ELSEVIER

Theoretical Computer Science 297 (2003) 367–384

Theoretical Computer Science

www.elsevier.com/locate/tcs

Rank inequalities and separation algorithms for packing designs and sparse triple systems

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Abstract

Combinatorial designs find numerous applications in computer science, and are closely related to problems in coding theory. Packing designs correspond to codes with constant weight; 4-sparse partial Steiner triple systems (4-sparse PSTSs) correspond to erasure-resilient codes that are useful in handling failures in large disk arrays (Chee, Colbourn, Ling, Discrete Appl. Math., to appear; Hellerstein, Gibson, Karp, Katz, Paterson, Algorithmica 12 (1994) 182–208). The study of polytopes associated with combinatorial problems has proven to be important for both algorithms and theory, but only recently the study of design polytopes has been pursued (Moura, Math. Appl. 368 (1996) 227–254; Moura, Ph.D. Thesis, University of Toronto, 1999; Moura, Proc. Seventh Annu. European Symp. Prague, 1999, Lecture Notes in Computer Science, vol. 1643, Springer, Berlin, 1999, pp. 462–475; Wengrzik, Master's Thesis, Universität Berlin, 1995; Zehendner, Doctoral Thesis, Universität Augsburg, 1986). In this article, we study polytopes associated with t-(v,k,λ) packing designs and with m-sparse PSTSs. Subpacking and l-sparseness inequalities are introduced and studied. They can be regarded as rank inequalities for the independence systems associated with these designs. Conditions under which subpacking inequalities define facets are derived; in particular, those which define facets for PSTSs are determined. For $m \geqslant 4$, the l-sparseness inequalities with $2 \leqslant l \leqslant m$ are proven to induce facets for the m-sparse PSTS polytope; this proof uses extremal families of PSTSs known as Erdös configurations. Separation algorithms for these inequalities are proposed. We incorporate some of the sparseness inequalities in a polyhedral algorithm, and determine maximal 4-sparse PSTS(v), $v \leqslant 16$. An upper bound on the size of m-sparse PSTSs is presented.

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PII: S0304-3975(02)00648-5

1. Introduction

In this article, polytopes associated with problems in combinatorial design and coding theories are investigated. An extended abstract of this work appeared in [22]; here we also include extensions of the results presented there, as well as a new section including separation algorithms. We start by defining the problems in which we are interested, and then describe their polytopes and motivations for this research. Throughout the paper, we denote by $\binom{V}{k}$ the family of sets $\{B \subseteq V : |B| = k\}$. Let $v \geqslant k \geqslant t$. A t-(v,k,λ) *design* is a pair $(V,\mathscr{B})$ where V is a v-set and $\mathscr{B}$ is a collection of k-subsets of V called *blocks* such that every t-subset of V is contained in exactly λ blocks of $\mathscr{B}$. Design theorists are concerned with the existence of these designs. A t-(v,k,λ) *packing design* is defined by replacing the condition "in exactly λ blocks" in the above definition by "in at most λ blocks". Our goal is to determine the *packing number*, denoted by $D_\lambda(v,k,t)$, which is the *maximum* number of blocks in a t-(v,k,λ) packing design. The existence of a t-(v,k,λ) design can be decided by checking whether the packing number $D_\lambda(v,k,t)$ is equal to $\lambda\binom{v}{t}/\binom{k}{t}$. Thus, the determination of the packing number is a more general problem and we will concentrate on it. Designs play a central role in the theory of error-correcting codes, and, in particular, t-$(v,k,1)$ packing designs correspond to constant weight codes of weight k, length v and minimum distance $2(k-t+1)$. For surveys on packing designs see [17,23]. Determining the packing number is a hard problem in general, although the problem has been solved for specific sets of parameters. For instance, the existence of Steiner triple systems (STSs), i.e. 2-$(v,3,1)$ designs, and the determination of the packing number for partial Steiner triple systems (PSTSs), i.e. 2-$(v,3,1)$ packing designs, have been settled. On the other hand, the study of triple systems is an active area of research with plenty of open problems (see [8]). Interesting problems arise in the study of STSs and PSTSs avoiding prescribed sub-configurations (see [8,10]). Let us denote by STS(v) the Steiner triple system (PSTS(v) for a partial one) on v points. A (p,l)-configuration in a (partial) Steiner triple system is a set of l blocks (of the (partial) Steiner triple system) spanning p elements. Let $m \geqslant 4$. A PSTS(v) is said to be m-sparse if it avoids every $(l+2,l)$-configuration for $4 \leqslant l \leqslant m$. Erdös (see 15) conjectured that for all $m \geqslant 4$ there exists an integer v_m such that for every admissible $v \geqslant v_m$ there exists an m-sparse STS(v). The 4-sparse PSTSs are the same as anti-Pasch ones, since Pasches are the only $(6,4)$-configurations. Brouwer [2] further conjectured that a 4-sparse (or anti-Pasch) STS(v) exists for every admissible parameter with the exceptions of $v = 7, 13$. After several constructions of 4-sparse STSs for various sets of parameters (see [8]), this conjecture has been finally settled [12]. Anti-mitre Steiner triple systems were first studied in [7]. The 5-*sparse* Steiner triple systems are the systems that are both anti-Pasch and anti-mitre. Although there are some results on 5-sparse STSs [7,16], the problem is far from settled. No m-sparse STS is known for $m \geqslant 6$. Again for the packing version of the problem, our objective is to determine the sparse packing number, denoted by $D(m,v)$, which is the *maximum* number of blocks in an m-sparse PSTS(v). The study of m-sparse PSTSs gives rise to interesting extremal problems in hypergraph theory; in addition, these designs have various applications in computer science. For instance, the 4-sparse (or anti-Pasch) PSTSs correspond to erasure-resilient codes that tolerate all 3-erasures and most

4-erasures, which are useful in applications for handling failures in large disk arrays [4,13].

Let $\mathscr{D}$ be the set of all packing designs of the same kind and with the same parameters (for instance, the set of all 2-$(10,3,1)$ packing designs or the set of all 5-sparse PSTS(10)). Let $P(\mathscr{D})$ be the polytope in $\mathbb{R}^{\binom{v}{k}}$ given by the convex hull of the incidence vectors of the packing designs in $\mathscr{D}$. Thus, determining the packing number associated with $\mathscr{D}$ amounts to solving the following optimization problem

$$\text{maximize} \sum_{B\in\binom{V}{k}} x_B$$

$$\text{subject to } x \in P(\mathscr{D}).$$

If we had a description of $P(\mathscr{D})$ in terms of linear inequalities, this problem could be solved via linear programming. Unfortunately, it is unlikely for us to find complete descriptions of polytopes for hard combinatorial problems. On the other hand, some very effective computational methods use partial descriptions of a problem's polytope [3]. Therefore, it is of great interest to find classes of facets for these polytopes. It is also important to design efficient separation algorithms for a class of facets. Given a point outside a polytope and a class of valid inequalities for the polytope, a separation algorithm determines an inequality that is violated by the point or decides one does not exist. This is fundamental in branch-and-cut or other polyhedral algorithms that work with partial descriptions of polytopes.

Polytopes for general t-(v,k,λ) packing designs were first discussed in [19]; their clique facets have been determined for all packings with $\lambda=1$ and $k-t\in\{1,2\}$ for all t and v [20]. A polyhedral algorithm for t-$(v,k,1)$ packings and designs was proposed and tested in [21]. A related work that employs incidence matrix formulations for 2-(v,k,λ) design polytopes can be found in [25].

In this paper, we present two new classes of inequalities: the subpacking and the sparseness inequalities. They are types of *rank inequalities* when one regards the packing designs as independence systems, as discussed in Section 2. In Section 3, we focus on the subpacking inequalities, which are valid inequalities for both t-(v,k,λ) packing designs and m-sparse PSTSs. We study conditions under which these inequalities induce facets for the packing design polytope. In Section 4, we introduce l-sparseness inequalities. Given $m\geqslant 4$, the l-sparseness inequalities, $2\leqslant l\leqslant m$, are valid for the m-sparse PSTS polytope, and we prove they are always facet-inducing. In Section 5, we propose separation algorithms for the rank inequalities under study. In Section 6, we show the results of our branch-and-cut algorithm for determining the sparse packing number for 4-sparse PSTS(v) with $v\leqslant 16$. The algorithm follows the lines of the one described in [21], but employs sparse facets. With these 4-sparse packing numbers at hand, we develop a simple bound that uses the previous packing number and Chvátal–Gomory cuts to give an upper bound on the next packing numbers. The algorithm and upper bound are used in the determination of 4-sparse packing numbers for $v\in\{10,11,12,13\}$, which were unknown before. Open problems are discussed in Section 7.

2. Independence systems, packing designs and their polytopes

In this section, we define some terminology about independence systems and collect some results we use from the independence system literature. We closely follow the notation in [14]. Throughout the section, we translate the concepts to the context of combinatorial designs.

Let $N=\{v_1,v_2,\dots,v_n\}$ be a finite set. An *independence system* on N is a family $\mathscr{I}$ of subsets of N closed under inclusion, i.e. satisfying the property: $J\in\mathscr{I}$ and $I\subseteq J$ implies $I\in\mathscr{I}$, for all $J\in\mathscr{I}$. Any set in $\mathscr{I}$ is called *independent* and any set outside $\mathscr{I}$ is called *dependent*. Any minimal (with respect to set inclusion) dependent set is called a *circuit*, and an independence system is characterized by its family of circuits, which we denote by $\mathscr{C}$. The *independence number* of $\mathscr{I}$, denoted by $\alpha(\mathscr{I})$, is the maximum size of an independent set in $\mathscr{I}$. Given a subset S of N, the *rank* of S is defined by $r(S)=\max\{|I|: I\in\mathscr{I}$ and $I\subseteq S\}$. Note that $\alpha(\mathscr{I})=r(N)$. For a subset $S\subseteq N$, let $\mathscr{I}_S=\{I\in\mathscr{I}: I\subseteq S\}$; it is easy to see that $\mathscr{I}_S$ is an independence system whose family of circuits is given by $\mathscr{C}_S=\{C\in\mathscr{C}: C\subseteq S\}$.

Let $\mathscr{I}$ be an independence system on N with $\mathscr{C}$ its family of circuits. If all the circuits in $\mathscr{C}$ have size 2, then $G=(N,\mathscr{C})$ forms a graph with N as the node set, $\mathscr{C}$ as the edge set and $\mathscr{I}$ forms the set of all independent (or stable) sets of G.

Remark 1 (Packing designs). Given t, v, k, λ, let $\mathscr{I}$ be the family of all t-(v,k,λ) packing designs on the same v-set V. Let, $N=\binom{V}{k}$, then $\mathscr{I}$ is clearly an independence system on N. The packing number is the independence number. Each circuit in $\mathscr{C}$ corresponds to $(\lambda+1)$ sets in $\binom{V}{k}$ containing a common t-subset of V. For $\lambda=1$, $\mathscr{C}$ is simply formed by the pairs sets in $\binom{V}{k}$ which intersect in at least t points, and the underlying graph is obvious.

Following the definition in [11], an *Erdös configuration of order n*, $n\geqslant 1$, in a (partial) STS is any $(n+2,n)$-configuration, which contains no $(l+2,l)$-configuration, $1<l<n$. In fact, this is equivalent to requiring that $4\leqslant l<n$, since there cannot be any $(4,2)$- or $(5,3)$-configurations in a PSTS.

Remark 2 (Sparse PSTSs). Let $\mathscr{I}$ be the independence system of the 2-$(v,3,1)$ packing designs on the same v-set V. Let $\mathscr{C}$ be its collection of circuits, namely, the family of all pairs of triples of V whose intersection has cardinality 2. Adding m-sparseness requirements to $\mathscr{I}$ amounts to removing from $\mathscr{I}$ the packing designs that are not m-sparse, and adding extra circuits to $\mathscr{C}$. The circuits to be added to $\mathscr{C}$ are precisely the Erdös configurations of order l, for all $4\leqslant l\leqslant m$.

Before we discuss valid inequalities for the independence system polytope, we recall some definitions. A *polyhedron* $P\subseteq\mathbb{R}^n$ is the set of points satisfying a finite set of linear inequalities. A *polytope* is a bounded polyhedron. A polyhedron $P\subseteq\mathbb{R}^n$ is of *dimension* k, denoted by $\dim P=k$, if the maximum number of affinely independent points in P is $k+1$. We say that P is *full dimensional* if $\dim P=n$. Let $d\in\mathbb{R}^n$ and $d_0\in\mathbb{R}$. An inequality $d^{\mathrm{T}}x\leqslant d_0$ is said to be *valid* for P if it is satisfied by all points of

P. A subset $F \subseteq P$ is called a *face* of P if there exists a valid inequality $d^{\mathrm{T}}x \leqslant d_0$ such that $F = P \cap \{x \in \mathbb{R}^n : d^{\mathrm{T}}x = d_0\}$; the inequality is said to *represent* or to *induce* the face F. A *facet* is a face of P with dimension $(\dim P) - 1$. If P is full dimensional (which can be assumed w.l.o.g. for independence systems), then each facet is determined by a unique (up to multiplication by a positive number) valid inequality. Moreover, the minimal system of inequalities representing P is given by the inequalities inducing its facets.

Consider again an independence system $\mathscr{I}$ on N. The *rank inequality* associated with a subset S of N is defined by

$$\sum_{i \in S} x_i \leqslant r(S) \tag{1}$$

and is obviously a valid inequality for the independence system polytope $P(\mathscr{I})$. Necessary or sufficient conditions for a rank inequality to induce a facet have been discussed [14]. We recall some definitions. A subset S of N is said to be *closed* if $r(S \cup \{i\}) \geqslant r(S)+1$ for all $i \in N \backslash S$. S is said to be *nonseparable* if $r(S) < r(T)+r(S \backslash T)$ for all nonempty proper subset T of S.

A necessary condition for (1) to induce a facet is that S be closed and nonseparable. This was observed by Laurent [14], and was stated by Balas and Zemel [1] for independent sets in graphs. A sufficient condition for (1) to induce a facet is given in the next theorem. Let $\mathscr{I}$ be an independence system on N and let S be a subset of N. Let $\mathscr{C}$ be the family of circuits of $\mathscr{I}$ and let $\mathscr{C}_S$ denote its restriction to S. The *critical graph* of $\mathscr{I}$ on S, denoted by $G_S(\mathscr{I})$, is defined as having S as its nodeset and with edges defined as follows: $i_1, i_2 \in S$ are adjacent if and only if the removal of all circuits of $\mathscr{C}_S$ containing $\{i_1, i_2\}$ increases the rank of S.

Theorem 1 (Laurent [14], Chvátal [15] for graphs). *Let $S \subseteq N$. If S is closed and the critical graph $G_S(\mathscr{I})$ is connected, then the rank inequality* (1) *associated with S induces a facet of the polytope $P(\mathscr{I})$.*

Proposition 1 (Laurent [14], Cornuejols and Sassano [9]). *The following are equivalent*:
1. *The rank inequality* (1) *induces a facet of $P(\mathscr{I})$.*
2. *S is closed and the rank inequality* (1) *induces a facet of $P(\mathscr{I}_S)$.*

3. Subpacking inequalities for t-(v, k, λ) packings

Let us denote by $P_{t,v,k,\lambda}$ the polytope associated with the t-(v,k,λ) packing designs on the same v-set V, and by $\mathscr{I}_{t,v,k,\lambda}$ the corresponding independence system on $N = \binom{V}{k}$. Let $S \subseteq V$. Then, it is clear that $r(\binom{S}{k}) = D_\lambda(|S|,k,t)$ and the rank inequality associated with $\binom{S}{k}$ is given by

$$\sum_{B \in \binom{S}{k}} x_B \leqslant D_\lambda(|S|,k,t). \tag{2}$$

We call this the *subpacking inequality* associated with S, which is clearly valid for $P_{t,v,k,\lambda}$. In this section, we investigate conditions for this inequality to be facet inducing. The next proposition gives a sufficient condition for a subpacking inequality not to induce a facet.

Proposition 2. *If there exists a t-(v,k,λ) design, then*

$$\sum_{B\in\binom{V}{k}} x_B \leqslant D_\lambda(v,k,t) \tag{3}$$

does not induce a facet of $P_{t,v,k,\lambda}$.

Proof. Since there exists a t-(v,k,λ) design, it follows that $D_\lambda(v,k,t)=\lambda\binom{v}{t}/\binom{k}{t}$. Then, Eq. (3) can be obtained by adding the clique facets: $\sum_{B\supseteq T} x_B \leqslant \lambda$, for all $T\subseteq V$, $|T|=t$. Thus, (3) cannot induce a facet. □

The next proposition addresses the extendibility of facet inducing subpacking inequalities from $P_{t,|S|,k,\lambda}$ to $P_{t,v,k,\lambda}$, $v\geqslant|S|$.

Proposition 3. *Let $S\subseteq V$. Then, the following are equivalent*:
1. *The subpacking inequality (2) induces a facet of $P_{t,v,k,\lambda}$.*
2. *The subpacking inequality (2) induces a facet of $P_{t,|S|,k,\lambda}$; and for all $B'\in\binom{V}{k}\setminus\binom{S}{k}$ there exist a t-$(|S|,k,\lambda)$ packing design $(S,\mathscr{B})$ with $|\mathscr{B}|=D_\lambda(|S|,k,t)$ such that $(S,\mathscr{B}\cup\{B'\})$ is a t-(v,k,λ) packing design.*

Proof. The last condition in 2 is equivalent to $\binom{S}{k}$ being closed for the independence system $\mathscr{I}_{t,v,k,\lambda}$; thus, the equivalence comes directly from Theorem 1. □

For the particular case of $k=t+1$, facet inducing subpacking inequalities are always extendible.

Proposition 4 (Guaranteed extendibility of a class of subpacking facets). *Let $k=t+1$. Then, the subpacking inequality*

$$\sum_{B\in\binom{S}{t+1}} x_B \leqslant D_\lambda(|S|,t+1,t) \tag{4}$$

associated with $S\subseteq V$ induces a facet for $P_{t,v,t+1,\lambda}$ if and only if it induces a facet for $P_{t,|S|,t+1,\lambda}$.

Proof. Let $L\in\binom{V}{t+1}\setminus\binom{S}{t+1}$ and $L^I=L\cap S$. Then $|L^I|\leqslant t$. Let $\mathscr{P}=(S,\mathscr{B})$ be any t-$(|S|,t+1,\lambda)$ packing design with $|\mathscr{B}|=D_\lambda(|S|,t+1,t)$. If the inequality (4) defines a facet of $P_{t,|S|,t+1,\lambda}$, by Proposition 2, $\mathscr{P}$ cannot be a t-design. Thus, there exists a t-subset $T\subseteq S$ covered at most $\lambda-1$ times by $\mathscr{P}$. Let π be any permutation on S such that $\pi(T)\supseteq L^I$. Let $\mathscr{P}'=\pi(\mathscr{P})$, and denote its blocks by $\mathscr{B}'$. Then $(V,\mathscr{B}'\cup\{L\})$

Table 1
Summary of facet inducing subpacking inequalities of $P_{2,v,3,1}$, for $|S| \geqslant 4$

$\|S\| \equiv$	Inequality	Facet inducing?
$1,3\,(\mathrm{mod}\,6)$	$\sum_{B\in\binom{S}{3}} x_B \leqslant \dfrac{\|S\|^2-\|S\|}{6}$	No
$0,2\,(\mathrm{mod}\,6)$	$\sum_{B\in\binom{S}{3}} x_B \leqslant \dfrac{\|S\|^2-2\|S\|}{6}$	No
$4\,(\mathrm{mod}\,6)$	$\sum_{B\in\binom{S}{3}} x_B \leqslant \dfrac{\|S\|^2-2\|S\|-2}{6}$	Yes
$5\,(\mathrm{mod}\,6)$	$\sum_{B\in\binom{S}{3}} x_B \leqslant \dfrac{\|S\|^2-\|S\|-8}{6}$	Yes

is a t-$(v,t+1,\lambda)$ packing design with $D_\lambda(|S|,t+1,t)+1$ blocks. Proposition 3 concludes the proof. □

The following theorem determines which subpacking inequalities induce facets for partial STSs (see Table 1).

Theorem 2 (Facet defining subpacking inequalities for PSTSs). *Let $v \geqslant 4$ and let $S \subseteq [1,v]$, $|S| \geqslant 4$. Then, the subpacking inequality associated with S induces a facet of $P_{2,v,3,1}$ if and only if $|S| \equiv 4,5\,(\mathrm{mod}\,6)$.*

Proof. Let $s = |S|$. By Proposition 4, the first part of the statement is equivalent to the subpacking inequality associated with S inducing a facet of $P_{2,s,3,1}$.

Case 1: $s \equiv 1,3\,(\mathrm{mod}\,6)$. Since there exists an STS(s), by Proposition 2, the subpacking inequality associated with S does not induce a facet of $P_{2,s,3,1}$.

Case 2: $s \equiv 0,2\,(\mathrm{mod}\,6)$. Let $e \in S$. Then, by a "derived packing" argument we conclude that the inequality

$$\sum_{B\in\binom{S}{3}:e\in B} x_B \leqslant \left\lfloor \frac{s-1}{2} \right\rfloor \tag{5}$$

is valid for $P_{2,s,3,1}$. Note that $\lfloor (s-1)/2 \rfloor = (s-2)/2$ and recall that $D_1(s,3,2) = (s^2-2s)/6$, for $s \equiv 0,2\,(\mathrm{mod}\,6)$. Thus, by adding inequalities (5) for all $e \in S$ we get $\sum_{B\in\binom{S}{3}} x_B \leqslant D_1(s,3,2)$, proving that the latter inequality is not facet-inducing.

Case 3: $s \equiv 4,5\ (\mathrm{mod}\,6)$. By Theorem 1, it is enough to show that the critical graph $G_{\binom{S}{3}}(\mathscr{I}_{2,s,3,1})$ is connected. For $s \equiv 4,5\,(\mathrm{mod}\,6)$, it is known that there exists a maximal PSTS(s), say $(S,\mathscr{B})$, which leaves pairs $\{a,b\}$ and $\{a,c\}$ uncovered for some (distinct) $a,b,c \in S$ (this comes from the study of the structure of the "leave graphs" of PSTSs; see [8]). Since $(S,\mathscr{B})$ is maximal, we know that pair $\{b,c\}$ must be covered by some triple $\{x,b,c\}\in\mathscr{B}$ for some $x\in S\backslash\{a,b,c\}$. Thus, there is an edge connecting $\{a,b,c\}$ and $\{x,b,c\}$ in the critical graph $G_{\binom{S}{3}}(\mathscr{I}_{2,s,3,1})$, since the removal of the circuit (edge) $\{\{a,b,c\},\{x,b,c\}\}$ from the independence system $\mathscr{I}_{2,s,3,1}$ would make $(S,\mathscr{B}\cup\{a,b,c\})$ independent, increasing the rank of S. Now, by permuting the elements of S, by the previous argument, we conclude that there exist an edge in $G_{\binom{S}{3}}(\mathscr{I}_{2,s,3,1})$ connecting every pair of triples B_1,B_2 with $|B_1\cap B_2|=2$. From this last observation, it is easy to check that $G_{\binom{S}{3}}(\mathscr{I}_{2,s,3,1})$ is connected. □

4. Sparseness facets for *m*-sparse PSTSs

Let us denote by $P_{m,v}$ the polytope associated with m-sparse PSTS(v) on the same v-set V, and by $\mathscr{I}_{m,v}$ the corresponding independence system. The main contribution of this section is a class of facet inducing inequalities for $P_{m,v}$, which we call l-sparseness inequalities, given by Theorem 3. We need a few lemmas.

Lemma 1 (Lefmann et al. [15, Lemma 2.3]). *Let l,r be positive integers. Then any $(l+2,l+r)$-configuration in an STS contains an $(l+2,l)$-configuration.*

It is folklore that Erdös configurations exist for every order $n\geqslant 4$. The next lemma gives a construction for such configurations (see Table 2).

Lemma 2 (Construction of an Erdös configuration, for any order $n\geqslant 4$). *Let $n\geqslant 4$ be even. Then,*

$$\mathscr{E}_n=\{\{a,x,(x+1)\bmod n\}:x\in[0,n-1],x\text{ even}\}$$
$$\cup\{\{b,y,(y+1)\bmod n\}:y\in[0,n-1],y\,\text{odd}\},$$
$$\mathscr{E}_{n+1}=\mathscr{E}_n\backslash\{\{b,n-1,0\}\}\cup\{\{c,n-1,0\},\{a,b,c\}\},$$

are Erdös configurations of orders n and $n+1$, respectively.

Proof. Let us first consider $\mathscr{E}_n$. By construction, $\mathscr{E}_n$ is an $(n+2,n)$-configuration; it remains to prove that it does not contain an $(l+2,l)$-configuration for $2\leqslant l\leqslant n-1$. Let $\mathscr{B}$ be a $(l+2,l)$-configuration contained in $\mathscr{E}_n$, and let $B=\bigcup_{A\in\mathscr{B}}A$. Let us first consider the case in which $\{a,b\}\subseteq B$. Let $r_i=|\{A\in\mathscr{B}:i\in A\}|$, for $i\in B$. By construction, $r_i\leqslant 2$, for $i\in[0,n-1]$. In addition, $\sum_{i\in B\backslash\{a,b\}}r_i=2l$, which implies, since $|B\backslash\{a,b\}|=l$, that $r_i=2$ for all $i\in B\backslash\{a,b\}$. Therefore, $\mathscr{B}=\mathscr{E}_n$, and so $l=n$, which concludes this case. Let us now consider the case in which $b\notin B$ (the case $a\notin B$ is equivalent). Then, every set in $\mathscr{B}$ must contain a, which implies that $|B|=2l+1$. Thus, since $|B|=l+2$, we conclude that $l=1$, which concludes this case. Let us now consider $\mathscr{E}_{n+1}$. We must show that any $(l+2,l)$-configuration contained in $\mathscr{B}$ is such that either $l=1$ or $l=n+1$. We can assume w.l.o.g. that $c\in B$, for otherwise, the configuration would appear in $\mathscr{E}_n\backslash\{\{b,n-1,0\}\}$, which implies by the first part that $l=1$. First, we consider the case in which $\{a,b,c\}\in\mathscr{B}$. Then, considering $\mathscr{E}_{n+1}\backslash\{\{a,b,c\}\}$, we conclude that

Table 2
Examples of Erdös configurations given by Lemma 2

$\mathscr{E}_4$	$\mathscr{E}_5$	$\mathscr{E}_6$	$\mathscr{E}_7$
$\{a,0,1\}$	$\{a,0,1\}$	$\{a,0,1\}$	$\{a,0,1\}$
$\{b,1,2\}$	$\{b,1,2\}$	$\{b,1,2\}$	$\{b,1,2\}$
$\{a,2,3\}$	$\{a,2,3\}$	$\{a,2,3\}$	$\{a,2,3\}$
$\{b,3,0\}$	$\{c,3,0\}$	$\{b,3,4\}$	$\{b,3,4\}$
	$\{a,b,c\}$	$\{a,4,5\}$	$\{a,4,5\}$
		$\{b,5,0\}$	$\{c,5,0\}$
			$\{a,b,c\}$

$2(l-1)=\sum_{i\in B\setminus\{a,b,c\}} r_i \leqslant 2(l+2-3)$, which implies that $r_i=2$ for all $i\in B\setminus\{a,b,c\}$, and therefore that $\mathscr{B}=\mathscr{E}_{n+1}$. Now assume that $\{a,b,c\}\notin\mathscr{B}$. If $b\notin B$ then looking at the sets of $\mathscr{B}$ that contain a, implies that $2(l-1)\leqslant l+2-2$, and so $l\leqslant 1$. On the other hand, if $\{a,b\}\subseteq B$, then $|B\setminus\{a,b,c\}|=l-1$, which implies $2l=\sum_{i\in B\setminus\{a,b,c\}} r_i\leqslant 2(l-1)$, which is a contradiction. □

Lemma 3. *Let $4\leqslant l\leqslant v-2$ and let T be an $(l+2)$-subset of V. Let $R\in\binom{V}{3}\setminus\binom{T}{3}$. Then, there exists an Erdös configuration $\mathscr{S}$ of order l on the points of T and a triple $S\in\mathscr{S}$, such that $\mathscr{S}\setminus\{S\}\cup\{R\}$ is an l-sparse* PSTS(v).

Proof. Let $\mathscr{S}$ be an Erdös configuration of order l on the points of T (Lemma 2 guarantees its existence). If $|R\cap T|\leqslant 1$, taking any $S\in\mathscr{S}$, the set $\mathscr{S}'=\mathscr{S}\setminus\{S\}\cup\{R\}$ is a 2-$(v,3,1)$ packing. Otherwise, if $|R\cap T|=2$, we will choose $\mathscr{S}$ such that the pair $R\cap T$ appears in a block, say S. Then, $\mathscr{S}'=\mathscr{S}\setminus\{S\}\cup\{R\}$ is a 2-$(v,3,1)$ packing. In either case, we claim $\mathscr{S}'$ does not contain an $(n+2,n)$-configuration, $4\leqslant n\leqslant l$. Indeed, if that was the case, the configuration, say $\mathscr{B}$, would contain R. Thus, $\mathscr{B}$ could not be a Pasch, since the element in $R\setminus S$ appears only once in the configuration. Since the Pasch is the only $(6,4)$-configuration in a PSTS, this implies $n\geqslant 5$. Moreover, since $R\nsubseteq T$ and $|R|=3$, $\mathscr{B}\setminus\{R\}$ would be a $(p,n-1)$-configuration with $4\leqslant n-1\leqslant p\leqslant n+1$. Thus, by Lemma 1, $\mathscr{B}\subseteq\mathscr{S}$ would contain a $(p,p-2)$-configuration for $p\geqslant 4$, which is a contradiction. □

The following theorem establishes that sparseness inequalities are facet-inducing.

Theorem 3 (l-sparseness facets). *Let $m\geqslant 4$. Then, for any $2\leqslant l\leqslant m$ and any $(l+2)$-subset T of V, the inequality*

$$s(T):\sum_{B\in\binom{T}{3}} x_B \leqslant l-1$$

induces a facet for $P_{m,v}$.

Proof. Inequalities $s(T)$ with $l \in \{2,3\}$ are facet-inducing for $P_{2,v,3,1}$ (see Table 1 for $|S| = 4,5$), and even though the inclusion $P_{m,v} \subseteq P_{2,v,3,1}$ is in general proper, it is easy to show they remain facet-inducing for $P_{m,v}$. Thus, we concentrate on $l \geqslant 4$. The validity of $s(T)$ comes from the definition of l-sparse PSTSs, i.e. the fact that $r(\binom{T}{3}) = l-1$ for $\mathscr{I}_{m,v}$. Lemma 3 implies that $\mathscr{I}_{m,v}$ is closed. Thus, by Theorem 1, it is sufficient to show that the critical graph $G_{\binom{T}{3}}(\mathscr{I}_{m,v})$ is connected. Let $\mathscr{E}$ be an Erdös configuration of order l on the points of T. There must be two triples in $\mathscr{E}$ whose intersection is a single point, call those triples B_1 and B_2. We claim $\mathscr{E}\backslash\{B_1\}$ and $\mathscr{E}\backslash\{B_2\}$ are m-sparse 2-$(v,3,1)$ packings. Indeed, $|\mathscr{E}\backslash\{B_i\}| = |\mathscr{E}| - 1 = l - 1$, and since $\mathscr{E}$ was $(l-1)$-sparse, so is $\mathscr{E}\backslash\{B_i\}$, $i = 1,2$. Thus, there exists an edge in the critical graph $G_{\binom{T}{3}}(\mathscr{I}_{m,v})$ connecting triples B_1 and B_2. By permuting T, we can show this is true for any pair of triples which intersect in one point. That is, there exists an edge in $G_{\binom{T}{3}}(\mathscr{I}_{m,v})$ connecting C_1 and C_2, for any $C_1, C_2 \in \binom{T}{3}$ with $|C_1 \cap C_2| = 1$. It is easy to check that this graph is connected. □

Remark 3. The following is an integer programming formulation for the optimization problem associated with $P_{m,v}$, in which all the inequalities are facet-inducing (see Theorem 3). Note that the 3-sparseness inequalities could have been omitted from the integer programming formulation, since for integral points they are implied by the 2-sparseness inequalities (the 2-sparseness inequalities guarantee that x is a PSTS, which implies it is 3-sparse).

$$
\begin{aligned}
&\text{maximize } \textstyle\sum_{B\in\binom{V}{3}} x_B\\
&\text{subject to}\\
&\textstyle\sum_{B\in\binom{T}{3}} x_B \leqslant 1 && \text{for all } T \subseteq V, |T| = 4 && \text{(2-sparseness inequalities)},\\
&\textstyle\sum_{B\in\binom{T}{3}} x_B \leqslant 2 && \text{for all } T \subseteq V, |T| = 5 && \text{(3-sparseness inequalities)},\\
&\textstyle\sum_{B\in\binom{T}{3}} x_B \leqslant 3 && \text{for all } T \subseteq V, |T| = 6 && \text{(4-sparseness inequalities)},\\
&\vdots && \vdots\\
&\textstyle\sum_{B\in\binom{T}{3}} x_B \leqslant m-1 && \text{for all } T \subseteq V, |T| = m+2 && \text{(m-sparseness inequalities)},\\
&\quad x \in \{0,1\}^{\binom{v}{3}}.
\end{aligned}
$$

5. Separation algorithms

In this section, we discuss separation algorithms for subpacking and sparseness inequalities. The following propositions examine the complexity of the trivial algorithm. After that, we propose an algorithm for l-sparseness inequalities with small l, which is more efficient for fractional points with small support.

For the purpose of complexity analysis of the separation algorithms, we consider the input size of the algorithm as the number of bits needed to represent a point $\bar{x} \in [0,1]^{\binom{v}{k}}$ up to a certain precision, say β bits. Let $supp(\bar{x})$ denote the support of $\bar{x}$, i.e. $supp(\bar{x}) := \{B \in \binom{V}{k} : \bar{x}_B \neq 0\}$. Using a vector representation, we need at most $\binom{v}{k}\beta$ bits to represent $\bar{x}$; using a sparse representation, we need at most $k \log v \, |supp(\bar{x})|\beta$ bits to represent $\bar{x}$.

Proposition 5 (Separation of subpacking inequalities). *Let C be a constant. Subpacking inequalities with $|S| \leqslant C$ can be separated in polynomial time.*

Proof. Let $\bar{x}$ be the point to be separated. Let $U = (\bigcup_{B \in supp(\bar{x})} B)$ and $u = |U|$. The violated inequalities can be detected by examining sets S with $S \subseteq U \subseteq V$ and $4 \leqslant |S| \leqslant C$. For fixed C, there are exactly $\sum_{s=4}^{C} \binom{u}{s}$ inequalities to check, which is in $O(u^C)$. Since $u \leqslant v$ and $u \leqslant k\,|supp(\bar{x})|$, whatever representation we use, the complexity of this algorithm is polynomial on the size of the input. $\square$

Proposition 6 (Separation of l-sparseness facets). *For constant $m \geqslant 2$, l-sparseness facets with $l \leqslant m$ can be separated in polynomial time.*

Proof. Let $\bar{x}$ be the point to be separated. Let $U = (\bigcup_{B \in supp(\bar{x})} B)$ and $u = |U|$. For fixed m, there are exactly $\sum_{i=4}^{m+2} \binom{u}{i} \in O(u^{m+2})$ inequalities to check, which is polynomial on the size of the input. $\square$

Although it takes polynomial time to check every sparseness inequality, this trivial method can be improved. Next, we propose a separation algorithm for l-sparseness inequalities for $l \in [2,5]$, which has complexity of the same order as the trivial algorithm for points with large support, but improves on this complexity when the support is significantly smaller than $\binom{v}{3}$. Note that 2-sparseness inequalities correspond to subpacking inequalities with $|S| = 4$ and that 3-sparseness inequalities correspond to subpacking inequalities with $|S| = 5$.

Denote by $W_{t,k}^{v}$ the $\binom{v}{t}$ by $\binom{v}{k}$ matrix representing the incidence of t-subsets on the k-subsets of a v-set. The linear programming relaxation for t-$(v,k,1)$ packings is given by $P' = \{x \in \mathbb{R}^{\binom{v}{k}} : W_{t,k}^{v} x \leqslant 1, 0 \leqslant x \leqslant 1\}$. Thus, it is natural to assume that the point to be separated by a separation algorithm is in P'.

Algorithms 1 and 2 (Fig. 1) were inspired by properties that obviously hold for integral $\bar{x}$. For instance, if $\bar{x}$ is the incidence vector of a PSTS and it violates a 4-sparseness inequality, then its support contains a Pasch and three distinct sets B_1, B_2, B_3 in the Pasch satisfy $|B_1 \cap B_2| = |B_1 \cap B_3| = 1$ and $|B_1 \cup B_2 \cup B_3| = 6$. However, it is not obvious that the same properties must hold for fractional $\bar{x}$ contained in the linear relaxation of the problem, and the correctness of the algorithms is nontrivial. The correctness of these algorithms is established by the following theorems.

Theorem 4 (Correctness of Algorithm 1). *Let $\bar{x} \in \{x \in \mathbb{R}^{\binom{v}{3}} : W_{2,3}^{v} x \leqslant 1, \quad 0 \leqslant x \leqslant 1\}$. Then, if $\bar{x}$ violates at least one subpacking inequality associated with some S with $|S| \in \{4,5\}$, then Algorithm 1 returns one such facet.*

Algorithm 1. Separation of subpacking inequalities, $|S| \in 4, 5$.
Let $\overline{x} \in \{x \in \mathbb{R}^{\binom{v}{3}} : W^v_{2,3}x \leq 1, 0 \leq x \leq 1\}$ be the point to be separated.
Let $F(\overline{x}) = \{B \in \binom{v}{3} : 0 < \overline{x}_B < 1\}$.
for every $B_1, B_2 \in F(\overline{x})$ do
  if $(|B_1 \cup B_2| = 4)$ then
    if $(\sum_{B\in\binom{B_1\cup B_2}{3}} x_B > 1)$ then return $(B_1 \cup B_2, 2)$.
  elseif $(|B_1 \cup B_2| = 5)$ then
    if $(\sum_{B\in\binom{B_1\cup B_2}{3}} x_B > 2)$ then return $(B_1 \cup B_2, 3)$.
return $(\emptyset, 0)$.

Algorithm 2. Separation of l-sparseness inequalities, $2 \leq l \leq 5$.
Let $\overline{x} \in \{x \in \mathbb{R}^{\binom{v}{3}} : W^v_{2,3}x \leq 1, 0 \leq x \leq 1\}$ be the point to be separated.
Run Algorithm 1 for $\overline{x}$; let R be its returned value.
if $(R \neq (\emptyset, 0))$ then return R
else
for every $B_1, B_2, B_3 \in supp(\overline{x})$ such that $|B_1 \cap B_2| = |B_1 \cap B_3| = 1$ do
  if $(|B_1 \cup B_2 \cup B_3| = 6)$ then
    if $(\sum_{B\in\binom{B_1\cup B_2\cup B_3}{3}} x_B > 3)$ then return $(B_1 \cup B_2 \cup B_3, 4)$.
  elseif $(|B_1 \cup B_2 \cup B_3| = 7)$ then
    if $(\sum_{B\in\binom{B_1\cup B_2\cup B_3}{3}} x_B > 4)$ then return $(B_1 \cup B_2 \cup B_3, 5)$.
return $(\emptyset, 0)$.

Fig. 1. Separation algorithms.

Theorem 5 (Correctness of Algorithm 2). *Let* $\bar{x} \in \{x \in \mathbb{R}^{\binom{v}{3}} : W^v_{2,3}x \leqslant 1,\ 0 \leqslant x \leqslant 1\}$. *Then, if* $\bar{x}$ *violates at least one l-sparseness facet for* $l \in [2,5]$, *then Algorithm* 2 *returns one such facet.*

Due to their tedious nature, the proofs of Theorems 4 and 5 are delayed to Section 5.1.

Proposition 7 (Running time of Algorithm 1). *Let* $\bar{x}$ *and* $f = |F(\bar{x})|$ *be as described in Algorithm 1. Let* n_1 *be the number of inequalities that are examined by the algorithm. Then,* $n_1 \in O(min\{f\,v^2, f^2\})$.

Proof. For each $B_1 \in F(\bar{x})$, we just have to check sets B_2 such that $|B_1 \cap B_2| \in \{1, 2\}$; there are at most $O(v^2)$ such sets. Also, at most $\binom{f}{2}$ pairs B_1, B_2 have to be looked at. □

Note that $f \in O(v^3)$ but it may be the case that $f \in o(v^3)$. So, Algorithm 1 is as fast as the trivial one (i.e. in $O(min\{v^5, f^2\})$), but improves on the trivial one in the latter case.

Proposition 8 (Running time of Algorithm 2). *Let* $\bar{x}$, $f = |F(\bar{x})|$ *and* $s = |supp(\bar{x})|$ *be as described in Algorithm 2. Let* n_2 *be the number of inequalities that are examined* *by the algorithm. If there exists a violated l-sparseness inequality for* $l \in \{2,3\}$, *then* $n_2 \in O(min\{fv^2, f^2\})$; *otherwise*, $n_2 \in O(min\{sv^4, s^3\})$.

Proof. The first part of the statement comes from Proposition 7. The second part comes from analysing the number of sets $B_1, B_2, B_3 \in supp(\bar{x})$ satisfying $|B_1 \cap B_2| = |B_1 \cap B_3| = 1$. For each $B_1 \in supp(\bar{x})$, there are no more than $3\binom{v-3}{2}^2 \in O(v^4)$ pairs $\{B_2, B_3\}$ satisfying this condition. This justifies the upper bound of $O(sv^4)$. The upper bound of s^3 comes from the existence of at most $\binom{s}{3}$ triples of sets in $supp(\bar{x})$ with the desired property. □

Similarly, $s \in O(v^3)$ but it may be the case that $s \in o(v^3)$. So, Algorithm 2 is as fast as the trivial one (i.e. in $O(min\{v^7, s^3\})$), but improves on the trivial one in the latter case.

In our branch-and-cut implementation, we experimentally observed that $s = |supp(\bar{x})|$ is in $\Theta(v^2)$ throughout the algorithm. In such a case, Algorithm 2 runs in $O(v^6)$, while the trivial algorithm for l-sparseness separation for $l \in [2,5]$ runs in $O(v^7)$.

5.1. Correctness of Algorithms 1 and 2

Lemma 4 (Moura [20, Proposition 4.1]). *Let* $\mathscr{C} \subseteq \binom{V}{3}$ *and the inequality* $\sum_{B \in \mathscr{C}} x_B \leqslant 1$ *be a clique facet for* $P_{2,v,3,1}$. *Then, either*

1. $\mathscr{C} = \binom{F}{3}$ *for some* F *with* $|F| = 4$ (*i.e. the clique inequality is a subpacking inequality associated with* F, *and a* 2*-sparseness inequality* $s(F)$), *or*
2. $\mathscr{C} = \{B \in \binom{V}{3} : B \supseteq T\}$ *for some* T *with* $|T| = 2$ (*i.e. the clique inequality is one of the inequalities in* $W^v_{2,3}x \leqslant 1$).

Proof of Theorem 4. Suppose $\bar{x}$ violates a subpacking inequality $p(S)$ for $|S| = 4$. First note that $\bar{x}_A + \bar{x}_B \leqslant 1$ for all distinct $A, B \in \binom{S}{3}$, since $|A \cap B| = 2$ and $\bar{x}$ satisfies $W^v_{2,3}\bar{x} \leqslant 1$. We claim that $\bar{x}_B < 1$ for all $B \in \binom{S}{3}$, for if there exists $B' \in \binom{S}{3}$ with $\bar{x}_{B'} = 1$ the previous inequality would imply that $\bar{x}_A = 0$ for all $A \in \binom{S}{3} \setminus \{B'\}$ and consequently $p(S)$ would not be violated. From the previous claim and the fact that $p(S)$ is violated, it follows that there exist distinct $B_1, B_2 \in \binom{S}{3} \cap F(\bar{x})$. In addition, we know that $B_1 \cup B_2 = S$. Therefore, if Algorithm 1 did not return another violated inequality before B_1, B_2 were selected in the main loop, it will return $(S, 2)$ indicating that $p(S)$ is violated (the 2 indicates 2-sparseness inequality).

Now, we assume that $\bar{x}$ violates no subpacking inequality $p(L)$ with $|L| = 4$, but it does violate a subpacking inequality $p(S)$ with $|S| = 5$, i.e. $\sum_{B \in \binom{S}{3}} \bar{x}_B > 2$. We claim that there exists $B_1, B_2 \in \binom{S}{3} \cap F(\bar{x})$ such that $|B_1 \cap B_2| = 1$ (which implies $B_1 \cup B_2 = S$). Indeed, for it is easy to see that $\bar{x}_B < 1$ for all $B \in \binom{S}{3}$, and that if $|B_1 \cap B_2| = 2$ for all $B_1, B_2 \in \binom{S}{3} \cap supp(\bar{x})$, the support of $p(S)$ would be contained in a clique inequality, but, by hypothesis and Lemma 4, $\bar{x}$ violates no clique inequality. Thus, if Algorithm 1 has not returned any other violated inequality before B_1, B_2 were selected in the main loop, it will return $(S,3)$ indicating that $p(S)$ is violated (the 3 indicates 3-sparseness). □

Lemma 5. *Let $\bar{x} \in \{x \in \mathbb{R}^{\binom{v}{3}} : W_{2,3}^{v} x \leqslant 1, 0 \leqslant x \leqslant 1\}$ and let $l \geqslant 3$. Suppose there exists no violated l'-sparseness inequality with $l' < l$ and there exists a violated l-sparseness inequality $s(T)$ associated with T. For any $e \in T$, let $\mathscr{B}_{T,e} = \{B \in \binom{T}{3} : e \in B\}$. Then, for any $e \in T$, we have:*
1. $\sum_{B \in \mathscr{B}_{T,e}} \bar{x}_B > 1$; *and*
2. *there exists $B_1, B_2 \in supp(\bar{x}) \cap \binom{T}{3}$ such that $|B_1 \cap B_2| = e$.*

Proof (Part 1). Since $s(T)$ is violated by $\bar{x}$, we have $\sum_{B \in \binom{T}{3}} \bar{x}_B > l - 1$. By hypothesis, $s(T \backslash \{e\})$ is not violated, so $\sum_{B \in \binom{T}{3} : B \notin \mathscr{B}_{T,e}} \bar{x}_B \leqslant l - 2$. So, from the previous two inequalities we get $\sum_{B \in \mathscr{B}_{T,e}} \bar{x}_B > 1$.

(Part 2). By Part 1, we have $|\mathscr{B}_{T,e} \cap supp(\bar{x})| \geqslant 2$, since $x_B \leqslant 1$ for all $B \in \binom{T}{3}$. Suppose that for all $B_1, B_2 \in \mathscr{B}_{T,e} \cap supp(\bar{x})$, $|B_1 \cap B_2| \geqslant 2$. Then, $\mathscr{B}_{T,e} \cap supp(\bar{x})$ is contained in the support of a clique inequality, and by Part 1, this clique inequality is violated. Thus, by Lemma 4, this contradicts the hypothesis that $\bar{x}$ satisfies $W_{2,3}^{v} \bar{x} \leqslant 1$ and that there exists no violated 2-sparseness inequality. □

Proof of Theorem 5. Suppose there exists a violated l'-sparseness inequality with $l' \in \{2,3,4,5\}$. Let t be the cardinality of the smallest T such that $s(T)$ is an l-sparseness inequality. Since $t = l + 2$, we have $t \in \{4,5,6,7\}$. If $t = 4, 5$, by the correctness of Algorithm 1, we know that one such violated inequality will be returned. It remains to analyse the cases $t = 6$ and 7. We have to show in each case that there exist sets $B_1, B_2, B_3 \in \binom{T}{3} \cap supp(\bar{x})$ such that $|B_1 \cap B_2| = |B_1 \cap B_3| = 1$ and $T = B_1 \cup B_2 \cup B_3$; let us call such $\{B_1, B_2, B_3\}$ a *perfect triple of sets.*

Case 1: $t = 6$ (there exists a violated 4-sparseness inequality but there exists no violated l-sparseness inequality for $l < 4$):

Assume w.l.o.g. that $T = [1,6]$. By Lemma 5 Part 2, we can assume w.l.o.g. that $C_1 := \{1,2,3\}, C_2 := \{1,4,5\} \in \mathscr{B}_{T,1} \cap supp(\bar{x})$. If there exists a set $D \in \mathscr{B}_{T,6} \cap supp(\bar{x})$ intersecting C_1 or C_2 in one point we would be done, since $\{C_1, C_2, D\}$ would form a perfect triple of sets. So, we assume the contrary, which implies $\mathscr{B}_{T,6} \cap \mathscr{B}_{T,1} \cap supp(\bar{x}) = \emptyset$ and that $\bar{x}_{B'} = 0$ for any $B' \in \mathscr{B}_{T,6} \cap \mathscr{B}_{T,1}$. Thus, by Lemma 5 Part 2, we conclude that there exist sets of the form $D_1 := \{6,a,b\}, D_2 := \{6,c,d\} \in \mathscr{B}_{T,6} \cap supp(\bar{x})$, for distinct $a,b,c,d \in [2,5]$. If D_j for $j = 1$ or $j = 2$ intersects either C_1 or C_2 in exactly one point, we are done, since $\{C_1, C_2, D_j\}$ would form a perfect triple of sets. Thus, we assume w.l.o.g. that $D_1 = \{6,2,3\}, D_2 = \{6,4,5\}$, and that there are no other sets in $\mathscr{B}_{T,6} \cap supp(\bar{x})$. Similarly, we can assume that there is no other set in $\mathscr{B}_{T,1} \cap supp(\bar{x})$ except C_1 and C_2. Now, since $\bar{x}$ satisfies $W_{2,3}^{v} \bar{x} \leqslant 1$, it follows that $\bar{x}_{C_1} + \bar{x}_{D_1} \leqslant 1$ and $\bar{x}_{C_2} + \bar{x}_{D_2} \leqslant 1$. This implies $\sum_{B \in \mathscr{B}_{T,6}} \bar{x}_B + \sum_{B \in \mathscr{B}_{T,1}} \bar{x}_B = \bar{x}_{C_1} + \bar{x}_{C_2} + \bar{x}_{D_1} + \bar{x}_{D_2} \leqslant 2$, but by Lemma 5 Part 1, $\sum_{B \in \mathscr{B}_{T,6}} \bar{x}_B + \sum_{B \in \mathscr{B}_{T,1}} \bar{x}_B > 2$, so we reached a contradiction.

Case 2: $t = 7$ (there exists a violated 5-sparseness inequality but there exists no violated l-sparseness inequality for $l < 5$):

Assume w.l.o.g. that $T=[1,7]$. By Lemma 5 Part 2, we can assume w.l.o.g. that $C_1:=\{1,2,3\}, C_2:=\{1,4,5\}\in\mathscr{B}_{T,1}\cap supp(\bar{x})$. If there exists $B'\in supp(\bar{x})\cap\binom{T}{3}$ with $\{6,7\}\subseteq B'$, we are done since $|B'\cap\{1,2,3,4,5\}|=1$ which implies that $\{C_1,C_2,B'\}$ is a perfect triple of sets. So, we can assume there exists no $B'\in supp(\bar{x})\cap\binom{T}{3}$ with $\{6,7\}\subseteq B'$. By Lemma 5 Part 2, there exists $D_1:=\{6,a,b\}, D_2:=\{6,c,d\}\in supp(\bar{x})\cap\binom{T}{3}$ with distinct a,b,c,d, and by the previous observation $a,b,c,d\in\{1,2,3,4,5\}$. Let e be the remaining element in $\{1,2,3,4,5\}\backslash\{a,b,c,d\}$. If there exists $D'\in supp(\bar{x})\cap\binom{T}{3}$ with $\{e,7\}\subseteq D'$, we would be done since $\{D_1,D_2,D'\}$ would be a perfect triple of sets. So, we can assume there exists no $D'\in supp(\bar{x})\cap\binom{T}{3}$ with $\{e,7\}\subseteq D'$. By Lemma 5 Part 2, and previous assumptions, there exist $E_1:=\{7,x_1,x_2\}, E_2:=\{7,y_1,y_2\}\in supp(\bar{x})\cap\binom{T}{3}$ with $\{x_1,x_2,y_1,y_2\}=\{a,b,c,d\}$.

Subcase (a) $E_1=\{7,a,c\}$ and $E_2=\{7,b,d\}$ (equivalently $E_1=\{7,a,d\}$ and $E_2=\{7,b,c\}$): Let $j\in\{1,2\}$ be such that $e\in C_j$. Note, that

$$C_j\in\{\{e,a,b\},\{e,a,c\},\{e,a,d\},\{e,b,c\},\{e,b,d\},\{e,c,d\}\}.$$

Now, for every possible C_j, we can find a perfect triple of sets, namely one of: $\{\{e,a,b\},D_2,E_1\}$, $\{\{e,a,c\},D_2,E_2\}$, $\{\{e,a,d\},D_1,E_1\}$, $\{\{e,b,c\},D_1,E_2\}$, $\{\{e,b,d\},D_1,E_1\}$, $\{\{e,c,d\},D_1,E_2\}$.

Subcase (b) $E_1=\{7,a,b\}$ and $E_2=\{7,c,d\}$: Now, if there exists $E'\in supp(\bar{x})\cap\binom{T}{3}$ with $\{e,6\}\subseteq E'$, we conclude that $E'=\{e,6,z\}$ for $z\in\{a,b,c,d\}$; thus, either $\{D_1,E_2,E'\}$ or $\{D_2,E_1,E'\}$ would be a perfect triple of sets. Otherwise, $\mathscr{B}_{T,e}\cap\mathscr{B}_{T,6}\cap supp(\bar{x})=\emptyset$. Thus, for all $B\in\mathscr{B}_{T,e}\cap supp(\bar{x})$ we have $|B\cap\{a,b,c,d\}|=2$. Now, if there exists $E'\in\mathscr{B}_{T,e}\cap supp(\bar{x})$ with $E'\in\{\{e,a,c\},\{e,b,c\},\{e,a,d\},\{e,b,d\}\}$, then we are done, since either $\{D_2,E_1,E'\}$ or $\{D_1,E_2,E'\}$ would form a perfect triple of sets. On the other hand, if this is not the case, then $\mathscr{B}_{T,e}\cap supp(\bar{x})=\{F_1:=\{e,a,b\},F_2:=\{e,c,d\}\}$. Similarly to the argument at the end of Case 1, if $(\mathscr{B}_{T,e}\cap supp(\bar{x}))\dot\cup(\mathscr{B}_{T,6}\cap supp(\bar{x}))\dot\cup(\mathscr{B}_{T,7}\cap supp(\bar{x}))=\{D_1,D_2,E_1,E_2,F_1,F_2\}$ we would reach the contradiction $3<\sum_{B\in\mathscr{B}_{T,e}\dot\cup\mathscr{B}_{T,6}\dot\cup\mathscr{B}_{T,7}}\bar{x}_B\leqslant 2$. Therefore, there must exist $F\in supp(\bar{x})$ with $|F\cap\{6,7\}|=1$ and $F\backslash\{6,7\}\in\{\{a,c\},\{b,d\},\{a,d\},\{b,c\}\}$. Again, we conclude that we can find a perfect triple of sets in $\{D_1,D_2,E_1,E_2,F_1,F_2,F\}$. $\square$

6. Using facets for lower and upper bounds

In this section, we illustrate some interesting uses of valid inequalities for packing design problems. Recall that $D(m,v)$ denotes the maximum size of an m-sparse PSTS(v). We show an upper bound on $D(m,v)$ based on valid subpacking inequalities for m-sparse PSTSs. We also display the results of an algorithm that uses 4-sparse facets to determine $D(4,v)$.

Proposition 9 (Upper bound for m-sparse number). *Let* $m\geqslant 4$. *Then*,

$$D(m,v)\leqslant U(m,v):=\left\lfloor\frac{D(m,v-1)v}{v-3}\right\rfloor.$$

Table 3
The anti-Pasch (4-sparse) PSTS number for small v

	Exact[a]	Upper bounds[b]	
v	$D(4,v)$	$D_1(v,3,2)$	$U(4,v)$
6	3	4	4
7	5	7	5
8	8	8	8
9	12	12	12
<u>10</u>	12	13	17
<u>11</u>	15	17	16
<u>12</u>	19	20	20
<u>13</u>	$\geqslant 24$	26	24
14	28	28	30
15	35	35	35
16	37	37	43

[a]Results from branch-and-cut algorithm.
[b]Upper bounds from known packing numbers and from Proposition 9.
To the best of our knowledge the determination of $D(4,v)$ for $v \in [10,13]$ are new results (see Table 4 for the designs obtained).

Proof. There are v rank inequalities of the form $\sum_{B\in\binom{T}{3}} x_B \leqslant D(m, v-1)$, for $T \in \binom{V}{v-1}$. Each triple appears in $v-3$ of these inequalities. Thus, adding these inequalities yields $\sum_{B\in\binom{V}{3}} x_B \leqslant (D(m,v-1)v)/(v-3)$. Since the left-hand side is integral, we take the floor function on the right-hand side. The inequality is valid for all $x \in P_{m,v}$, in particular when x is the incidence vector of a maximal m-sparse STS(v), in which case the left-hand side is equal to $D(m,v)$. $\square$

In Table 3, we show values for $D(4,v)$ obtained by our algorithm. To the general algorithm presented in [21], we added 4-sparse inequalities. Due to their large number, the 4-sparse inequalities were not included in the original integer programming formulation, but were added (using a variation of Algorithm 2) whenever violated during the branch-and-cut algorithm. For $v=13$, it was not possible to solve the problem to optimality but a solution of size 24 was obtained; since this matches the upper bound $U(4,13)$, we conclude $D(4,13)=24$. All other cases in Table 3 were solved to optimality by the algorithm. See Table 4 for optimal anti-Pasch PSTS(v) for $(v) \in [10,13]$.

7. Conclusion and open problems

In this article, we initiate the study of new classes of facet-inducing inequalities for the packing designs and m-sparse PSTS polytopes. We also design and analyse separation algorithms for some subclasses. In Section 6, we exemplify how the knowledge of some of these facets can be used in algorithms for constructing designs as well as

Table 4
Anti-Pasch (4-sparse) PSTS of maximum number of blocks

v	b	An anti-Pasch PSTS(v) obtained by our branch-and-cut algorithm
10	12	$\{0,1,2\},\{0,4,8\},\{0,7,9\},\{1,3,9\},\{1,4,6\}\ \{1,5,7\}$ $\{2,3,5\},\{2,4,7\},\{2,6,9\},\{3,7,8\},\{4,5,9\},\{5,6,8\}$
11	15	$\{0,1,3\},\{0,2,5\},\{0,4,7\},\{0,6,10\},\{1,2,6\},\{1,4,10\}\{1,5,9\}\{2,3,7\}$ $\{2,4,9\},\{2,8,10\},\{3,4,8\},\{3,5,10\},\{3,6,9\},\{6,7,8\},\{7,9,10\}$
12	19	$\{0,1,9\},\{0,2,11\},\{0,3,7\},\{0,4,6\},\{0,8,10\},\{1,2,7\},\{1,3,4\}$ $\{1,6,10\},\{1,8,11\},\{2,3,5\},\{2,6,8\},\{2,9,10\},\{3,6,11\},\{4,5,8\}$ $\{4,7,10\},\{4,9,11\},\{5,6,9\},\{5,10,11\},\{7,8,9\}$
13	24	$\{0,1,8\},\{0,2,6\},\{0,3,11\},\{0,4,12\},\{0,5,9\},\{0,7,10\},\{1,2,5\},\{1,3,7\}$ $\{1,4,6\},\{1,9,11\},\{1,10,12\},\{2,4,10\},\{2,8,11\}\{2,9,12\},\{3,4,8\},\{3,6,12\}$ $\{3,9,10\},\{4,5,11\},\{4,7,9\},\{5,6,7\},\{5,8,12\},\{6,8,9\},\{6,10,11\},\{7,11,12\}$

in the derivation of upper bounds for the packing number. Now, we list some open problems and directions for further research:

1. Discover other facet-inducing inequalities for design polytopes and investigate the use of these inequalities in cutting-plane proofs (in the lines proposed by Chvátal 6) of non-existence for designs or of new upper bounds for packing numbers.
2. Generalize Algorithm 2 to deal with arbitrary l-sparseness inequalities for bounded l.
3. Prove, using integer programming techniques, the correctness of Erdös' conjecture that for any $m \geqslant 4$ there exists an integer v_m such that for every admissible $v \geqslant v_m$ there exists an m-sparse STS(v). More precisely, show that, for v large enough, the equations given in Remark 3 do not cut every point of the $\binom{v}{3}$-dimensional hypercube that lie in the hyperplane $\sum_{B\in\binom{v}{3}} x_B = v(v-1)/6$. Note that this is the fundamental avoidance question for STSs, so it is likely to be hard to answer.

Acknowledgements

At the time of the submission of this article, the author was a postdoctoral fellow at the department of Computer Science of the University of Toronto and at the Fields Institute for Research in Mathematical Sciences, supported by the Natural Sciences and Engineering Research Council of Canada PDF. The author would like to thank an anonymous referee for the careful reading and useful comments.

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