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Lower curvature bounds and cohomogeneity one manifolds ☆

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Abstract

We shall discuss Riemannian metrics of fixed diameter and controlled lower curvature bound. As in [34], we give a general construction of invariant metrics on homogeneous vector bundles of cohomogeneity one, which implies, in particular, that any cohomogeneity one manifold admits invariant metrics of almost nonnegative sectional curvature. This provides positive evidence for a conjecture by Grove and Ziller [24] which states that any cohomogeneity one manifold should have invariant metrics of nonnegative curvature.

MSC: 53C20; 57R60

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1. Introduction

One of the classical problems of differential geometry is the investigation of manifolds which admit (complete) Riemannian metrics with given lower curvature bounds, and the study of relations between the existence of such metrics and the topology and geometry of the underlying manifold. Despite many efforts during the past decades, this problem is still far from being understood. While certain topological obstructions for the existence of metrics with positive, nonnegative or almost nonnegative sectional curvature are known, general methods for the construction of such metrics are rare, leaving an enormous gap between the known examples and those manifolds for which all known obstructions vanish.

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Quite recently, K. Grove and W. Ziller discovered a large class of new examples of closed manifolds admitting Riemannian metrics of nonnegative sectional curvature. These manifolds all admit a cohomogeneity one action, i.e., a smooth action by a compact Lie group whose principal orbit has codimension one. In [24], Grove and Ziller showed that any such cohomogeneity one manifold admits an invariant metric of nonnegative sectional curvature if it has two singular orbits of codimension two, where we call a metric invariant if the Lie group acts by isometries. This class already contains many interesting new examples. In fact, Grove and Ziller conjectured that any cohomogeneity one manifold admits an invariant metric with nonnegative sectional curvature.

There is some positive evidence for this conjecture. Apart from the aforementioned special case considered in [24], Grove and Ziller showed in [25] that any cohomogeneity one manifold admits an invariant metric of *nonnegative Ricci curvature* and in fact an invariant metric of *positive Ricci curvature* if it is closed and its fundamental group is finite.

Moreover, in [34] W. Tuschmann and this author showed that any cohomogeneity one manifold admits invariant metrics of almost nonnegative sectional curvature, i.e., for every $\varepsilon > 0$ there is a metric g_ε on M such that $\mathrm{Sec}(M, g_\varepsilon) \cdot \mathrm{diam}(M, g_\varepsilon)^2 > -\varepsilon$. This is equivalent to saying that in the Gromov–Hausdorff topology M can be collapsed to a single point under a lower curvature bound.

While there are examples of simply connected closed manifolds with positive Ricci curvature which do not admit metrics of almost nonnegative curvature (cf. [23,35]), there are neither obstructions nor examples known which tell the class of closed simply connected manifolds with *almost nonnegative sectional curvature* from the class of such manifolds with *nonnegative sectional curvature*. In this sense, the result from [34] is indeed significant support for the above mentioned conjecture.

Following this introduction, we shall recall some standard methods of constructing manifolds of nonnegative sectional curvature, including compact homogeneous manifolds and biquotients. In Section 3, we shall discuss metrics on homogeneous vector bundles, generalizing some ideas of Cheeger, and discuss when such bundles admit invariant metrics with normal homogeneous collar. In the following section, we shall apply these results to cohomogeneity one manifolds, describing the aforementioned results in greater detail. In Section 5, we shall give some applications, describing examples of manifolds with nonnegative or almost nonnegative sectional curvature. Finally, in Section 6 we give a survey of known obstructions for the existence of metrics of almost nonnegative curvature in order to put these results into a broader context.

2. Nonnegative curvature: standard techniques and examples

2.1. General construction methods

The first almost trivial observation is that the Riemannian product $(M_1 \times M_2, g_1 + g_2)$ of two nonnegatively curved Riemannian manifolds (M_i, g_i) has itself nonnegative sectional curvature.

Another standard fact which is of great importance in this context is *O'Neill's formula*. For this, consider a submersion $\pi : M \to N$ between two Riemannian manifolds, i.e., a surjective map for which the differential $d\pi_p$ is an epimorphism for all $p \in M$. Define the vertical and horizontal distributions on M as $\mathcal{V} := \ker(d\pi)$ and $\mathcal{H} := \mathcal{V}^\perp$, and call sections of $\mathcal{V}$ and $\mathcal{H}$ vertical and horizontal vector fields, respectively. Then we say that $\pi : M \to N$ is a *Riemannian submersion* if the restriction $d\pi_p : \mathcal{H}_p \to T_{\pi(p)}N$ is an isometry w.r.t. the Riemannian metrics on each space. Then we have the following

Proposition 2.1. *Let $\pi : (M, g_1) \to (N, g_2)$ be a Riemannian submersion. Let $p \in M$ and $x, y \in T_{\pi(p)}N$. Let $\bar{x}, \bar{y} \in \mathcal{H}_p$ be the unique tangent vectors with $d\pi(\bar{x}) = x$ and $d\pi(\bar{y}) = y$. Then*

$$R_N(x, y; y, x) = R_M(\bar{x}, \bar{y}; \bar{y}, \bar{x}) + \frac{3}{4}\|A(\bar{x}, \bar{y})\|^2_{g_1}, \tag{1}$$

where $A : \Lambda^2\mathcal{H} \to \mathcal{V}$ is the tensor given by $A(\bar{x}, \bar{y}) = [\overline{X}, \overline{Y}]_{\mathcal{V}}$, where $\overline{X}, \overline{Y}$ are horizontal vector fields with $\overline{X}_p = \bar{x}$ and $\overline{Y}_p = \bar{y}$.

Here we use the convention $R(x, y; z, w) := g(R(x, y)z, w)$, so that $R(x, y; y, x) = \mathrm{Sec}(x, y)\|x \wedge y\|^2_g$. As an immediate consequence, we obtain

Corollary 2.2. *If $\pi : (M, g_1) \to (N, g_2)$ is a Riemannian submersion and (M, g_1) has nonnegative sectional curvature, then so does (N, g_2).*

As a further important standard formula we state the curvature for a warped product metric.

Proposition 2.3. *Let (M, g) be a Riemannian manifold, and let $\widetilde{M} := I \times M$ where $I \subset \mathbb{R}$ is an interval. For some smooth function $f : I \to \mathbb{R}^+$, we define the metric $\tilde{g}$ on $\widetilde{M}$ by the formula*

$$\tilde{g} = dt^2 + f(t)^2 g,$$

using t as the parameter for I. Then the curvature tensor $\widetilde{R}$ of $\tilde{g}$ satisfies

$$\widetilde{R}(c\partial_t + x, y; y, c\partial_t + x) = -c^2 f'' f \|y\|_g + f^2\bigl(R(x, y; y, x) - f'^2\|x \wedge y\|^2_g\bigr),$$

where R denotes the curvature tensor of g and for $x, y \in TM$. Thus, if $C_0 := \inf(\mathrm{Sec}(M, g))$, then $(\widetilde{M}, \tilde{g})$ has nonnegative (positive, respectively) sectional curvature iff $f'' \leqslant 0$ and $f'^2 \leqslant C_0$ ($f'' < 0$ and $f'^2 < C_0$, respectively).

2.2. Compact Lie groups

Let G be a compact Lie group and choose any right invariant Riemannian metric on G, i.e., such that all right translations $R_h : G \to G$, $g \mapsto gh$ are isometries. Moreover, let V be a finite dimensional vector space on which G acts, i.e., such that there is a Lie group homomorphism $\rho : G \to \mathrm{Aut}(V)$. As usual, we abbreviate $\rho(g)x$ by gx for $g \in G$ and $x \in V$. Let $(\, , \,)$ be any inner product on V. Then the inner product on V given by

$$\langle x, y\rangle := \int_G (gx, gy)\, dg \tag{2}$$

is G-invariant; indeed, since $R^*_{h^{-1}} dg = dg$ due to the right invariance, we have $\langle hx, hy\rangle = \int_G (ghx, ghy)\, dg = \int_G (gx, gy) R^*_{h^{-1}}\, dg = \langle x, y\rangle$. Therefore, we obtain a morphism $\rho : G \to O(V, \langle \, , \, \rangle)$, whence its differential yields a linear map $d\rho : \mathfrak{g} \to \mathfrak{so}(V, \langle \, , \, \rangle)$, so that $d\rho(x)$ is skew-symmetric w.r.t. $\langle \, , \, \rangle$ for all $x \in \mathfrak{g}$.

In particular, since G acts on its Lie algebra $\mathfrak{g}$ via the adjoint representation, we conclude that there is an Ad_G-invariant inner product on $\mathfrak{g}$. The corresponding left invariant metric on G is then evidently biinvariant, i.e., both the left and the right translations of G are isometries. Since the differential

$\mathrm{ad} := d(\mathrm{Ad}) : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ is given by the Lie bracket, its skew symmetry reads

$$\big\langle [x, y], z\big\rangle + \big\langle y, [x, z]\big\rangle = 0 \quad \text{for all } x, y, z \in \mathfrak{g}. \tag{3}$$

From (3) it is now immediate to verify that the connection on G given by

$$\nabla_x y := \frac{1}{2}[x, y] \quad \text{for all left invariant vector fields } x, y \in \mathfrak{g}$$

is the Levi-Civita connection of any biinvariant metric, and whence the sectional curvature satisfies

$$\mathrm{Sec}(x, y) = \frac{1}{4}\big\langle [x, y], [x, y]\big\rangle \geqslant 0, \quad \text{where } x, y \in \mathfrak{g} \text{ is an orthonormal pair,}$$

so that we have the following

Proposition 2.4. *Let G be a compact Lie group. Then the sectional curvature of any biinvariant metric on G is nonnegative.*

2.3. Compact homogeneous spaces

Let M be a closed manifold, and suppose that the compact Lie group G acts transitively on M. If we fix $p \in M$ and let $H := \mathrm{Stab}_p \subset G$ be the stabilizer of p, then H is also compact, and we can naturally identify M with the set of left cosets G/H. In particular, there is a natural submersion map $\pi : G \to M \cong G/H$. We fix a biinvariant metric $\langle\ ,\ \rangle$ on $\mathfrak{g}$ and thus have the orthogonal decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}.$$

Then any other inner product on $\mathfrak{g}$ is of the form

$$g_\varphi(x, y) := \langle x, \varphi y\rangle, \tag{4}$$

where $\varphi : \mathfrak{g} \to \mathfrak{g}$ is a linear map which is symmetric and positive definite w.r.t. $\langle\ ,\ \rangle$.

It is now easy to see that there is a unique Riemannian metric on M such that the natural projection $\pi : (G, g) \to M$ becomes a Riemannian submersion iff $g(\mathfrak{h}, \mathfrak{m}) = 0$ and the restriction $g|_{\mathfrak{m}}$ is Ad_H-invariant. Conversely, any G-invariant metric on M is obtained by this procedure, so that there is a one-to-one correspondence between G-invariant Riemannian metrics on M and Ad_H-invariant inner products on $\mathfrak{m} \subset \mathfrak{g}$. In particular, if we choose $g = \langle\ ,\ \rangle$ then the induced metric on G is called a *normal homogeneous metric*, and from Corollary 2.2 and Proposition 2.4 we obtain

Proposition 2.5. *Let $M = G/H$ be a compact homogeneous space. Then every normal homogeneous metric on M is G-invariant and has nonnegative sectional curvature.*

In general, if $\varphi : \mathfrak{m} \to \mathfrak{m}$ is an $\mathrm{Ad}(H)$-equivariant linear map which is symmetric and positive definite w.r.t. $\langle\ ,\ \rangle$, then we can extend it to $\mathfrak{g}$ by setting $\varphi|_{\mathfrak{h}} = \mathrm{Id}_{\mathfrak{h}}$ and define g_φ on G by (4). By abuse of notation, we denote the induced submersion metric on $M = G/H$ also by g_φ. Now, if we let

$$\pi^{\pm}(x, y) := \frac{1}{2}\big([x, \varphi y] \pm [y, \varphi x]\big),$$

then the Levi-Civita connection ∇^φ and the curvature tensor R^φ of g_φ have been calculated in [33] to satisfy

$$\nabla^\varphi_x y = -\frac{1}{2}[x, y]_{\mathfrak{m}} + \varphi^{-1}\pi^+(x, y),$$

$$R^\varphi(x, y; y, x) = \langle \pi^-(x, y), [x, y]\rangle - \frac{3}{4}\langle \varphi[x, y]_{\mathfrak{m}}, [x, y]_{\mathfrak{m}}\rangle + \langle \pi^+(x, y), \varphi^{-1}\pi^+(x, y)\rangle - \langle \pi^+(x, x), \varphi^{-1}\pi^+(y, y)\rangle. \quad (5)$$

An interesting question is to determine the invariant metrics of *positive sectional curvature*. These spaces are well known due to the work of Berger [7], Aloff and Wallach [1,37] and Berard-Bergery [6]. We shall not give the classification here, but we would like to point out that other than the compact rank one symmetric spaces which obviously have positive sectional curvature, such homogeneous spaces exist only in dimensions at most 24. Indeed, in dimensions larger than 24 the compact rank one symmetric spaces are the only known closed manifolds with positive sectional curvature.

2.4. Biquotients

Let G be a compact Lie group as before, and let $H \subset G \times G$ be a closed subgroup which hence acts on G via

$$(h_1, h_2) \cdot g := h_1 g h_2^{-1}.$$

An easy calculation shows that this action is free iff for all $e \neq (h_1, h_2) \in H$, h_1 and h_2 are not conjugate in G. If this is the case, then the quotient space $G /\!\!/ H$ is a manifold and is called a *biquotient space*. Evidently, there is a projection map $\pi : G \to G /\!\!/ H$. Moreover, any biinvariant Riemannian metric on G induces a (unique) submersion metric on $G /\!\!/ H$ whence by Corollary 2.2 and Proposition 2.4, we get

Proposition 2.6. *Let $M = G /\!\!/ H$ be a biquotient. Then M carries a Riemannian metric of nonnegative sectional curvature.*

The biquotients are also of interest as a source of new examples of nonnegatively curved manifolds with "interesting" topology, as well as for manifolds of positive sectional curvature, as the following examples illustrate.

Examples.

1. $G = Sp(2)$ and $H = \{(\mathrm{diag}(q, q), \mathrm{diag}(q, 1)), q \in Sp(1)\}$.
   Clearly, if $q \neq 1$, these two matrices are not conjugate, whence $G /\!\!/ H$ is a biquotient and hence admits a Riemannian metric of nonnegative sectional curvature. In fact, one can show that $G /\!\!/ H$ is an *exotic seven dimensional sphere*, i.e., it is homeomorphic but not diffeomorphic to the standard sphere. This was historically the first example for a nonnegatively curved exotic sphere [22].
2. The Eschenburg spaces and the Bazaĭkin spaces
   (a) $G = SU(3)$ and $H = T^2 = \{(\mathrm{diag}(z, w, zw), \mathrm{diag}(1, 1, z^2w^2)), z, w \in U(1)\}$.
   (b) $G = SU(3)$ and $H = S^1_{p,q,r,s} = \{(\mathrm{diag}(z^p, z^q, z^{-(p+q)}), \mathrm{diag}(z^r, z^s, z^{-(r+s)})), z \in U(1)\}$ with $p, q, r, s \in \mathbb{Z}$.

(c) $G = SU(5)$ and

$$H = \left\{ \left(\operatorname{diag}\left(z^{p_1}, \ldots, z^{p_5}\right), \begin{pmatrix} A & \\ & z^{p_1+\cdots+p_5} \end{pmatrix} \right), \ A \in Sp(2),\ z \in U(1) \right\}$$

with $p_i \in \mathbb{Z}$.

One verifies that $G /\!/ H$ is a biquotient in the following cases: in case (a); in (b) e.g. if the sets $\{p, q, -(p+q)\}$ and $\{r, s, -(r+s)\}$ are relatively prime; in (c) if all p_i are odd and $\gcd(p_{\sigma(1)} + p_{\sigma(2)}, p_{\sigma(3)} + p_{\sigma(4)}) = 2$ for all $\sigma \in S_5$. Moreover, it has been shown in [5,15,16] that the submersion metric has *positive sectional curvature* in the following cases: in case (a); in case (b) if $p, q, -(p+q) \notin [m, M]$ where $m = \min\{r, s, -(r+s)\}$ and $M = \max\{r, s, -(r+s)\}$; in case (c) if all $p_i > 0$. Finally, it has also been shown in these references that infinitely many of these examples are not homotopy equivalent to any homogeneous space, so that these are examples of positively curved manifolds which are topologically distinct from the homogeneous ones.

3. Homogeneous vector bundles

Let G/K be a compact homogeneous space, and suppose there is a representation $\imath : K \to \operatorname{Aut}(V)$ on some finite dimensional vector space V, which by (2) we may assume to be orthogonal as K is compact. Then we can associate the *homogeneous vector bundle*

$$D := G \times_K V,$$

i.e., the set of equivalence classes under the relation on $G \times V$ given by $(gh, v) \sim (g, hv)$ for all $g \in G$, $h \in K$ and $v \in V$. Thus, we can regard D as the orbit space of $G \times V$ under the "diagonal action" $h \cdot (g, v) := (gh^{-1}, hv)$ of K, and since this action is free, it follows that for any K-invariant metric on $G \times V$ we get a (unique) metric on D for which the submersion $G \times V \to D$ is Riemannian.

Note that there is a canonical action of G on D, and the cohomogeneity of the principal orbit of this action equals the cohomogeneity of the principal orbit of the action of K on V.

Let us assume that G and hence K act by cohomogeneity one. Since K acts orthogonally, it leaves all spheres centered at the origin invariant, whence K has cohomogeneity one iff it acts transitively on the unit sphere $S^n \subset V$. In particular, we can write the unit sphere $S^n = K/H$ as a homogeneous space where $H \subset K$ is the stabilizer of some unit vector in V.

Note that the norm function $r : V \to \mathbb{R}$, $v \mapsto \|v\|$ is K-invariant and hence induces a function $r : D \to \mathbb{R}$, and for $R \in \mathbb{R}$, we let

$$D_R := r^{-1}\left([0, R]\right) \subset D. \tag{6}$$

Moreover, the level sets of r are precisely the G-orbits of D.

Evidently, D carries a G-invariant metric of nonnegative sectional curvature. Indeed, by Corollary 2.2 and Proposition 2.4 we can choose the submersion metric induced by the Riemannian product of a biinvariant metric on G and any K-invariant metric on V with nonnegative curvature.

We fix once and for all a biinvariant inner product $\langle\ ,\ \rangle$ on $\mathfrak{g}$, and choose subspaces $\mathfrak{m}_1, \mathfrak{m}_2 \subset \mathfrak{g}$ such that

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}_1 \oplus \mathfrak{m}_2, \quad \text{and} \quad \mathfrak{k} = \mathfrak{h} \oplus \mathfrak{m}_1 \tag{7}$$

are orthogonal decompositions w.r.t. $\langle\ ,\ \rangle$. Recall the one-to-one correspondence between K-invariant Riemannian metrics g_φ on $S^n = K/H$ and symmetric bilinear maps $\varphi : \mathfrak{m}_1 \to \mathfrak{m}_1$ described in Section 2.3 and suppose that the metric on V can be written in polar coordinates as

$$g_V = dr^2 + g_{\varphi}(r), \tag{8}$$

with the norm function $r : V \to \mathbb{R}$ from above and a one-parameter family of symmetric maps $\varphi(r) : \mathfrak{m}_1 \to \mathfrak{m}_1$. Then we have the following lemma.

Lemma 3.1 [12]. *Let $D \to G/K$ be a homogeneous disc bundle of cohomogeneity one. Let g_λ be the* Ad_K*-invariant metric on G induced by $\psi : \mathfrak{m} \to \mathfrak{m}$ such that $\psi|_{\mathfrak{m}_1} = \lambda\, \mathrm{Id}_{\mathfrak{m}_1}$ for some $\lambda \in \mathbb{R}$ and $\psi|_{\mathfrak{m}_2} = \mathrm{Id}_{\mathfrak{m}_2}$, and let g_V be a K-invariant metric on V of the form* (8)*. Then the metric on the G-orbit $r^{-1}(t_0) \subset D$ of the corresponding submersion metric is induced by the map $\varphi : \mathfrak{m}_1 \oplus \mathfrak{m}_2 \to \mathfrak{m}_1 \oplus \mathfrak{m}_2$ with*

$$\varphi|_{\mathfrak{m}_2} = \mathrm{Id}_{\mathfrak{m}_2}, \quad \textit{and} \quad \varphi|_{\mathfrak{m}_1} = \lambda\varphi(t_0)\big(\varphi(t_0) + \lambda\, \mathrm{Id}_{\mathfrak{m}_1}\big)^{-1}. \tag{9}$$

Proof. For $X \in \mathfrak{m}_1$, we denote the vector field on S^n induced by the K-action by X_*. Consider the diagonal action of K on $G \times S^n(t_0)$. At the point $(e, t_0 e_0)$, the tangent space to the fiber $\mathcal{V}$ and its orthogonal complement $\mathcal{H}$ are given as

$$\begin{aligned}
\mathcal{V} &= \big\{(A, 0) \mid A \in \mathfrak{h}\big\} \oplus \big\{(X, -X_*) \mid X \in \mathfrak{m}_1\big\}, \quad \text{and}\\
\mathcal{H} &= \big\{(Y, 0) \mid Y \in \mathfrak{m}_2\big\} \oplus \big\{(\varphi X, \lambda X_*) \mid X \in \mathfrak{m}_1\big\}.
\end{aligned}$$

Indeed, $g_{G\times V}((\varphi X, \lambda X_*), (X, -X_*)) = \lambda\langle\varphi X, X\rangle + \langle\varphi(\lambda X), -X\rangle = 0$. Thus, the horizontal lift of a tangent vector on G/H is given by

$$\begin{aligned}
\overline{X} &= \big(\varphi(\varphi + \lambda\, \mathrm{Id})^{-1} X, \lambda(\varphi + \lambda\, \mathrm{Id})^{-1} X_*\big) && \text{for } X \in \mathfrak{m}_1, \quad \text{and}\\
\overline{Y} &= (Y, 0) && \text{for } Y \in \mathfrak{m}_2,
\end{aligned}$$

whence for $X \in \mathfrak{m}_1$ and $Y \in \mathfrak{m}_2$ we have $g(\overline{Y}, \overline{Y}) = \langle Y, Y\rangle$, $g(\overline{X}, \overline{Y}) = 0$, and

$$\begin{aligned}
g(\overline{X}, \overline{X}) &= \lambda\big\langle\varphi(\varphi + \lambda\, \mathrm{Id})^{-1} X, \varphi(\varphi + \lambda\, \mathrm{Id})^{-1} X\big\rangle + \big\langle\varphi(\lambda(\varphi + \lambda\, \mathrm{Id})^{-1} X), \lambda(\varphi + \lambda\, \mathrm{Id})^{-1} X\big\rangle\\
&= \big\langle\lambda\varphi^2(\varphi + \lambda\, \mathrm{Id})^{-2} X, X\big\rangle + \big\langle\lambda^2\varphi(\varphi + \lambda\, \mathrm{Id})^{-2} X, X\big\rangle\\
&= \big\langle\lambda\varphi(\varphi + \lambda\, \mathrm{Id})^{-1} X, X\big\rangle,
\end{aligned}$$

and the claim follows. □

This lemma can be used in different ways to construct G-invariant metrics on D with nonnegative sectional curvature. For example, we can impose the condition that outside of some compact set, the metrics are product metrics.

Corollary 3.2 [12]. *Let $D \to G/K$ be a homogeneous disc bundle over the compact homogeneous space G/K on which G acts with cohomogeneity one. Then D carries a G-invariant metric of nonnegative sectional curvature such that for some $t_0 > 0$, $r^{-1}(t_0, \infty)$ is isometric to $(t_0, \infty) \times (G/H, g_1)$ where g_1 is an arbitrary G-invariant metric on the principal orbit G/H.*

Proof. Choose a K-invariant metric on V of the form $g_V = dr^2 + f(r)^2 g_0$ where g_0 denotes the standard metric on S^n. By Proposition 2.3 we can do this such that g_V has nonnegative sectional curvature and $f \equiv c_0$ on (t_0, ∞) for some $t_0, c_0 > 0$. Then the submersion metric on D has also nonnegative curvature, and by the lemma, the metric on $G/H = r^{-1}(t)$ is fixed for all $t > t_0$. □

Corollary 3.3 [12]. *Let X be a compact rank one symmetric space and let $-X$ be the same space with the opposite orientation. Then there exists a Riemannian metric of nonnegative sectional curvature on $M = X\# \pm X$. Moreover, this metric can be chosen such that for $M = X\# - X$ its isometry group acts with cohomogeneity one, while for $M = X\#X$, this is true only for the local isometry group.*

Proof. Let $X = G/K$ be a compact rank one symmetric space such that $K = \mathrm{Stab}_p$, some $p \in X$. Then it is known that K acts transitively on the unit sphere $S^n \subset T_pX$, and that $D := X\backslash\{p\}$ is a homogeneous vector bundle over some rank one symmetric space of lower dimension. Thus, K acts on D by cohomogeneity one, and hence there is a K-invariant Riemannian metric of nonnegative sectional curvature on D which is a product metric on $r^{-1}(t_0, \infty)$.

Now $r^{-1}[0, t_0 + 1]$ is the complement of an open neighborhood of $p \in X$, and hence we can glue together two such complements along their boundary to obtain a smooth metric on $X\# - X$. The same is true if we change the orientation of X before the glueing process, thus we also obtain a smooth metric on $X\#X$. Evidently, these metrics have nonnegative sectional curvature. Moreover the action of K on D induces a local action of K on $X\# \pm X$, and this action has cohomogeneity one. In the case $M = X\# - X$, this action is globally defined. □

In order to generalize this idea of Cheeger to glue together metrics on two homogeneous disc bundles D_1 and D_2 which close to the boundary are isometric to a product of an interval and a fixed homogeneous metric, one has to overcome the difficulty that in general, even if the principal orbits of the D_i are equivalent as homogeneous spaces, their bundle structures are distinct. That is, the homogeneous metric close to the collar cannot be chosen arbitrarily in order to do the glueing.

Thus, given a homogeneous disc bundle of cohomogeneity one, it is natural to look for homogeneous metrics which close to the collar are isometric to the product of an interval and a fixed *normal homogeneous* metric on the principal orbit. We shall call such a metric a *metric with normal homogeneous collar*.

To construct such metrics, we fix a biinvariant metric Q and the Q-orthogonal decomposition (7). For $\varepsilon > 0$, define the map $\psi_\varepsilon : \mathfrak{m} \to \mathfrak{m}$ by $\psi_\varepsilon|_{\mathfrak{m}_1} = (1+\varepsilon)\,\mathrm{Id}_{\mathfrak{m}_1}$ and $\psi_\varepsilon|_{\mathfrak{m}_2} = \mathrm{Id}_{\mathfrak{m}_2}$ which induces a left invariant metric g_ε on G. Moreover, choose a K-invariant metric g_V on V which takes the form (8) with a one-parameter family of symmetric maps $\varphi(t) : \mathfrak{m}_1 \to \mathfrak{m}_1$ such that $\varphi(t) = \mu^2\,\mathrm{Id}_{\mathfrak{m}_1}$ for all $t \geqslant t_0$ and some constant $\mu > 0$. By Proposition 2.3, this can be done such that this metric has nonnegative sectional curvature. According to (9), the submersion metric induced from the submersion $(G \times V, g_\varepsilon + g_V) \to D$ takes the form

$$g = dt^2 + g_{\varphi(t)}, \quad \text{where}$$

$$\varphi(t)|_{\mathfrak{m}_2} = \mathrm{Id}_{\mathfrak{m}_2}, \quad \text{and} \quad \varphi(t)|_{\mathfrak{m}_1} = \frac{(1+\varepsilon)\mu^2}{\mu^2 + (1+\varepsilon)}\,\mathrm{Id}_{\mathfrak{m}_1} \quad \text{for all } t \geqslant t_0.$$

In particular, if $\mu^2 = (1+\varepsilon)/\varepsilon$, then this metric has a normal homogeneous collar and, by Corollary 2.2, it has nonnegative sectional curvature provided the curvature of (G, g_ε) is nonnegative.

Unfortunately, g_ε will in general have some negative curvature for any $\varepsilon > 0$. However, there is a special case where this approach works.

Theorem 3.4 [24]. *Let $D \to G/K$ be a homogeneous vector bundle of cohomogeneity one and of rank $\leqslant 2$. Then there exists a G-invariant metric of nonnegative sectional curvature on D with normal homogeneous collar, i.e., such that for some $t_0 > 0$, this metric on $r^{-1}(t_0, \infty)$ is isometric to $(t_0, \infty) \times (G/H, g_Q)$ where g_Q is a normal homogeneous metric on the principal orbit G/H.*

Proof. By our discussion above, it suffices to show that (G, g_ε) has nonnegative sectional curvature for some $\varepsilon > 0$. Now, using the decomposition (7), we have $\dim \mathfrak{m}_1 \leqslant 1$ so that $[\mathfrak{m}_1, \mathfrak{m}_1] = 0$. Therefore, if $x = x_1 + x_2$ and $y = y_1 + y_2$ with $x_i, y_i \in \mathfrak{m}_i$ then (5) yields

$$R(x, y; y, x) = \left\| [x_2, y_2]_{\mathfrak{h}} \right\|^2 + \frac{1}{4} \left\| [x_2, y_2]_{\mathfrak{m}_2} + (1+\varepsilon)\big([x_1, y_2] + [x_2, y_1]\big) \right\|^2 + \frac{1}{4}(1 - 3\varepsilon) \left\| [x_2, y_2]_{\mathfrak{m}_1} \right\|^2$$

which is nonnegative for sufficiently small $\varepsilon > 0$ (in fact, $\varepsilon \leqslant \frac{1}{3}$ suffices). □

The conclusion of this theorem is not true for homogeneous bundles of higher rank. That is, there are homogeneous disc bundles of cohomogeneity one for which there is no invariant metric of nonnegative sectional curvature with normal homogeneous collar (cf. Example 3.6).

In order to describe homogeneous metrics on disc bundles, we note that the complement of the zero section of D is of the form $(0, \infty) \times G/H$ where G/H is the principal orbit and the first factor is induced by $r : D \to (0, \infty)$. On this set, we may assume that the metric takes the form

$$g = dt^2 + g_{\varphi(t)}, \tag{10}$$

where $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ and the metric $g_{\varphi(t)}$ on G/H is induced by $\varphi(t) : \mathfrak{m} \to \mathfrak{m}$ as in (4). Then the connection and the curvature of g has been calculated in [25] and [34] as follows.

Proposition 3.5. *Let $M = I \times G/H$ and $g = dt^2 + g_{\varphi(t)}$ be as above, and let $c \in \mathbb{R}$, $x, y \in T_{eH}G/H \cong \mathfrak{m}$. Then*

$$\nabla_x y = \nabla^\varphi_x y - \langle S_t x, \varphi y\rangle \partial_t, \qquad \nabla_{\partial_t} x = \nabla_x \partial_t = S_t x, \qquad \nabla_{\partial_t} \partial_t = 0,$$

$$\begin{aligned} R(c\partial_t + x, y; y, c\partial_t + x) = {} & R^{\varphi(t)}(x, y; y, x) - \frac{1}{4}\big(\langle \dot\varphi x, x\rangle \langle \dot\varphi y, y\rangle - \langle \dot\varphi x, y\rangle^2\big) \\ & + \frac{1}{2} c\big(3\langle \dot\varphi [x, y], y\rangle + 4\big(\langle S_t y, \pi^+(x, y)\rangle - \langle S_t x, \pi^+(y, y)\rangle\big)\big) \\ & - \frac{1}{4} c^2 \big\langle (2\ddot\varphi - \dot\varphi \varphi^{-1} \dot\varphi) y, y \big\rangle, \end{aligned}$$

where $S_t : \mathfrak{m} \to \mathfrak{m}$ is given as $S_t := \frac{1}{2}\varphi^{-1}\dot\varphi$ and where ∇^φ and R^φ are the connection and the curvature of $(G/H, g_\varphi)$, respectively.

Example 3.6. [1] Let $G = SU(3)$, and fix the biinvariant inner product

$$Q(A, B) := -\frac{1}{2}\operatorname{tr} AB^*$$

on $\mathfrak{g} = \mathfrak{su}(3)$. Up to multiples, this is the only biinvariant inner product. Moreover, let $K = S(U(2) \cdot U(1))$, whence $G/K = \mathbb{CP}^2$, and consider the irreducible representation of K on $V = \mathbb{R}^3$ which is determined by the fact that the center of K acts trivially on V. Then the principal orbit of $D := G \times_K V$ is G/T where T consists of all diagonal matrices. As a T-module, we have the decomposition

$$\mathfrak{su}(3) = \mathfrak{t} \oplus \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3, \qquad \mathfrak{k} = \mathfrak{t} \oplus \mathfrak{m}_1,$$

where each $\mathfrak{m}_i$ consists of all matrices with non-zero entries only in two fixed positions off the diagonal. As T-modules, the $\mathfrak{m}_i$ are irreducible and pairwise inequivalent, hence every T-equivariant map $\varphi : \mathfrak{m} \to \mathfrak{m}$ must have the $\mathfrak{m}_i$ as eigenspaces. Whence, off the 0-section of D, any G-invariant metric must be of the form

$$g = dt^2 + g_{\varphi(t)}, \qquad \varphi(t)|_{\mathfrak{m}_i} = f_i(t)^2 \operatorname{Id}_{\mathfrak{m}_i}.$$

Let $c \subset M$ be the geodesic which is pointwise fixed by T, whence $\dot{c} = \partial_t$. Consider the subalgebras $\mathfrak{k}_i := \mathfrak{t} \oplus \mathfrak{m}_i \subset \mathfrak{g}$, and let $K_i \subset G$ be the corresponding subgroups. Then the orbits $M_i := K_i \cdot c \subset M$ are totally geodesic by Proposition 3.5, and for $i = 2, 3$ we have $M_i \cong \mathbb{R} \times S^2$ where K_i acts transitively on the second factor. Thus, any K_i-invariant metric on M_i must be of the form $dt^2 + f_i(t)^2 g_0$ where g_0 is the standard metric on S^2, and the nonnegativity of the curvature of this metric implies that $f_i'' \leqslant 0$, whence f_i is constant.

If we could find a metric of nonnegative sectional curvature with normal homogeneous collar on D, then we would have that g is given as above with $f_i^2 \equiv c_0 > 0$ for $i = 2, 3$, and $f_1(t)^2 \equiv c_0$ for $t \geqslant t_0$.

Now we choose the elements $x_1, y_1 \in \mathfrak{m}_1$ and $x_2, y_2 \in \mathfrak{m}_2 \oplus \mathfrak{m}_3$ as follows:

$$x_1 := \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \qquad y_1 := \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix},$$

$$x_2 := \begin{pmatrix} 0 & i & 0 \\ i & 0 & i \\ 0 & i & 0 \end{pmatrix}, \qquad y_2 := \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}.$$

One verifies that $[x_1, y_2] + [x_2, y_1] = 0$ and $[x_1, y_1] = -[x_2, y_2]$. Thus, if we let $x := x_1 + sx_2$ and $y := y_1 + sy_2$, some $s \in \mathbb{R}$, then (5) and Proposition 3.5 imply that

$$R(x, y; y, x) = 8/c_0\big(c_0 f_1^2\big(1 - (f_1')^2\big) - f_1^2\big(3c_0 - f_1^2\big)s^2 + c_0^2 s^4\big),$$

and, assuming that $f_1^2 \leqslant 3c_0$, this expression is nonnegative for all $s \in \mathbb{R}$ iff

$$4c_0^3 f_1^2\big(1 - (f_1')^2\big) - f_1^4\big(3c_0^2 - f_1^2\big)^2 \geqslant 0$$

which is equivalent to saying that

$$4c_0^3 (f_1')^2 \leqslant 4c_0^3 - f_1^2\big(3c_0 - f_1^2\big)^2 = \big(4c_0 - f_1^2\big)\big(c_0 - f_1^2\big)^2,$$

[1] This example has been communicated to us by W. Ziller.

or

$$\left((\log|\sqrt{c_0}-f_1|)'\right)^2 \leqslant \frac{1}{4c_0^3}\left(4c_0-f_1^2\right)\left(\sqrt{c_0}+f_1\right)^2.$$

Since $f(t)^2 < c_0$ for small t and the right hand side of this inequality is uniformly bounded for $f_1^2 \leqslant c_0$, it follows that $f_1(t)^2 < c_0$ for all $t \in \mathbb{R}$, contradicting our assumption.

Of course, this example can be generalized to all G whose Lie algebra contains a subalgebra isomorphic to $\mathfrak{su}(3)$ by choosing $\mathfrak{m}_1$ to lie in that Lie algebra and the principal orbit to be G/T with $T \subset G$ a maximal torus.

Since this example shows that we can in general not expect invariant metrics on homogeneous disc bundles of cohomogeneity one with normal homogeneous collar to have nonnegative sectional curvature, the question remains what can be said about lower curvature bounds of such metrics.

Let us again consider submersion metrics $(G \times V, g_Q + g_V) \to D$ where g_Q denotes any biinvariant metric on G, and g_V is a K-invariant metric on V, which we write in polar coordinates as

$$g_V = dr^2 + g_{\varphi(r)},$$

where $g_{\varphi(r)}$ is a K-invariant metric on the sphere of radius r, $S^n(r) \subset V$. The first difficulty we have to overcome is that in general, the normal homogeneous metric on $S^n = K/H$ does not coincide with the standard metric; rather, these metrics are some *Berger metrics*, linked to the shrinking of certain fibers of the Hopf fibrations. By the smoothness, $\frac{1}{r^2} g_{\varphi(r)}$ must converge to the standard metric as $r \to 0$, while by (9) and the fact that we want a metric with normal homogeneous collar, $g_{\varphi(r)}$ must be normal homogeneous for sufficiently large r. Thus, we need some transition from the round metric to the normal homogeneous metric while maintaining the lower curvature bound. This has been achieved in [34] by the following theorem whose proof is omitted here.

Theorem 3.7. *Let $K \subset O(n+1)$ be a Lie subgroup which acts transitively on $S^n \subset \mathbb{R}^{n+1}$, and let g_Q be a normal homogeneous metric on S^n induced by some* Ad_K*-invariant inner product Q on $\mathfrak{k}$. Let $r(x) := \|x\|$ be the radius function on $\mathbb{R}^{n+1}$.*

Then there exists a K-invariant metric g on the unit ball $B_1(0) \subset \mathbb{R}^{n+1}$ with positive sectional curvature, and an $\varepsilon > 0$, such that on $r^{-1}(1-\varepsilon, 1)$ we have $g = dr^2 + f(r)^2 g_Q$ where $f : (1-\varepsilon, 1) \to \mathbb{R}$ satisfies $f > 0$, $f' > 0$.

By Proposition 2.3, it follows that $f'' < 0$ and $f'^2 < \inf \mathrm{Sec}(S^n, g_Q)$ on $(1-\varepsilon, 1)$ whence we can extend this metric to a nonnegatively curved metric g_V on all on $V = \mathbb{R}^{n+1}$ which outside of $B_1(0)$ has the form

$$g_V = dt^2 + c_0^2 t^2 g_Q \quad \text{for some } c_0 > 0.$$

By Corollary 2.2, the corresponding sumbersion metric on D has nonnegative sectional curvature, and by (9), it can be written on $D \backslash D_1$ (cf. (6)) in the form

$$g = dt^2 + g_{\varphi(t)}, \quad \text{where } \varphi(t)|_{\mathfrak{m}_2} = \mathrm{Id}_{\mathfrak{m}_2} \quad \text{and} \quad \varphi(t)|_{\mathfrak{m}_1} = \frac{c_0^2 t^2}{1 + c_0^2 t^2} \mathrm{Id}_{\mathfrak{m}_1}. \tag{11}$$

Since $\frac{c_0^2 t^2}{1+c_0^2 t^2} < 1$, it follows that this metric will not have a normal homogeneous collar, which was to be expected in view of Example 3.6. But we shall show now that we can change this metric to one with normal homogeneous collar with arbitrarily little negative sectional curvature.

Proposition 3.8 [34]. *Let $H \subset K \subset G$ be compact Lie groups, let Q be a biinvariant inner product on $\mathfrak{g}$ and $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}_1 \oplus \mathfrak{m}_2$ be the decomposition from* (7). *Suppose that $(K/H, g_Q)$ has positive sectional curvature, and let $C > 0$ be the pinching constant, i.e.,*

$$C := \frac{\inf \mathrm{Sec}(K/H, g_Q)}{\sup \mathrm{Sec}(K/H, g_Q)}.$$

For some $R_0 \in \mathbb{R}$ and $\varepsilon > 0$, let $f : [R_0, R_0 + \varepsilon) \to \mathbb{R}$ be a smooth function with

$$0 < f < 1, \quad f' > 0 \quad \textit{and} \quad f'' < -\frac{9f(f')^2}{4 - 3f^2}. \tag{12}$$

Moreover, suppose that

$$\delta := \frac{4(f')^2(R_0)}{Cf^2(R_0)} < \sup \mathrm{Sec}(K/H, g_Q). \tag{13}$$

Then there is a smooth extension $f : [R_0, R) \to \mathbb{R}$ such that $f \equiv 1$ near R, where

$$R - R_0 \leqslant \frac{4}{\sqrt{C\delta}} \frac{1 - f(R_0)^2}{f(R_0)^2} + 1, \tag{14}$$

and such that the metric $g = dt^2 + g_{\varphi(t)}$ on $[R_0, R) \times G/H$ with $\varphi(t) : \mathfrak{m} \to \mathfrak{m}$ given by

$$\varphi(t)|_{\mathfrak{m}_2} = \mathrm{Id}_{\mathfrak{m}_2} \quad \textit{and} \quad \varphi(t)|_{\mathfrak{m}_1} = f(t)^2 \, \mathrm{Id}_{\mathfrak{m}_1}$$

satisfies

$$\mathrm{Ric}(g) \geqslant 0, \qquad \mathrm{Sec}(g) \geqslant -\delta.$$

Moreover, if $\dim K/H > 0$ and $\mathfrak{z}(\mathfrak{g}) \cap \mathfrak{m}_2 = 0$ then there exist points where the Ricci curvature is positive.

Proof. We relate f to a function μ by the equations

$$\mu := \frac{f}{\sqrt{4 - 3f^2}}, \quad \text{whence } f = \frac{2\mu}{\sqrt{1 + 3\mu^2}}, \tag{15}$$

and notice that (12) is equivalent to the conditions $0 < \mu < 1$, $\dot{\mu} > 0$ and $\ddot{\mu} < 0$. Thus, we can extend μ to a smooth function $\mu : [R_0, R) \to \mathbb{R}$ in such a way that $\ddot{\mu} \leqslant 0$ and $\mu \equiv 1$ near R, and define $f : [R_0, R) \to \mathbb{R}$ according to (15). Evidently, this can be done on an interval of length $R - R_0 \leqslant \frac{1 - \mu(R_0)}{\dot{\mu}(R_0)} + 1$, and it is straightforward to verify (14) from (13) and (15).

Given $x = x_1 + x_2$ and $y = y_1 + y_2$ with $x_i, y_i \in \mathfrak{m}_i$, we let $B^{ii} := [x_i, y_i]$ and $B^{12} := \frac{1}{2}([x_1, y_2] + [x_2, y_1]) \in \mathfrak{m}_2$. By hypothesis, we can find constants $C_1 \geqslant C_2 > 0$ such that $1/C_2 \geqslant \mathrm{Sec}(K/H, g_Q) \geqslant 1/C_1$ and $C = C_2/C_1$. Now $R^{g_Q}(x_1, y_1; y_1, x_1) = \langle B^{11}_{\mathfrak{h}}, B^{11}_{\mathfrak{h}} \rangle + \frac{1}{4} \langle B^{11}_{\mathfrak{m}_1}, B^{11}_{\mathfrak{m}_1} \rangle$ by (5), whence

$$C_2 \left(\langle B^{11}_{\mathfrak{h}}, B^{11}_{\mathfrak{h}} \rangle + \frac{1}{4} \langle B^{11}_{\mathfrak{m}_1}, B^{11}_{\mathfrak{m}_1} \rangle \right) \leqslant \|x_1 \wedge y_1\|^2 \leqslant C_1 \left(\langle B^{11}_{\mathfrak{h}}, B^{11}_{\mathfrak{h}} \rangle + \frac{1}{4} \langle B^{11}_{\mathfrak{m}_1}, B^{11}_{\mathfrak{m}_1} \rangle \right),$$

and from there,

$$\frac{1}{4}C_2\langle B^{11}, B^{11}\rangle \leqslant \|x_1 \wedge y_1\|^2 \leqslant C_1\langle B^{11}, B^{11}\rangle. \tag{16}$$

Now Proposition 3.5 implies

$$\begin{aligned} &R(c\partial_t + x, y; y, c\partial_t + x) \\ &\quad = R^{g_{\varphi(t)}}(x, y; y, x) - f^2(f')^2\|x_1 \wedge y_1\|^2 - 3cff'\langle B^{22}, y_1\rangle - c^2 ff''\langle y_1, y_1\rangle. \end{aligned} \tag{17}$$

We decompose $B^{22} = v + w + B_2^{22}$ with $B_2^{22} \in \mathfrak{m}_2$, $v, w \in \mathfrak{k}$ such that $\langle v, y_1\rangle = 0$ and w is a multiple of y_1; if $y_1 = 0$ then we set $w = 0$. Moreover, we let $B_{\mathfrak{k}}^{22} := v + w$. Since $\langle B^{11}, y_1\rangle = \langle [x_1, y_1], y_1\rangle = 0$, it follows that

$$\langle B^{11}, B_{\mathfrak{k}}^{22}\rangle = \langle B^{11}, v\rangle, \quad \langle B^{22}, y_1\rangle = \langle w, y_1\rangle \quad \text{and} \quad \langle B_{\mathfrak{k}}^{22}, B_{\mathfrak{k}}^{22}\rangle = \langle v, v\rangle + \langle w, w\rangle. \tag{18}$$

Then (5) and (18) yields

$$\begin{aligned} R^{g_{\varphi(t)}}(x, y; y, x) &= \frac{3}{4}f^2\big\|[x, y]_{\mathfrak{h}}\big\|^2 + \frac{1}{4}\big\|B_2^{22} + 2f^2B^{12}\big\|^2 \\ &\quad + \frac{1}{4}f^2\langle B^{11}, B^{11}\rangle + \frac{1}{2}f^2(3 - 2f^2)\langle B^{11}, B_{\mathfrak{k}}^{22}\rangle + \left(1 - \frac{3}{4}f^2\right)\langle B_{\mathfrak{k}}^{22}, B_{\mathfrak{k}}^{22}\rangle \\ &\geqslant \frac{1}{4}f^2\langle B^{11}, B^{11}\rangle + \frac{1}{2}f^2(3 - 2f^2)\langle B^{11}, v\rangle + \left(1 - \frac{3}{4}f^2\right)\langle v, v\rangle \\ &\quad + \left(1 - \frac{3}{4}f^2\right)\langle w, w\rangle. \end{aligned}$$

Substituting this and (16) into (17) yields

$$\begin{aligned} &R(c\partial_t + x, y; y, c\partial_t + x) \\ &\quad \geqslant \frac{1}{4}f^2(1 - 4C_1(f')^2)\langle B^{11}, B^{11}\rangle + \frac{1}{2}f^2(3 - 2f^2)\langle B^{11}, v\rangle + \left(1 - \frac{3}{4}f^2\right)\langle v, v\rangle \\ &\quad + \left(1 - \frac{3}{4}f^2\right)\langle w, w\rangle - 3cff'\langle w, y_1\rangle - c^2ff''\langle y_1, y_1\rangle. \end{aligned} \tag{19}$$

From (13) and (15) we deduce that $f'' \leqslant 0$, $f \leqslant 1$ and $4C_1(f')^2 < 1$. Thus, from (19) a straightforward calculation now yields that this metric has nonnegative Ricci curvature. Moreover, if $f'' < 0$, then the Ricci curvature is positive, unless $\mathfrak{m}_1 = 0$ (in which case $\mathrm{Ric}(\partial_t) = 0$) or $\mathfrak{z}(\mathfrak{g}) \cap \mathfrak{m}_2 \neq 0$ (in which case $\mathrm{Ric}(x) = 0$ for any $x \in \mathfrak{z}(\mathfrak{g}) \cap \mathfrak{m}_2$). Since there are points where $f'' < 0$, the assertion about points with positive Ricci curvature follows.

Next, observe that $f^4\|x_1 \wedge y_1\|^2 = \|x_1 \wedge y_1\|_g^2 \leqslant \|(c\partial_t + x) \wedge y\|_g^2$, whence in order to guarantee that $\mathrm{Sec}(g) \geqslant -\delta$, it suffices to show that

$$R(c\partial_t + x, y; y, c\partial_t + x) + \delta f^4\|x_1 \wedge y_1\|^2 \geqslant 0. \tag{20}$$

But by (16) and (19), we have the following estimate:

$$\begin{aligned} &R(c\partial_t + x, y; y, c\partial_t + x) + \delta f^4\|x_1 \wedge y_1\|^2 \\ &\quad \geqslant \frac{1}{4}C_1f^2(C\delta f^2 - 4(f')^2)\langle B^{11}, B^{11}\rangle + \frac{1}{4}f^2\langle B^{11}, B^{11}\rangle + \frac{1}{2}f^2(3 - 2f^2)\langle B^{11}, v\rangle \\ &\quad + \left(1 - \frac{3}{4}f^2\right)\langle v, v\rangle + \left(1 - \frac{3}{4}f^2\right)\langle w, w\rangle + 3cff'\langle w, y_1\rangle - c^2ff''\langle y_1, y_1\rangle. \end{aligned} \tag{21}$$

Note that $C\delta f^2 - 4(f')^2 \geqslant 0$; indeed, by (13) this holds at R_0, and moreover, $(f'/f)' \leqslant 0$ as $f'' \leqslant 0$. Thus, the first row on the right of (21) is nonnegative.

The second and third row on the right of (21) are nonnegative if the quadratic polynomials

$$p_1(x) = \frac{1}{4}f^2x^2 + \frac{1}{2}f^2\big(3 - 2f^2\big)x + \left(1 - \frac{3}{4}f^2\right),$$
$$p_2(x) = -ff''x^2 + 3ff_1'x + \left(1 - \frac{3}{4}f^2\right)$$

are nonnegative for all $x \in \mathbb{R}$. The discriminants d_i of p_i are given by

$$d_1 = \frac{1}{4}f^2\left(1 - \frac{3}{4}f^2\right) - \frac{1}{16}f^4\big(3 - 2f^2\big)^2 = \frac{1}{4}f^2\big(1 - f^2\big)^3 \geqslant 0,$$
$$d_2 = -ff''\left(1 - \frac{3}{4}f^2\right) - \frac{9}{4}f^2\big(f_1'\big)^2 = -\frac{4\mu\ddot{\mu}}{(1+3\mu^2)^3} \geqslant 0,$$

and since $p_i(0) = 1 - \frac{3}{4}f^2 > 0$, $p_i(x) \geqslant 0$ for all $x \in \mathbb{R}$ follows. □

As a consequence, we now obtain the following

Theorem 3.9 [25,34]. *Let $D \to G/K$ be a homogeneous vector bundle with cohomogeneity one. For every $\delta > 0$, there exists an invariant metric g_δ on D_R (cf. (6)) for some $R = R(\delta)$ with normal homogeneous collar such that*

$$\mathrm{Sec}(D_R, g_\delta) \geqslant -\delta, \qquad \mathrm{diam}(D_R, g_\delta) = \mathrm{O}\big(\delta^{-1/6}\big), \qquad \mathrm{Ric}(D_R, g_\delta) \geqslant 0.$$

Moreover, if the rank of D is at least two and if $\pi_1(G/K)$ is finite then there exist points of positive Ricci curvature.

Here, $\mathrm{O}(\delta^p)$ denotes any function of δ such that $\limsup_{\delta\to 0} |\delta^{-p}\mathrm{O}(\delta^p)| < \infty$.

Proof. First of all, we note that any normal homogeneous metric on the sphere K/H has positive sectional curvature [7], and we assume that $\delta < \sup\ \mathrm{Sec}(K/H, g_Q)$. By (11), we can for any $R_0 > 1$ and $\varepsilon > 0$ construct an invariant metric on $D_{R_0+\varepsilon}$ with nonnegative sectional curvature such that it is given in the form needed in Proposition 3.8 with $f(t) = c_0t/\sqrt{1 + c_0^2t^2}$. One verifies that (12) holds, and we define $R_0 = R_0(\delta)$ by the equation

$$\delta = \frac{4(f')^2(R_0)}{Cf^2(R_0)} = \frac{4}{CR_0^2(1 + c_0^2R_0^2)^2},$$

so that $R_0 = \mathrm{O}(\delta^{-1/6})$ and (13) is satisfied. Thus, by Proposition 3.8 there is an invariant metric on $(R_0, R) \times G/H \cong D_R \backslash D_{R_0}$ which can be glued together with the metric on $D_{R_0+\varepsilon}$ to obtain an invariant metric on D_R with normal homogeneous collar and the asserted curvature bounds.

Furthermore, (14) implies that $R - R_0 \leqslant \frac{4}{\sqrt{C\delta c_0^2 R_0^2}} + 1 = \mathrm{O}(\delta^{-1/6})$, so that $R = \mathrm{O}(\delta^{-1/6})$ as well.

Since on $D_R \backslash D_1$ this metric is of the form $g = dt^2 + g_{\varphi(t)}$, it follows that the curves $t \mapsto (t, p)$ are unit speed geodesics, and since the metric on D_1 is independent of δ, we have $\mathrm{diam}(D_R, g_\delta) \leqslant \mathrm{diam}(D_1, g_\delta) + 2(R(\delta) - 1) = \mathrm{O}(\delta^{-1/6})$ as claimed.

Finally, note that the condition $\mathfrak{z}(\mathfrak{g}) \cap \mathfrak{m}_2 = 0$ is equivalent to the condition that $\pi_1(G/K)$ is finite. □

4. Cohomogeneity one manifolds

A connected manifold M is said to have *cohomogeneity one* if it supports a smooth action by a compact Lie group G such that the orbit space is one dimensional. The topological structure of cohomogeneity one manifolds is well understood in principle. Namely, there are exactly the following four cases.

1. $M/G = \mathbb{R}$. In this case $M = \mathbb{R} \times M_0$ where $M_0 = G/H$ is a compact homogeneous space, and G acts trivially on the first factor,
2. $M/G = \mathbb{R}_+ = [0, \infty)$. In this case M is a homogeneous disc bundle of cohomogeneity one over some compact homogeneous space, i.e., $M = G \times_K \mathbb{R}^n$ where $K \subset O(n)$ acts transitively on the unit sphere,
3. $M/G = S^1 = \mathbb{R}/\mathbb{Z}$. Then $M = (\mathbb{R} \times G/H)/\mathbb{Z}$ where $\mathbb{Z}$ acts on $\mathbb{R}$ by translation and on the compact homogeneous space G/H by an element in the normalizer of H in G. Since the action of such an element has finite order, it follows that M is finitely covered by $S^1 \times G/H$ and hence is locally homogeneous,
4. $M/G = [a, b]$. In this case, M is obtained by glueing together two homogeneous disc bundles along their common boundary, i.e., $M = (G \times_{K_-} D_-) \cup (G \times_{K_+} D_+)$ where $D_\pm$ is the unit disc in the vector space on which $K_\pm$ acts orthogonally and with cohomogeneity one.

The most interesting kind of cohomogeneity one manifolds from the topological point of view is the last one. Moreover, from our discussion of homogeneous vector bundles of cohomogeneity one in the preceding section, we can construct G-invariant Riemannian metrics on these spaces by glueing together metrics with normal homogeneous collar along their common boundary, just like in the proof of Corollary 3.3. Thus, we immediately obtain the following results.

Corollary 4.1. *Let (M, G) be a cohomogeneity one manifold with two singular orbits, i.e., such that $M/G = [a, b]$.*

1. [24] *If both singular orbits have codimension at most two, then M admits a G-invariant Riemannian metric of nonnegative sectional curvature.*
2. [25] *M admits a G-invariant metric of nonnegative Ricci curvature. Moreover, if $\pi_1(M)$ is finite then M admits a G-invariant metric of positive Ricci curvature.*
3. [34] *M admits G-invariant metrics of* almost nonnegative curvature, *i.e., for every $\varepsilon > 0$ there is a G-invariant metric g_ε on M such that* $\mathrm{Sec}(M, g_\varepsilon) \cdot \mathrm{diam}(M, g_\varepsilon)^2 > -\varepsilon$.

Proof. All of these follow immediately from the above description of cohomogeneity one manifolds and Theorems 3.4 and 3.9, except for the second part of the second statement.

Namely, for this one shows that if there are no points of positive Ricci curvature on M then either M has infinite fundamental group, or M is a Seifert type bundle whose orbit space base and generic fiber both admit metrics with positive Ricci curvature, whence it admits a G-invariant metric of positive Ricci curvature by [31].

On the other hand, if the G-invariant metric of nonnegative Ricci curvature *has* points of positive Ricci curvature, then the existence of an invariant metric of positive Ricci curvature follows from the deformation results in [3,14,38]. We refer to [25] for details. □

It has been conjectured in [24] that *any* cohomogeneity one manifold should support an invariant metric of nonnegative sectional curvature. Since this follows from the standard constructions discussed in Section 2.1 in the cases where M/G is the line, the half line or the circle, we may restrict our attention to the case where there are two singular orbits. Thus, the above results can be viewed as steps into the direction of proving this conjecture.

However, if this conjecture is correct, it cannot be proven in a way analogous to Corollary 4.1 by glueing together normal homogeneous metrics on homogeneous disc bundles. For example, if we consider the adjoint action of $SU(3)$ on the unit sphere $S^7 \subset \mathfrak{su}(3)$, then one verifies easily that this action has cohomogeneity one and has two singular orbits. Moreover, the normal bundles of the singular orbits are precisely the bundles considered in Example 3.6 and thus do not admit invariant metrics of nonnegative sectional curvature with normal homogeneous collar. On the other hand, S^7 carries an invariant metric of constant positive sectional curvature, whence this example illustrates the limits of the "glueing method" used to show Corollary 4.1.

5. Applications

5.1. Principal bundles, vector bundles and sphere bundles

Definition 5.1. Let M be a manifold with a (smooth) action by a compact Lie group G, and let $P \to M$ be a principal H-bundle where H is a compact Lie group. We say that the action of G on M *lifts to* P if there is an action of $\widetilde{G} \times H$ on P where $\widetilde{G} \to G$ is a (finite) cover extending the action of H on P and such that the induced action of $\widetilde{G}$ on $P/H = M$ coincides with the given one.

Observe that the induced action of $\widetilde{G} \times H$ on P has the same cohomogeneity as the action of G on M, and the number and codimensions of the singular orbits is the same for both actions.

Not every group action admits a lift to any principal bundle. However, if H is abelian then such a lift almost always exists. More precisely, the following is known.

Proposition 5.2 [26]. *Let M be a closed smooth manifold on which a compact connected Lie group G acts smoothly, and let $\pi : P \to M$ be a principal T^k bundle over M where T^k is the k-dimensional torus. If $H^1(M, \mathbb{Z})$ is trivial or if G is semisimple, then the action of G lifts to P.*

Of course, this proposition implies, in particular, that any principal torus bundle over any simply connected cohomogeneity one manifold is again of cohomogeneity one, whence the statements of Corollary 4.1 hold.

Another interesting class of principal bundles which admit a cohomogeneity one action has been considered in [24]. Namely, we have the following

Proposition 5.3 [24]. *Any principal H-bundle $P \to S^4$ with $H = SO(3), SO(4), Sp(1)$ or $Sp(1) \times Sp(1)$ admits a cohomogeneity one action with two singular orbits of codimension two, and whence an invariant metric of nonnegative sectional curvature.*

For this, one considers the action of $SO(3)$ on $S^4 \subset \mathbb{R}^5$ induced by the (unique) five-dimensional irreducible representation of $SO(3)$, and notes that this action has cohomogeneity one and two singular orbits of codimension two. Then one proves that this action lifts to any principal bundle with one of these structure groups by a direct investigation, using the topological classification of these principal bundles.

Corollary 5.4 [24]. *Every vector bundle and every sphere bundle over S^4 admits a complete metric of nonnegative sectional curvature whose isometry group acts with cohomogeneity one. In particular, all Milnor spheres* (*i.e.*, 10 *of the* 14 *unoriented seven dimensional exotic spheres which are S^3-bundles over S^4*) *admit metrics of nonnegative sectional curvature.*

The proof uses the fact that every vector bundle over S^4 of rank > 4 is the direct sum of a rank four vector bundle and a trivial one. Moreover, every vector bundle of rank $\leqslant 2$ is trivial, whence the structure group of any nontrivial vector bundle over S^4 can be reduced to $SO(3)$ or $SO(4)$. Whence, the associated principal bundle $P \to S^4$ carries an invariant metric of nonnegative sectional curvature. The total space E of any vector bundle or sphere bundle can thus be written as

$$E = P \times_H \mathbb{R}^{n+1}, \quad \text{or} \quad E = P \times_H S^n$$

where $H = SO(3)$ and $n = 2$, or $n \geqslant 3$ and $H = SO(4)$ acts trivially on the second summand of $\mathbb{R}^{n+1} = \mathbb{R}^4 \oplus \mathbb{R}^{n-3}$. Whence the submersion metric on E induced by the product metric on $P \times \mathbb{R}^{n+1}$ ($P \times S^n$, respectively) has nonnegative sectional curvature by Corollary 2.2.

Similar arguments also lead to the following statements.

Corollary 5.5 [24]. *Every vector bundle and every sphere bundle over S^5 admits a complete metric of nonnegative sectional curvature.*

Every rank three vector bundle and 88 *of the* 144 *rank four vector bundles over S^7 and the corresponding sphere bundles admit complete metrics of nonnegative sectional curvature.*

For general cohomogeneity one manifolds, the statement of the existence of almost nonnegatively curved metrics on associated vector bundles follows from the following result.

Theorem 5.6 [17]. *Let $M \to B$ be a fiber bundle for which the fiber F, the structure group G and the base B are compact. If B carries metrics of almost nonnegative sectional curvature and F carries a G-invariant metric of nonnegative sectional curvature then M carries metrics of almost nonnegative sectional curvature.*

Thus, all compact homogeneous fiber bundles—in particular, all principal bundles and all sphere bundles—over a cohomogeneity one manifold carry metrics of almost nonnegative sectional curvature.

5.2. Brieskorn manifolds and the Kervaire spheres

Particularly interesting examples of closed cohomogeneity one manifolds are given by the odd-dimensional Brieskorn manifolds (see [8,10,27,30]). Given an integer $d \geqslant 1$, the Brieskorn manifolds $W^{2n-1}(d)$ are the $2n-1$ dimensional real algebraic submanifolds of $\mathbb{C}^{n+1}$ defined by the equations

$$z_0^d + z_1^2 + \cdots + z_n^2 = 0 \quad \text{and} \quad |z_0|^2 + |z_1|^2 + \cdots + |z_n|^2 = 1.$$

The manifolds $W^{2n-1}(d)$ are invariant under the standard linear action of $O(n)$ on the $(z_1, \ldots, z_n)$ coordinates, and the action of S^1 via the diagonal matrices of the form $\operatorname{diag}(e^{2i\theta}, e^{di\theta}, \ldots, e^{di\theta})$. The resulting action of the product group $S^1 \times O(n)$ has cohomogeneity one [28], whence by Corollary 4.1, all Brieskorn manifolds $W^{2n-1}(d)$, $2 \leqslant d \in \mathbb{Z}$, admit $S^1 \times O(n)$-invariant metrics of almost nonnegative sectional curvature and invariant metrics of positive Ricci curvature.

The topology of the Brieskorn manifolds is fairly well understood. In particular, it is known [27] that for $n, d \geqslant 3$ odd, $W^{2n-1}(d)$ is homeomorphic to a sphere which bounds a parallelizable manifold. Indeed, if $d \equiv \pm 1 \bmod 8$ then the manifolds $W^{2n-1}(d)$ are diffeomorphic to the standard $2n-1$ sphere, while for $d \equiv \pm 3 \bmod 8$, $W^{2n-1}(d)$ is diffeomorphic to the *Kervaire sphere* K^{2n-1}, which is a topological sphere obtained as the boundary manifold of the plumbing of two copies of the tangent disc bundle of S^n [10].[2]

Moreover, the Kervaire sphere K^{2n-1} is an exotic sphere, i.e., homeomorphic but not diffeomorphic to the standard sphere, if $n+1$ is not a power of 2 [9].[3]

Recall that the orbit space of a free action of a nontrivial finite cyclic group on a homotopy sphere is called a *homotopy real projective space* if this group has order two, and a *homotopy lens space* otherwise. Notice that homotopy real projective spaces are always homotopy equivalent to standard real projective spaces [36], whereas a corresponding statement for homotopy lens spaces does in general not hold.

Suppose again that $n \geqslant 3$ and $d \geqslant 1$ are odd. For $m \geqslant 2$ define an action of $\mathbb{Z}_m$ on $\mathbb{C}^{n+1}$ by $\alpha(z_0, z_1, \ldots, z_n) := (\alpha^2 z_0, \alpha^d z_1, \ldots \alpha^d z_n)$, where α is a primitive mth root of unity generating $\mathbb{Z}_m \subset S^1$. One verifies that if m and d are relatively prime, then this action induces a free action on $W^{2n-1}(d)$. Since $W^{2n-1}(d)$ is a homotopy sphere, the quotient $Q_m^{2n-1}(d) := W^{2n-1}(d)/\mathbb{Z}_m$ is a homotopy real projective space for $m = 2$ and a homotopy lens space for $m \geqslant 3$. Moreover, the action of $S^1 \times SO(n)$ on $W^{2n-1}(d)$ descends to the quotient which is therefore again a cohomogeneity one manifold.

The orbit spaces of these free cyclic group actions of on the Brieskorn spheres $W^{2n-1}(d)$ have been extensively studied (see [2,11,19–21,27,29,32]). Combining these results with Corollary 4.1, we obtain the following.

Corollary 5.7 [34]. *The following closed manifolds admit metrics of almost nonnegative sectional curvature and of positive Ricci curvature with an isometry group of cohomogeneity one*:

1. *all Kervaire spheres* (*cf. also* [4]),
2. *quotients of the Kervaire spheres by a free action of* $\mathbb{Z}_m$ *for any integer* $m \geqslant 3$; *these quotients are homotopy lens spaces which are differentiably distinct from the standard ones in those odd dimensions in which the Kervaire sphere is exotic,*
3. *quotients of the Kervaire spheres by a free* $\mathbb{Z}_2$*-action*; *indeed, for any integer* $k \geqslant 1$ *this results in at least* 4^k *oriented diffeomorphism types of homotopy* $\mathbb{RP}^{4k+1}$.

Since there are exactly four oriented diffeomorphism types of $\mathbb{RP}^5$, the last statement for $k = 1$ implies that all of them are obtained as quotients of the Kervaire sphere $K^5 = W^5(d)$. In this case, the singular orbits have codimension two, so that Corollary 4.1 implies

Corollary 5.8 [24]. *All four oriented diffeomorphism types of homotopy* $\mathbb{RP}^5$ *admit metrics of nonnegative sectional curvature with an isometry group of cohomogeneity one.*

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[2] Indeed, the Kervaire spheres are generators of the group of homotopy spheres which bound a parallelizable manifold.
[3] Whether or not K^{2n-1} is diffeomorphic to the standard sphere is unknown if $n + 1 = 2^k$ and $k \geqslant 6$.

6. Obstructions for almost nonnegative curvature

We end this report by stating some of the obstructions which are known for a closed manifold to have almost nonnegative sectional curvature. It would lead to far to give a comprehensive list, but we shall list the most important ones which are easy to formulate.

Let M be a closed smooth n-dimensional manifold. If M admits metrics of almost nonnegative sectional curvature, then:

1. [23] For any field of coefficients the total Betti number of M must be bounded above by a constant depending only on n.
2. [39] A finite cover of M must fibre over a $b_1(\mathrm{M})$-dimensional torus, and if $b_1(\mathrm{M}) = n$, then M must be diffeomorphic to a torus. (The latter statement also holds when M supports metrics of almost nonnegative Ricci curvature [13].)
3. [17] If M has infinite fundamental group, then the Euler characteristic of M must vanish.
4. [17] If the fundamental group of M is finite, for some universal constant C which depends only on n the diameters of M and its universal Riemannian covering $\widetilde{M}$ must satisfy the inequality $\mathrm{diam}(\widetilde{M}) < C \cdot \mathrm{diam}(M)$.
5. [17] The fundamental group of M must be almost nilpotent, i.e., it must contain a nilpotent subgroup Λ of finite index. Moreover, Λ is generated by at most n elements and the degree of nilpotency of Λ is not greater than n.
6. [18] If M is spin, the $\widehat{A}$-genus of M must be bounded by $|\widehat{A}(M)| \leqslant 2^{\frac{n-1}{2}}$.
   (This condition already holds if M has almost nonnegative Ricci curvature.)

Note that for simply connected M, all of these obstructions with the exception of the first and the last are automatically satisfied.

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