Forced Vibration of Edge-Cracked Functionally Graded Beams Due to a Transverse Moving Load

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Abstract

This paper presents an analytical study on the forced flexural vibration of functionally graded beams with open edge cracks under a combined action of an axial compressive force and a concentrated transverse load moving along the longitudinal direction. The cracked section is modeled as a rotational spring whose sectional flexibility is calculated through fracture mechanics. The forced vibration response is determined by employing modal series expansion technique. Analytical dynamic deflections are obtained for cantilever, hinged-hinged, and clamped-clamped beams whose material properties follow an exponential through-thickness variation. A parametric study is conducted to examine the effects of cracks, material property gradient, axial compression, and the speed of the moving load.

Keywords: Forced vibration, Functionally graded materials, Euler-Bernoulli beam theory, Edge crack, Moving load

1. Introduction

Functionally graded materials (FGMs) have attracted increasing research efforts over the past ten years due to their outstanding properties. Numerous studies on the dynamic behavior and fracture of FGM structures have been reported (e.g. Erdogan and Wu, 1997; Praveen and Reddy, 1998; Woo et al., 2006). Investigations concerning the effect of crack defects on the dynamic behavior of FGM structures, however, are very limited in number. Sridhar et al. (2006) analyzed wave propagation in FGM beams and layered structures containing embedded horizontal or vertical edge cracks using pseudospectral finite element method. Yang and Chen (2008) analytically discussed the influence of open edge cracks on the free vibration and buckling of FGM beams. Ke et al. (2009) studied the flexural vibration and elastic buckling of a cracked FGM Timoshenko beam with different boundary conditions. Most recently, Yang et al. (2010) examine the nonlinear dynamic response of a FGM plate containing a through-width surface crack.
When subjected to a moving load or mass, a beam structure produces larger deflections and higher stresses than it does under an equivalent load applied statically. Such a structure is of practical importance, especially in transportation system and in the design of machining process. Bilello and Bergman (2004) carried out a theoretical and experimental study on the response of a damaged Euler-Bernoulli beam traversed by a moving mass. The effective mass distribution of the beam and the convective acceleration terms were considered to correctly evaluate the beam-moving mass interaction force. Lin and Chang (2006) obtained an analytical solution of the forced response of a cantilever beam with a crack subjected to a concentrated moving load by using the equivalent rotational spring model, transfer matrix method, and modal series expansion technique. Simsek and Kocaturk (2009) investigated the free vibration characteristics and the dynamic behavior of an FGM simply-supported beam under a concentrated moving harmonic load. By using Timoshenko beam theory with the von-Karman’s non-linear strain-displacement relationships, Simsek (2010) further examined the non-linear dynamic analysis of an FGM beam with pinned-pinned supports due to a moving harmonic load.

This paper investigates the forced vibration of slender FGM beams with open edge cracks under a combined action of an axial compression and a concentrated transverse moving load. The classical Bernoulli-Euler beam theory, rotational spring model and modal expansion theory are used to obtain the dynamic response of cantilever, hinged-hinged, and clamped-clamped FGM beams with single or multiple cracks. A parametric study is conducted to demonstrate the effects of material property gradient, the location and total number of cracks, the axial compressive force, the moving speed of the concentrated load, the slenderness ratio, and boundary condition on the dynamic behavior of cracked FGM beams.

2. The Rotational Spring Model

Figure 1 shows an FGM beam of length \( L \), thickness \( h \), and containing an open edge crack of depth \( a \) located at a distance \( L_1 \) from the left end which is taken as the origin of the \( x-z \) coordinate system as shown in. The beam is subjected to an axial compressive force \( P \) and a concentrated transverse load \( F \) moving at a constant speed \( v \) from the left end of the beam to the right end.

![Figure 1: an axially compressed fgm beam with an open edge crack under a moving load.](image)

The Young’s modulus \( E(z) \), shear modulus \( \nu(z) \) and mass density \( \rho(z) \) of the beam vary exponentially in the thickness direction according to

\[
E(z) = E_0 e^{\beta z} \quad \nu(z) = \nu_0 e^{\beta z} \quad \rho(z) = \rho_0 e^{\beta z}
\]  

(1)
where \( E_0 \), \( v_0 \), and \( \rho_0 \) are Young’s modulus, shear modulus and mass density at the mid-plane \((z = 0)\) of the beam. \( \beta \) is a constant describing the material property gradient in the thickness direction, and \( \beta = 0 \) corresponds to an isotropic homogeneous beam. Poisson’s ratio \( \nu \) is taken as a constant since its influence on stress intensity factors (SIFs) is limited (Erdogan and Wu, 1997).

It is assumed that the crack is perpendicular to the beam surface and always remains open. The cracked beam is treated as two sub-beams connected by an elastic rotational spring at the cracked section with bending flexibility (Broek, 1986)

\[
\frac{1 - \nu^2}{E(a)} K_I = \frac{M_I^2}{2} \frac{dG}{da}
\]

(2)

where \( M_I \) is the bending moment at the cracked section, the bending stiffness \( k_T = 1/G \). \( K_I \) is the stress intensity factor (SIF) under mode I bending load, \( E(a) \) is Young’s modulus at the crack tip. Based on the data given by Erdogan and Wu (1997), the magnitude of SIF can be obtained by using Lagrange interpolation technique

\[
K_I = \frac{6M_I}{h^2} \sqrt{\frac{\pi h}{E_a}} \theta(\zeta), \quad (\zeta \leq 0.7)
\]

(3)

where \( \theta(\zeta) \) is given by Ke et al. (2009)

3. Governing Equations

Based on the Kirchhoff-Love hypothesis, the displacement components in the \( x \)- and \( z \)-axes of an arbitrary point, denoted by \( \bar{u}(x,z,t) \) and \( \bar{w}(x,z,t) \), respectively, take the form of

\[
\bar{u}(x,z,t) = u(x,t) - z \frac{\partial \bar{w}}{\partial x}, \quad \bar{w}(x,z,t) = w(x,t)
\]

(4)

where \( u(x,t) \) and \( w(x,t) \) are displacement components in the mid-plane, and \( t \) is time. The axial force \( N \), bending moment \( M \), and transverse shear force \( Q \) are related to the normal strain \( \varepsilon_0 = \frac{\partial u}{\partial x} \) and flexural curvature \( k_x = \frac{\partial^2 w}{\partial x^2} \) by

\[
\begin{bmatrix}
N \\
M
\end{bmatrix} = \begin{bmatrix}
A_{11} & B_{11} \\
B_{11} & D_{11}
\end{bmatrix} \begin{bmatrix}
\varepsilon_0 \\
-k_x
\end{bmatrix}, \quad Q = \frac{\partial M}{\partial x} = B_{11} \frac{\partial \varepsilon_0}{\partial x} - D_{11} \frac{\partial k_x}{\partial x}
\]

(5)

The dimensionless equations of motion for the FGM beam can be derived as follows

\[
\begin{align}
\frac{\partial^2 \bar{U}}{\partial \xi^2} + \gamma \frac{\partial^3 \bar{W}}{\partial \xi^3} &= 0 \\
\frac{\partial^4 \bar{W}}{\partial \xi^4} + p \frac{\partial^2 \bar{W}}{\partial \xi^2} + \frac{\partial^2 \bar{W}}{\partial T^2} &= f \delta(\xi - VT)
\end{align}
\]

(6a, 6b)

where in Equations (5) and (6)

\[
\bar{W} = \frac{w}{L}, \quad \bar{U} = \frac{u}{L}, \quad \xi = \frac{x}{L}, \eta = \frac{z}{h}, \Delta_1 = L_1/L, \quad T = \frac{t}{L^2 \sqrt{\frac{I_1}{I}}}, \quad V = vd^2 \sqrt{\frac{I_1}{d}}, \quad \rho(z)dz = \int_{-h/2}^{h/2} \rho(z)dz
\]
\[ d = D_{11} - \frac{B_{11}^2}{A_{11}}, \quad \gamma = \frac{B_{11}}{A_{11}L}, \quad p = \frac{PL^2}{d}, \quad f = \frac{FL^2}{d}, \quad (A_{11}, B_{11}, D_{11}) = \int_{-h/2}^{h/2} \frac{E(z)}{1 - \mu^2} (1, z, z^2) dz \]  

(7)

At the cracked section \((\xi = \Delta_1)\), compatibility condition enforces the continuity of transverse and axial displacements, axial and shear forces across the crack, and requires that the discontinuity in the slope be proportional to the bending moment transmitted by the cracked section, i.e.,

\[ \ddot{U}_1 = \ddot{U}_2, \quad \ddot{W}_1 = \ddot{W}_2, \quad N_1 = N_2, \quad M_1 = M_2, \quad Q_1 - P \frac{d\ddot{W}_1}{d\xi} = Q_2 - P \frac{d\ddot{W}_2}{d\xi}, \]

\[ k_T \frac{d\ddot{W}_1}{d\xi} - k_T \frac{d\ddot{W}_2}{d\xi} = M_1 \]

(8)

For the cracked cantilever FGM beam, the boundary conditions require

\[ \ddot{U}_1 = 0, \quad \ddot{W}_1 = 0, \quad \frac{d\ddot{W}_1}{d\xi} = 0 \text{ at } \xi = 0; \quad N_2 = 0, \quad M_2 = 0, \quad Q_2 - P \frac{d\ddot{W}_2}{d\xi} = 0 \text{ at } \xi = 1 \]

(9)

In Equations (8) and (9), subscript \(i = 1, 2\) refer to the left sub-beam and right sub-beam divided by an open edge crack.

4. Forced Vibration Response

The modal expansion technique is employed to determine the forced response of the cracked beam. The dynamic deflection of each sub-beam can be expressed as

\[ \ddot{W}_i(\xi, T) = \sum_{k=1}^{n} W_{ik}(\xi)q_k(T) \]

(10)

where \(n\) is the total number of truncated terms, \(q_k(T)\) are the generalized coordinates, and \(W_{ik}(\xi)\) \((i = 1, 2)\) represent the normalized mode shapes of the \(i\)th sub-beam

\[ W_{ik}(\xi) = e_{i1}^k \sin(\alpha_{ik}\xi) + e_{i2}^k \cos(\alpha_{ik}\xi) + e_{i3}^k \sinh(\beta_{ik}\xi) + e_{i4}^k \cosh(\beta_{ik}\xi) \]

(11)

Substituting Equation (10) into Equation (6b), multiplying by \(W_{ij}(\xi)\) and integrating along each sub-beam of the beam, one has

\[ \int_{0}^{\Delta_1} \sum_{k=1}^{n} [W_{1k}^{(4)}(\xi)q_k(T) + pW_{1k}^{(2)}(\xi)q_k(T) + W_{1k}(\xi)\dot{q}_k(T)]W_{1j}(\xi)d\xi \]

\[ + \int_{0}^{\Delta_1} \sum_{k=1}^{n} [W_{2k}^{(4)}(\xi)q_k(T) + pW_{2k}^{(2)}(\xi)q_k(T) + W_{2k}(\xi)\dot{q}_k(T)]W_{2j}(\xi)d\xi \]

\[ = \int_{0}^{\Delta_1} f(\xi - VT)W_{1j}(\xi)d\xi + \int_{0}^{\Delta_1} f(\xi - VT)W_{2j}(\xi)d\xi \]

(12)

The above equation can be simplified to be
\[
\sum_{k=1}^{n} [\ddot{q}_k(T) + \omega_k^2 q_k(T)] = \int_{0}^{\Delta_1} \delta(\xi - VT)W_{1j}(\xi)d\xi + \int_{0}^{\Delta_1} \delta(\xi - VT)W_{2j}(\xi)d\xi
\]

Making use of the orthogonality relationship of normalized mode shapes, Equation (13) becomes

\[
\ddot{q}_k(T) + \omega_k^2 q_k(T) = Q_k(T) \quad (k = 1, 2, \cdots, n)
\]

where

\[
Q_k(T) = \begin{cases} 
  f[e_{11}^k \sin(\alpha VT) + e_{12}^k \cos(\alpha VT) + e_{13}^k \sinh(\beta VT) + e_{14}^k \cosh(\beta VT)] & (T \leq \frac{\Delta_1L}{V}) \\
  f[e_{21}^k \sin(\alpha VT) + e_{22}^k \cos(\alpha VT) + e_{23}^k \sinh(\beta VT) + e_{24}^k \cosh(\beta VT)] & (T > \frac{\Delta_1L}{V}) 
\end{cases}
\]

If the beam is initially at rest before the concentrated load starts to move from the left end, then

\[
q_k(T) = \frac{1}{\omega_k} \int_{0}^{T} \sin \omega_k(T-\tau)Q_k(\tau)d\tau
\]

\[
= \frac{f}{\omega_k} \int_{0}^{T} \sin \omega_k(T-\tau)[e_{11}^k \sin(\alpha V\tau) + e_{12}^k \cos(\alpha V\tau) + e_{13}^k \sinh(\beta V\tau) + e_{14}^k \cosh(\beta V\tau)]d\tau 
\]

\[
= \frac{f}{\omega_k} \left\{ \int_{0}^{\frac{\Delta_1}{V}} \sin \omega_k(T-\tau)[e_{11}^k \sin(\alpha V\tau) + e_{12}^k \cos(\alpha V\tau) + e_{13}^k \sinh(\beta V\tau) + e_{14}^k \cosh(\beta V\tau)]d\tau 
\right\}
\]

The forced responses of FGM cracked beams can then be determined from Equation (10).

### 5. Numerical Results and Discussion

Unless otherwise stated, it is assumed that the magnitude of the transverse moving load is not high so that the cracked section of the beam will not be torn, the crack depth \( a/h = 0.2 \), slenderness ratio \( L/h = 20 \); Young’s modulus ratio \( E_2/E_1 = 5.0, 1.0 \), and \( 0.2 \) where \( E_2/E_1 = 1.0 \) corresponds to a homogeneous beam. The beam is 100% aluminum at the top surface with \( E_1 = 70 \) GPa; \( \mu_1 = 0.33 \); \( \rho_1 = 2780 \) kg/m³. The beam height is kept constant while the beam length may be varied.

Figures 2-5 give dynamic deflections of cracked FGM beams, normalized by the static tip deflection of a perfect homogeneous cantilever under a point load at its free end, i.e., \( FL^3/3d_0 \). Note that the horizontal
axis $\xi$ is the distance the point load has traveled from the left end and virtually represents the time the load has traveled since the moving speed is constant.

\begin{figure}[h!]
\centering
\includegraphics[width=\textwidth]{fig2.png}
\caption{Dynamic tip deflections of fgm cantilevers: effect of material property gradient.}
\end{figure}

\begin{figure}[h!]
\centering
\includegraphics[width=\textwidth]{fig3.png}
\caption{Dynamic tip deflections of fgm cantilevers: effect of moving load speed.}
\end{figure}

Figure 2 demonstrates the effects of both axial compressive force and Young’s modulus ratio $E_2/E_1$ on the dynamic tip deflection of cracked inhomogeneous cantilevers ($L/h = 20, \Delta_1 = 0.5$) when a point load moves at a speed of $V/V_0 = 0.4$. Figures 3 and 4 investigate, respectively, how the dynamic tip deflection is influenced by the moving speed of the point load and the crack location. It is observed that
the deflection increases with a decrease in Young’s modulus ratio and an increase in the axial compression but decreases at a higher value of $V/V_0$. This is because the beam does not have enough time to reach its maximum deflection when the load moves at a faster speed. The crack location, however, has very little effect on the dynamic tip deflection.

Figure 4: Dynamic tip deflections of FGM cantilevers: Effect of crack location.

Figure 5 compares the dynamic deflections at the mid-point of hinged-hinged and clamped-clamped inhomogeneous beams ($E_2/E_1 = 0.2$, $L/h = 20$, $V/V_0 = 0.4$) containing an edge crack ($\Delta_1 = 0.5$) with and without an axial compression. As expected, the hinged-hinged beam has much higher deflections than the clamped-clamped beam. The dynamic deflection is much more significantly influenced by the axial compressive load than the edge crack, although both weakening the bending stiffness and consequently resulting in higher bending deformation.
6. Conclusions

Forced vibration of FGM Euler-Bernoulli beams with open edge cracks subjected to an axial force and a concentrated moving load is analytically investigated using the rotational spring model and the modal expansion technique. It is found that the dynamic deflection increases due to the presence of edge crack and the axial compressive force and is much more affected by the axial compression than by the edge crack. In addition, the beam with a smaller modulus ratio $E_2/E_1$ has a higher dynamic deflection.

References