 2.4].

**Lemma 2.2.** *If $(\alpha, \beta, \gamma) \in \mathcal{F}_+$ then*

$$\left|\mathrm{Cov}(X_{k_1,\ell_1}, X_{k_2,\ell_2})\right| \le C_{\alpha,\beta}\sqrt{k_1 + \ell_1 + k_2 + \ell_2}$$

*with some constant $C_{\alpha,\beta} > 0$.*
*If $(\alpha, \beta, \gamma) \in \mathcal{E}_1$ or $(\alpha, \beta, \gamma) \in \mathcal{E}_2$ then*

$$\left|\mathrm{Cov}(X_{k_1,\ell_1}, X_{k_2,\ell_2})\right| \le (k_1 \wedge k_2)\frac{|\gamma|^{|\ell_1-\ell_2|}}{1-\gamma^2} \quad \text{or} \quad \left|\mathrm{Cov}(X_{k_1,\ell_1}, X_{k_2,\ell_2})\right| \le (\ell_1 \wedge \ell_2)\frac{|\gamma|^{|k_1-k_2|}}{1-\gamma^2},$$

*respectively.*
*If $(\alpha, \beta, \gamma) \in \mathcal{V}$ then*

$$\mathrm{Cov}(X_{k_1,\ell_1}, X_{k_2,\ell_2}) = (k_1 \wedge k_2)(\ell_1 \wedge \ell_2)\,\alpha^{|k_1-k_2|}\beta^{|\ell_1-\ell_2|}.$$

For $n \in \mathbb{N}$ let us introduce the piecewise constant random fields

$$Y_{1,0}^{(n)}(s,t) := X_{[ns]+1,[nt]}, \qquad Y_{0,1}^{(n)}(s,t) := X_{[ns],[nt]+1}, \qquad Y_{0,0}^{(n)}(s,t) := X_{[ns],[nt]},$$

for $0 \le s, t \in \mathbb{R}$.

The following result is a natural, but non-trivial generalization of Proposition 2.5 of [4].

**Proposition 2.3.** *If $(\alpha, \beta, \gamma) \in \mathcal{F}_{++}$ then there exists a constant $K_{\alpha,\beta} > 0$ such that*

$$\left|\mathrm{Cov}\left(Y_{i,j}^{(n)}(s_1,t_1), Y_{0,0}^{(n)}(s_2,t_2)\right) - \mathrm{Cov}\left(Y_{0,0}^{(n)}(s_1,t_1), Y_{0,0}^{(n)}(s_2,t_2)\right)\right| \le K_{\alpha,\beta}$$

*for all $n \in \mathbb{N}$, $0 < s_1, t_1, s_2, t_2 \in \mathbb{R}$, with $(i,j) \in \{(0,1),(1,0)\}$.*

In the proof of Proposition 2.3 we make use of the following lemmas. Lemma 2.4 is an obvious generalization of Theorem 2.4 of [4], while Lemma 2.5 can be easily obtained from a generalization of Theorem 2.6 of [4] using Taylor series expansion.

**Lemma 2.4.** *Let $k, \ell \in \mathbb{N}$, let $0 < \mu, \nu < 1$ be real numbers and let $\xi_k^{(\nu)}$ and $\eta_\ell^{(\mu)}$ be independent binomial random variables with parameters $(k, \nu)$ and $(\ell, \mu)$, respectively. Further, let $S_{k,\ell}^{(\nu,\mu)} := \xi_k^{(\nu)} + \eta_\ell^{(\mu)}$ and let*

$$m_{k,\ell} := \mathsf{E}S_{k,\ell}^{(\nu,\mu)}, \qquad b_{k,\ell} := \mathrm{Var}\left(S_{k,\ell}^{(\nu,\mu)}\right), \qquad x_{j,k,\ell} := (j - m_{k,\ell})/\sqrt{b_{k,\ell}}.$$

*Then for all $k, \ell \in \mathbb{N}$ and $j \in \{0, 1, \dots, k+\ell\}$, we have*

$$\left|\mathsf{P}\left(S_{k,\ell}^{(\nu,\mu)} = j\right) - \frac{1}{\sqrt{2\pi\, b_{k,\ell}}}\exp\left(-x_{j,k,\ell}^2/2\right)\right| \le \frac{C_{\mu,\nu}}{b_{k,\ell}},$$

*where $C_{\mu,\nu} > 0$ is a constant depending only on $\mu$ and $\nu$ (and not depending on $k, \ell, j$).*

**Lemma 2.5.** *Using notations of Lemma 2.4, for $j \in \{0, 1, \dots, k+\ell-1\}$ let*

$$\Delta_{j,k,\ell} := \left(\mathsf{P}\left(S_{k,\ell}^{(\nu,\mu)} = j+1\right) - \mathsf{P}\left(S_{k,\ell}^{(\nu,\mu)} = j\right)\right) + \frac{x_{j,k,\ell}}{\sqrt{2\pi}\, b_{k,\ell}}\exp\left(-x_{j,k,\ell}^2/2\right).$$

*Then there exists a constant $C_{\mu,\nu} > 0$ depending only on $\mu$ and $\nu$ (and not depending on $k, \ell, j$) such that*

$$\left|\Delta_{j,k,\ell}\right| \le \frac{C_{\mu,\nu}}{b_{k,\ell}^{3/2}}.$$

**Corollary 2.6.** *Let $0 < \mu, \nu < 1$ be real numbers. There exists a constant $C_{\mu,\nu} > 0$ such that for all $k, \ell \in \mathbb{N}$ and $j \in \{0, 1, \dots, k+\ell-1\}$ we have*

$$\left|\mathsf{P}\left(S_{k,\ell}^{(\nu,\mu)} = j+1\right) - \mathsf{P}\left(S_{k,\ell}^{(\nu,\mu)} = j\right)\right| \le \frac{C_{\mu,\nu}}{b_{k,\ell}}.$$

**Proof of Proposition 2.3.** Without loss of generality we may assume $(i, j) = (1, 0)$. Let

$$\omega^{(n)}_{\alpha,\beta}(s_1, t_1, s_2, t_2) := \operatorname{Cov}\left(Y^{(n)}_{1,0}(s_1, t_1), Y^{(n)}_{0,0}(s_2, t_2)\right) - \operatorname{Cov}\left(Y^{(n)}_{0,0}(s_1, t_1), Y^{(n)}_{0,0}(s_2, t_2)\right).$$

Consider first the case $[ns_1] \ge [ns_2]$ and $[nt_1] \ge [nt_2]$. Obviously, one may assume $[ns_2] \ge 1$ and $[nt_2] \ge 1$. From the definition of random fields $Y^{(n)}_{1,0}$ and $Y^{(n)}_{0,0}$ with the help of Lemma 2.1 and (2.1) we obtain

$$\omega^{(n)}_{\alpha,\beta}(s_1, t_1, s_2, t_2) = \sum_{k=0}^{[ns_2]-1}\sum_{\ell=0}^{[nt_2]-1}(1-\alpha)\Big(\mathsf{P}\left(S^{(\alpha,1-\beta)}_{[ns_1]-[ns_2]+k,[nt_1]-[nt_2]+\ell} = [ns_1]-[ns_2]+k+1\right) - \mathsf{P}\left(S^{(\alpha,1-\beta)}_{[ns_1]-[ns_2]+k,[nt_1]-[nt_2]+\ell} = [ns_1]-[ns_2]+k\right)\Big)\mathsf{P}\left(S^{(\alpha,1-\beta)}_{k,\ell} = k\right).$$

Hence, one can use the local versions of the CLT given in Lemmas 2.4 and 2.5 yielding approximation

$$\omega^{(n)}_{\alpha,\beta}(s_1, t_1, s_2, t_2) \approx \widetilde{E}^{(n)}_{\alpha,\beta}(s_1, t_1, s_2, t_2) := -\frac{1-\alpha}{2\pi}\sum_{k=1}^{[ns_2]-1}\sum_{\ell=1}^{[nt_2]-1} f_1(b_{k,\ell}, a_{k,\ell})$$
$$= -\frac{1-\alpha}{2\pi}\int_1^{[ns_2]}\int_1^{[nt_2]} f_1(b_{[y],[z]}, a_{[y],[z]})\,\mathrm{d}z\,\mathrm{d}y,$$

where

$$f_1(u, v) := \frac{v + g_{\alpha,\beta}}{u^{1/2}(u + q_{\alpha,\beta})^{3/2}}\exp\left(-\frac{v^2}{2u}\right)\exp\left(-\frac{(v + g_{\alpha,\beta})^2}{2(u + q_{\alpha,\beta})}\right),$$

and

$$\begin{aligned} b_{k,\ell} &= \alpha(1-\alpha)k + \beta(1-\beta)\ell, \quad q_{\alpha,\beta} := \alpha(1-\alpha)([ns_1]-[ns_2]) + \beta(1-\beta)([nt_1]-[nt_2]) \ge 0, \\ a_{k,\ell} &:= (1-\alpha)k - (1-\beta)\ell, \qquad g_{\alpha,\beta} := (1-\alpha)([ns_1]-[ns_2]) - (1-\beta)([nt_1]-[nt_2]). \end{aligned} \tag{2.3}$$

Using Lemmas 2.4 and 2.5, as for $z \ge 0$ we have $z\exp(-z) \le 1$, direct calculations show that for the error

$$\widetilde{\Delta}^{(n)}_{\alpha,\beta}(s_1, t_1, s_2, t_2) := \omega^{(n)}_{\alpha,\beta}(s_1, t_1, s_2, t_2) - \widetilde{E}^{(n)}_{\alpha,\beta}(s_1, t_1, s_2, t_2)$$

we have

$$\left|\widetilde{\Delta}^{(n)}_{\alpha,\beta}(s_1, t_1, s_2, t_2)\right| \le C_{\alpha,\beta}\left(\widetilde{\Delta}^{(n,1)}_{\alpha,\beta}(s_1, t_1, s_2, t_2) + \widetilde{\Delta}^{(n,2)}_{\alpha,\beta}(s_1, t_1, s_2, t_2) + \widetilde{\Delta}^{(n,3)}_{\alpha,\beta}(s_1, t_1, s_2, t_2)\right),$$

where $C_{\alpha,\beta}$ is a positive constant and

$$\widetilde{\Delta}^{(n,1)}_{\alpha,\beta}(s_1, t_1, s_2, t_2) := \sum_{k=1}^{[ns_2]-1}\sum_{\ell=1}^{[nt_2]-1}\frac{1}{b_{k,\ell}^{5/2}} + \sum_{k=1}^{[ns_2]-1}\left(\frac{1}{b_{k,1}^2} + \frac{\alpha^k}{b_{k,0}}\right) + \sum_{\ell=1}^{[nt_2]-1}\left(\frac{1}{b_{1,\ell}^2} + \frac{\beta^\ell}{b_{0,\ell}}\right) + 1,$$
$$\widetilde{\Delta}^{(n,2)}_{\alpha,\beta}(s_1, t_1, s_2, t_2) := \sum_{k=2}^{[ns_2]-1}\sum_{\ell=2}^{[nt_2]-1}\frac{1}{b_{k,\ell}^2}\exp\left(-\frac{a_{k,\ell}^2}{2b_{k,\ell}}\right),$$
$$\widetilde{\Delta}^{(n,3)}_{\alpha,\beta}(s_1, t_1, s_2, t_2) := \sum_{k=2}^{[ns_2]-1}\sum_{\ell=2}^{[nt_2]-1}\frac{1}{b_{k,\ell}(b_{k,\ell} + q_{\alpha,\beta})}\exp\left(-\frac{(a_{k,\ell} + g_{\alpha,\beta})^2}{4(b_{k,\ell} + q_{\alpha,\beta})}\right).$$

Long but straightforward calculations yield the existence of a constant $0 < K^{(1)}_{\alpha,\beta} < \infty$ not depending on $s_1, t_1, s_2, t_2$ and $n$ such that

$$\left|\widetilde{\Delta}^{(n)}_{\alpha,\beta}(s_1, t_1, s_2, t_2)\right| \le K^{(1)}_{\alpha,\beta}. \tag{2.4}$$

Further, let

$$\Delta^{(n)}_{\alpha,\beta}(s_1, t_1, s_2, t_2) := \widetilde{E}^{(n)}_{\alpha,\beta}(s_1, t_1, s_2, t_2) - E^{(n)}_{\alpha,\beta}(s_1, t_1, s_2, t_2),$$

where

$$E^{(n)}_{\alpha,\beta}(s_1, t_1, s_2, t_2) := -\frac{1-\alpha}{2\pi}\int_1^{[ns_2]}\int_1^{[nt_2]} f_1(b_{y,z}, a_{y,z})\,\mathrm{d}z\,\mathrm{d}y.$$

Obviously,

$$\left|\Delta^{(n)}_{\alpha,\beta}(s_1, t_1, s_2, t_2)\right| \le \frac{1-\alpha}{2\pi}\left(\Delta^{(n,1)}_{\alpha,\beta}(s_1, t_1, s_2, t_2) + \Delta^{(n,2)}_{\alpha,\beta}(s_1, t_1, s_2, t_2)\right),$$

where

$$\Delta^{(n,1)}_{\alpha,\beta}(s_1,t_1,s_2,t_2) := \int_1^{[ns_2]}\int_1^{[nt_2]} \left|f_1(b_{[y],[z]},a_{[y],[z]}) - f_1(b_{[y],[z]},a_{y,z})\right| \,\mathrm{d}z\,\mathrm{d}y,$$

$$\Delta^{(n,2)}_{\alpha,\beta}(s_1,t_1,s_2,t_2) := \int_1^{[ns_2]}\int_1^{[nt_2]} \left|f_1(b_{[y],[z]},a_{y,z}) - f_1(b_{y,z},a_{y,z})\right| \,\mathrm{d}z\,\mathrm{d}y.$$

Again, tedious but straightforward calculations show the existence of a constant $0 < K^{(2)}_{\alpha,\beta} < \infty$ such that

$$\left|\widetilde{\Delta}^{(n)}_{\alpha,\beta}(s_1,t_1,s_2,t_2)\right| \le K^{(2)}_{\alpha,\beta}. \tag{2.5}$$

Finally, one can show the existence of a constant $0 < K^{(3)}_{\alpha,\beta} < \infty$ such that

$$\left|E^{(n)}_{\alpha,\beta}(s_1,t_1,s_2,t_2)\right| \le K^{(3)}_{\alpha,\beta}, \tag{2.6}$$

which together with (2.4) and (2.5) implies the statement of the proposition in the case $[ns_1] \ge [ns_2]$ and $[nt_1] \ge [nt_2]$. For more details, see [6].

By symmetry, case $[ns_1] < [ns_2]$, $[nt_1] < [nt_2]$ can be handled in the same way as case $[ns_1] \ge [ns_2]$, $[nt_1] \ge [nt_2]$, while in case $[ns_1] < [ns_2]$, $[nt_1] \ge [nt_2]$ we have

$$\omega^{(n)}_{\alpha,\beta}(s_1,t_1,s_2,t_2) = \sum_{k=0}^{[ns_1]-1}\sum_{\ell=0}^{[nt_2]-1}(1-\alpha)\left(\mathsf{P}\left(S^{(\alpha,1-\beta)}_{k,[nt_1]-[nt_2]+\ell} = k+1\right) - \mathsf{P}\left(S^{(\alpha,1-\beta)}_{k,[nt_1]-[nt_2]+\ell} = k\right)\right)$$
$$\times \mathsf{P}\left(S^{(\alpha,1-\beta)}_{[ns_2]-[ns_1]+k,\ell} = [ns_2]-[ns_1]+k\right) + \sum_{\ell=0}^{[nt_2]-1}\beta^{\ell}.$$

Thus, local versions of the CLT given in Lemmas 2.4 and 2.5 yield approximation

$$\omega^{(n)}_{\alpha,\beta}(s_1,t_1,s_2,t_2) \approx E^{(n)}_{\alpha,\beta}(s_1,t_1,s_2,t_2) := -\frac{1-\alpha}{2\pi}\int_1^{[ns_1]}\int_1^{[nt_2]} f_2(b_{y,z},a_{y,z})\,\mathrm{d}z\,\mathrm{d}y,$$

where

$$f_2(u,v) := \frac{v - g_{2,\alpha,\beta}}{(u+\alpha g_{1,\alpha,\beta})^{1/2}(u+\beta g_{2,\alpha,\beta})^{3/2}} \exp\left(-\frac{(v+g_{1,\alpha,\beta})^2}{2(u+\alpha g_{1,\alpha,\beta})}\right)\exp\left(-\frac{(v-g_{2,\alpha,\beta})^2}{2(u+\beta g_{2,\alpha,\beta})}\right), \tag{2.7}$$

and

$$g_{1,\alpha,\beta} := (1-\alpha)\left([ns_2]-[ns_1]\right), \qquad g_{2,\alpha,\beta} := (1-\beta)\left([nt_1]-[nt_2]\right).$$

Using similar ideas as in case $[ns_1] \ge [ns_2]$, $[nt_1] \ge [nt_2]$ one can show that the error of the approximation is bounded with a bound not depending on $s_1, t_1, s_2, t_2$ and $n$, and one can verify (2.6) that completes the proof in case $[ns_1] < [ns_2]$, $[nt_1] \ge [nt_2]$. Finally, case $[ns_1] \ge [ns_2]$, $[nt_1] < [nt_2]$ follows by symmetry. For more details, see [6]. □

**Proposition 2.7.** *Let $0 < s_1, t_1, s_2, t_2 \in \mathbb{R}$ and let $(q_1,q_2), (r_1,r_2) \in \{(0,1),(1,0),(0,0)\}$.*
*If $(\alpha,\beta,\gamma) \in \mathcal{F}_{++}$ then*

$$\frac{1}{n^{1/2}}\mathrm{Cov}\left(Y^{(n)}_{q_1,q_2}(s_1,t_1), Y^{(n)}_{r_1,r_2}(s_2,t_2)\right) \to \frac{((1-\alpha)s_1)^{1/2}\wedge((1-\beta)t_1)^{1/2}}{\pi^{1/2}(\alpha+\beta)^{1/2}(1-\alpha)(1-\beta)}, \quad \text{if } s_1 = s_2 \text{ and } t_1 = t_2,$$

*otherwise, if $(1-\alpha)(s_1-s_2) \ne (1-\beta)(t_1-t_2)$ it tends to 0, as $n \to \infty$. Moreover, convergence to 0 has an exponential rate.*
*If $(\alpha,\beta,\gamma) \in \mathcal{E}_{1+}$ then*

$$\lim_{n\to\infty}\frac{1}{n}\mathrm{Cov}\left(Y^{(n)}_{q_1,q_2}(s_1,t_1), Y^{(n)}_{r_1,r_2}(s_2,t_2)\right) = \begin{cases}(s_1\wedge s_2)\beta^{|q_2-r_2|}(1-\gamma^2)^{-1}, & \text{if } t_1 = t_2,\\ 0, & \text{otherwise,}\end{cases}$$

*while for $(\alpha,\beta,\gamma) \in \mathcal{E}_{2+}$ we have*

$$\lim_{n\to\infty}\frac{1}{n}\mathrm{Cov}\left(Y^{(n)}_{q_1,q_2}(s_1,t_1), Y^{(n)}_{r_1,r_2}(s_2,t_2)\right) = \begin{cases}(t_1\wedge t_2)\alpha^{|q_1-r_1|}(1-\gamma^2)^{-1}, & \text{if } s_1 = s_2,\\ 0, & \text{otherwise.}\end{cases}$$

*Moreover, convergences to 0 in both cases have exponential rates.*
*If $(\alpha,\beta,\gamma) \in \mathcal{V}_{+}$ then*

$$\lim_{n\to\infty}\frac{1}{n^2}\mathrm{Cov}\left(Y^{(n)}_{q_1,q_2}(s_1,t_1), Y^{(n)}_{r_1,r_2}(s_2,t_2)\right) = (s_1\wedge s_2)(t_1\wedge t_2).$$

The proof of the above Proposition is strongly based on the following Lemma that is an obvious generalization of Theorem 2.3 of [4]. The statement of the Lemma can be obtained from Hoeffding's inequality, see [19].

**Lemma 2.8.** *Using notations of Lemma 2.4 let*

$$\theta := \frac{\nu k + \mu \ell}{k + \ell} \quad \textit{and} \quad I_\theta(x) := \begin{cases} x \log \dfrac{x}{\theta} + (1-x) \log \dfrac{1-x}{1-\theta}, & x \in [0,1], \\ \infty, & \textit{otherwise}. \end{cases}$$

*Then for $x \neq \theta$ we have $I_\theta(x) > I_\theta(\theta) = 0$, and*

$$P\left(S_{k,\ell}^{(\nu,\mu)} \geq (k+\ell)x\right) \leq \exp\left(-(k+\ell) I_\theta(x)\right), \quad \textit{for all } x > \theta,$$
$$P\left(S_{k,\ell}^{(\nu,\mu)} \leq (k+\ell)x\right) \leq \exp\left(-(k+\ell) I_\theta(x)\right), \quad \textit{for all } x < \theta.$$

**Proof of Proposition 2.7.** Let $(\alpha, \beta, \gamma) \in \mathcal{F}_{++}$ and let $0 < s, t \in \mathbb{R}$. By Theorem 1.1 of [1] we have

$$\lim_{n\to\infty} \frac{1}{n^{1/2}} \mathrm{Var}\left(Y_{0,0}^{(n)}(s,t)\right) = \frac{((1-\alpha)s)^{1/2} \wedge ((1-\beta)t)^{1/2}}{\pi^{1/2}(\alpha+\beta)^{1/2}(1-\alpha)(1-\beta)}.$$

Now, as e.g.

$$\begin{aligned} \mathrm{Cov}\left(Y_{1,0}^{(n)}(s,t), Y_{0,1}^{(n)}(s,t)\right) &= \mathrm{Cov}\left(Y_{0,1}^{(n)}(s,t), Y_{0,0}^{(n)}(s+1/n,t)\right) - \mathrm{Cov}\left(Y_{0,0}^{(n)}(s,t), Y_{0,0}^{(n)}(s+1/n,t)\right) \\ &\quad + \mathrm{Cov}\left(Y_{1,0}^{(n)}(s,t), Y_{0,0}^{(n)}(s,t)\right) - \mathrm{Var}\left(Y_{0,0}^{(n)}(s,t), Y_{0,0}^{(n)}(s,t)\right) + \mathrm{Var}\left(Y_{0,0}^{(n)}(s,t)\right), \end{aligned}$$

by Proposition 2.3 the limit of $n^{-1/2}\mathrm{Cov}\left(Y_{q_1,q_2}^{(n)}(s_1,t_1), Y_{r_1,r_2}^{(n)}(s_2,t_2)\right)$ as $n \to \infty$ equals the limit of $n^{1/2}\mathrm{Var}\left(Y_{0,0}^{(n)}(s,t)\right)$ for all $(q_1,q_2), (r_1,r_2) \in \{(0,1),(1,0),(0,0)\}$.

Assume first $s_1 > s_2$, $t_1 > t_2$ that implies $[ns_1] + q_1 \geq [ns_2] + r_1$ and $[nt_1] + q_2 \geq [nt_2] + r_2$ if $n \in \mathbb{N}$ is large enough. In this case (2.1) and Lemma 2.1 imply

$$\mathrm{Cov}\left(Y_{q_1,q_2}^{(n)}(s_1,t_1), Y_{r_1,r_2}^{(n)}(s_2,t_2)\right) = \sum_{k=0}^{[ns_2]+r_1-1} \sum_{\ell=0}^{[nt_2]+r_2-1} P\left(S_{k,\ell}^{(\alpha,1-\beta)} = k\right) P\left(S_{g_{1,n}+k,g_{2,n}+\ell}^{(\alpha,1-\beta)} = g_{1,n}+k\right),$$

where

$$g_{1,n} := |[ns_1] - [ns_2] + q_1 - r_1|, \qquad g_{2,n} := |[nt_1] - [nt_2] + q_2 - r_2|.$$

We are going to apply Lemma 2.8 for the terms of the above sum. Let

$$\begin{aligned} \theta_n &:= \frac{\alpha(g_{1,n}+k) + (1-\beta)(g_{2,n}+\ell)}{g_{1,n}+k+g_{2,n}+\ell} \to \frac{\alpha(s_1-s_2)+(1-\beta)(t_1-t_2)}{s_1-s_2+t_1-t_2} =: \theta, \\ \omega_n &:= \frac{g_{1,n}+k}{g_{1,n}+k+g_{2,n}+\ell} - \theta_n \to \frac{(1-\alpha)(s_1-s_2)-(1-\beta)(t_1-t_2)}{s_1-s_2+t_1-t_2} =: \omega, \end{aligned}$$

as $n \to \infty$. If $(1-\alpha)(s_1-s_2) > (1-\beta)(t_1-t_2)$ then $\omega > 0$. Hence, for sufficiently large $n \in \mathbb{N}$ we have $\omega_n \geq \omega/2 > 0$ and in this way

$$P\left(S_{g_{1,n}+k,g_{2,n}+\ell}^{(\alpha,1-\beta)} = g_{1,n}+k\right) \leq P\left(S_{g_{1,n}+k,g_{2,n}+\ell}^{(\alpha,1-\beta)} \geq (g_{1,n}+k+g_{2,n}+\ell)(\theta_n+\omega/2)\right)$$

for all $k \in \{0,\ldots,[ns_2]+r_1-1\}$ and $\ell \in \{0,\ldots,[nt_2]+r_2-1\}$. Further, for sufficiently large $n \in \mathbb{N}$ and for all $k \in \{0,\ldots,[ns_2]+r_1-1\}$ and $\ell \in \{0,\ldots,[nt_2]+r_2-1\}$

$$g_{1,n}+k+g_{2,n}+\ell = [ns_1]-[ns_2]+[nt_1]-[nt_2]+q_1-r_1+q_2-r_2+k+\ell \geq (s_1-s_2+t_1-t_2)n/2$$

holds, so Lemma 2.8 yields

$$P\left(S_{g_{1,n}+k,g_{2,n}+\ell}^{(\alpha,1-\beta)} \geq (g_{1,n}+k+g_{2,n}+\ell)(\theta_n+\omega/2)\right) \leq \exp\left(-n(s_1-s_2+t_1-t_2)I_{\theta_n}(\theta_n+\omega/2)/2\right).$$

Since $\omega > 0$ implies $I_{\theta_n}(\theta_n+\omega/2) > 0$, with the help of the above inequality we obviously obtain

$$n^{-1/2}\mathrm{Cov}\left(Y_{q_1,q_2}^{(n)}(s_1,t_1), Y_{r_1,r_2}^{(n)}(s_2,t_2)\right) \to 0 \tag{2.8}$$

in exponential rate as $n \to \infty$. If $(1-\alpha)(s_1-s_2) < (1-\beta)(t_1-t_2)$ then $\omega < 0$. Hence, for sufficiently large $n \in \mathbb{N}$ we have $\omega_n \le \omega/2 < 0$ and in this way

$$P\left(S^{(\alpha,1-\beta)}_{g_{1,n}+k,g_{2,n}+\ell} = g_{1,n}+k\right) \le P\left(S^{(\alpha,1-\beta)}_{g_{1,n}+k,g_{2,n}+\ell} \le (g_{1,n}+k+g_{2,n}+\ell)(\theta_n+\omega/2)\right)$$

for all $k \in \{0,\dots,[ns_2]+r_1-1\}$ and $\ell \in \{0,\dots,[nt_2]+r_2-1\}$. Using again Lemma 2.8 we obtain

$$P\left(S^{(\alpha,1-\beta)}_{g_{1,n}+k,g_{2,n}+\ell} \le (g_{1,n}+k+g_{2,n}+\ell)(\theta_n+\omega/2)\right) \le \exp\left(-n(s_1-s_2+t_1-t_2)I_{\theta_n}(\theta_n+\omega/2)/2\right),$$

which directly implies (2.8).

Case $s_1 < s_2$, $t_1 < t_2$ follows by symmetry. The remaining cases can be handled in the same way as the earlier ones.

Now, let $(\alpha,\beta,\gamma) \in \mathcal{E}_{1+}$. Obviously,

$$\operatorname{Cov}\left(Y^{(n)}_{q_1,q_2}(s_1,t_1), Y^{(n)}_{r_1,r_2}(s_2,t_2)\right) = (([ns_1]+q_1) \wedge ([ns_2]+r_1))\,\beta^{|[nt_1]-[nt_2]+q_2-r_2|}\frac{1-\beta^{([nt_1]+q_2)\wedge([nt_2]+r_2)}}{1-\beta^2},$$

that immediately implies the statement of the Proposition. Case $(\alpha,\beta,\gamma) \in \mathcal{E}_{2+}$ can be handled in the same way.

Finally, in case $(\alpha,\beta,\gamma) \in \mathcal{V}_+$ the statement directly follows from Lemma 2.2. □

## 3. Proof of Proposition 1.6

According to the results of the Introduction in the following sections we may assume $\alpha \ge 0$, $\beta \ge 0$ and $\gamma \ge 0$ if $\alpha\beta\gamma \ge 0$ and $\alpha > 0$, $\beta > 0$ and $\gamma < 0$ if $\alpha\beta\gamma < 0$. In this case $\Psi_{\alpha,\beta}$ equals the three-by-three matrix of ones denoted by $\mathbf{1}$,

$$\Sigma_{\alpha,\beta} = \begin{bmatrix} 1 & \alpha\beta & \beta \\ \alpha\beta & 1 & \alpha \\ \beta & \alpha & 1 \end{bmatrix},$$

and under the conditions of Proposition 1.6 the coefficients $G(k-i,\ell-j;\alpha,\beta,\gamma)$ in representation (1.5) of $X_{k,\ell}$ are non-negative.

Obviously,

$$\frac{1}{n^2}\mathsf{E}B_n = \frac{1}{n^2}\sum_{(k,\ell)\in R_n}\mathsf{E}\left(\begin{bmatrix} X_{k-1,\ell} \\ X_{k,\ell-1} \\ X_{k-1,\ell-1}\end{bmatrix}\begin{bmatrix} X_{k-1,\ell} \\ X_{k,\ell-1} \\ X_{k-1,\ell-1}\end{bmatrix}^\top\right)$$
$$= \int_0^1\int_0^1 \begin{bmatrix} \operatorname{Var}Y^{(n)}_{0,1}(s,t) & \operatorname{Cov}\left(Y^{(n)}_{0,1}(s,t), Y^{(n)}_{1,0}(s,t)\right) & \operatorname{Cov}\left(Y^{(n)}_{0,1}(s,t), Y^{(n)}_{0,0}(s,t)\right) \\ \operatorname{Cov}\left(Y^{(n)}_{0,1}(s,t), Y^{(n)}_{1,0}(s,t)\right) & \operatorname{Var}Y^{(n)}_{1,0}(s,t) & \operatorname{Cov}\left(Y^{(n)}_{1,0}(s,t), Y^{(n)}_{0,0}(s,t)\right) \\ \operatorname{Cov}\left(Y^{(n)}_{0,1}(s,t), Y^{(n)}_{0,0}(s,t)\right) & \operatorname{Cov}\left(Y^{(n)}_{1,0}(s,t), Y^{(n)}_{0,0}(s,t)\right) & \operatorname{Var}Y^{(n)}_{0,0}(s,t) \end{bmatrix} \mathrm{d}s\,\mathrm{d}t.$$

By Lemma 2.2 if $(\alpha,\beta,\gamma) \in \mathcal{F}_{++}$ then

$$n^{-1/2}\left|\operatorname{Cov}\left(Y^{(n)}_{q_1,q_2}(s,t), Y^{(n)}_{r_1,r_2}(s,t)\right)\right| \le C_{\alpha,\beta}n^{-1/2}\left(2[ns]+2[nt]+2\right)^{1/2} \le C_{\alpha,\beta}(2s+2t+2)^{1/2},$$

where $C_{\alpha,\beta}$ is a positive constant, while in case $(\alpha,\beta,\gamma) \in \mathcal{E}_{1+} \cup \mathcal{E}_{2+}$ we have

$$n^{-1}\left|\operatorname{Cov}\left(Y^{(n)}_{q_1,q_2}(s,t), Y^{(n)}_{r_1,r_2}(s,t)\right)\right| \le \frac{C_{\alpha,\beta}}{n}\frac{[ns]+[nt]}{1-\gamma^2} \le \frac{C_{\alpha,\beta}(s+t)}{1-\gamma^2},$$

$(q_1,q_2), (r_1,r_2) \in \{(0,1),(1,0),(0,0)\}$. As both upper bounds are integrable on the unit square $[0,1]\times[0,1]$, the dominated convergence theorem applies. Hence, if $(\alpha,\beta,\gamma) \in \mathcal{F}_{++}$ by Proposition 2.7 we obtain

$$\lim_{n\to\infty}\frac{1}{n^{5/2}}\mathsf{E}B_n = \frac{1}{\sqrt{\pi(\alpha+\beta)(1-\alpha)(1-\beta)}}\int_0^1\int_0^1 ((1-\alpha)s)^{1/2} \wedge ((1-\beta)t)^{1/2}\,\mathrm{d}s\,\mathrm{d}t\,\mathbf{1} = \sigma^2_{\alpha,\beta}\mathbf{1}, \tag{3.1}$$

while in case $(\alpha,\beta,\gamma) \in \mathcal{E}_{1+} \cup \mathcal{E}_{2+}$ we have

$$\lim_{n\to\infty}\frac{1}{n^3}\mathsf{E}B_n = \frac{1}{1-\gamma^2}\int_0^1\int_0^1 (s\mathbb{1}_{\{\alpha=1\}} + t\mathbb{1}_{\{\beta=1\}})\mathrm{d}s\,\mathrm{d}t\,\Sigma_{\alpha,\beta} = \frac{1}{2(1-\gamma^2)}\Sigma_{\alpha,\beta}, \tag{3.2}$$

where $\mathbb{1}_H$ denotes the indicator of a set $H$.

Besides (3.1) and (3.2) to prove the first two statements of Proposition 1.6 one has to show

$$\frac{1}{n^\tau}\mathrm{Var}\left(\sum_{(k,\ell)\in R_n} X_{k-1+q_1,\ell-1+q_2}X_{k-1+r_1,\ell-1+r_2}\right) \tag{3.3}$$
$$= \frac{1}{n^\tau}\sum_{(k_1,\ell_1),(k_2,\ell_2)\in R_n} \mathrm{Cov}\left(X_{k_1-1+q_1,\ell_1-1+q_2}X_{k_1-1+r_1,\ell_1-1+r_2}, X_{k_2-1+q_1,\ell_2-1+q_2}X_{k_2-1+r_1,\ell_2-1+r_2}\right)$$
$$= \frac{1}{n^{\tau-4}}\int_0^1\int_0^1\int_0^1\int_0^1 \mathrm{Cov}\left(Y^{(n)}_{q_1,q_2}(s_1,t_1)Y^{(n)}_{r_1,r_2}(s_1,t_1), Y^{(n)}_{q_1,q_2}(s_2,t_2)Y^{(n)}_{r_1,r_2}(s_2,t_2)\right)\,\mathrm{d}s_1\,\mathrm{d}t_1\,\mathrm{d}s_2\,\mathrm{d}t_2 \to 0$$

as $n\to\infty$, where $\{q_1,q_2\},\{r_1,r_2\}\in\{(0,1),(1,0),(0,0)\}$ and

$$\tau := \begin{cases} 5, & \text{if } (\alpha,\beta,\gamma)\in\mathcal{F}_{++};\\ 6, & \text{if } (\alpha,\beta,\gamma)\in\mathcal{E}_{1+}\cup\mathcal{E}_{2+}.\end{cases} \tag{3.4}$$

Using Lemma 2.8 of [4] we have

$$\int_0^1\int_0^1\int_0^1\int_0^1 \mathrm{Cov}\left(Y^{(n)}_{q_1,q_2}(s_1,t_1)Y^{(n)}_{r_1,r_2}(s_1,t_1), Y^{(n)}_{q_1,q_2}(s_2,t_2)Y^{(n)}_{r_1,r_2}(s_2,t_2)\right)\,\mathrm{d}s_1\,\mathrm{d}t_1\,\mathrm{d}s_2\,\mathrm{d}t_2$$
$$\le M_4\int_0^1\int_0^1\int_0^1\int_0^1 \mathrm{Cov}\left(Y^{(n)}_{q_1,q_2}(s_1,t_1), Y^{(n)}_{q_1,q_2}(s_2,t_2)\right)\mathrm{Cov}\left(Y^{(n)}_{r_1,r_2}(s_1,t_1), Y^{(n)}_{r_1,r_2}(s_2,t_2)\right)\,\mathrm{d}s_1\,\mathrm{d}t_1\,\mathrm{d}s_2\,\mathrm{d}t_2$$
$$+ M_4\int_0^1\int_0^1\int_0^1\int_0^1 \mathrm{Cov}\left(Y^{(n)}_{q_1,q_2}(s_1,t_1), Y^{(n)}_{r_1,r_2}(s_2,t_2)\right)\mathrm{Cov}\left(Y^{(n)}_{r_1,r_2}(s_1,t_1), Y^{(n)}_{q_1,q_2}(s_2,t_2)\right)\,\mathrm{d}s_1\,\mathrm{d}t_1\,\mathrm{d}s_2\,\mathrm{d}t_2,$$

which by Lemma 2.2, Proposition 2.7 and by the dominated convergence theorem implies (3.3).

Finally, let

$$S_{n,1} := \sum_{(k,\ell)\in R_n}\left(X_{k,\ell-1}-X_{k-1,\ell-1}\right)^2, \qquad S_{n,3} := \sum_{(k,\ell)\in R_n}\left(X_{k,\ell-1}-X_{k-1,\ell-1}\right)X_{k-1,\ell-1},$$
$$S_{n,2} := \sum_{(k,\ell)\in R_n}\left(X_{k-1,\ell}-X_{k-1,\ell-1}\right)^2, \qquad S_{n,4} := \sum_{(k,\ell)\in R_n}\left(X_{k-1,\ell}-X_{k-1,\ell-1}\right)X_{k-1,\ell-1},$$
$$S_{n,5} := \sum_{(k,\ell)\in R_n}\left(X_{k,\ell-1}-X_{k-1,\ell-1}\right)\left(X_{k-1,\ell}-X_{k-1,\ell-1}\right), \tag{3.5}$$
$$T_n := \sum_{(k,\ell)\in R_n} X^2_{k-1,\ell-1}.$$

Observe, that $T_n$ is exactly entry (3, 3) of the matrix $B_n$. In case $(\alpha,\beta,\gamma)\in\mathcal{V}_+$

$$X_{k,\ell} = \sum_{i=1}^{k}\sum_{j=1}^{\ell}\varepsilon_{i,j}, \tag{3.6}$$

so by the continuous mapping theorem (CMT) [14]

$$\frac{1}{n^4}T_n = \int_0^1\int_0^1\left(\frac{1}{n}Y^{(n)}_{0,0}(s,t)\right)^2\mathrm{d}s\,\mathrm{d}t \xrightarrow{\mathcal{D}} \int_0^1\int_0^1 \mathcal{W}^2(s,t)\,\mathrm{d}s\,\mathrm{d}t \quad \text{as } n\to\infty \tag{3.7}$$

follows from Donsker's theorem [29]

$$\frac{1}{n}Y^{(n)}_{0,0}(s,t) = \frac{1}{n}\sum_{i=1}^{[ns]}\sum_{j=1}^{[nt]}\varepsilon_{i,j} \xrightarrow{\mathcal{D}} \mathcal{W}(s,t) \quad \text{as } n\to\infty. \tag{3.8}$$

Further, by (3.6) we have

$$X_{k,\ell}-X_{k,\ell-1} = \sum_{i=1}^{k}\varepsilon_{i,\ell} \quad\text{and}\quad X_{k,\ell}-X_{k-1,\ell} = \sum_{j=1}^{\ell}\varepsilon_{k,j}. \tag{3.9}$$

Using the independence of the error terms $\varepsilon_{i,j}$ short calculation shows

$$\mathsf{E}S_{n,1} = \mathsf{E}S_{n,2} = n^2(n-1)/2, \qquad \mathrm{Var}\left(S_{n,1}\right) = \mathrm{Var}\left(S_{n,2}\right) = O(n^5)$$

implying

$$n^{-3}S_{n,1} \xrightarrow{L_2} 1/2 \quad \text{and} \quad n^{-3}S_{n,2} \xrightarrow{L_2} 1/2 \quad \text{as } n \to \infty. \tag{3.10}$$

Applying again the independence of $\varepsilon_{i,j}$ it is not difficult to verify

$$\mathsf{E}S_{n,3} = \mathsf{E}S_{n,4} = 0, \qquad \mathrm{Var}\left(S_{n,3}\right) = \mathrm{Var}\left(S_{n,4}\right) = O(n^6)$$

and

$$\mathsf{E}S_{n,5} = 0, \qquad \mathrm{Var}\left(S_{n,5}\right) = O(n^4).$$

Hence, for all $\delta > 0$ we have

$$n^{-3-\delta}S_{n,3} \xrightarrow{L_2} 0, \qquad n^{-3-\delta}S_{n,4} \xrightarrow{L_2} 0 \quad \text{and} \quad n^{-2-\delta}S_{n,5} \xrightarrow{L_2} 0 \quad \text{as } n \to \infty. \tag{3.11}$$

Obviously,

$$\sum_{(k,\ell)\in R_n} X_{k-1,\ell}^2 = S_{n,2} + 2S_{n,2} + T_n, \qquad \sum_{(k,\ell)\in R_n} X_{k-1,\ell}X_{k-1,\ell-1} = S_{n,2} + T_n,$$
$$\sum_{(k,\ell)\in R_n} X_{k,\ell-1}^2 = S_{n,1} + 2S_{n,2} + T_n, \qquad \sum_{(k,\ell)\in R_n} X_{k,\ell-1}X_{k-1,\ell-1} = S_{n,1} + T_n,$$
$$\sum_{(k,\ell)\in R_n} X_{k,\ell-1}X_{k-1,\ell} = S_{n,3} + S_{n,4} + S_{n,5} + T_n,$$

so by (3.10) and (3.11) each entry of $n^{-4}B_n$ has the same limit in distribution, which completes the proof of Proposition 1.6. □

## 4. Proof of Proposition 1.7

To prove the first two statements of Proposition 1.7 first we show that $(A_n)_{n\geq 1}$ is a square integrable three dimensional martingale with respect to filtration $(\mathcal{F}_n)_{n\geq 1}$, where $\mathcal{F}_n$ denotes the $\sigma$-algebra generated by the random variables $\{\varepsilon_{k,\ell} : (k,\ell) \in R_n\}$. In order to do this consider the following decomposition of $A_n - A_{n-1}$, where $A_0 := 0$. Let $A_n^{(i)}$, $i = 1, 2, 3$, denote the components of $A_n$. By representation (1.5),

$$A_n^{(1)} - A_{n-1}^{(1)} = \sum_{(k,\ell)\in R_n\setminus R_{n-1}} \varepsilon_{k,\ell} \sum_{(i,j)\in R_{k-1,\ell}} G(k-1-i, \ell-j; \alpha, \beta, \gamma)\varepsilon_{i,j},$$
$$A_n^{(2)} - A_{n-1}^{(2)} = \sum_{(k,\ell)\in R_n\setminus R_{n-1}} \varepsilon_{k,\ell} \sum_{(i,j)\in R_{k,\ell-1}} G(k-i, \ell-1-j; \alpha, \beta, \gamma)\varepsilon_{i,j},$$
$$A_n^{(3)} - A_{n-1}^{(3)} = \sum_{(k,\ell)\in R_n\setminus R_{n-1}} \varepsilon_{k,\ell} \sum_{(i,j)\in R_{k-1,\ell-1}} G(k-1-i, \ell-1-j; \alpha, \beta, \gamma)\varepsilon_{i,j}.$$

Collecting first the terms containing only $\varepsilon_{i,j}$ with $(i,j) \in R_n \setminus R_{n-1}$, and then the rest, we obtain the decomposition

$$A_n - A_{n-1} = A_{n,1} + \sum_{(k,\ell)\in R_n\setminus R_{n-1}} \varepsilon_{k,\ell}A_{n,2,k,\ell}, \tag{4.1}$$

where $A_{n,1} = \left(A_{n,1}^{(1)}, A_{n,1}^{(2)}, 0\right)^\top$ and $A_{n,2,k,\ell} = \left(\widetilde{A}_{n,2,k-1,\ell}, \widetilde{A}_{n,2,k,\ell-1}, \widetilde{A}_{n,2,k-1,\ell-1}\right)^\top$ with

$$A_{n,1}^{(1)} := \sum_{(k,\ell)\in R_n\setminus R_{n-1}} \varepsilon_{k,\ell} \sum_{(i,j)\in R_{k-1,\ell}\setminus R_{n-1}} G(k-1-i, \ell-j; \alpha, \beta, \gamma)\varepsilon_{i,j} = \sum_{k=2}^{n}\sum_{i=1}^{k-1} \alpha^{k-1-i}\varepsilon_{i,n}\varepsilon_{k,n}, \tag{4.2}$$

$$A_{n,1}^{(2)} := \sum_{(k,\ell)\in R_n\setminus R_{n-1}} \varepsilon_{k,\ell} \sum_{(i,j)\in R_{k,\ell-1}\setminus R_{n-1}} G(k-i, \ell-1-j; \alpha, \beta, \gamma)\varepsilon_{i,j} = \sum_{\ell=2}^{n}\sum_{j=1}^{\ell-1} \beta^{\ell-1-j}\varepsilon_{n,j}\varepsilon_{n,\ell}, \tag{4.3}$$

$$\widetilde{A}_{n,2,k,\ell} := \sum_{(i,j)\in R_{k,\ell}\cap R_{n-1}} G(k-i, \ell-j; \alpha, \beta, \gamma)\varepsilon_{i,j}. \tag{4.4}$$

The first two components of $A_{n,1}$ are quadratic forms of the variables $\{\varepsilon_{i,j} : (i,j) \in R_n \setminus R_{n-1}\}$, hence $A_{n,1}$ is independent of $\mathcal{F}_{n-1}$. Besides this the terms $\widetilde{A}_{n,2,k,\ell}$ are linear combinations of the variables $\{\varepsilon_{i,j} : (i,j) \in R_{n-1}\}$, thus vectors $A_{n,2,k,\ell}$ are measurable with respect to $\mathcal{F}_{n-1}$. Consequently,

$$\mathsf{E}(A_n - A_{n-1} \mid \mathcal{F}_{n-1}) = \mathsf{E}A_{n,1} + \sum_{(k,\ell)\in R_n\setminus R_{n-1}} A_{n,2,k,\ell}\mathsf{E}(\varepsilon_{k,\ell} \mid \mathcal{F}_{n-1}) = 0.$$

Hence $(A_n)_{n\geq 1}$ is a square integrable martingale with respect to the filtration $(\mathcal{F}_n)_{n\geq 1}$.

By the Martingale Central Limit Theorem [20], in order to prove the first two statements of Proposition 1.7, it suffices to show that the conditional variances of the martingale differences converge in probability and to verify the conditional Lindeberg condition. To be precise, the statements are consequences of the following two propositions.

**Proposition 4.1.** *If* $(\alpha,\beta,\gamma)\in\mathcal{F}_{++}$ *then*

$$\frac{1}{n^{5/2}}\sum_{m=1}^{n}\mathsf{E}\left((A_m-A_{m-1})(A_m-A_{m-1})^\top \mid \mathcal{F}_{m-1}\right)\xrightarrow{\mathsf{P}}\sigma^2_{\alpha,\beta}\mathbf{1}\quad \text{as } n\to\infty.$$

*If* $(\alpha,\beta,\gamma)\in\mathcal{E}_{1+}\cup\mathcal{E}_{2+}$ *then*

$$\frac{1}{n^{3}}\sum_{m=1}^{n}\mathsf{E}\left((A_m-A_{m-1})(A_m-A_{m-1})^\top \mid \mathcal{F}_{m-1}\right)\xrightarrow{\mathsf{P}}\frac{1}{2(1-\gamma^2)}\Sigma_{\alpha,\beta}\quad \text{as } n\to\infty.$$

**Proposition 4.2.** *For all* $\delta>0$,

$$\frac{1}{n^{\tau/2}}\sum_{m=1}^{n}\mathsf{E}\left(\|A_m-A_{m-1}\|^2\mathbb{1}_{\{\|A_m-A_{m-1}\|\geq\delta n^{\tau/4}\}} \mid \mathcal{F}_{m-1}\right)\xrightarrow{\mathsf{P}}0$$

*as* $n\to\infty$, *where* $\tau$ *is defined by (3.4), i.e.*

$$\tau:=\begin{cases}5, & \text{if } (\alpha,\beta,\gamma)\in\mathcal{F}_{++};\\ 6, & \text{if } (\alpha,\beta,\gamma)\in\mathcal{E}_{1+}\cup\mathcal{E}_{2+}.\end{cases}$$

**Proof of Proposition 4.1.** Let $U_m:=\mathsf{E}\left((A_m-A_{m-1})(A_m-A_{m-1})^\top \mid \mathcal{F}_{m-1}\right)$. First we show that if $(\alpha,\beta,\gamma)\in\mathcal{F}_{++}$

$$\lim_{n\to\infty}\frac{1}{n^{5/2}}\sum_{m=1}^{n}\mathsf{E}U_m=\sigma^2_{\alpha,\beta}\mathbf{1}, \tag{4.5}$$

while in case $(\alpha,\beta,\gamma)\in\mathcal{E}_{1+}\cup\mathcal{E}_{2+}$ we have

$$\lim_{n\to\infty}\frac{1}{n^{3}}\sum_{m=1}^{n}\mathsf{E}U_m=\frac{1}{2(1-\gamma^2)}\Sigma_{\alpha,\beta}. \tag{4.6}$$

Obviously,

$$A_m-A_{m-1}=\sum_{(k,\ell)\in R_m\setminus R_{m-1}}\varepsilon_{k,\ell}\begin{bmatrix}X_{k-1,\ell}\\ X_{k,\ell-1}\\ X_{k-1,\ell-1}\end{bmatrix},$$

and by representation (1.5) and independence of the $\varepsilon_{i,j}$, the terms in the summation have zero mean and they are mutually uncorrelated. Since for all $\{q_1,q_2\},\{r_1,r_2\}\in\{(0,1),(1,0),(0,0)\}$ products $X_{k-1+q_1,\ell-1+q_2}X_{k-1+r_1,\ell-1+r_2}$ and $\varepsilon_{k,\ell}$ are independent we obtain

$$\begin{aligned}\mathsf{E}U_m&=\mathsf{E}(A_m-A_{m-1})(A_m-A_{m-1})^\top\\ &=\sum_{(k,\ell)\in R_m\setminus R_{m-1}}\mathsf{E}\left(\begin{bmatrix}X_{k-1,\ell}\\ X_{k,\ell-1}\\ X_{k-1,\ell-1}\end{bmatrix}\begin{bmatrix}X_{k-1,\ell}\\ X_{k,\ell-1}\\ X_{k-1,\ell-1}\end{bmatrix}^\top\right)\mathsf{E}\varepsilon_{k,\ell}=\mathsf{E}B_m-\mathsf{E}B_{m-1},\end{aligned} \tag{4.7}$$

where $B_0$ equals the three-by-three matrix of zeros. Consequently, (4.5) and (4.6) follow from (3.1) and (3.2), respectively.

By decomposition (4.1) and by the measurability of $A_{m,2,k,\ell}$ with respect to $\mathcal{F}_{m-1}$ one can derive

$$\begin{aligned}U_m&=\mathsf{E}\left(A_{m,1}A_{m,1}^\top \mid \mathcal{F}_{m-1}\right)+\sum_{(k,\ell)\in R_m\setminus R_{m-1}}\mathsf{E}\left(A_{m,1}\varepsilon_{k,\ell} \mid \mathcal{F}_{m-1}\right)A_{m,2,k,\ell}^\top\\ &\quad+\sum_{(k,\ell)\in R_m\setminus R_{m-1}}A_{m,2,k,\ell}\mathsf{E}\left(A_{m,1}^\top\varepsilon_{k,\ell} \mid \mathcal{F}_{m-1}\right)\\ &\quad+\sum_{(k_1,\ell_1)\in R_m\setminus R_{m-1}}\ \sum_{(k_2,\ell_2)\in R_m\setminus R_{m-1}}A_{m,2,k_1,\ell_1}A_{m,2,k_2,\ell_2}^\top\mathsf{E}\left(\varepsilon_{k_1,\ell_1}\varepsilon_{k_2,\ell_2} \mid \mathcal{F}_{m-1}\right).\end{aligned}$$

By the independence of $A_{m,1}$ and $\{\varepsilon_{k,\ell}:(k,\ell)\in R_m\setminus R_{m-1}\}$ from $\mathcal{F}_{m-1}$, and by $\mathsf{E}(A_{m,1}\varepsilon_{k,\ell})=(0,0,0)^\top$, one obtains

$$U_m=\mathsf{E}A_{m,1}A_{m,1}^\top+\sum_{(k,\ell)\in R_m\setminus R_{m-1}}A_{m,2,k,\ell}A_{m,2,k,\ell}^\top. \tag{4.8}$$

This means that to complete the proof of the proposition we have to show that for all $\{q_1, q_2\}, \{r_1, r_2\} \in \{(0, 1), (1, 0), (0, 0)\}$

$$\lim_{n\to\infty} \frac{1}{n^\tau} \mathrm{Var}\left(\sum_{m=1}^{n} \sum_{(k,\ell)\in R_m\setminus R_{m-1}} \widetilde{A}_{m,2,k-1+q_1,\ell-1+q_2}\widetilde{A}_{m,2,k-1+r_1,\ell-1+r_2}\right) = 0, \tag{4.9}$$

where $\tau$ is defined by (3.4). Obviously,

$$\begin{aligned}
&\mathrm{Var}\left(\sum_{m=1}^{n} \sum_{(k,\ell)\in R_m\setminus R_{m-1}} \widetilde{A}_{m,2,k-1+q_1,\ell-1+q_2}\widetilde{A}_{m,2,k-1+r_1,\ell-1+r_2}\right)\\
&= \sum_{m_1=1}^{n} \sum_{(k_1,\ell_1)\in R_{m_1}\setminus R_{m_1-1}} \sum_{m_2=1}^{n} \sum_{(k_2,\ell_2)\in R_{m_2}\setminus R_{m_2-1}} \mathrm{Cov}\left(\widetilde{A}_{m_1,2,k_1-1+q_1,\ell_1-1+q_2}\widetilde{A}_{m_1,2,k_1-1+r_1,\ell_1-1+r_2},\right.\\
&\qquad\qquad \left.\widetilde{A}_{m_2,2,k_2-1+q_1,\ell_2-1+q_2}\widetilde{A}_{m_2,2,k_2-1+r_1,\ell_2-1+r_2}\right),
\end{aligned}$$

and using Lemma 2.8 of [4] we have

$$\begin{aligned}
&\mathrm{Cov}\left(\widetilde{A}_{m_1,2,k_1-1+q_1,\ell_1-1+q_2}\widetilde{A}_{m_1,2,k_1-1+r_1,\ell_1-1+r_2}, \widetilde{A}_{m_2,2,k_2-1+q_1,\ell_2-1+q_2}\widetilde{A}_{m_2,2,k_2-1+r_1,\ell_2-1+r_2}\right)\\
&\le M_4 \mathrm{Cov}\left(\widetilde{A}_{m_1,2,k_1-1+q_1,\ell_1-1+q_2}, \widetilde{A}_{m_2,2,k_2-1+q_1,\ell_2-1+q_2}\right)\mathrm{Cov}\left(\widetilde{A}_{m_1,2,k_1-1+r_1,\ell_1-1+r_2}, \widetilde{A}_{m_2,2,k_2-1+r_1,\ell_2-1+r_2}\right)\\
&\quad + M_4 \mathrm{Cov}\left(\widetilde{A}_{m_1,2,k_1-1+q_1,\ell_1-1+q_2}, \widetilde{A}_{m_2,2,k_2-1+r_1,\ell_2-1+r_2}\right)\mathrm{Cov}\left(\widetilde{A}_{m_1,2,k_1-1+r_1,\ell_1-1+r_2}, \widetilde{A}_{m_2,2,k_2-1+q_1,\ell_2-1+q_2}\right).
\end{aligned}$$

Moreover, by (4.4) and representation (1.5)

$$\begin{aligned}
\mathrm{Cov}\left(\widetilde{A}_{m_1,2,k_1,\ell_1}, \widetilde{A}_{m_2,2,k_2,\ell_2}\right) &= \sum_{(i,j)\in R_{k_1\wedge k_2,\ell_1\wedge \ell_2}\cap R_{m_1\wedge m_2-1}} G(k_1-i,\ell_1-j;\alpha,\beta,\gamma)G(k_2-i,\ell_2-j;\alpha,\beta,\gamma)\\
&\le \mathrm{Cov}\left(X_{k_1,\ell_1}, X_{k_2,\ell_2}\right).
\end{aligned}$$

Furthermore,

$$\begin{aligned}
&\sum_{m_1=1}^{n} \sum_{(k_1,\ell_1)\in R_{m_1}\setminus R_{m_1-1}} \sum_{m_2=1}^{n} \sum_{(k_2,\ell_2)\in R_{m_2}\setminus R_{m_2-1}} \mathrm{Cov}\left(X_{k_1-1+q_1,\ell_1-1+q_2}, X_{k_2-1+q_1,\ell_2-1+q_2}\right)\\
&\qquad \times \mathrm{Cov}\left(X_{k_1-1+r_1,\ell_1-1+r_2}, X_{k_2-1+r_1,\ell_2-1+r_2}\right)\\
&\quad = \sum_{(k_1,\ell_1),(k_2,\ell_2)\in R_n} \mathrm{Cov}\left(X_{k_1-1+q_1,\ell_1-1+q_2}, X_{k_2-1+q_1,\ell_2-1+q_2}\right)\mathrm{Cov}\left(X_{k_1-1+r_1,\ell_1-1+r_2}, X_{k_2-1+r_1,\ell_2-1+r_2}\right).
\end{aligned}$$

Hence,

$$\begin{aligned}
&\mathrm{Var}\left(\sum_{m=1}^{n} \sum_{(k,\ell)\in R_m\setminus R_{m-1}} \widetilde{A}_{m,2,k-1+q_1,\ell-1+q_2}\widetilde{A}_{m,2,k-1+r_1,\ell-1+r_2}\right) \le M_4\\
&\quad \times \sum_{(k_1,\ell_1),(k_2,\ell_2)\in R_n} \Big(\mathrm{Cov}\left(X_{k_1-1+q_1,\ell_1-1+q_2}, X_{k_2-1+q_1,\ell_2-1+q_2}\right)\mathrm{Cov}\left(X_{k_1-1+r_1,\ell_1-1+r_2}, X_{k_2-1+r_1,\ell_2-1+r_2}\right)\\
&\quad + \mathrm{Cov}\left(X_{k_1-1+q_1,\ell_1-1+q_2}, X_{k_2-1+r_1,\ell_2-1+r_2}\right)\mathrm{Cov}\left(X_{k_1-1+r_1,\ell_1-1+r_2}, X_{k_2-1+q_1,\ell_2-1+q_2}\right)\Big),
\end{aligned}$$

so (4.9) can be proved in a similar way as (3.3). □

**Proof of Proposition 4.2.** Since

$$\mathbb{1}_{\{\|A_m - A_{m-1}\|\ge \delta n^{\tau/4}\}} \le \delta^{-2} n^{-\tau/2}\|A_m - A_{m-1}\|^2,$$

to prove Proposition 4.2 it suffices to show

$$\frac{1}{n^\tau}\sum_{m=1}^{n} \mathsf{E}\left(\|A_m - A_{m-1}\|^4 \mid \mathcal{F}_{m-1}\right) \xrightarrow{\mathsf{P}} 0 \quad \text{as } n\to\infty, \tag{4.10}$$

where $\tau$ is defined by (3.4). By the decomposition (4.1) of $A_m - A_{m-1}$ and by the inequality $(x+y)^4 \le 2^3(x^4+y^4)$ for $x, y \in \mathbb{R}$,

$$\|A_m - A_{m-1}\|^4 \le 2^3\|A_{m,1}\|^4 + 2^3\left\|\sum_{(k,\ell)\in R_m\setminus R_{m-1}} \varepsilon_{k,\ell}A_{m,2,k,\ell}\right\|^4.$$

By the independence of $A_{m,1}$ and $\mathcal{F}_{m-1}$, we have $\mathsf{E}\left(\|A_{m,1}\|^4 \mid \mathcal{F}_{m-1}\right) = \mathsf{E}\|A_{m,1}\|^4$. Applying the measurability of $A_{m,2,k,\ell}$ with respect to $\mathcal{F}_{m-1}$, we obtain

$$\mathsf{E}\left(\left\|\sum_{(k,\ell)\in R_m\setminus R_{m-1}} \varepsilon_{k,\ell}A_{m,2,k,\ell}\right\|^4 \mid \mathcal{F}_{m-1}\right) \le \left((M_4-3)^+ + 3\right)\left(\sum_{(k,\ell)\in R_m\setminus R_{m-1}} \|A_{m,2,k,\ell}\|^2\right)^2.$$

Hence, in order to prove (4.10), it suffices to show

$$\lim_{n\to\infty}\frac{1}{n^\tau}\sum_{m=1}^n \mathsf{E}\|A_{m,1}\|^4 = 0, \tag{4.11}$$

$$\lim_{n\to\infty}\frac{1}{n^\tau}\sum_{m=1}^n \mathsf{E}\left(\sum_{(k,\ell)\in R_m\setminus R_{m-1}} \|A_{m,2,k,\ell}\|^2\right)^2 = 0. \tag{4.12}$$

Using (4.2) and (4.3) it is easy to see that

$$\|A_{m,1}\|^4 \le 2\left(\sum_{k=2}^m\sum_{i=1}^{k-1}\alpha^{k-1-i}\varepsilon_{i,m}\varepsilon_{k,m}\right)^4 + 2\left(\sum_{\ell=2}^m\sum_{j=1}^{\ell-1}\beta^{\ell-1-j}\varepsilon_{m,j}\varepsilon_{m,\ell}\right)^4.$$

If $0<\alpha,\beta<1$ then by Lemma 12 of [2] we have $\mathsf{E}\|A_{m,1}\|^4 = O(m^2)$, while for $\alpha=1$ or $\beta=1$ a short calculation shows that $\mathsf{E}\|A_{m,1}\|^4 = O(m^4)$. Hence, (4.11) is satisfied for both possible values of $\tau$.

Furthermore,

$$\begin{aligned}\mathsf{E}\left(\sum_{(k,\ell)\in R_m\setminus R_{m-1}} \|A_{m,2,k,\ell}\|^2\right)^2 = \sum_{(k_1,\ell_1)(k_2,\ell_2)\in R_m\setminus R_{m-1}} &\mathsf{E}\left(\left(\widetilde{A}^2_{m,2,k_1-1,\ell_1} + \widetilde{A}^2_{m,2,k_1,\ell_1-1} + \widetilde{A}^2_{m,2,k_1-1,\ell_1-1}\right)\right.\\ &\left.\times\left(\widetilde{A}^2_{m,2,k_2-1,\ell_2} + \widetilde{A}^2_{m,2,k_2,\ell_2-1} + \widetilde{A}^2_{m,2,k_2-1,\ell_2-1}\right)\right).\end{aligned}$$

From Lemma 2.8 of [4] follows

$$\mathsf{E}\left(\widetilde{A}^2_{m,2,k_1,\ell_1}\widetilde{A}^2_{m,2,k_2,\ell_2}\right) \le 3M_4\mathsf{E}\widetilde{A}^2_{m,2,k_1,\ell_1}\mathsf{E}\widetilde{A}^2_{m,2,k_2,\ell_2},$$

while using (4.4) and representation (1.5) one can easily see $\mathsf{E}\widetilde{A}^2_{m,2,k,\ell} \le \operatorname{Var} X_{k,\ell}$. As by Lemma 2.2 there exists a positive constant $C_{\alpha,\beta}$ such that

$$\operatorname{Var} X_{k,\ell} \le \begin{cases} C_{\alpha,\beta}\sqrt{k+\ell}, & \text{if } (\alpha,\beta,\gamma)\in\mathcal{F}_{++};\\ C_{\alpha,\beta}(k+\ell), & \text{if } (\alpha,\beta,\gamma)\in\mathcal{E}_{1+}\cup\mathcal{E}_{2+},\end{cases}$$

short calculation shows

$$\mathsf{E}\left(\sum_{(k,\ell)\in R_m\setminus R_{m-1}} \|A_{m,2,k,\ell}\|^2\right)^2 = O\left(m^{\tau-2}\right),$$

which implies (4.12). □

Now, consider the case $(\alpha,\beta,\gamma)\in\mathcal{V}_+$. Let

$$Z_n := \sum_{(k,\ell)\in R_n} X_{k-1,\ell-1}\varepsilon_{k,\ell},$$

and $C_n^{(1)}$ and $C_n^{(2)}$ be the random sequences defined by (1.3) and (1.4), respectively. Using Eq. (1.2) which in this case takes form $\Delta_1\Delta_2 X_{k,\ell} = \varepsilon_{k,\ell}$ with $\Delta_1 X_{k,\ell} := X_{k,\ell} - X_{k-1,\ell}$, $\Delta_2 X_{k,\ell} := X_{k,\ell} - X_{k,\ell-1}$, from (3.8) and CMT we obtain

$$\begin{aligned}\frac{1}{n^2}Z_n &= \sum_{(k,\ell)\in R_n}\left(\frac{1}{n}Y^{(n)}\left(\frac{k-1}{n},\frac{\ell-1}{n}\right)\right)\Delta_1\Delta_2\left(\frac{1}{n}Y^{(n)}\left(\frac{k}{n},\frac{\ell}{n}\right)\right)\\ &= \int_0^1\int_0^1\left(\frac{1}{n}Y^{(n)}_{0,0}(s,t)\right)\left(\frac{1}{n}Y^{(n)}_{0,0}(\mathrm{d}s,\mathrm{d}t)\right) \xrightarrow{\mathcal{D}} \int_0^1\int_0^1 \mathcal{W}(s,t)\mathcal{W}(\mathrm{d}s,\mathrm{d}t) \quad \text{as } n\to\infty.\end{aligned}$$

Further, using the independence of the error terms $\varepsilon_{i,j}$ and (3.9) short calculation shows

$$\mathsf{E}C_n^{(1)} = \mathsf{E}C_n^{(2)} = 0, \qquad \operatorname{Var}\left(C_n^{(1)}\right) = \operatorname{Var}\left(C_n^{(2)}\right) = O(n^3).$$

Hence, for all $\delta > 0$ we have

$$n^{-3/2-\delta} C_n^{(1)} \xrightarrow{L_2} 0 \quad \text{and} \quad n^{-3/2-\delta} C_n^{(2)} \xrightarrow{L_2} 0 \quad \text{as } n \to \infty. \tag{4.13}$$

Obviously,

$$A_n - Z_n(1,1,1)^\top = \left(C_n^{(1)}, C_n^{(2)}, 0\right)^\top,$$

that together with (4.13) completes the proof. □

## 5. Proof of Proposition 1.8

According to the results of the Introduction it suffices to consider the case $\alpha \geq 0$, $\beta \geq 0$ and $\gamma \geq 0$ if $\alpha\beta\gamma \geq 0$ and $\alpha > 0$, $\beta > 0$ and $\gamma < 0$ if $\alpha\beta\gamma < 0$.

Consider the following expression of $\det(B_n)$

$$\det(B_n) = \sum_{(k_1,\ell_1)\in R_n} \sum_{(k_2,\ell_2)\in R_n} \sum_{(k_3,\ell_3)\in R_n} W_{k_1,\ell_1,k_2,\ell_2,k_3,\ell_3},$$

where

$$\begin{aligned} W_{k_1,\ell_1,k_2,\ell_2,k_3,\ell_3} := {} & 2X_{k_1-1,\ell_1}X_{k_1-1,\ell_1-1}X_{k_2,\ell_2-1}X_{k_2-1,\ell_2-1}X_{k_3-1,\ell_3}X_{k_3,\ell_3-1} \\ & + X_{k_1-1,\ell_1}^2 X_{k_2,\ell_2-1}^2 X_{k_3-1,\ell_3-1}^2 - X_{k_1,\ell_1-1}^2 X_{k_2-1,\ell_2}X_{k_2-1,\ell_2-1}X_{k_3-1,\ell_3}X_{k_3-1,\ell_3-1} \\ & - X_{k_1-1,\ell_1}^2 X_{k_2,\ell_2-1}X_{k_2-1,\ell_2-1}X_{k_3,\ell_3-1}X_{k_3-1,\ell_3-1} - X_{k_1-1,\ell_1-1}^2 X_{k_2,\ell_2-1}X_{k_2-1,\ell_2}X_{k_3,\ell_3-1}X_{k_3,\ell_3-1}. \end{aligned}$$

Short calculation shows that

$$\begin{aligned} W_{k_1,\ell_1,k_2,\ell_2,k_3,\ell_3} = {} & \left(X_{k_1,\ell_1-1} - X_{k_1-1,\ell_1-1}\right)^2 \left(X_{k_2-1,\ell_2} - X_{k_2-1,\ell_2-1}\right)^2 X_{k_3-1,\ell_3-1}^2 \\ & + 2\left(X_{k_1,\ell_1-1} - X_{k_1-1,\ell_1-1}\right)\left(X_{k_1-1,\ell_1} - X_{k_1-1,\ell_1-1}\right)\left(X_{k_2,\ell_2-1} - X_{k_2-1,\ell_2-1}\right) \\ & \times \left(X_{k_3-1,\ell_3} - X_{k_3-1,\ell_3-1}\right) X_{k_2-1,\ell_2-1}X_{k_3-1,\ell_3-1} \\ & - \left(X_{k_1,\ell_1-1} - X_{k_1-1,\ell_1-1}\right)\left(X_{k_1-1,\ell_1} - X_{k_1-1,\ell_1-1}\right)\left(X_{k_2,\ell_2-1} - X_{k_2-1,\ell_2-1}\right) \\ & \times \left(X_{k_2-1,\ell_2} - X_{k_2-1,\ell_2-1}\right) X_{k_3-1,\ell_3-1}^2 - \left(X_{k_1,\ell_1-1} - X_{k_1-1,\ell_1-1}\right)^2 \left(X_{k_2-1,\ell_2} - X_{k_2-1,\ell_2-1}\right) \\ & \times \left(X_{k_3-1,\ell_3} - X_{k_3-1,\ell_3-1}\right) X_{k_2-1,\ell_2-1}X_{k_3-1,\ell_3-1} - \left(X_{k_1-1,\ell_1} - X_{k_1-1,\ell_1-1}\right)^2 \\ & \times \left(X_{k_2,\ell_2-1} - X_{k_2-1,\ell_2-1}\right)\left(X_{k_3,\ell_3-1} - X_{k_3-1,\ell_3-1}\right) X_{k_2-1,\ell_2-1}X_{k_3-1,\ell_3-1}. \end{aligned} \tag{5.1}$$

First let $(\alpha,\beta,\gamma) \in \mathcal{F}_{++}$. Using notations (3.5) introduced in Section 3, by representation (5.1) we have

$$\begin{aligned} n^{-13/2}\det(B_n) = {} & \left(n^{-2}S_{n,1}\right)\left(n^{-2}S_{n,2}\right)\left(n^{-5/2}T_n\right) + 2\left(n^{-2}S_{n,5}\right)\left(n^{-9/4}S_{n,3}\right)\left(n^{-9/4}S_{n,4}\right) \\ & - \left(n^{-2}S_{n,5}\right)^2\left(n^{-5/2}T_n\right) - \left(n^{-2}S_{n,1}\right)\left(n^{-9/4}S_{n,4}\right)^2 - \left(n^{-2}S_{n,2}\right)\left(n^{-9/4}S_{n,3}\right)^2. \end{aligned} \tag{5.2}$$

Obviously,

$$\mathsf{E}S_{n,3} = n^2 \int_0^1 \int_0^1 \operatorname{Cov}\left(Y_{1,0}^{(n)}(s,t), Y_{0,0}^{(n)}(s,t)\right) - \operatorname{Var}\left(Y_{0,0}^{(n)}(s,t)\right) \mathrm{d}s\,\mathrm{d}t,$$

and with the help of (2.1), Lemmas 2.1 and 2.4, Corollary 2.6 and Lemma 2.8 of [4] one can show

$$\begin{aligned} \operatorname{Var}\left(S_{n,3}\right) \leq {} & n^4 \int_0^1\int_0^1\int_0^1\int_0^1 \left|\operatorname{Cov}\left(Y_{1,0}^{(n)}(s_1,t_1), Y_{0,0}^{(n)}(s_2,t_2)\right) - \operatorname{Cov}\left(Y_{0,0}^{(n)}(s_1,t_1), Y_{0,0}^{(n)}(s_2,t_2)\right)\right| \\ & \times \left|\operatorname{Cov}\left(Y_{1,0}^{(n)}(s_2,t_2), Y_{0,0}^{(n)}(s_1,t_1)\right) - \operatorname{Cov}\left(Y_{0,0}^{(n)}(s_2,t_2), Y_{0,0}^{(n)}(s_1,t_1)\right)\right| \mathrm{d}s_1\,\mathrm{d}t_1\,\mathrm{d}s_2\,\mathrm{d}t_2 \\ & + n^4 \int_0^1\int_0^1\int_0^1\int_0^1 \left(\left|\operatorname{Cov}\left(Y_{1,0}^{(n)}(s_1,t_1), Y_{0,0}^{(n)}(s_2,t_2)\right) - \operatorname{Cov}\left(Y_{0,0}^{(n)}(s_1,t_1), Y_{0,0}^{(n)}(s_2,t_2)\right)\right|\right. \\ & + \left.\left|\operatorname{Cov}\left(Y_{1,0}^{(n)}(s_1,t_1), Y_{0,0}^{(n)}(s_2+1/n,t_2)\right) - \operatorname{Cov}\left(Y_{0,0}^{(n)}(s_1,t_1), Y_{0,0}^{(n)}(s_2+1/n,t_2)\right)\right|\right) \\ & \times \left|\operatorname{Cov}\left(Y_{0,0}^{(n)}(s_1,t_1), Y_{0,0}^{(n)}(s_2,t_2)\right)\right| \mathrm{d}s_1\,\mathrm{d}t_1\,\mathrm{d}s_2\,\mathrm{d}t_2 + n^4(M_4-3)^+ C_{\alpha,\beta}, \end{aligned} \tag{5.3}$$

where $C_{\alpha,\beta}$ is a positive constant. In this way Propositions 2.3 and 2.7 and the dominated convergence theorem imply

$$\lim_{n\to\infty} n^{-9/4}\mathsf{E}S_{n,3} = 0 \quad \text{and} \quad \lim_{n\to\infty} n^{-9/2}\operatorname{Var}\left(S_{n,3}\right) = 0,$$

and the same result can be proved for $S_{n,4}$. Hence,

$$n^{-9/4}S_{n,3} \xrightarrow{\mathrm{L}_2} 0 \quad \text{and} \quad n^{-9/4}S_{n,4} \xrightarrow{\mathrm{L}_2} 0 \quad \text{as } n \to \infty. \tag{5.4}$$

Further,

$$\mathsf{E}S_{n,1} = n^2 \int_0^1 \int_0^1 \mathsf{E}\left(Y_{1,0}^{(n)}(s,t) - Y_{0,0}^{(n)}(s,t)\right)^2 \mathrm{d}s\,\mathrm{d}t,$$

and using representation (1.5) and Lemma 2.1 with notations of Lemma 2.4 we obtain

$$\mathsf{E}\left(Y_{1,0}^{(n)}(s,t) - Y_{0,0}^{(n)}(s,t)\right)^2 = \frac{1-\beta^{2[nt]}}{1-\beta^2} + (1-\alpha)^2 \sum_{k=0}^{[ns]-1}\sum_{\ell=0}^{[nt]-1}\left(\mathsf{P}\left(S_{k,\ell}^{(\alpha,1-\beta)} = k+1\right) - \mathsf{P}\left(S_{k,\ell}^{(\alpha,1-\beta)} = k\right)\right)^2.$$

Obviously, $\mathsf{E}\left(Y_{1,0}^{(n)}(s,t) - Y_{0,0}^{(n)}(s,t)\right)^2$ is a monotone increasing sequence and by Proposition 2.3 it has an upper bound independent of $s$, $t$ and $n$. Hence,

$$\lim_{n\to\infty} \mathsf{E}\left(Y_{1,0}^{(n)}(s,t) - Y_{0,0}^{(n)}(s,t)\right)^2 = \frac{1}{1-\beta^2} + (1-\alpha)^2 \varrho_{\alpha,\beta}^{(1)} > 0. \tag{5.5}$$

Similarly to (5.3) one can show

$$\begin{aligned}\mathrm{Var}\left(S_{n,1}\right) \le\ & n^4(M_4-3)^+ C_{\alpha,\beta} + 2n^4 \int_0^1\int_0^1\int_0^1\int_0^1 \Big(\Big|\mathrm{Cov}\left(Y_{1,0}^{(n)}(s_1,t_1), Y_{0,0}^{(n)}(s_2+1/n,t_2)\right) \\ & - \mathrm{Cov}\left(Y_{0,0}^{(n)}(s_1,t_1), Y_{0,0}^{(n)}(s_2+1/n,t_2)\right)\Big| \\ & + \Big|\mathrm{Cov}\left(Y_{1,0}^{(n)}(s_1,t_1), Y_{0,0}^{(n)}(s_2,t_2)\right) - \mathrm{Cov}\left(Y_{0,0}^{(n)}(s_1,t_1), Y_{0,0}^{(n)}(s_2,t_2)\right)\Big|\Big)^2 \mathrm{d}s_1\,\mathrm{d}t_1\,\mathrm{d}s_2\,\mathrm{d}t_2,\end{aligned} \tag{5.6}$$

where $C_{\alpha,\beta}$ is a positive constant. Again, Propositions 2.3 and 2.7, the dominated convergence theorem and (5.5) imply

$$\lim_{n\to\infty} n^{-2}\mathsf{E}S_{n,1} = \frac{1}{1-\beta^2} + (1-\alpha)^2\varrho_{\alpha,\beta}^{(1)} \quad \text{and} \quad \lim_{n\to\infty} n^{-4}\mathrm{Var}\left(S_{n,1}\right) = 0,$$

and a similar result can be proved for $S_{n,2}$. Hence,

$$n^{-2}S_{n,1} \xrightarrow{\mathrm{L}_2} \frac{1}{1-\beta^2} + (1-\alpha)^2\varrho_{\alpha,\beta}^{(1)} = \kappa_{\alpha,\beta}^{(1)} \quad \text{and} \quad n^{-2}S_{n,2} \xrightarrow{\mathrm{L}_2} \frac{1}{1-\alpha^2} + (1-\beta)^2\varrho_{\beta,\alpha}^{(1)} = \kappa_{\beta,\alpha}^{(1)} \tag{5.7}$$

as $n \to \infty$.

Finally,

$$\mathsf{E}S_{n,5} = n^2 \int_0^1\int_0^1 \mathsf{E}\left(Y_{1,0}^{(n)}(s,t) - Y_{0,0}^{(n)}(s,t)\right)\left(Y_{0,1}^{(n)}(s,t) - Y_{0,0}^{(n)}(s,t)\right)\mathrm{d}s\,\mathrm{d}t,$$

while for $\mathrm{Var}\left(S_{n,5}\right)$ one can find a result similar to (5.6). Using again representation (1.5) and Lemma 2.1 we obtain

$$\mathsf{E}\left(Y_{1,0}^{(n)}(s,t) - Y_{0,0}^{(n)}(s,t)\right)\left(Y_{0,1}^{(n)}(s,t) - Y_{0,0}^{(n)}(s,t)\right) = (1-\alpha)(1-\beta)\mathcal{V}_{\alpha,\beta}^{(n)}(s,t),$$

where

$$\mathcal{V}_{\alpha,\beta}^{(n)}(s,t) := \sum_{k=0}^{[ns]-1}\sum_{\ell=0}^{[nt]-1}\left(\mathsf{P}\left(S_{k,\ell}^{(\alpha,1-\beta)} = k+1\right) - \mathsf{P}\left(S_{k,\ell}^{(\alpha,1-\beta)} = k\right)\right)\left(\mathsf{P}\left(S_{\ell,k}^{(\beta,1-\alpha)} = \ell+1\right) - \mathsf{P}\left(S_{\ell,k}^{(\beta,1-\alpha)} = \ell\right)\right).$$

By Proposition 2.3 $\mathsf{E}\left(Y_{1,0}^{(n)}(s,t) - Y_{0,0}^{(n)}(s,t)\right)\left(Y_{0,1}^{(n)}(s,t) - Y_{0,0}^{(n)}(s,t)\right)$ is bounded with a bound independent of $s$, $t$ and $n$, but one has to show that $\mathcal{V}_{\alpha,\beta}^{(n)}(s,t)$ has a limit as $n \to \infty$. In order to prove this we show that for fixed $s$ and $t$ values $\mathcal{V}_{\alpha,\beta}^{(n)}(s,t)$ is a Cauchy sequence.

Let $n, m \in \mathbb{N}$, $n > m$, $0 < s, t < 1$, and without loss of generality we may assume $[ms] \ge 1$ and $[mt] \ge 1$. The local version of the CLT given in Lemma 2.5 yields approximation

$$\mathcal{V}_{\alpha,\beta}^{(n)}(s,t) - \mathcal{V}_{\alpha,\beta}^{(m)}(s,t) \approx \mathcal{D}_{\alpha,\beta}^{(n,m)}(s,t) := -\frac{1}{2\pi}\sum_{k=1}^{[ms]-1}\sum_{\ell=[mt]}^{[nt]-1} f(b_{k,\ell}, a_{k,\ell}) - \frac{1}{2\pi}\sum_{k=[ms]}^{[ns]-1}\sum_{\ell=1}^{[nt]-1} f(b_{k,\ell}, a_{k,\ell}),$$

where

$$f(u,v) = \frac{v^2}{u^3}\exp\left(-\frac{v^2}{2u}\right),$$

while $a_{k,\ell}$ and $b_{k,\ell}$ are given in (2.3). Using Lemma 2.5 one can easily show that for the error of the approximation we have

$$\begin{aligned}\left|\mathcal{V}^{(n)}_{\alpha,\beta}(s,t)-\mathcal{V}^{(m)}_{\alpha,\beta}(s,t)-\mathcal{D}^{(n,m)}_{\alpha,\beta}(s,t)\right| &\leq C_{\alpha,\beta}\left(\sum_{k=1}^{[ms]-1}\sum_{\ell=[mt]}^{[nt]-1}\frac{1}{b^{5/2}_{k,\ell}}+\sum_{k=[ms]}^{[ns]-1}\sum_{\ell=1}^{[nt]-1}\frac{1}{b^{5/2}_{k,\ell}}+\sum_{\ell=[mt]}^{[nt]-1}\frac{1}{b^{2}_{0,\ell}}+\sum_{k=[ms]}^{[ns]-1}\frac{1}{b^{2}_{k,0}}\right)\\ &\leq \frac{8C_{\alpha,\beta}}{\alpha\beta(1-\alpha)(1-\beta)}\left(\frac{1}{b^{1/2}_{1,[mt]}}+\frac{1}{b^{1/2}_{[ms],1}}+\frac{1}{b_{0,[mt]}}+\frac{1}{b_{[ms],0}}\right)\to 0 \qquad (5.8)\end{aligned}$$

as $m,n\to\infty$, where $C_{\alpha,\beta}$ is a positive constant. Further, as

$$F(u,v):=\int f(u,v)\,\mathrm{d}v=-\frac{v}{2u^2}\exp\left(-\frac{v^2}{u}\right)+\frac{\sqrt{\pi}}{4u^{3/2}}\widetilde{\Phi}\left(\frac{v}{u^{1/2}}\right),\quad \text{so } |F(u,v)|\leq\frac{1}{u^{3/2}},$$

using the notations of the proof of Proposition 2.3 we obtain

$$\begin{aligned}\left|\mathcal{D}^{(n,m)}_{\alpha,\beta}(s,t)\right| &\leq \frac{2}{\pi}\int_1^{[ms]}\int_{[mt]}^{[nt]} f(b_{y,z},a_{y,z})\,\mathrm{d}z\,\mathrm{d}y+\frac{2}{\pi}\int_{[ms]}^{[ns]}\int_1^{[nt]} f(b_{y,z},a_{y,z})\,\mathrm{d}z\,\mathrm{d}y\\ &\leq \frac{2H_{\alpha,\beta}}{\pi b^{-1/2}_{1,1}}\left(\int_{b_{1,[mt]}}^{b_{[ms],[nt]}}\int_{a_{1,[mt]}}^{a_{[ms],[mt]}} f(u,v)\,\mathrm{d}v\,\mathrm{d}u+\int_{b_{[ms],1}}^{b_{[ns],[nt]}}\int_{a_{[ms],[nt]}}^{a_{[ns],1}} f(u,v)\,\mathrm{d}v\,\mathrm{d}u\right)\\ &\leq \frac{4H_{\alpha,\beta}}{\pi b^{-1/2}_{1,1}}\left(\int_{b_{1,[mt]}}^{b_{[ms],[nt]}}\frac{1}{u^{3/2}}\,\mathrm{d}u+\int_{b_{[ms],1}}^{b_{[ns],[nt]}}\frac{1}{u^{3/2}}\,\mathrm{d}u\right)\leq\frac{8H_{\alpha,\beta}}{\pi b^{-1/2}_{1,1}}\left(\frac{1}{b^{1/2}_{1,[mt]}}+\frac{1}{b^{1/2}_{[ms],1}}\right)\to 0\end{aligned}$$

as $m,n\to\infty$, where $H_{\alpha,\beta}:=\left((\alpha+\beta)(1-\alpha)(1-\beta)b^{1/2}_{1,1}\right)^{-1}$, which together with (5.8) proves that $\mathcal{V}^{(n)}_{\alpha,\beta}(s,t)$ is Cauchy.

In this way

$$\lim_{n\to\infty}\mathsf{E}\left(Y^{(n)}_{1,0}(s,t)-Y^{(n)}_{0,0}(s,t)\right)\left(Y^{(n)}_{0,1}(s,t)-Y^{(n)}_{0,0}(s,t)\right)=\varrho^{(2)}_{\alpha,\beta},$$

so by Propositions 2.3 and 2.7 and the dominated convergence theorem we have

$$\lim_{n\to\infty} n^{-2}\mathsf{E}S_{n,5}=(1-\alpha)(1-\beta)\varrho^{(2)}_{\alpha,\beta}\quad\text{and}\quad \lim_{n\to\infty} n^{-4}\mathrm{Var}\left(S_{n,5}\right)=0.$$

Hence,

$$n^{-2}S_{n,5}\xrightarrow{\mathsf{L}_2}(1-\alpha)(1-\beta)\varrho^{(2)}_{\alpha,\beta}=\kappa^{(2)}_{\alpha,\beta}\quad\text{as } n\to\infty. \qquad (5.9)$$

By representation (5.2), Proposition 1.6 and limits (5.4), (5.7) and (5.9) imply

$$\begin{aligned}n^{-13/2}\det(B_n)&\xrightarrow{\mathsf{P}}\sigma^2_{\alpha,\beta}\left(\kappa^{(1)}_{\alpha,\beta}\kappa^{(1)}_{\beta,\alpha}-\left(\kappa^{(2)}_{\alpha,\beta}\right)^2\right)=\sigma^2_{\alpha,\beta}\det\left(\mathcal{K}_{\alpha,\beta}\right)\\ &=\sigma^2_{\alpha,\beta}\left(\frac{1}{(1-\alpha^2)(1-\beta^2)}+\frac{(1-\alpha)}{(1+\alpha)}\varrho^{(1)}_{\alpha,\beta}+\frac{(1-\beta)}{(1+\beta)}\varrho^{(1)}_{\beta,\alpha}+(1-\alpha)^2(1-\beta)^2\left(\varrho^{(1)}_{\alpha,\beta}\varrho^{(1)}_{\beta,\alpha}-\left(\varrho^{(2)}_{\alpha,\beta}\right)^2\right)\right)\end{aligned}$$

as $n\to\infty$, which is the first statement of Proposition 1.8. Observe, the positivity of the limit of $n^{-13/2}\det(B_n)$ follows from the non-negativity of $\varrho^{(1)}_{\alpha,\beta}\varrho^{(1)}_{\beta,\alpha}-\left(\varrho^{(2)}_{\alpha,\beta}\right)^2$ that is a trivial consequence of the Cauchy–Schwarz inequality.

Now, let $(\alpha,\beta,\gamma)\in\mathcal{E}_{1+}$, so

$$X_{k,\ell}=\sum_{i=1}^{k}\sum_{j=1}^{\ell}\beta^{\ell-j}\varepsilon_{i,j}.$$

In this way

$$X_{k,\ell-1}-X_{k-1,\ell-1}=\sum_{j=1}^{\ell-1}\beta^{\ell-1-j}\varepsilon_{k,j}\quad\text{and}\quad X_{k-1,\ell}-X_{k-1,\ell-1}=\sum_{i=1}^{k-1}\varepsilon_{i,\ell}-(1-\beta)X_{k-1,\ell-1}.$$

Using the independence of the error terms $\varepsilon_{i,j}$ short straightforward calculations show

$$\mathsf{E}S_{n,1}=\frac{n^2}{1-\gamma^2}+\frac{n(1-\gamma^{2n})}{(1-\gamma^2)^2},\qquad \mathsf{E}S_{n,3}=\mathsf{E}S_{n,5}=0,$$

$$\mathsf{E}S_{n,2}=\frac{n^2(n-1)}{2}+(1+\gamma)^2\mathsf{E}T_n,\qquad \mathsf{E}S_{n,4}=-(1+\gamma)\mathsf{E}T_n,$$

and

$$\operatorname{Var}(S_{n,1}) = O(n^2), \qquad \operatorname{Var}(S_{n,3}) = O(n^3), \qquad \operatorname{Var}(S_{n,5}) = O(n^4),$$
$$\operatorname{Var}(S_{n,2}) = O(n^5), \qquad \operatorname{Var}(S_{n,4}) = O(n^5).$$

Hence

$$n^{-2}S_{n,1} \xrightarrow{\mathsf{L}_2} \frac{1}{1-\gamma^2}, \qquad n^{-3}S_{n,2} \xrightarrow{\mathsf{L}_2} \frac{1}{1-\gamma}, \qquad n^{-3}S_{n,4} \xrightarrow{\mathsf{L}_2} -\frac{1}{2(1-\gamma)}, \tag{5.10}$$

and for all $\delta > 0$

$$n^{-3/2-\delta}S_{n,3} \xrightarrow{\mathsf{L}_2} 0, \qquad n^{-2-\delta}S_{n,5} \xrightarrow{\mathsf{L}_2} 0 \tag{5.11}$$

as $n \to \infty$. As by representation (5.1) we have

$$\begin{aligned} n^{-8}\det(B_n) &= \left(n^{-2}S_{n,1}\right)\left(n^{-3}S_{n,2}\right)\left(n^{-3}T_n\right) + 2\left(n^{-5/2}S_{n,5}\right)\left(n^{-5/2}S_{n,3}\right)\left(n^{-3}S_{n,4}\right) \\ &\quad - \left(n^{-5/2}S_{n,5}\right)^2\left(n^{-3}T_n\right) - \left(n^{-2}S_{n,1}\right)\left(n^{-3}S_{n,4}\right)^2 - \left(n^{-3}S_{n,2}\right)\left(n^{-5/2}S_{n,3}\right)^2. \end{aligned}$$

Proposition 1.6 and limits (5.10) and (5.11) imply the second statement of Proposition 1.8. Case $(\alpha, \beta, \gamma) \in \mathcal{E}_{2+}$ can be handled in the same way.

Finally, consider the case $(\alpha, \beta, \gamma) \in \mathcal{V}_+$. As by representation (5.1) we have

$$\begin{aligned} n^{-10}\det(B_n) &= \left(n^{-3}S_{n,1}\right)\left(n^{-3}S_{n,2}\right)\left(n^{-4}T_n\right) + 2\left(n^{-3}S_{n,5}\right)\left(n^{-7/2}S_{n,3}\right)\left(n^{-7/2}S_{n,4}\right) \\ &\quad - \left(n^{-3}S_{n,5}\right)^2\left(n^{-4}T_n\right) - \left(n^{-3}S_{n,1}\right)\left(n^{-7/2}S_{n,4}\right)^2 - \left(n^{-3}S_{n,2}\right)\left(n^{-7/2}S_{n,3}\right)^2, \end{aligned}$$

the last statement of Proposition 1.8 is a direct consequence of Slutsky's lemma, (3.7), (3.10) and (3.11). □

## 6. Proof of Proposition 1.9

To prove Proposition 1.9 we are going to apply the same ideas as in the proof of Proposition 1.7. Consider first the cases $(\alpha, \beta, \gamma) \in \mathcal{F}_{++}$ and $(\alpha, \beta, \gamma) \in \mathcal{E}_{1+} \cup \mathcal{E}_{2+}$. As $(A_n)_{n\geq 1}$ is a three dimensional square integrable martingale with respect to the filtration $(\mathcal{F}_n)_{n\geq 1}$, random sequence

$$C_n - C_{n-1} = C_{n,1} + \sum_{(k,\ell)\in R_n\setminus R_{n-1}} \varepsilon_{k,\ell} C_{n,2,k,\ell} \tag{6.1}$$

is a two dimensional martingale difference with respect to the same filtration, where

$$C_{n,1} := \begin{bmatrix} A_{n,1}^{(1)} \\ A_{n,1}^{(2)} \end{bmatrix}, \qquad C_{n,2,k,\ell} = \begin{bmatrix} C_{n,2,k,\ell}^{(1)} \\ C_{n,2,k,\ell}^{(2)} \end{bmatrix} := \begin{bmatrix} \widetilde{A}_{n,2,k-1,\ell} - \widetilde{A}_{n,2,k-1,\ell-1} \\ \widetilde{A}_{n,2,k,\ell-1} - \widetilde{A}_{n,2,k-1,\ell-1} \end{bmatrix},$$

with $A_{n,1}^{(1)}, A_{n,1}^{(2)}$ and $\widetilde{A}_{n,2,k,\ell}$ defined by (4.2)–(4.4), respectively. Here $C_{n,1}$ is independent of $\mathcal{F}_{n-1}$, while $C_{n,2,k,\ell}$ is measurable with respect to it. However, representation (6.1) is also valid in the case $(\alpha, \beta, \gamma) \in \mathcal{V}_+$, when

$$\sum_{(k,\ell)\in R_n\setminus R_{n-1}} \varepsilon_{k,\ell} C_{n,2,k,\ell}^{(1)} = \sum_{\ell=1}^{n-1}\sum_{i=1}^{n-1} \varepsilon_{i,\ell}\varepsilon_{n,\ell}, \qquad \sum_{(k,\ell)\in R_n\setminus R_{n-1}} \varepsilon_{k,\ell} C_{n,2,k,\ell}^{(2)} = \sum_{k=1}^{n-1}\sum_{j=1}^{n-1} \varepsilon_{k,j}\varepsilon_{k,n}.$$

Hence, $C_n - C_{n-1}$ is a martingale difference in this case, too. This means that according to the Martingale Central Limit Theorem the statement of Proposition 1.9 follows from the propositions below.

**Proposition 6.1.** *If* $(\alpha, \beta, \gamma) \in \mathcal{F}_{++}$ *then*

$$\frac{1}{n^2}\sum_{m=1}^{n} \mathsf{E}\left((C_m - C_{m-1})(C_m - C_{m-1})^\top \mid \mathcal{F}_{m-1}\right) \xrightarrow{\mathsf{P}} \mathcal{K}_{\alpha,\beta} \quad \text{as } n \to \infty.$$

*If* $(\alpha, \beta, \gamma) \in \mathcal{E}_{1+}$

$$\frac{1}{n^3}\sum_{m=1}^{n} \mathsf{E}\left(\left(C_m^{(1)} - C_{m-1}^{(1)}\right)^2 \mid \mathcal{F}_{m-1}\right) \xrightarrow{\mathsf{P}} \frac{1}{1-\gamma}, \qquad \frac{1}{n^2}\sum_{m=1}^{n} \mathsf{E}\left(\left(C_m^{(2)} - C_{m-1}^{(2)}\right)^2 \mid \mathcal{F}_{m-1}\right) \xrightarrow{\mathsf{P}} \frac{1}{1-\gamma^2}$$

*as* $n \to \infty$.

*If* $(\alpha, \beta, \gamma) \in \mathcal{E}_{2+}$ *then*

$$\frac{1}{n^2}\sum_{m=1}^{n} \mathsf{E}\left(\left(C_m^{(1)} - C_{m-1}^{(1)}\right)^2 \mid \mathcal{F}_{m-1}\right) \xrightarrow{\mathsf{P}} \frac{1}{1-\gamma^2}, \qquad \frac{1}{n^3}\sum_{m=1}^{n} \mathsf{E}\left(\left(C_m^{(2)} - C_{m-1}^{(2)}\right)^2 \mid \mathcal{F}_{m-1}\right) \xrightarrow{\mathsf{P}} \frac{1}{1-\gamma}$$

*as* $n \to \infty$.

*If* $(\alpha, \beta, \gamma) \in \mathcal{V}_+$ *then*

$$\frac{1}{n^3}\sum_{m=1}^{n} \mathsf{E}\left((C_m - C_{m-1})(C_m - C_{m-1})^\top \mid \mathcal{F}_{m-1}\right) \xrightarrow{\mathsf{P}} \frac{1}{2}\mathcal{I}_2 \quad \text{as } n \to \infty.$$

**Proposition 6.2.** *If* $(\alpha, \beta, \gamma) \in \mathcal{F}_{++}$ *then for all* $\delta > 0$

$$\frac{1}{n^2}\sum_{m=1}^{n} \mathsf{E}\left(\|C_m - C_{m-1}\|^2 \mathbb{1}_{\{\|C_m - C_{m-1}\| \geq \delta n\}} \mid \mathcal{F}_{m-1}\right) \xrightarrow{\mathsf{P}} 0 \quad \text{as } n \to \infty.$$

*If* $(\alpha, \beta, \gamma) \in \mathcal{E}_{1+} \cup \mathcal{E}_{2+}$ *then for all* $\delta > 0$

$$\frac{1}{n^3}\sum_{m=1}^{n} \mathsf{E}\left(|C_m^{(i)} - C_{m-1}^{(i)}|^2 \mathbb{1}_{\left\{|C_m^{(i)} - C_{m-1}^{(i)}| \geq \delta n^{3/2}\right\}} \mid \mathcal{F}_{m-1}\right) \xrightarrow{\mathsf{P}} 0$$

$$\frac{1}{n^2}\sum_{m=1}^{n} \mathsf{E}\left(|C_m^{(j)} - C_{m-1}^{(j)}|^2 \mathbb{1}_{\left\{|C_m^{(j)} - C_{m-1}^{(j)}| \geq \delta n\right\}} \mid \mathcal{F}_{m-1}\right) \xrightarrow{\mathsf{P}} 0$$

*as* $n \to \infty$, *where*

$$(i, j) := \begin{cases} (1, 2), & \text{if } (\alpha, \beta, \gamma) \in \mathcal{E}_{1+}; \\ (2, 1), & \text{if } (\alpha, \beta, \gamma) \in \mathcal{E}_{2+}. \end{cases}$$

*If* $(\alpha, \beta, \gamma) \in \mathcal{V}_+$ *then for all* $\delta > 0$

$$\frac{1}{n^3}\sum_{m=1}^{n} \mathsf{E}\left(\|C_m - C_{m-1}\|^2 \mathbb{1}_{\{\|C_m - C_{m-1}\| \geq \delta n^{3/2}\}} \,\Big|\, \mathcal{F}_{m-1}\right) \xrightarrow{\mathsf{P}} 0 \quad \text{as } n \to \infty.$$

**Proof of Proposition 6.1.** The proof is very similar to the proof of Proposition 4.1. The details can be found in [6]. □

**Proof of Proposition 6.2.** Similarly to the proof of Proposition 4.2 it suffices to show that if $(\alpha, \beta, \gamma) \in \mathcal{F}_{++}$ then

$$\frac{1}{n^4}\sum_{m=1}^{n} \mathsf{E}\left(\|C_m - C_{m-1}\|^4 \mid \mathcal{F}_{m-1}\right) \xrightarrow{\mathsf{P}} 0 \quad \text{as } n \to \infty,$$

if $(\alpha, \beta, \gamma) \in \mathcal{E}_{1+} \cup \mathcal{E}_{2+}$ then

$$\frac{1}{n^6}\sum_{m=1}^{n} \mathsf{E}\left(|C_m^{(i)} - C_{m-1}^{(i)}|^4 \mid \mathcal{F}_{m-1}\right) \xrightarrow{\mathsf{P}} 0, \qquad \frac{1}{n^4}\sum_{m=1}^{n} \mathsf{E}\left(|C_m^{(j)} - C_{m-1}^{(j)}|^4 \mid \mathcal{F}_{m-1}\right) \xrightarrow{\mathsf{P}} 0$$

as $n \to \infty$, where

$$(i, j) := \begin{cases} (1, 2), & \text{if } (\alpha, \beta, \gamma) \in \mathcal{E}_{1+}; \\ (2, 1), & \text{if } (\alpha, \beta, \gamma) \in \mathcal{E}_{2+}, \end{cases}$$

while for $(\alpha, \beta, \gamma) \in \mathcal{V}_+$

$$\frac{1}{n^6}\sum_{m=1}^{n} \mathsf{E}\left(\|C_m - C_{m-1}\|^4 \mid \mathcal{F}_{m-1}\right) \xrightarrow{\mathsf{P}} 0 \quad \text{as } n \to \infty.$$

The details can be found in [6]. □

## 7. Proof of Proposition 1.10

Let $(\alpha, \beta, \gamma) \in \mathcal{F}_{++}$. Using notations (3.5) introduced in Section 3 and definitions (1.3) and (1.4) after short calculation we obtain

$$n^{-11/2}\overline{B}_n A_n = \left(n^{-9/2} Q_n^{(1)}\right)\left(n^{-1} C_n\right) + \left(n^{-17/4} Q_n^{(2)}\right)(0, 0, 1)\left(n^{-5/4} A_n\right), \tag{7.1}$$

where $Q_n^{(1)}$ is a three-by-two matrix with entries

$$Q_{n,1,1}^{(1)} := S_{n,1}T_n - S_{n,3}^2, \qquad Q_{n,2,2}^{(1)} := S_{n,2}T_n - S_{n,4}^2,$$
$$Q_{n,1,2}^{(1)} = Q_{n,2,1}^{(1)} := S_{n,3}S_{n,4} - S_{n,5}T_n,$$
$$Q_{n,3,1}^{(1)} := \left(S_{n,3} + S_{n,5}\right)S_{n,3} - \left(S_{n,1} + S_{n,3}\right)S_{n,4} + \left(S_{n,5} - S_{n,1}\right)T_n,$$
$$Q_{n,3,2}^{(1)} := \left(S_{n,4} + S_{n,5}\right)S_{n,4} - \left(S_{n,2} + S_{n,4}\right)S_{n,3} + \left(S_{n,5} - S_{n,2}\right)T_n,$$

and $Q_n^{(2)} = \left(Q_{n,1}^{(2)}, Q_{n,2}^{(2)}, Q_{n,3}^{(2)}\right)^\top$ with

$$Q_{n,1}^{(2)} := S_{n,3}S_{n,5} - S_{n,4}S_{n,1}, \qquad Q_{n,2}^{(2)} := S_{n,4}S_{n,5} - S_{n,3}S_{n,2},$$
$$Q_{n,3}^{(2)} := S_{n,1}S_{n,2} - S_{n,5}^2 + \left(S_{n,2} - S_{n,5}\right) S_{n,3} + \left(S_{n,1} - S_{n,5}\right) S_{n,4}.$$

Now, Proposition 1.6 and limits (5.4), (5.7) and (5.9) imply

$$\begin{aligned} n^{-9/2}Q_{n,3,1}^{(1)} &= \left(n^{-9/4}S_{n,3} + n^{-9/4}S_{n,5}\right)\left(n^{-9/4}S_{n,3}\right) - \left(n^{-9/4}S_{n,1} + n^{-9/4}S_{n,3}\right)\left(n^{-9/4}S_{n,4}\right) \\ &\quad + \left(n^{-2}S_{n,5} - n^{-2}S_{n,1}\right)\left(n^{-5/2}T_n\right) \xrightarrow{\mathsf{P}} \sigma_{\alpha,\beta}^2 \left(\kappa_{\alpha,\beta}^{(2)} - \kappa_{\alpha,\beta}^{(1)}\right), \\ n^{-17/4}Q_{n,2}^{(2)} &= \left(n^{-17/4}S_{n,1}\right)\left(n^{-17/8}S_{n,2}\right) - \left(n^{-17/8}S_{n,5}\right)^2 + \left(n^{-2}S_{n,2} - n^{-2}S_{n,5}\right)\left(n^{-9/4}S_{n,3}\right) \\ &\quad + \left(n^{-2}S_{n,1} - n^{-2}S_{n,5}\right)\left(n^{-9/4}S_{n,4}\right) \xrightarrow{\mathsf{P}} 0 \end{aligned} \tag{7.2}$$

as $n \to \infty$, and using the same ideas one can find the limits of the remaining entries of $Q_n^{(1)}$ and coordinates of $Q_n^{(2)}$. In this way

$$n^{-9/2}Q_n^{(1)} \xrightarrow{\mathsf{P}} \sigma_{\alpha,\beta}^2 \mathcal{H}^\top \overline{\mathcal{K}}_{\alpha,\beta} \quad \text{and} \quad n^{-17/4}Q_n^{(2)} \xrightarrow{\mathsf{P}} (0,0,0)^\top$$

as $n \to \infty$, that together with (7.1), Slutsky's lemma and Propositions 1.7 and 1.9 implies the first statement of Proposition 1.10.

Further, let $(\alpha, \beta, \gamma) \in \mathcal{E}_{1+}$. As in this case $\overline{\Sigma}_{\alpha,\beta} = (1-\gamma^2)(0,1,-1)^\top(0,1,-1)$, short calculation shows

$$\begin{aligned} n^{-7}\overline{B}_nA_n &= \left(n^{-7}\overline{B}_nA - \left(2(1-\gamma^2)\right)^{-2}\overline{\Sigma}_{\alpha,\beta}\right)\left(n^{-1}A_n\right) + \left(\left(2(1-\gamma^2)\right)^{-2}\overline{\Sigma}_{\alpha,\beta}\right)\left(n^{-1}A_n\right) \\ &= \left(n^{-11/2}Q_n^{(1)}\right)\left(n^{-3/2}A_n\right) + \left(n^{-6}Q_n^{(2)} - \left(4(1-\gamma^2)\right)^{-1}(0,1,-1)^\top\right)\left(n^{-1}C_n^{(2)}\right) \\ &\quad + \left(4(1-\gamma^2)\right)^{-1}(0,1,-1)^\top\left(n^{-1}C_n^{(2)}\right), \end{aligned} \tag{7.3}$$

where now $Q_n^{(1)}$ is a three-by-three matrix with entries

$$\begin{aligned} &Q_{n,1,1}^{(1)} := S_{n,1}T_n - S_{n,3}^2, \qquad Q_{n,2,2}^{(1)} = Q_{n,3,3}^{(1)} := 0, \\ &Q_{n,1,2}^{(1)} = Q_{n,2,1}^{(1)} := S_{n,3}S_{n,4} - S_{n,5}T_n, \\ &Q_{n,1,3}^{(1)} = Q_{n,3,1}^{(1)} := \left(S_{n,3} + S_{n,5}\right)S_{n,3} - \left(S_{n,1} + S_{n,3}\right)S_{n,4} + \left(S_{n,5} - S_{n,1}\right)T_n, \\ &Q_{n,2,3}^{(1)} := S_{n,4}S_{n,5} - \left(S_{n,2} + S_{n,4}\right)S_{n,3} + S_{n,5}T_n, \\ &Q_{n,3,2}^{(1)} := S_{n,1}S_{n,2} + \left(S_{n,2} - S_{n,5}\right)S_{n,3} + \left(S_{n,1} - S_{n,5}\right)S_{n,4} + \left(S_{n,1} - S_{n,5}\right)T_n \\ &\qquad\qquad - \left(S_{n,3} + S_{n,5}\right)S_{n,3} + \left(S_{n,1} + S_{n,3}\right)S_{n,4} - S_{n,5}^2, \end{aligned}$$

and $Q_n^{(2)} = \left(0, Q_{n,2}^{(2)}, -Q_{n,3}^{(2)}\right)^\top$ with

$$\begin{aligned} &Q_{n,2}^{(2)} := S_{n,2}T_n - S_{n,4}^2, \\ &Q_{n,3}^{(2)} := S_{n,1}S_{n,2} + \left(S_{n,2} - S_{n,5}\right)S_{n,3} + \left(S_{n,1} - S_{n,5}\right)S_{n,4} + \left(S_{n,1} - S_{n,5}\right)T_n + \left(S_{n,2} - S_{n,5}\right)T_n \\ &\qquad - \left(S_{n,4} + S_{n,5}\right)S_{n,4} - \left(S_{n,3} + S_{n,5}\right)S_{n,3} + \left(S_{n,1} + S_{n,3}\right)S_{n,4} + \left(S_{n,2} + S_{n,4}\right)S_{n,3} - S_{n,5}^2. \end{aligned}$$

Using Proposition 1.6 and limits (5.10) and (5.11), similarly to (7.2) one can show that

$$n^{-11/2}Q_{n,i,j}^{(1)} \xrightarrow{\mathsf{P}} 0, \quad i,j = 1,2,3,$$

and

$$n^{-6}Q_{n,2}^{(2)} \xrightarrow{\mathsf{P}} \left(4(1-\gamma^2)\right)^{-1}, \qquad n^{-6}Q_{n,3}^{(2)} \xrightarrow{\mathsf{P}} \left(4(1-\gamma^2)\right)^{-1}$$

as $n \to \infty$, that together with (7.3), Slutsky's lemma and Propositions 1.7 and 1.9 implies the second statement of Proposition 1.10.

Finally, if $(\alpha, \beta, \gamma) \in \mathcal{E}_{2+}$ we have $\overline{\Sigma}_{\alpha,\beta} = (1-\gamma^2)(1,0,-1)^\top(1,0,-1)$. Hence, similarly to the previous case one can prove that the limiting distribution of $n^{-7}\overline{B}_nA_n$ equals that of $\left(4(1-\gamma^2)\right)^{-1}(1,0,-1)^\top\left(n^{-1}C_n^{(1)}\right)$ which completes the proof. □

## 8. Proof of Theorem 1.1

Cases $(\alpha, \beta, \gamma) \in \mathcal{F}_{++}$ and $(\alpha, \beta, \gamma) \in \mathcal{E}_{1+} \cup \mathcal{E}_{2+}$ are direct consequences of Propositions 1.9 and 1.10.

Consider now the case $(\alpha, \beta, \gamma) \in \mathcal{V}_{+}$. Using notations (3.5) introduced in Section 3 let

$$S_n := \begin{bmatrix} S_{n,1} & 0 \\ 0 & S_{n,2} \\ -S_{n,1} & -S_{n,2} \end{bmatrix}.$$

As by Proposition 1.9

$$n^{-3/2} C_n \xrightarrow{\mathcal{D}} \mathcal{N}(0, \mathcal{I}_2/2) \quad \text{as } n \to \infty, \tag{8.1}$$

to prove the last statement of Theorem 1.1 it suffices to show

$$n^{3/2} \left( B_n^{-1} A_n - \frac{S_n C_n}{S_{n,1} S_{n,2}} \right) \xrightarrow{\mathsf{P}} (0, 0, 0)^\top \quad \text{as } n \to \infty. \tag{8.2}$$

Further, denote by $R_{n,i,j}$, $i, j = 1, 2, 3$, the entries of the matrix $\overline{B}_n - T_n S_n H$. Short calculations show that

$$\begin{aligned}
&R_{n,1,1} := -S_{n,3}^2, \qquad R_{n,2,2} := -S_{n,4}^2, \qquad R_{n,1,2} = R_{n,2,1} := S_{n,3} S_{n,4} - T_n S_{n,5}, \\
&R_{n,1,3} = R_{n,3,1} := S_{n,3} \left( S_{n,3} + S_{n,5} \right) - S_{n,4} \left( S_{n,1} + S_{n,3} \right) + T_n S_{n,5}, \\
&R_{n,2,3} = R_{n,3,2} := S_{n,4} \left( S_{n,4} + S_{n,5} \right) - S_{n,3} \left( S_{n,2} + S_{n,4} \right) + T_n S_{n,5}, \\
&R_{n,3,3} := \left( S_{n,2} + S_{n,4} - S_{n,3} - S_{n,5} \right) \left( S_{n,1} + S_{n,3} - S_{n,4} - S_{n,5} \right) \\
&\qquad + \left( S_{n,3} + S_{n,4} + S_{n,5} \right) \left( S_{n,1} + S_{n,2} - 2S_{n,5} \right) - 2 T_n S_{n,5}.
\end{aligned}$$

Now, Slutsky's lemma together with (3.7), (3.10) and (3.11) implies

$$\begin{aligned}
n^{-13/2} R_{n,3,3} &= n^{-13/4} \left( S_{n,2} + S_{n,4} - S_{n,3} - S_{n,5} \right) n^{-13/2} \left( S_{n,1} + S_{n,3} - S_{n,4} - S_{n,5} \right) \\
&\quad + n^{-13/2} \left( S_{n,3} + S_{n,4} + S_{n,5} \right) n^{-13/2} \left( S_{n,1} + S_{n,2} - 2S_{n,5} \right) - 2 n^{-4} T_n n^{-5/2} S_{n,5} \xrightarrow{\mathsf{P}} 0
\end{aligned}$$

as $n \to \infty$, and obviously the same result can be proved for the remaining 8 entries of $\overline{B}_n - T_n S_n H$. Combining this result with Proposition 1.7 we obtain

$$n^{-17/2} \left( \overline{B}_n A_n - T_n S_n C_n \right) \xrightarrow{\mathsf{P}} (0, 0, 0)^\top \quad \text{as } n \to \infty. \tag{8.3}$$

Using again Slutsky's lemma together with (3.7), (3.10) and (3.11) from (5.2) we have

$$n^{-10} \left( T_n S_{n,1} S_{n,2} - \det(B_n) \right) \xrightarrow{\mathsf{P}} 0 \quad \text{as } n \to \infty,$$

that together with (3.10) and (8.1) gives us

$$\begin{aligned}
\frac{1}{n^{17/2}} \left( T_n S_n C_n - \frac{S_n C_n}{S_{n,1} S_{n,2}} \det(B_n) \right) &= \frac{(n^{-3} S_n)(n^{-3/2} C_n)}{(n^{-3} S_{n,1})(n^{-3} S_{n,2})} \\
&\quad \times \frac{1}{n^{10}} \left( T_n S_{n,1} S_{n,2} - \det(B_n) \right) \xrightarrow{\mathsf{P}} (0, 0, 0)^\top
\end{aligned} \tag{8.4}$$

as $n \to \infty$. In this way (8.2) follows from Proposition 1.8 and limits (8.3) and (8.4). □


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## References


[1] S. Baran, On the variances of a spatial unit root model, Lith. Math. J. 51 (2011) 122–140.
[2] S. Baran, G. Pap, M.V. Zuijlen, Asymptotic inference for an unstable spatial AR model, Statistics 38 (2004) 465–482.
[3] S. Baran, G. Pap, M.V. Zuijlen, Asymptotic inference for a nearly unstable sequence of stationary spatial AR models, Statist. Probab. Lett. 69 (2004) 53–61.
[4] S. Baran, G. Pap, M.V. Zuijlen, Asymptotic inference for unit roots in spatial triangular autoregression, Acta Appl. Math. 96 (2007) 17–42.
[5] S. Baran, G. Pap, Asymptotic behaviour of the least squares estimator in a nearly unstable sequence of stationary spatial AR models, J. Multivariate Anal. 100 (2009) 686–698.
[6] S. Baran, G. Pap, Parameter estimation in a spatial unit root autoregressive model, arXiv:1102.3318v3, 2011.
[7] S. Basu, G.C. Reinsel, A note on properties of spatial Yule–Walker estimators, J. Statist. Comput. Simul. 41 (1992) 243–255.
[8] S. Basu, G.C. Reinsel, Properties of the spatial unilateral first-order ARMA model, Adv. Appl. Probab. 25 (1993) 631–648.
[9] S. Basu, G.C. Reinsel, Regression models with spatially correlated errors, J. Amer. Statist. Assoc. 89 (1994) 88–99.
[10] H. Bateman, A. Erdélyi, Higher Transcendental Functions, vol. 1, Mc Graw-Hill, New York, 1953.
[11] B.B. Bhattacharyya, T.M. Khalil, G.D. Richardson, Gauss–Newton estimation of parameters for a spatial autoregression model, Statist. Probab. Lett. 28 (1996) 173–179.
[12] B.B. Bhattacharyya, G.D. Richardson, L.A. Franklin, Asymptotic inference for near unit roots in spatial autoregression, Ann. Statist. 25 (1997) 1709–1724.
[13] J.E. Besag, On the correlation structure of some two dimensional stationary processes, Biometrika 59 (1972) 43–48.
[14] P. Billingsley, Convergence of Probability Measures, Wiley, New York, 1968.
[15] N.H. Chan, C.Z. Wei, Asymptotic inference for nearly nonstationary AR(1) processes, Ann. Statist. 15 (1987) 1050–1063.
[16] B.R. Cullis, A.C. Gleeson, Spatial analysis of field experiments—an extension to two dimensions, Biometrics 47 (1991) 1449–1460.
[17] Y. Davydov, V. Paulauskas, On estimation of parameters for spatial autoregressive model, Stat. Inference Stoch. Process. 11 (2008) 237–247.
[18] M.G. Genton, H.L. Koul, Minimum distance inference in unilateral autoregressive lattice processes, Statist. Sinica 18 (2008) 617–631.
[19] W. Hoeffding, Probability inequalities for sums of bounded random variables, J. Amer. Statist. Assoc. 58 (1963) 13–30.
[20] J. Jacod, A.N. Shiryaev, Limit Theorems for Stochastic Processes, Springer-Verlag, Berlin, 1987.
[21] A.K. Jain, Advances in mathematical models for image processeng, Proc. IEEE 69 (1981) 502–528.
[22] R.J. Martin, A subclass of lattice processes applied to a problem in planar sampling, Biometrika 66 (1979) 209–217.
[23] R.J. Martin, The use of time-series models and methods in the analysis of agricultural field trials, Comm. Statist. Theory Methods 19 (1990) 55–81.
[24] V. Paulauskas, On unit roots for spatial autoregressive models, J. Multivariate Anal. 98 (2007) 209–226.
[25] P.C.B. Phillips, Towards a unified asymptotic theory for autoregression, Biometrika 74 (1987) 535–547.
[26] D. Tjøstheim, Statistical spatial series modelling, Adv. Appl. Probab. 10 (1978) 130–154.
[27] D. Tjøstheim, Autoregressive modelling and spectral analysis of array data in the plane, IEEE Trans. Geosci. Remote Sens. 19 (1981) 15–24.
[28] D. Tjøstheim, Statistical spatial series modelling II: some further results on unilateral processes, Adv. Appl. Probab. 15 (1983) 562–584.
[29] M.J. Wichura, Inequalities with applications to the weak convergence of random processes with multi-dimensional time parameters, Ann. Math. Statist. 40 (1969) 681–687.
[30] P. Whittle, On stationary processes in the plane, Biometrika 41 (1954) 434–449.