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On a Matrix Partition Conjecture

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In 1977, Ganter and Teirlinck proved that any $2t \times 2t$ matrix with $2t$ nonzero elements can be partitioned into four submatrices of order t of which at most two contain nonzero elements. In 1978, Kramer and Mesner conjectured that any $mt \times nt$ matrix with kt nonzero elements can be partitioned into mn submatrices of order t of which at most k contain nonzero elements. We show that this conjecture is true for some values of m, n, t and k but that it is false in general.

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We show how a conjecture of Erdös which was proved by Olson can be used to prove the conjecture when $m = 2$, thereby obtaining a generalization of the theorem of Ganter and Teirlinck. We also prove the conjecture for $k = 3$. Finally, we consider a generalization to matrices of higher dimension.

1. INTRODUCTION

The following theorem is proved by Ganter and Teirlinck in [4].

THEOREM 1.1. *Every $2t \times 2t$ matrix with $2t$ nonzero elements can be partitioned into four submatrices of order t of which at most two contain nonzero elements.*

This theorem is best possible in the sense that there exist $2t$ by $2t$ matrices with $2t + 1$ nonzero elements such that in every partition into four submatrices of order t, at least three contain nonzero elements. For example, if $t = 3$ the matrix

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

cannot be partitioned into four submatrices of order 3 of which only two contain nonzero elements.

In 1978, Kramer and Mesner conjectured the following.[1]

Conjecture. Let m, n, t and k be positive integers. Then every $mt \times nt$ matrix with kt nonzero elements can be partitioned into mn submatrices of order t of which at most k contain nonzero elements.

For notational convenience we denote the assertion of this conjecture by $KM(m, n, k, t)$. Clearly, $KM(m, n, k, 1)$ is true. In addition, $KM(m, n, k, t)$ is true if $k = 1$ or $k \geq mn$, and it is also true if $k = 2$ by Theorem 1.1. However, as can be shown by exhaustive checking, the

[1]This conjecture was stated in the talk "On the Distribution of Nonzero Elements in Certain Sparse Matrices" given by D. M. Mesner at the 9th Southeastern Conference held at Florida Atlantic University in 1978.

partitioned matrix

$$\left[\begin{array}{cc|cc|cc|cc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

cannot be repartitioned into submatrices of order 2 of which at most six contain nonzero elements, and hence $KM(4,4,6,2)$ is false. Although $KM(m,n,k,t)$ is not always true, we show that it is true in situations other than that given in Theorem 1.1.

We now introduce a function related to the conjecture. Let $f(m,n,k,t)$ denote the largest number N such that each $mt \times nt$ matrix with N nonzero elements can be partitioned into submatrices of order t of which at most k contain nonzero elements. Clearly $f(m,n,mn,t) = mnt^2$. The assertion $KM(m,n,k,t)$ is equivalent to $f(m,n,k,t) \geq kt$. The above example shows that $f(4,4,6,2) \leq 11$.

Problem. Determine $f(m,n,k,t)$.

In Section 2, we show that $KM(m,n,k,t)$ is true if $k = mn - 1$ or $mn - 2$, and that $KM(2,2,k,t)$ is true for all k and t. We also make a connection $KM(m,n,k,t)$ and the Zarankiewicz problem, and show that the conjecture does not in general give the correct order of magnitude for $f(m,n,k,t)$. In Section 3, we discuss some connections with a conjecture of Erdös proved by Olson, and show that $KM(2,n,k,t)$ is always true. We also prove that $KM(m,n,3,t)$ is always true. In Section 4, we discuss a family of counterexamples to the conjecture. In Section 5, we consider a generalization of Theorem 1.1 to matrices of higher dimension.

Throughout we view our matrix A as an adjacency matrix of a bipartite graph $G(U,V;E)$ where U is the set of vertices corresponding to the rows of A, V is the set of vertices corresponding to the columns of A, and E is the set of edges determined by the nonzero elements in A. In this context the conjecture can be formulated in terms of graph homomorphisms. Recall that a graph G' is a *homomorphic image* of a graph G provided G' can be obtained from G by a sequence of identifications of nonadjacent vertices. Suppose that $|U| = mt$ and $|V| = nt$. Then $f(m,n,k,t)$ is the largest number N such that every bipartite graph $G(U,V;E)$ with $|E| = N$ has a homomorphic image $G'(U',V';E')$ where each vertex of U' is obtained by identifying t vertices of U, each vertex of V' is obtained by identifying t vertices of V and $|E'| \le k$.

2. $KM(m, n, k, t)$ for $k \le 2$ and for k Large

We first prove the following lemma.

Lemma 2.1. *Let c, M, N and t be positive integers with $c \le M$. Assume that $q_1, q_2, \ldots, q_{Mt}$ is a nondecreasing sequence of nonnegative integers with $\sum_{i=1}^{Mt} q_i \le (MN - c)t$. Then $\sum_{i=1}^{ct} q_i \le ct(N-1)$.*

Proof. Suppose to the contrary that $\sum_{i=1}^{ct} q_i \ge 1 + ct(N-1)$. Then $q_{ct} \ge N$, implying that $q_i \ge N$ for all i with $ct \le i \le Mt$. Hence

$$\begin{aligned}\sum_{i=1}^{Mt} q_i &= \sum_{i=1}^{ct} q_i + \sum_{i=ct+1}^{Mt} q_i \\ &\ge 1 + ct(N-1) + (Mt - ct)N = 1 + (MN - c)t,\end{aligned}$$

contradicting our assumption. ∎

The following theorem shows that $KM(m, n, k, t)$ is true if k is either very small or very large.

Theorem 2.2. *If $k \le 2$ or $k \ge mn - 2$, then $KM(m, n, k, t)$ is true.*

Proof. Let A be an mt by nt matrix with kt nonzero elements. If $k = 1$, then the nonzero elements of A are contained in a submatrix of order t. If $k = 2$, the nonzero elements of A are contained in a submatrix of order $2t$ to which we can then apply Theorem 1.1. If $k \ge mn$, the conclusion holds trivially.

Let q_i be the number of nonzero elements in row i of A and rearrange rows so that $q_1 \le q_2 \le \cdots \le q_{Mt}$. If $k = mn - 1$, then we apply Lemma 2.1 with $M = m$, $N = n$ and $c = 1$ and obtain a $t \times nt$ submatrix with at most $(n-1)t$ nonzero elements and hence a $t \times t$ submatrix of zeros. Now assume that $k = mn - 2$. Applying Lemma 2.1 with $M = m$, $N = n$ and $c = 2$ we obtain a $2t \times nt$ submatrix A' with at most $2t(n-1) = (2n-2)t$ nonzero elements. Let q'_j be the number of nonzero elements in column j of A' and rearrange columns so that $q'_1 \le q'_2 \le \cdots \le q'_{nt}$. Applying Lemma 2.1 to this sequence with $M = n$, $N = 2$, and $c = 2$, we obtain a $2t \times 2t$ submatrix of A' with at most $2t$ nonzero elements. Application of Theorem 1.1 completes the proof. ∎

If $m = n = 2$, then Theorem 2.2 gives the following.

COROLLARY 2.3. *$KM(2, 2, k, t)$ is true for all k and t.*

We now show that Corollary 2.3 is best possible for each $k = 1, 2$ and 3 in the sense that the conclusion need not hold if the number of nonzero elements is $kt + 1$. Let A be a $2t \times 2t$ matrix. If A contains $t + 1$ nonzero entries, then A can be partitioned into four submatrices of order t of which only one contains nonzero elements if and only if either no row contains more than one nonzero element or no column contains more than non nonzero element. There are many combinatorially different matrices A with $2t + 1$ nonzero elements that cannot be partitioned into four submatrices of order t of which only two contain nonzero elements. For example, if the $2t + 1$ nonzero elements of A occupy at least $t + 1$ rows and at least $t + 1$ columns, and the bipartite graph of A has a connected component which contains either $t + 1$ vertices corresponding to rows or $t + 1$ vertices corresponding to columns,[2] then at least three submatrices of order t are required to contain all the nonzero elements.

In contrast to the above two cases, we now show that there is a unique matrix A (up to permutations of rows and columns) such that A is a $2t \times 2t$ matrix with $3t + 1$ nonzero elements which cannot be partitioned into four $t \times t$ matrices of which at most three contain nonzero elements. Our discussion is in terms of a bipartite graph $G(U, V; E)$ with $3t + 1$ edges which does not contain an empty subgraph $H(U', V'; E' = \varnothing)$ where $|U'| = |V'| = t$, $U' \subset U$ and $V' \subset V$. This is equivalent to the condition that for each subset W of cardinality t of U we have $|N(W)| > t$, where $N(W)$ is the set of all neighbors of vertices of W. We claim that G is isomorphic to a disjoint union of a cycle C_{2t+2} of length $2t + 2$ and a matching of size $t - 1$. Let $U = \{u_1, \dots, u_{2t}\}$. After relabeling we can assume that $\deg(u_i) \le \deg(u_{i+1})$ for $1 \le i \le 2t - 1$. Now $\deg(u_1) + \cdots + \deg(u_t) > t$, since otherwise $|N(u_1, \dots, u_t)| \le t$. Thus $\deg(u_t) \ge 2$ which implies $\deg(u_{2t}) \le 2$ and hence $0 \le \deg(u_i) \le 2$ for all i. Suppose U has $x \ge 1$ vertices of degree 0. Then we have $t - 1 - 2x$ vertices of degree 1 and $t + 1 + x$ vertices of degree 2. We have $|N(W)| \le |V| = 2t$ for all subsets W of U and hence for W equal to the subset consisting of all vertices of U of degree 2. As there are $t + 1 + x$ such vertices, two of them, say u and v, have a common neighbor, i.e., $|N(u, v)| \le 3$. Let U_i be the set of vertices of degree i, $(i = 0, 1)$ and let U_2 be a set of $x + 1$ vertices of degree 2 such that u and v are in U_2. Let $W = U_0 \cup U_1 \cup U_2$. Then $|N(W)| \le |N(U_0)| + |N(U_1)| + |N(U_2)| \le x \cdot 0 + (t - 1 - 2x) \cdot 1 + (2x + 1) \le t$, a contradiction as $|W| = t$. Thus all vertices in U (and in V) are of degree 1 or 2. So, G consists of cycles and paths. If there were

[2]For instance, the bipartite graph of A is, except for isolated vertices, a tree of order $2t + 2$ with $t + 1$ vertices corresponding to rows and $t + 1$ vertices corresponding to columns.

a cycle of length at most t, then the vertices of the cycle in U together with a suitable number of vertices of U of degree 1 would form a set W with $|W| = t$ and $|N(W)| \leq t$, a contradiction. Suppose that G has a path of length at least two. Then there is a path $P = uvw \cdots z$ of length at least two with the initial vertex u from U. If $\deg(w) = 2$, we take $W = U_1 \cup \{w\}$ where U_1 is the set of all vertices of U of degree 1 and obtain a contradiction, since $|U_1| = t - 1$ and $|N(W)| \leq t$. If $\deg(w) = 1$, then P is of length 2 and we obtain a contradiction by choosing as the t-th vertex of W an arbitrary vertex of degree 2, since in this case $|N(U_1)| \leq t - 2$. Our conclusion now follows by noting that if $G(U, V; E)$ is isomorphic to the union of a cycle of length $2t + 2$ and a matching of size $t - 1$, then $|N(W)| > t$ is valid for any subset W of cardinality t of U.

Corollary 2.3 and the above discussion imply that $f(2, 2, k, t) = kt$ for $1 \leq k \leq 3$. This fact might suggest that whenever $KM(m, n, k, t)$ is true, then it is best possible in the sense that the number kt is the largest number of nonzero entries for which the conclusion holds. We show that this is not the case. In addition we prove that $KM(m, n, k, t)$ is true for $k = mn - p$ where p is small in comparison to mn. We obtain these results by making a connection with the famous Zarankiewicz problem.

To make the relation between $KM(m, n, k, t)$ and the Zarankiewicz problem transparent, we state the latter in the following form: *Let $1 \leq c \leq a$ and $1 \leq d \leq b$. Determine $Z(a, b; c, d)$, the smallest number N such that each $a \times b$ matrix with N zeros contains a $c \times d$ zero submatrix.*[3] We clearly have

$$f(m, n, mn - rs, t) \geq (mt)(nt) - Z(mt, nt; rt, st) \text{ with equality if } r = s = 1. \quad (1)$$

A result of Čulik [2] (see also Exercise 13, p. 361, in [1]) asserts that

$$Z(a, b; c, d) = (d - 1)a + (c - 1)\binom{b}{d} + 1, \qquad \text{if } a \geq (c - 1)\binom{b}{d}.$$

A result of Reiman [7] (see also part (i) of Thm. 2.6, p. 312, in [1]) asserts that

$$Z(a, b; c, 2) \leq \frac{a + \sqrt{a^2 + 4(c - 1)ab(b - 1)}}{2} + 1;$$

in particular,

$$Z(a, a; 2, 2) \leq \frac{a + a\sqrt{4a - 3}}{2} + 1.$$

[3]Usually the Zarankiewicz problem is formulated with the zeros and nonzeros interchanged.

Applying these results we obtain the following.

THEOREM 2.4. *We have*

(i) $f(m, n, mn - p, t) \geq tmn$ *for* $mt \geq (pt - 1)\binom{nt}{t}$ *and* $n \geq 2$, *and*

(ii) $f(m, n, mn - p, 2) \geq 2mn$ *for* $p \leq n/4$ *and* $n \leq m$.

In particular, $f(n, n, n^2 - 1, 2) \geq 4n^2 - o(n^2)$.

Thus $KM\ (m, n, k, t)$ is true for the values given in the theorem, but it is not best possible. Indeed, Theorem 2.4 shows that for the case $m = n, t = 2$, and $k = n^2 - 1$, $KM(m, n, k, t)$ does not provide the correct order of magnitude for $f(m, n, k, t)$.

Corollary 2.3 and (1) with $m = n = 2$ and $r = s = 1$ imply that $Z(2t, 2t; t, t) = 4t^2 - 3t$.

3. $KM\ (m, n, k, t)$ FOR $m = 2$ AND FOR $k = 3$

The following theorem was conjectured by Erdös [3] and proved by Olson [6] (see also [5]).

THEOREM 3.1. *If* $\alpha_1, \ldots, \alpha_{2t-1}$ *is a sequence (repetitions allowed) of elements in the elementary Abelian group* $Z_t \times Z_t$, *then some subsequence has sum* $(0, 0)$.

Olson's theorem is a key ingredient in the proof of the following generalization of Theorem 1.1.

THEOREM 3.2. $KM\ (2, n, k, t)$ *is true for all* n, k, *and* t.

Proof. We prove the theorem by induction on n. If $n = 1$, the theorem clearly holds. Now assume that $n > 1$. Let A be a $2t \times nt$ matrix with kt nonzero elements. Let q_i be the number of nonzero elements in the column i of A $(i = 1, \ldots, nt)$, where we may assume $q_1 \leq \cdots \leq q_{nt}$. First suppose that $k \geq n + 1$. Then $\sum_{i=1}^{nt} q_i = kt \geq (n + 1)t$ implies that $q_{(n-1)t+1} + \cdots + q_{nt} \geq 2t$. Otherwise $q_{(n-1)t+1} = 1$, and so $q_i \leq 1$ for $i = 1, \ldots, (n - 1)t$, and hence

$$\begin{aligned}\sum_{i=1}^{nt} q_i &= \sum_{i=1}^{(n=1)t} q_i + \sum_{i=(n-1)t+1}^{nt} q_i \\ &\leq (n - 1)t + (2t - 1) = (n + 1)t - 1,\end{aligned}$$

a contradiction. Let A' be the submatrix obtained from A by omitting its last t columns. Then A' has at most $(k - 2)t$ nonzero elements and by the induction hypothesis, there is a partition of A' into $t \times t$ submatrices of which at most $(k-2)$ have nonzero elements. This partition together with the two $t \times t$ submatrices determined by the last t columns of A yields the desired conclusion. Now suppose that $k \le n-1$. Then $q_1 = \cdots = q_t = 0$ and we apply the induction hypothesis to the submatrix A' comprising the last $(n-1)t$ columns of A. Finally we suppost that $k = n$. Let $C_1, \dots, C_s$ be the connected components of the bipartite graph $G(U, V; E)$ corresponding to the matrix A. For $i = 1, \dots, s$ let (r_i, c_i) be the ordered pair consisting of the number of vertices of C_i in U and V, respectively. Then C_i has at least $r_i + c_i - 1$ edges so that $(r_1 + c_1 - 1) + \cdots + (r_s + c_s - 1) \le nt$ which, since $r_1 + \cdots + r_s = 2t$ and $c_1 + \cdots + c_s = nt$, implies that $s \ge 2t$. We now interpret the integers r_i and s_i modulo t. Since $s \ge 2t$, Theorem 3.1 implies that there is a *proper* subset J of $\{1, \dots, s\}$ such that the sum of (r_j, c_j) over J equals (et, ft) for some integers $e \le 2$ and $f \le n$. If $e = 0$, then there are ft components each consisting of one vertex in V (equivalently ft columns of A containing only zeros), and we apply the induction hypothesis to the submatrix A' obtained by deleting those zero columns. If $e = 2$, then $f < n$ (otherwise J could not be a proper subset of $\{1, \dots, s\}$) implying that there are $(n-f)t$ components each consisting of one vertex in V (equivalently, $(n-f)t$ columns in A containing only zeros), and we apply the induction hypothesis to the submatrix A' obtained by deleting those zero columns. If $e = 1$, we get that there are permutation matrices P and Q such that PAQ is a direct sum of matrices of size $t \times ft$ and $t \times (n-f)t$. Therefore we have at least $f + (n-f) = n$ zero $t \times t$ submatrices, and the proof of the theorem is complete. ▮

We next show that the conjecture is true for $k = 3$. First we prove the following lemma which shows that $KM\ (k, k, k, t)$ is true for matrices whose nonzero elements are sufficiently spread out.

LEMMA 3.3. *Let A be a $kt \times kt$ matrix with kt nonzero elements and assume that A has at most $2t-1$ zero rows and columns. The A can be partitioned into k^2 submatrices of order t of which at most k contain nonzero elements.*

Proof. We prove the lemma by induction on k. If $k \le 2$, the lemma follows from Theorem 2.2. Now assume that $k \ge 3$. Let $C_1, \dots, C_s$ be the connected components of the bipartite graph $G(U, V; E)$ corresponding to the matrix A. For $i = 1, \dots, s$ let (r_i, c_i) be the ordered pair consisting of the number of vertices of C_i in U and V, respectively. Then C_i has at least $r_i + c_i - 1$ edges, and since $G(U, V; E)$ has kt edges, we have

$$kt \ge \sum_{i=1}^{s} (r_i + c_i - 1) = 2kt - s.$$

Thus the number of components satisfies $s \geq kt$. Let w_i equal the number of edges of C_i, $(1 \leq i \leq s)$ and let $N = \{i: w_i \neq 0\}$. The hypothesis of the lemma implies that the number of trivial components of $G(U, V; E)$ is at most $2t - 1$, and hence $|N| \geq kt - (2t - 1) \geq t + 1$ since $k \geq 3$. We now interpret the integers w_i modulo t. Since every sequence of t integers modulo t contains by the pigeon-hole principle a subsequence which sums to zero modulo t, it follows that there is a *proper* subset J of N such that $\sum_{i \in J} w_i = et$ for some positive integer $e < k$. This implies that there exists permutation matrices P and Q such that $PAQ = A_1 \oplus A_2$ where A_1 is an $et \times et$ matrix with et nonzero elements (corresponding to the edges of the components C_i with i in J) and A_2 is a $(k - e)t \times (k - e)t$ matrix with $(k - e)t$ nonzero elements. Since each of the matrices A_1 and A_2 can contain at most $2t - 1$ zero rows and columns, the lemma now follows by induction. ∎

THEOREM 3.4. *KM $(m, n, 3, t)$ is true for all m, n and t.*

Proof. Let A be an $mt \times nt$ matrix with $3t$ nonzero elements. There exists a $3t$ by $3t$ submatrix B of A containing all the nonzero elements of A. If B either has at least t zero rows or at least t zero columns, the theorem follows from Theorem 3.2. Otherwise B has at most $t - 1$ zero rows and at most $t - 1$ zero columns, and the theorem follows from Lemma 3.3. ∎

4. COUNTEREXAMPLES

In this section, we construct for each integer $t \geq 2$ a counterexample to $KM(m, n, k, t)$. Using our identification of matrices with bipartite graphs, we formulate our constructions as bipartite graphs.

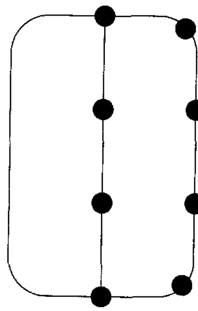
In the next theorem we identify a class of matrices for which the bound on the number of submatrices of order t containing nonzero elements as stated in the conjecture is tight. First we prove the following lemma.

LEMMA 4.1. *Let A be a t by nt matrix with kt nonzero entries. Assume that the associated bipartite graph $G(U, V; E)$ has no cycles. Then for each partition of A into n submatrices of order t, at least k submatrices contain nonzero elements.*

Proof. Since $G(U, V; E)$ has no cycles, it is a forest with kt edges. Thus, since $|U| = t$, the number of nonzero columns of A_i is at least $(k - 1)t + 1$. Hence for any partition of A into n submatrices of order t, at least k of the submatrices contain nonzero entries. ∎

THEOREM 4.2. *Let A be an mt by nt matrix with kt nonzero entries such that the number of nonzero entries in each row is congruent to l modulo t for*

T



B

FIG. 1. $G(1,3,5)$.

some integer l. Assume that the associated bipartite graph $G(U,V;\ E)$ has no cycles of length $2t$ or less. Then for each partition of A into mn submatrices of order t, at least k submatrices contain nonzero elements.

Proof. Fix any partition of A into $t \times nt$ submatrices $A_1, \dots, A_m$. Consider any submatrix A_i. Since the number of nonzero entries in each row of A_i is l modulo t, the number of nonzero entries of A_i is $q_i t$ for some integer q_i. Since $G(U,V;\ E)$ has no cycles of length at most $2t$, the bipartite graph $G(U_i, V;\ E_i)$ associated with A_i has no cycles and hence is a forest with $q_i t$ edges. Thus by Lemma 4.1, for any partition of A_i into n submatrices of order t, at least q_i of the submatrices contain nonzero entries. It follows that for any partition of A into mn $t \times t$ submatrices at least $k = q_1 + \cdots + q_m$ submatrices contain nonzero entries. ∎

We remark that the proof of Theorem 4.2 implies that if k submatrices of order t contain all nonzero entries of A, then exactly q_i of them are contained in A_i for each i.

Let $G(p_1, \dots, p_s)$ denote a graph consisting of a top vertex T and a bottom vertex B and disjoint paths of lengths $p_1, \dots, p_s$ joining T and B, called the *strands* of $G(p_1, \dots, p_s)$. The graph $G(1, 3, 5)$ is drawn in Fig. 1. The number of edges of $G(p_1, \dots, p_s)$ equals $p_1 + \cdots + p_s$, and the number of vertices is $p_1 + \cdots + p_s - s + 2$. Each of the vertices T and B has degree equal to s and all other vertices have degree equal to 2. If all p_i are odd, then $G(p_1, \dots, p_s)$ is a bipartite graph $G(U,V;\ E)$ with $|U| = |V|$ and without loss of generality we assume that $B \in U$ and $T \in V$,

THEOREM 4.3. *Let t be an integer with $t \geq 2$. Let s be an integer with $s \geq 3t^2 - 3t + 2$ and $s \equiv 2 \pmod{t}$. Let $p_1, \dots, p_s$ be distinct odd integers* *such that each* $p_i \geq t$ *and* $p_1 + \cdots + p_s - s + 2 \equiv 0 \pmod{2t}$. *Then the matrix* A *associated with the bipartite graph* $G(p_1, \ldots, p_s)$ *is a counterexample to* $KM(n, n, k, t)$ *for*

$$n = \frac{p_1 + \cdots + p_s - s + 2}{2t} \quad \text{and} \quad k = \frac{p_1 + \cdots + p_s}{t}.$$

Proof. The hypotheses imply that n and k are integers and that both the rows and columns of the adjacency matrix A of the bipartite graph $G(p_1, \ldots, p_s)$ satisfy the assumptions of Theorem 4.2. Thus for each partition of A into n^2 submatrices A_{ij} $(1 \leq i, j \leq n)$ of order t, at least k submatrices contain nonzero elements. Assume to the contrary that there exists a partition of A for which exactly k submatrices contain nonzero elements. Let $U_1, \ldots, U_n$ and $V_1, \ldots, V_n$ be the corresponding partition of the vertices of U and V, respectively, where $B \in U_1$ and $T \in V_1$. For each i, we speak of the vertices in U_i as being *matched* by the partition. Similarly, the vertices in each V_j are matched by the partition. Since $G(p_1, \ldots, p_s)$ is bipartite, the number of edges incident with vertices in U_1 is $s + 2(t-1) = s + 2t - 2$. It follows from the remark after the proof of Theorem 4.2 that the number of submatrices A_{1j} which contain nonzero elements is $x = (s + 2t - 2)/t$ and since $s \geq 3t^2 - 3t + 2$, we have $x \geq 3t - 1$. Without loss of generality we assume that these submatrices are $A_{11}, \ldots, A_{1x}$. Let $V' = V_1 \cup \cdots \cup V_x$. The number of strands which contain no vertex of U_1 different from B is $w \geq s - (t-1) = s - t + 1$. Let W be the w vertices of these strands which are adjacent to B. Then $W \subseteq V'$ and $|V' \setminus W| = xt - w \leq 3t - 3$. Thus the number of V_j with $1 \leq j \leq x$ which have a nonempty intersection with $V' \setminus W$ is at most $3t - 3$. Since $x \geq 3t - 1$, there exist integers e_0 and f_0 with $1 \leq e_0 < f_0 \leq x$ such that $V_{e_0} \cup V_{f_0} \subseteq W$.

Now consider the nt by t submatrix of A determined by the columns in V_{e_0}. Since $T \notin V_{e_0}$, this submatrix has exactly $2t$ nonzero entries, and it follows from the remark after the proof of Theorem 4.2 that there are exactly two nonzero submatrices of order t in the columns V_{e_0}. Since B is adjacent to each of the vertices in V_{e_0} and B is an element of U_1, one of these submatrices is A_{1e_0}. Let the other submatrix be $A_{e_1e_0}$. Since the other t vertices adjacent to V_{e_0} are not matched to B, they must be matched together and thus form the set U_{e_1}. A similar argument shows that the set of t vertices of $V \setminus V_{e_0}$ which are adjacent to U_{e_1} are matched together and thus form a set V_{e_2}. Continuing like this, we eventually determine a set U_{e_i} such that T is adjacent to U_{e_i}. Since the strands have distinct lengths, T is adjacent to exactly one vertex in U_{e_i}. Thus the set V_1 which contains T also contains the $t - 1$ other vertices adjacent to U_{e_i} not in $V_{e_{i-1}}$. Repeating this argument beginning with V_{f_0}, we obtain $t - 1$

vertices on the strands through V_{f_0} which are also contained in V_1. This implies that $|V_1| \geq 2t - 1 > t$, a contradiction. ∎

5. Matrix Partitioning in Higher Dimensions

In this section we consider a generalization of Theorem 1.1 to matrices of higher dimension.

Let d and n be positive integers. A *d-dimensional matrix of order n* is an array

$$A = [a_{i_1 i_2 \cdots i_d}] \ (1 \leq i_1, i_2, \ldots, i_d \leq n).$$

Let t and k be positive integers. Then we denote by $g(d,t,k)$ the maximum number M such that every d-dimensional $2t \times 2t \times \cdots \times 2t$ matrix with M nonzero elements can be partitioned into 2^d $t \times \cdots \times t$ submatrices of which at most k contain nonzero elements. Clearly $g(d,t,1) = t$ and $g(d,t,2^d) = (2t)^d$ for all t and d.

The main result of this section is the somewhat surprising fact that the number of nonzero elements which can always be stuffed into at most two $t \times t \times \cdots \times t$ submatrices, decreases from $2t$ when $d = 2$, to only $t + 1$ when the dimension $d \geq t + 2$.

Theorem 5.1. *For $d \geq 2$ and arbitrary t, $g(d,t,2) = t + \lceil t/(d-1) \rceil$.*

Proof. As before, we employ a graph-theoretical formulation of our problem. Let $A_1, \ldots, A_d$ be pairwise disjoint sets. Denote by $G(A_1, \ldots, A_d;\ E)$ a d-uniform hypergraph with vertex set $V(G) = \cup_{i=1}^{d} A_i$ where each edge e of E has the property that $|e \cap A_i| = 1$ for each $i = 1, \ldots, d$. We associate with a d-dimensional $2t \times 2t \times \cdots \times 2t$ matrix $A = [a_{j_1 j_2 \cdots j_d}]$ a d-uniform hypergraph $G(A_1, \ldots, A_d; E)$ where $|A_i| = 2t$ and $a_i = \{v_{1,i}, \ldots, v_{2t,i}\}$, $(i = 1, 2, \ldots, d)$ as follows: an edge $e = \{v_{j_1,1} v_{j_2,2}, \ldots, v_{j_d,d}\}$ belongs to E if and only if $a_{j_1 j_2 \cdots j_d} = 1$. To show that a matrix A can be partitioned into $t \times t \times \cdots \times t$ submatrices of which at most k contain nonzero elements is equivalent to showing that each A_i can be partitioned into two sets $B_{i,1}$ and $B_{i,2}$ of cardinality t, such that at most k of the 2^d subgraphs induced by vertex sets $B_{1,i_1} \cup B_{2,i_2} \cup \cdots \cup B_{d,i_d}$, $(1 \leq i_j \leq 2, j = 1, \ldots, d)$ contain an edge.

We prove the latter assertion by induction on d. If $d = 2$, we get $g(2,t,2) = 2t$ by Theorem 1.1. Now assume that $d > 2$. Let p_i be the number of vertices of A_i of positive degree. First we prove $g(d,t,2) < t + \lceil t/(d-1) \rceil + 1$ by construction a special hypergraph. Let $G(A_1, \ldots, A_d;\ E)$ with $|A_i| = 2t$, $(i = 1, \ldots, d)$ be a d-uniform hypergraph with $t + \lceil t/(d-1) \rceil + 1$ edges, where E is defined recursively as follows. The edge e_1 is an arbitrary edge. The kth edge is defined so that $e_k \cap e_i = \emptyset$ for $i = 1, \dots, k-2$, and $|e_k \cap e_{k-1}| = 1$, where the vertex v in $e_k \cap e_{k-1}$ belongs to A_j for j the unique integer satisfying $1 \le j \le d$ and $j \equiv k \pmod{d}$. Clearly, $G(A_1, \dots, A_d; E)$ is a connected hypergraph and it is a matter of routine calculation to show that $p_i > t$ for $i = 1, \dots, d$. If there were a partition of each A_i into $B_{i,1}$ and $B_{i,2}$ such that at most two of the 2^d induced subgraphs of G contain an edge, then we could employ a notation such that these two nonempty subgraphs are induced either by:

(i) $B_{1,1} \cup B_{2,1} \cup \cdots \cup B_{d,1}$ and $B_{1,2} \cup B_{2,2} \cup \cdots \cup B_{d,2}$, *or by*

(ii) $B_{1,1} \cup B_{2,1} \cup \cdots \cup B_{d,1}$ and $B_{1,1} \cup \cdots \cup B_{j,1} \cup B_{j+1,2} \cup \cdots \cup B_{d,2}$ *for some* $j \ge 1$.

However, case (i) cannot happen as G is connected, and case (ii) is excluded as $p_i > t$ for all i.

To prove the reverse inequality $g(d,t,2) \ge t + \lceil t/(d-1) \rceil$, we suppose that $G(A_1, \dots, A_d; E)$ has at most $t + \lceil t/(d-1) \rceil$ edges. We consider two cases.

Case (a). There is an i with $1 \le i \le d$ such that $p_i \le t$. Without loss of generality we assume that $p_1 \le t$. Form a $(d-1)$-uniform hypergraph $G'(A_1, \dots, A_{d-1}; E')$ where $e' = (u, v, \dots, w)$ belongs to E' provided there is a vertex z from A_d such that $e = (u, v, \dots, w, z)$ is an edge of E. Denote by $e(G)$ the number of edges of G. Thus $e(G') \le e(G) = t + \lceil t/(d-1) \rceil$ and by the induction hypothesis there is a partition B' of each of the sets $A_1, \dots, A_{d-1}$ into two parts such that at most two of the induced subgraphs are nonempty. To get a desired partition B of $A_1, \dots, A_d$ it is sufficient to extend B' by partitioning A_d into two sets $B_{d,1}$ and $B_{d,2}$, where both $B_{d,1}$ and $B_{d,2}$ have cardinality t and $B_{d,1}$ contains all vertices of A_d of nonzero degree.

Case (b). $p_i > t$ for all $i = 1, \dots, d$. Since a connected d-uniform hypergraph H with $e(H)$ edges has at most $e(H)(d-1) + 1$ vertices of nonzero degree, our hypergraph G in this case has at least two connected components (containing at least one edge). Suppose $H(C_1, \dots, C_d; E')$ is a connected component of G with $|C_i| > t$ for at least one of $i = 1, \dots, d$. Put $m = \min\{|C_i| : i = 1, \dots, d\}$. Clearly $m \le t$, and without loss of generality we assume that $|C_1| = m$. Then $|C_1| + \cdots + |C_d| \ge t + 1 + (d-1)m$ which in turn implies that H has at least $m + \lceil t/(d-1) \rceil$ edges. Hence, $e(G) - e(H) \le t - m$, which contradicts $p_1 > t$ as $p_1 \le |C_1| + t - m = t$. Therefore, $|C_i| \le t$ for $i = 1, \dots, d$ and each connected component H of G. We show that there is a partition of the A_i's into $B_{i,1}$ and $B_{i,2}$ such that each edge of G belongs either to a subgraph induced by $B_{1,1} \cup B_{2,1} \cup \cdots \cup B_{d,1}$ or induced by $B_{1,2} \cup B_{2,2} \cup \cdots \cup B_{d,2}$. If there is a connected component $H(C_1, \ldots, C_d; E')$ of G with at least $\lceil t/(d-1) \rceil$ edges, then (having in mind that $|C_i| \leq t$ for $i = 1, \ldots, d$) we arrive at the required partition by letting $B_{i,1}$ be a superset of C_i and letting $B_{i,2}$ contain all the vertices of $A_i - C_i$ of nonzero degree (there are at most t of them as $E - E'$ has at most t edges). Finally, we need to take care of the case when all connected components of G have less than $t + \lceil t/(d-1) \rceil$ edges. Then we form a subgraph H of G by taking a union of connected components $H_1, \ldots, H_s$ such that $\lceil t/(d-1) \rceil \leq e(H) = e(H_1) + e(H_2) + \cdots + e(H_s) \leq t$. Again, $e(H) \leq t$ implies $|C_i| \leq t$ for each $i = 1, \ldots, t$ and we can proceed as in the previous case. ∎

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