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Euler characteristics and Möbius algebras of p-subgroup categories☆

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ABSTRACT

Let G be a finite group and p a prime number. We compute the Euler characteristic in the sense of Leinster for some categories of nonidentity p-subgroups of G. The p-subgroup categories considered include the Brown poset $\mathcal{S}_G^*$, the transporter category $\mathcal{T}_G^*$, the linking category $\mathcal{L}_G^*$, the Frobenius, or fusion, category $\mathcal{F}_G^*$, and the orbit category $\mathcal{O}_G^*$ of all nonidentity p-subgroups of G.

1. Introduction

Tom Leinster [23] defined the Euler characteristic of a finite category $\mathcal{C}$ as follows:

- A *weighting* for $\mathcal{C}$ is a function $k_\mathcal{C}^\bullet \colon \mathrm{Ob}(\mathcal{C}) \to \mathbf{Q}$ such that $\sum_b |\mathcal{C}(a,b)| k_\mathcal{C}^b = 1$ for all objects a of $\mathcal{C}$.
- A *coweighting* for $\mathcal{C}$ is a function $k_\bullet^\mathcal{C} \colon \mathrm{Ob}(\mathcal{C}) \to \mathbf{Q}$ such that $\sum_a k_a^\mathcal{C} |\mathcal{C}(a,b)| = 1$ for all objects b of $\mathcal{C}$.
- If $\mathcal{C}$ admits both a weighting and a coweighting, the rational number

$$\sum_b k_\mathcal{C}^b = \chi(\mathcal{C}) = \sum_a k_a^\mathcal{C}$$

is the Euler characteristic of $\mathcal{C}$.

The purpose of this paper is to determine weightings, coweightings, and Euler characteristics of p-subgroup categories associated to finite groups. For a fixed finite group G and a fixed prime number p, we consider

$\mathcal{S}_G$: the poset of all p-subgroups of G ordered by inclusion
$\mathcal{T}_G$: the transporter category of all p-subgroups of G
$\mathcal{L}_G$: the linking category of all p-subgroups G [6]
$\mathcal{F}_G$: the Frobenius category of all p-subgroups of G [28,7]
$\mathcal{O}_G$: the orbit category of all p-subgroups of G
$\tilde{\mathcal{F}}_G$: the exterior quotient of the Frobenius category $\mathcal{F}_G$ [28, 1.3, 4.8].

If $\mathcal{C}$ is any of these categories, then

- $\mathcal{C}^*$ denotes the full subcategory of $\mathcal{C}$ generated by all *nonidentity* p-subgroups
- $\mathcal{C}^{\mathrm{ea}}$ denotes the full subcategory of $\mathcal{C}^*$ generated by all *elementary abelian* p-subgroups
- $\mathcal{C}^{\mathrm{sc}}$ denotes the full subcategory of $\mathcal{C}^*$ generated by all *p-selfcentralizing* p-subgroups (Definition 8.1).

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For instance $\mathcal{S}_G^*$ is the Brown poset of all nonidentity p-subgroups and $\mathcal{F}_G^*$ the Frobenius category of all nonidentity p-subgroups of G.

All of these finite categories have weightings and coweightings and therefore Euler characteristics (Corollary 2.15).

In our first theorem we determine *coweightings* for the six nonidentity p-subgroup categories. The coweightings are expressed in terms Möbius functions. Recall that the Möbius function of finite groups is recursively defined by $\mu(1) = 1$ and $\sum_{1 \leq K \leq G} \mu(K) = 0$ when $G > 1$ [34]. The Möbius function vanishes on all p-groups but the elementary abelian ones [18]. We write $|\mathcal{C}(a)|$ for the size of the endomorphism monoid of the object a in the finite category $\mathcal{C}$.

Theorem 1.1. *The six nonidentity p-subgroup categories $\mathcal{S}_G^*$, $\mathcal{T}_G^*$, $\mathcal{L}_G^*$, $\mathcal{F}_G^*$, $\widetilde{\mathcal{F}}_G^*$, and $\mathcal{O}_G^*$ have the following coweightings and Euler characteristics:*

(1) *The poset $\mathcal{S}_G^*$ has coweighting and Euler characteristic*

$$k_K^{\mathcal{S}^*} = -\mu(K), \qquad \chi(\mathcal{S}_G^*) = \sum_{K>1} -\mu(K).$$

The sum runs over all nonidentity p-subgroups K of G.

(2) *Let $\mathcal{C}$ be any of the categories the categories $\mathcal{T}_G$, $\mathcal{L}_G$ or $\mathcal{F}_G$. The category $\mathcal{C}^*$ has coweighting and Euler characteristic*

$$k_K^{\mathcal{C}^*} = \frac{1}{|G : N_G(K)|} \frac{-\mu(K)}{|\mathcal{C}(K)|}, \qquad \chi(\mathcal{C}^*) = \sum_{[K]} \frac{-\mu(K)}{|\mathcal{C}(K)|}.$$

The sum runs over all conjugacy classes $[K]$ of nonidentity p-subgroups K of G.

(3) *$\widetilde{\mathcal{F}}_G^*$ has the same coweighting as $\mathcal{F}_G^*$, and $\chi(\widetilde{\mathcal{F}}_G^*) = \chi(\mathcal{F}_G^*)$.*

(4) *The orbit category $\mathcal{O}_G^*$ has coweighting and Euler characteristic*

$$k_K^{\mathcal{O}^*} = \frac{-\mu(K)}{|G|} + \begin{cases} \frac{p-1}{p}\frac{|K|}{|G|} & K \text{ is cyclic} \\ 0 & K \text{ is noncyclic} \end{cases}, \qquad \chi(\mathcal{O}_G^*) = \chi(\mathcal{T}_G^*) + \frac{p-1}{p} \sum_{[C] \text{ cyclic}} \frac{1}{|\mathcal{O}_G(C)|}.$$

The sum runs over all conjugacy classes $[C]$ of nonidentity cyclic p-subgroups C of G.

Let $\mathcal{C}^*$ be any of the five categories $\mathcal{C}^* = \mathcal{S}_G^*, \mathcal{T}_G^*, \mathcal{L}_G^*, \mathcal{F}_G^*$, or $\widetilde{\mathcal{F}}_G^*$. Then

$$k_K^{\mathcal{C}^*} \neq 0 \iff K \text{ is elementary abelian}$$

for any nonidentity p-subgroup K of G. Thus the support of the coweighting for $\mathcal{C}^*$ is the set nonidentity elementary abelian p-subgroups of G.

If $\mathcal{C}$ is a finite category with Euler characteristic, write

$$\widetilde{\chi}(\mathcal{C}) = \chi(\mathcal{C}) - 1 \tag{1.2}$$

for the *reduced* Euler characteristic of $\mathcal{C}$.

In the next theorem we determine *weightings* for p-subgroup categories. This approach reveals that it is possible to compute global Euler characteristics of p-subgroup categories from data that are p-local in the group theoretic sense [14, Definition 5.2]. Any weighting, $k_{\mathcal{C}}^\bullet$, for a p-subgroup category $\mathcal{C}$ restricts to a weighting on the full subcategory $\mathcal{C}^*$ of nonidentity p-subgroups, and $\chi(\mathcal{C}) - \chi(\mathcal{C}^*) = k_{\mathcal{C}}^1$ (Remark 2.6).

Theorem 1.3. *The five p-subgroup categories $\mathcal{S}_G$, $\mathcal{T}_G$, $\mathcal{O}_G$, $\mathcal{F}_G$, and $\widetilde{\mathcal{F}}_G^{sc}$ have the following weightings and Euler characteristics:*

(1) *The poset $\mathcal{S}_G$ has weighting and Euler characteristic*

$$k_{\mathcal{S}}^H = -\widetilde{\chi}(\mathcal{S}_{\mathcal{O}_G(H)}^*), \qquad 1 = \chi(\mathcal{S}_G) = \sum_H -\widetilde{\chi}(\mathcal{S}_{\mathcal{O}_G(H)}^*).$$

In the formula for the Euler characteristic, the sum runs over all p-subgroups H of G.

(2) *The transporter category $\mathcal{T}_G$ has weighting and Euler characteristic*

$$k_{\mathcal{T}}^H = \frac{-\widetilde{\chi}(\mathcal{S}_{\mathcal{O}_G(H)}^*)}{|G|}, \qquad \frac{1}{|G|} = \chi(\mathcal{T}_G) = \sum_{[H]} \frac{-\widetilde{\chi}(\mathcal{S}_{\mathcal{O}_G(H)}^*)}{|\mathcal{T}_G(H)|}.$$

In the formula for the Euler characteristic, the sum runs over all conjugacy classes $[H]$ of p-subgroups H of G.

(3) *The Frobenius category $\mathcal{F}_G$ has weighting and Euler characteristic*

$$k_{\mathcal{F}}^H = \sum_{x \in C_{N_G(H)}(H)} \frac{-\widetilde{\chi}(\mathcal{S}_{C_{N_G(H)}(x)/H}^*)}{|G|}, \qquad 1 = \chi(\mathcal{F}_G) = \sum_{[H]} \sum_{x \in C_{N_G(H)}(H)} \frac{-\widetilde{\chi}(\mathcal{S}_{C_{N_G(H)}(x)/H}^*)}{|N_G(H)|}.$$

In the formula for the Euler characteristic, the first sum runs over all conjugacy classes $[H]$ of p-subgroups H of G.

(4) *The orbit category $\mathcal{O}_G$ has weighting and Euler characteristic*

$$k_{\mathcal{O}}^H = \frac{-\widetilde{\chi}(\mathcal{S}^*_{\mathcal{O}_G}(H))}{|G:H|}, \qquad \frac{1+(p-1)\sum_C |C|}{p|G|} = \chi(\mathcal{O}_G) = \sum_{[H]\in[\mathcal{T}_G]} \frac{-\widetilde{\chi}(\mathcal{S}^*_{\mathcal{O}_G}(H))}{|\mathcal{O}_G(H)|}.$$

In the formula for the Euler characteristic, the sum to the left runs over all cyclic p-subgroups C of G and the sum to the right over all conjugacy classes [H] of p-subgroups H of G.

(5) *The category $\widetilde{\mathcal{F}}_G^{\mathrm{sc}}$ has weighting and Euler characteristic*

$$k^H_{\widetilde{\mathcal{F}}^{\mathrm{sc}}} = \frac{1}{|G:N_G(H)|}\frac{-\widetilde{\chi}(\mathcal{S}^*_{\widetilde{\mathcal{F}}_G^{\mathrm{sc}}(H)})}{|\widetilde{\mathcal{F}}_G^{\mathrm{sc}}(H)|}, \qquad \chi(\widetilde{\mathcal{F}}_G^{\mathrm{sc}}) = \sum_{[H]} \frac{-\widetilde{\chi}(\mathcal{S}^*_{\widetilde{\mathcal{F}}_G^{\mathrm{sc}}(H)})}{|\widetilde{\mathcal{F}}_G^{\mathrm{sc}}(H)|}.$$

In the formula for the Euler characteristic, the sum runs over the set of conjugacy classes [H] of p-selfcentralizing p-subgroups H of G.

In the context Theorem 1.3 we would like to comment on the p-subgroup poset conjecture

$$\mathcal{S}_G^* \not\simeq * \iff O_pG = 1$$

made by Quillen in 1978 [29, Conjecture 2.9]: The nonidentity p-subgroup poset for a finite group is noncontractible if and only if the group's p-core is trivial. Quillen proved the implication '$\Longrightarrow$' [29, Proposition 2.4] and also the implication '$\Longleftarrow$' under the additional assumption that G be solvable [29, Corollary 12.2]. Aschbacher and Smith established '$\Longleftarrow$' for a larger class of groups including all p-solvable or simple finite groups [2, Theorems 0.5 and 0.7].

It is tempting to state a stronger form

$$\chi(\mathcal{S}_G^*) \neq 1 \iff O_pG = 1 \tag{1.4}$$

of the Quillen conjecture. The implication '$\Longrightarrow$' is true by Quillen, but the validity of implication '$\Longleftarrow$' remains open. It is known to hold for all p-solvable groups with abelian Sylow p-subgroups [20, Theorem A].

A p-subgroup H of G is p-radical if $O_p\mathcal{O}_G(H) = 1$ and $\mathcal{F}_G$-radical if $O_p\widetilde{\mathcal{F}}_G(H) = 1$ (Definition 3.18). If we imagine that the strong form of the Quillen conjecture (1.4) is true, then

$$k_{\mathcal{C}}^H \neq 0 \iff \begin{cases} H \text{ is } p\text{-radical} & \mathcal{C} = \mathcal{S}_G, \mathcal{T}_G, \mathcal{O}_G \\ H \text{ is } \mathcal{F}_G\text{-radical} & \mathcal{C} = \widetilde{\mathcal{F}}_G^{\mathrm{sc}} \end{cases}$$

for any p-subgroup H of G. (We stress that the implication '$\Longrightarrow$' is valid but '$\Longleftarrow$' is only conjectural.) This would mean that the support of the weightings for $\mathcal{S}_G, \mathcal{T}_G, \mathcal{O}_G$ is the set of p-radical p-subgroups and that the support for the weighting on $\widetilde{\mathcal{F}}_G^{\mathrm{sc}}$ is the set of $\mathcal{F}_G$-radical p-subgroups.

In any case, Theorems 1.1 and 1.3 show that the p-subgroup categories retain information, perceived by the weighting or the coweighting, about group theoretic characteristics (elementary abelian, p-radical, $\mathcal{F}_G$-radical) of their objects.

In Theorem 6.1 we prove that $-\widetilde{\chi}(\mathcal{S}_\bullet^*)$ and $-\widetilde{\chi}(\mathcal{F}_\bullet^*)$ are multiplicative functions of finite groups in the sense that

$$-\widetilde{\chi}(\mathcal{S}^*_{\prod_{i=1}^n G_i}) = \prod_{i=1}^n -\widetilde{\chi}(\mathcal{S}^*_{G_i}), \qquad -\widetilde{\chi}(\mathcal{F}^*_{\prod_{i=1}^n G_i}) = \prod_{i=1}^n -\widetilde{\chi}(\mathcal{F}^*_{G_i})$$

when $G_1, \ldots, G_n$ are finite groups. The reduced Euler characteristic $-\widetilde{\chi}(\mathcal{S}_\bullet^*)$ of the nonidentity p-subgroup poset vanishes on any finite group with a nontrivial *normal* p-subgroup [29, Proposition 2.4]. The reduced Euler characteristic $-\widetilde{\chi}(\mathcal{F}_\bullet^*)$ of the nonidentity p-Frobenius category vanishes on any finite group with a nontrivial *central* p-subgroup (Proposition 5.1).

We finish this paper with two sections on Möbius algebras. The classical Burnside algebra of a finite group is the Möbius algebra for the (set of isomorphism classes in the) orbit category of the group. Not only the orbit category, but also the other subgroup categories have associated Möbius algebras. We shall work out the product in these Möbius algebras and show some integrality results.

For the sake of quick reference we list here the notation that we are using throughout this paper:

- p is a fixed prime number
- n_p is the p-part of the integer n, the highest power of p dividing n, and $n_{p'} = n/n_p$ is the p'-part of n
- G is a finite group
- $H \leq K$ means that H is a subgroup of K
- $\Phi(G)$, the Frattini subgroup of G, is the intersection of all maximal subgroups of G [14, p. 18]
- O_pG, the p-core of G, is the greatest normal p-subgroup of G [14, p. 19]
- O^pG is the smallest normal p-power index subgroup of G [14, p. 19]
- $\mathcal{C}$ is a finite category, $\mathcal{C}(a, b)$ is the set of morphisms from object a to object b, and $\mathcal{C}(a) = \mathcal{C}(a, a)$ is the monoid of endomorphisms of a

- $\mathrm{Ob}(\mathcal{C})$ is the set of objects of $\mathcal{C}$
- $[\mathcal{C}]$ is the set of isomorphism classes of objects of $\mathcal{C}$, and $[a] \in [\mathcal{C}]$ the isomorphism class of $a \in \mathrm{Ob}(\mathcal{C})$.

Finally, we would like to thank the anonymous referees for many useful comments.

1.1. Subgroup categories

This subsection contains precise definitions of the p-subgroup categories occurring in this paper. Fix a finite group G and a prime number p. $\mathcal{S}_G$ is the poset of all p-subgroups of G ordered by inclusion. In other words, $\mathcal{S}_G$ is the category whose objects are all p-subgroups of G with one morphism $H \to K$ whenever $H \leq K$ and no morphisms otherwise. The objects of the finite categories $\mathcal{T}_G$, $\mathcal{L}_G$, $\mathcal{F}_G$, $\widetilde{\mathcal{F}}_G$, and $\mathcal{O}_G$ are all p-subgroups of G. For any two p-subgroups, H and K, of G, the morphism sets are

$$\begin{array}{ll} \mathcal{T}_G(H,K) = N_G(H,K) & \mathcal{L}_G(H,K) = O^pC_G(H)\backslash N_G(H,K) \\ \mathcal{F}_G(H,K) = C_G(H)\backslash N_G(H,K) & \mathcal{O}_G(H,K) = N_G(H,K)/K \\ \widetilde{\mathcal{F}}_G(H,K) = C_G(H)\backslash N_G(H,K)/K. & \end{array}$$

Here $N_G(H,K) = \{g \in G \mid H^g \leq K\}$ denotes the transporter set. Composition in any of these categories is induced from group multiplication in G. The morphisms in $\mathcal{F}_G(H,K)$ are restrictions to H of inner automorphisms of G, morphisms in $\mathcal{O}_G(H,K)$ are right G-maps $H\backslash G \to K\backslash G$, and morphisms in $\widetilde{\mathcal{F}}_G(H,K)$ are K-conjugacy classes of restrictions to H of inner automorphisms of G. The endomorphism groups in these categories of the p-subgroup H of G are $\mathcal{S}_G^*(H) = 1$, $\mathcal{T}_G(H) = N_G(H)$, $\mathcal{L}_G(H) = O^pC_G(H)\backslash N_G(H)$, $\mathcal{F}_G(H) = C_G(H)\backslash N_G(H)$, $\mathcal{O}_G(H) = N_G(H)/H$, and $\widetilde{\mathcal{F}}_G(H) = C_G(H)\backslash N_G(H)/H$. The five categories $\mathcal{T}_G$, $\mathcal{L}_G$, $\mathcal{F}_G$, $\mathcal{O}_G$, and $\widetilde{\mathcal{F}}_G$ are related by a commutative diagram

$$\begin{array}{ccccccc} \mathcal{T}_G & \longrightarrow & \mathcal{L}_G & \longrightarrow & \mathcal{F}_G & \longrightarrow & \widetilde{\mathcal{F}}_G \\ \downarrow & & & & & \nearrow & \\ \mathcal{O}_G & & & & & & \end{array}$$

of functors.

2. Euler characteristics

In this section we review the relevant parts of Tom Leinster's concept of Euler characteristic of a finite category $\mathcal{C}$ [23] supplemented by a few of our own observations.

2.1. The Euler characteristic of a square matrix

Let S be a finite set and $\zeta : S \times S \to \mathbf{Q}$ a rational function on $S \times S$. Equivalently, $\zeta = \big(\zeta(a,b)\big)_{a,b \in S}$ is a square matrix with rows and columns indexed by the finite set S and with rational entries $\zeta(a,b) \in \mathbf{Q}$, $a, b \in S$.

Definition 2.1 (*[23, Definition 1.10]*)**.** A weighting for ζ is a column vector $(k^\bullet)$ and a coweighting for ζ is a row vector $(k_\bullet)$ solving the linear equations

$$\big(\zeta(a,b)\big)\begin{pmatrix} \vdots \\ k^b \\ \vdots \end{pmatrix} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}, \qquad \begin{pmatrix} \cdots & k_a & \cdots \end{pmatrix}\big(\zeta(a,b)\big) = \begin{pmatrix} 1 & \cdots & 1 \end{pmatrix}.$$

If ζ admits both a weighting $k^\bullet$ and a coweighting $k_\bullet$, the sum of the values of the weighting

$$\sum_{b\in S} k^b = \sum_{b\in S}\left(\sum_{a\in S} k_a\zeta(a,b)\right)k^b = \sum_{a\in S} k_a\left(\sum_{b\in S}\zeta(a,b)k^b\right) = \sum_{a\in S} k_a \tag{2.2}$$

equals the sum of the values of the coweighting.

Definition 2.3 (*[23, Definition 2.2]*)**.** The square matrix ζ *has Euler characteristic* if it admits both a weighting and a coweighting. Its Euler characteristic is then the sum

$$\sum_{b\in S} k^b = \chi(\zeta) = \sum_{a\in S} k_a$$

of all the values of any weighting $k^\bullet$ or any coweighting $k_\bullet$.

As usual, we let δ stand for Kronecker's δ-function

$$\delta(a,b)=\begin{cases}1 & a=b\\ 0 & a\neq b\end{cases},\qquad a,b\in \mathrm{Ob}(\mathcal{C}).$$

Suppose that the square matrix ζ is invertible. Let $\mu = \big(\mu(a,b)\big)_{a,b\in S}$ denote the inverse of ζ. The *Möbius inversion formula*

$$\forall a,c\in S\colon \sum_b \zeta(a,b)\mu(b,c)=\delta(a,c)=\sum_b \mu(a,b)\zeta(b,c) \tag{2.4}$$

simply expresses that ζ and μ are inverse matrices. In this case, the vectors

$$\big(k^a\big)=\big(\mu(a,b)\big)_{a,b\in S}\begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix}=\Big(\sum_{b\in S}\mu(a,b)\Big)\qquad \big(k_b\big)=\begin{pmatrix}1 & \dots & 1\end{pmatrix}\big(\mu(a,b)\big)_{a,b\in S}=\Big(\sum_{a\in S}\mu(a,b)\Big)$$

are, respectively, the *unique* weighting and *unique* coweighting for ζ. The weighting is represented by row sums and the coweighting by column sums. The Euler characteristic of ζ is the sum

$$\chi(\zeta)=\sum_{a,b\in S}\mu(a,b)$$

of all the entries in the inverse matrix.

2.2. The Euler characteristic of a finite category

Define the ζ-matrix for the finite category $\mathcal{C}$ to be the square matrix

$$\zeta(\mathcal{C})=\big(|\mathcal{C}(a,b)|\big)_{a,b\in\mathrm{Ob}(\mathcal{C})}$$

tabulating the number of morphisms between pairs of objects of $\mathcal{C}$. We say that the category $\mathcal{C}$ admits a weighting, admits a coweighting, or has Euler characteristic if its ζ-matrix does. This means that a weighting for $\mathcal{C}$ is a function $k^\bullet_{\mathcal{C}}\colon \mathrm{Ob}(\mathcal{C})\to\mathbf{Q}$, and a coweighting is a function $k^{\mathcal{C}}_\bullet\colon \mathrm{Ob}(\mathcal{C})\to\mathbf{Q}$ such that

$$\forall a\in\mathrm{Ob}(\mathcal{C})\colon \sum_{b\in\mathrm{Ob}(\mathcal{C})}|\mathcal{C}(a,b)|k^b_{\mathcal{C}}=1,\qquad \forall b\in\mathrm{Ob}(\mathcal{C})\colon \sum_{a\in\mathrm{Ob}(\mathcal{C})}k^{\mathcal{C}}_a|\mathcal{C}(a,b)|=1. \tag{2.5}$$

The Euler characteristic of $\mathcal{C}$ is the Euler characteristic of its ζ-matrix

$$\chi(\mathcal{C})=\sum_{b\in\mathrm{Ob}(\mathcal{C})}k^b_{\mathcal{C}}=\sum_{a\in\mathrm{Ob}(\mathcal{C})}k^{\mathcal{C}}_a=\chi(\zeta(\mathcal{C}))$$

provided that $\mathcal{C}$ admits both a weighting and a coweighting. We say that $\mathcal{C}$ has Möbius inversion if its ζ-matrix is invertible. The Möbius function for $\mathcal{C}$ is then defined as the inverse $\mu(\mathcal{C})=\zeta(\mathcal{C})^{-1}$ of the ζ-matrix. In this case,

$$k^\bullet_{\mathcal{C}}=\sum_{b\in\mathrm{Ob}(\mathcal{C})}\mu(\mathcal{C})(\bullet,b),\qquad k^{\mathcal{C}}_\bullet=\sum_{a\in\mathrm{Ob}(\mathcal{C})}\mu(\mathcal{C})(a,\bullet),\qquad \chi(\mathcal{C})=\sum_{a,b\in\mathrm{Ob}(\mathcal{C})}\mu(\mathcal{C})(a,b)$$

are, respectively, the unique weighting, the unique coweighting, and the Euler characteristic of $\mathcal{C}$.

Remark 2.6. Let $\mathcal{C}$ be a (finite) category and $\mathcal{I}$ and $\mathcal{J}$ two full subcategories. If

$$a\in\mathrm{Ob}(\mathcal{I}),\quad \mathcal{C}(a,b)\neq\emptyset\Longrightarrow b\in\mathrm{Ob}(\mathcal{I}),\qquad \mathcal{C}(a,b)\neq\emptyset,\quad b\in\mathrm{Ob}(\mathcal{J})\Longrightarrow a\in\mathrm{Ob}(\mathcal{J})$$

holds for all $a,b\in\mathrm{Ob}(\mathcal{C})$, then $\mathcal{I}$ is a *left ideal* and $\mathcal{J}$ a *right ideal*, of $\mathcal{C}$. Clearly,

$$\mathcal{I}\text{ is a left ideal }\iff \mathcal{C}-\mathcal{I}\text{ is a right ideal}$$

where $\mathcal{C}-\mathcal{I}$ is the full subcategory of $\mathcal{C}$ generated by all objects of $\mathcal{C}$ not objects of $\mathcal{I}$.

- Weightings for $\mathcal{C}$ restrict to weightings on left ideals of $\mathcal{C}$
- Coweightings for $\mathcal{C}$ restrict to coweightings on right ideals of $\mathcal{C}$.
- Möbius functions for $\mathcal{C}$ restrict to Möbius functions on left or right ideals of $\mathcal{C}$.

These items are easy consequences of the defining relations for weightings, coweightings and Möbius functions (2.5) (2.4). For the third item one uses that $\mu(a,b)\neq 0\Longrightarrow|\zeta(a,b)|\neq 0$ [23, Theorem 4.1].

If $\mathcal{C}$ is a p-subgroup category, then $\mathcal{C}^{\mathrm{sc}}$ and $\mathcal{C}^*$ are left ideals, and $\mathcal{C}^{\mathrm{ea}}$ a right ideal, of $\mathcal{C}$. (Definitions of these subcategories can be found in the Introduction.)

Example 2.7 (*[23, Examples 1.1.c]*). Suppose that $\mathcal{C}$ has Euler characteristic. If $\mathcal{C}$ has a terminal element 1, then $k^\bullet_{\mathcal{C}} = \delta(\bullet, 1)$ is a weighting with value 1 concentrated at the terminal element because

$$\sum_b \zeta(a,b)\delta(b,1) = \zeta(a,1) = 1$$

for all $a \in \mathrm{Ob}(\mathcal{C})$. The Euler characteristic of $\mathcal{C}$ is $\chi(\mathcal{C}) = \sum_a \delta(a,1) = \delta(1,1) = 1$.

Dually, if $\mathcal{C}$ has an initial element 0, then $k_\bullet^{\mathcal{C}} = \delta(0, \bullet)$ is a coweighting concentrated, with value 1, at the initial element. Again, the Euler characteristic $\chi(\mathcal{C}) = 1$.

Lemma 2.8 (*[23, Proposition 2.4]*). *Let $\mathcal{C}$ and $\mathcal{D}$ be finite categories.*

(1) *$\mathcal{C}$ has Euler characteristic if and only if its opposite category $\mathcal{C}^{\mathrm{op}}$ has, and then $\chi(\mathcal{C}) = \chi(\mathcal{C}^{\mathrm{op}})$.*
(2) *If both $\mathcal{C}$ and $\mathcal{D}$ have Euler characteristics and there is an adjunction $\mathcal{C} \rightleftarrows \mathcal{D}$, then $\chi(\mathcal{C}) = \chi(\mathcal{D})$.*
(3) *If $\mathcal{C}$ and $\mathcal{D}$ are equivalent then $\mathcal{C}$ has Euler characteristic if and only if $\mathcal{D}$ has Euler characteristic, and then $\chi(\mathcal{C}) = \chi(\mathcal{D})$.*

Lemma 2.9. *Let $\mathcal{C}$ be a full subcategory of $\mathcal{D}$ and suppose that both categories have Euler characteristics.*

(1) *If $\mathrm{Ob}(\mathcal{C})$ contains the support of some weighting $k^\bullet_{\mathcal{D}}$ on $\mathcal{D}$, then the restriction $k^\bullet_{\mathcal{D}}|\mathrm{Ob}(\mathcal{C})$ is a weighting for $\mathcal{C}$ and $\chi(\mathcal{C}) = \chi(\mathcal{D})$.*
(2) *If $\mathrm{Ob}(\mathcal{C})$ contains the support of some coweighting $k_\bullet^{\mathcal{D}}$ on $\mathcal{D}$, then the restriction $k_\bullet^{\mathcal{D}}|\mathrm{Ob}(\mathcal{C})$ is a coweighting for $\mathcal{C}$ and $\chi(\mathcal{C}) = \chi(\mathcal{D})$.*

Proof. We shall only prove item (1) as (2) can be handled similarly. The assumption is that $\forall b \in \mathrm{Ob}(\mathcal{D})\colon k^b_{\mathcal{D}} \neq 0 \Longrightarrow b \in \mathrm{Ob}(\mathcal{C})$. For any $a \in \mathrm{Ob}(\mathcal{C})$

$$1 = \sum_{b \in \mathrm{Ob}(\mathcal{D})} \zeta(a,b) k^b_{\mathcal{D}} = \sum_{b \in \mathrm{Ob}(\mathcal{C})} \zeta(a,b) k^b_{\mathcal{D}}.$$

This shows that the restriction of $k^\bullet_{\mathcal{D}}$ to $\mathrm{Ob}(\mathcal{C})$ is indeed a weighting for $\mathcal{C}$ (2.5). The Euler characteristic of $\mathcal{C}$ is $\chi(\mathcal{C}) = \sum_{b \in \mathrm{Ob}(\mathcal{C})} k^b_{\mathcal{D}} = \sum_{b \in \mathrm{Ob}(\mathcal{D})} k^b_{\mathcal{D}} = \chi(\mathcal{D})$. □

2.3. *The Euler characteristic of a finite poset*

Any finite *poset*, $\mathcal{S}$, has Möbius inversion and Euler characteristic [34]. The Möbius function μ for $\mathcal{S}$ is the map $\mu\colon \mathrm{Ob}(\mathcal{S}) \times \mathrm{Ob}(\mathcal{S}) \to \mathbf{Z}$ defined by

- $\mu(a,b) = 0$ when $a \not\leq b$
- $\mu(a,a) = 1$
- $\sum_{b \in [a,c]} \mu(b,c) = \delta(a,c) = \sum_{b \in [a,c]} \mu(a,b)$ when $a \leq c$.

The equations of the third item are the Möbius inversion formulas (2.4) for posets. The value of the Möbius function $\mu(a,b) = \widetilde{\chi}((a,b))$, $a < b$, depends only on the open interval (a,b) from a to b and not on the whole poset [34, Proposition 3.8.6], [23, Corollary 1.5].

Example 2.10. Suppose that the poset $\mathcal{S}$ contains a least element, 0. Then the reduced Euler characteristic (1.2) $\widetilde{\chi}(\mathcal{S}) = 0$ by Example 2.7. Let $\mathcal{S}^*$ be the subposet of all elements $\neq 0$. For any element b different from the least element,

$$\sum_{a \in (0,b]} \mu(a,b) = -\mu(0,b) + \sum_{a \in [0,b]} \mu(a,b) = -\mu(0,b) + \delta(0,b) = -\mu(0,b). \tag{2.11}$$

The functions

$$k^a_{\mathcal{S}} = \sum_b \mu(a,b), \qquad k_b^{\mathcal{S}^*} = \sum_{a \in (0,b]} \mu(a,b) = -\mu(0,b)$$

are, respectively, a weighting for $\mathcal{S}$ and a coweighting for $\mathcal{S}^*$. The weighting for $\mathcal{S}$ restricts to a weighting for the left ideal $\mathcal{S}^*$ (Remark 2.6). The Euler characteristic and the opposite of the reduced Euler characteristic of $\mathcal{S}^*$ are

$$\chi(\mathcal{S}^*) = \sum_{b \in \mathcal{S}^*} -\mu(0,b), \qquad -\widetilde{\chi}(\mathcal{S}*) = 1 - \chi(\mathcal{S}^*) = \sum_{b \in \mathcal{S}} \mu(0,b). \tag{2.12}$$

2.4. The Euler characteristic of $[\mathcal{C}]$

As mentioned in the Introduction, $[\mathcal{C}]$ denotes the set of isomorphism classes of $\mathcal{C}$-objects. Let $[\zeta(\mathcal{C})]\colon [\mathcal{C}] \times [\mathcal{C}] \to \mathbf{Q}$ be the function induced by the ζ-function $\zeta(\mathcal{C})\colon \mathrm{Ob}(\mathcal{C}) \times \mathrm{Ob}(\mathcal{C}) \to \mathbf{Q}$ for $\mathcal{C}$.

We say that $[\mathcal{C}]$ admits a weighting, a coweighting, or has Euler characteristic if its ζ-matrix $[\zeta(\mathcal{C})]$ does. This means that a weighting for $[\mathcal{C}]$ is a rational function $k^{\bullet}_{[\mathcal{C}]}\colon [\mathcal{C}] \to \mathbf{Q}$ and a coweighting is a rational function $k^{[\mathcal{C}]}_{\bullet}\colon [\mathcal{C}] \to \mathbf{Q}$ such that

$$\forall [a] \in [\mathcal{C}]\colon \sum_{[b]\in[\mathcal{C}]} [\zeta(\mathcal{C})]([a],[b])k^{[b]}_{[\mathcal{C}]} = 1, \qquad \forall [b] \in [\mathcal{C}]\colon \sum_{[a]\in[\mathcal{C}]} k^{[\mathcal{C}]}_{[a]}[\zeta(\mathcal{C})]([a],[b]) = 1. \tag{2.13}$$

The Euler characteristic of $[\mathcal{C}]$ is

$$\sum_{[b]\in[\mathcal{C}]} k^{[b]}_{[\mathcal{C}]} = \chi([\mathcal{C}]) = \sum_{[a]\in[\mathcal{C}]} k^{[\mathcal{C}]}_{[a]}$$

provided that $[\mathcal{C}]$ admits both a weighting and a coweighting.

We say that $[\mathcal{C}]$ has Möbius inversion if its ζ-matrix $[\zeta(\mathcal{C})]$ is invertible and then

$$k^{[a]}_{[\mathcal{C}]} = \sum_{[b]\in[\mathcal{C}]} [\mu]([a],[b]), \qquad k^{[\mathcal{C}]}_{[b]} = \sum_{[a]\in[\mathcal{C}]} [\mu]([a],[b]), \qquad \chi([\mathcal{C}]) = \sum_{[a],[b]\in[\mathcal{C}]} [\mu]([a],[b])$$

is the unique weighting, the unique coweighting, and the Euler characteristic of $[\mathcal{C}]$, where $\big([\mu]([a],[b])\big)_{[a],[b]\in[\mathcal{C}]}$ denotes the inverse matrix of $[\zeta(\mathcal{C})]$.

Clearly, if $\mathcal{C}$ has a weighting, $k^{\bullet}_{\mathcal{C}}$, and a coweighting, $k^{\mathcal{C}}_{\bullet}$, then $[\mathcal{C}]$ has weighting $k^{[b]}_{[\mathcal{C}]} = \sum_{c\in[b]} k^{c}_{\mathcal{C}}$, and coweighting $k^{[\mathcal{C}]}_{[a]} = \sum_{c\in[a]} k^{\mathcal{C}}_{c}$, and $\chi(\mathcal{C}) = \chi([\mathcal{C}])$. The next proposition is about the converse.

Proposition 2.14. *Suppose that $[\mathcal{C}]$ has a weighting, $k^{\bullet}_{[\mathcal{C}]}$, and a coweighting, $k^{[\mathcal{C}]}_{\bullet}$. Then the functions*

$$k^{b}_{\mathcal{C}} = |[b]|^{-1}k^{[b]}_{[\mathcal{C}]}, \qquad k^{\mathcal{C}}_{a} = |[a]|^{-1}k^{[\mathcal{C}]}_{[a]}$$

are a weighting and a coweighting for $\mathcal{C}$, and $\chi([\mathcal{C}]) = \chi(\mathcal{C})$.

Proof. We shall use Eq. (2.13). Let a be any object of $\mathcal{C}$. Since the function $k^{\bullet}_{\mathcal{C}}$ is constant on isomorphism classes of objects we find that

$$\sum_b \zeta(a,b)k^b_{\mathcal{C}} = \sum_b [\zeta]([a],[b])k^b_{\mathcal{C}} = \sum_{[b]} [\zeta]([a],[b])k^b_{\mathcal{C}}|[b]| = \sum_{[b]} [\zeta]([a],[b])k^{[b]}_{[\mathcal{C}]} = 1.$$

A symmetric argument shows that $k^{\mathcal{C}}_{\bullet}$ is a coweighting for $\mathcal{C}$. Thus $\mathcal{C}$ has Euler characteristic $\chi(\mathcal{C}) = \sum_b k^b_{\mathcal{C}} = \sum_{[b]} |[b]|k^b_{\mathcal{C}} = \sum_{[b]} k^{[b]}_{[\mathcal{C}]} = \chi([\mathcal{C}])$. □

Corollary 2.15. *Let $\mathcal{C}$ be any of the six p-subgroup categories $\mathcal{S}_G$, $\mathcal{T}_G$, $\mathcal{L}_G$, $\mathcal{F}_G$, $\mathcal{O}_G$, $\widetilde{\mathcal{F}}_G$. Then*

- *$[\mathcal{C}]$ has Möbius inversion*
- *$\mathcal{C}$ admits a weighting, a coweighting, and $\mathcal{C}$ has Euler characteristic.*

The same conclusions hold for any full subcategory of $\mathcal{C}$.

Proof. The corollary obviously holds for the poset $\mathcal{S}_G$ (where $[\mathcal{S}_G] = \mathcal{S}_G$).

Let now $\mathcal{C}$ be any of the other five subgroup categories $\mathcal{T}_G$, $\mathcal{L}_G$, $\mathcal{F}_G$, $\mathcal{O}_G$, $\widetilde{\mathcal{F}}_G$. The set of isomorphism classes of object of $\mathcal{C}$ is the set of p-subgroups of G [21, p. 426] (Lemma 2.16). Suppose that G contains two conjugate but distinct subgroups. Then the ζ-matrix for $\mathcal{C}$ has two identical rows. Thus $\mathcal{C}$ does not have Möbius inversion in general.

However, equip the set of p-subgroups of G with a linear order extending the partial order given $[H] \leq [K] \iff \zeta([H],[K]) \neq \emptyset$. Then $[H] > [K] \Longrightarrow \zeta([H],[K]) = \emptyset$. The ζ-matrix $[\zeta(\mathcal{C})]$ for $[\mathcal{C}]$ with this linear order is upper triangular with diagonal entries $|\mathcal{C}(H)| \geq 1$. Thus $[\zeta]$ has Möbius inversion. By Proposition 2.14, the category $\mathcal{C}$ has a weighting and a coweighting.

This argument applies equally well to any full subcategory of $\mathcal{C}$. □

Lemma 2.16. *Let H and K be two p-subgroups of G. Then H and K are isomorphic in $\widetilde{\mathcal{F}}_G$ if and only if they are isomorphic in $\mathcal{F}_G$.*

Proof. Suppose that H and K are isomorphic in $\widetilde{\mathcal{F}}_G$. Then there exist $x \in N_G(H,K)$, $y \in N_G(K,H)$ so that conjugation by xy is an inner automorphism of H and conjugation by yx is an inner automorphism of K. By replacing y by another element of yH, if necessary, we obtain that $yx \in C_G(H)$. Then $yx = xy^x \in C_G(H)^x = C_G(K)$. This means that xy represents the identity of $\mathcal{F}_G(H)$ and yx represents the identity of $\mathcal{F}_G(K)$. □

2.5. The Euler characteristic of a homotopy orbit category

Let $\mathcal{S}$ be a finite category with a G-action. (This means that there is a functor from G to the category of finite categories taking the single object of G to $\mathcal{S}$.) The *homotopy orbit category*, $\mathcal{S}_{hG}$, is the Grothendieck construction on the G-action on $\mathcal{S}$: The category with the same set of objects as $\mathcal{S}$ and with morphism sets

$$\mathcal{S}_{hG}(a,b) = \coprod_{g\in G} S(ag,b), \qquad a,b\in \mathrm{Ob}(\mathcal{S}), \tag{2.17}$$

of size $|\mathcal{S}_{hG}(a,b)| = \sum_{g\in G}|S(ag,b)| = \sum_{g\in G}|S(a,bg^{-1})|$. The composition in $\mathcal{S}_{hG}$ is defined as $(g,\varphi)\cdot(h,\psi) = (gh,\varphi h\cdot\psi)$ for $g,h\in G$ and $\varphi\in\mathcal{S}(ag,b)$, $\psi\in\mathcal{S}(bh,c)$ for objects a,b,c of $\mathcal{S}$.

Theorem 2.18. *Let $\mathcal{F}$ be a finite category with the same objects as $\mathcal{S}$. Suppose that $d_\bullet, t^\bullet\colon \mathrm{Ob}(\mathcal{S})\to \mathbf{Z}_+$ are positive integral functions such that $d_a|\mathcal{F}(a,b)|t^b = |\mathcal{S}_{hG}(a,b)|$ for all $a,b\in\mathrm{Ob}(S)$.*

(1) *If $m^\bullet\colon \mathrm{Ob}(\mathcal{S})\to\mathbf{Q}$ is a rational function so that $\sum_b|\mathcal{S}(a,b)|m^b = d_a$ for all $a\in\mathrm{Ob}(\mathcal{S})$ and $d_\bullet$ is G-invariant, then $|G|^{-1}t^\bullet m^\bullet$ is a weighting for $\mathcal{F}$.*
(2) *If $m_\bullet\colon \mathrm{Ob}(\mathcal{S})\to\mathbf{Q}$ is a rational function so that $\sum_a m_a|\mathcal{S}(a,b)| = t^b$ for all $b\in\mathrm{Ob}(\mathcal{S})$ and $t^\bullet$ is G-invariant, then $|G|^{-1}m_\bullet d_\bullet$ is a coweighting for $\mathcal{F}$.*
(3) *Suppose that $\mathcal{S}$ has Möbius inversion and μ is the Möbius function. If $d_\bullet$ is G-invariant then $k^a = |G|^{-1}\sum_b t^a\mu(a,b)d_b$ is a weighting for $\mathcal{F}$, and if $t^\bullet$ is G-invariant then $k_b = |G|^{-1}\sum_a t^a\mu(a,b)d_b$ a coweighting for $\mathcal{F}$.*

Proof. (1) The proofs of (1) and (2) are dual to each other.
(2) For every $b\in\mathrm{Ob}(\mathcal{S})$,

$$\sum_a m_a d_a|\mathcal{F}(a,b)| = \sum_a m_a|\mathcal{S}_{hG}(a,b)|(t^b)^{-1} = \sum_{g\in G}\sum_a m_a|\mathcal{S}(a,bg^{-1})|(t^b)^{-1} = \sum_{g\in G} t^{bg^{-1}}(t^b)^{-1} = |G|$$

as $t^\bullet$ is G-invariant so that $t^{bg^{-1}} = t^b$ for all $g\in G$.
(3) If $m^a = \sum_b\mu(a,b)d_b$ then $\sum_b|\mathcal{S}(a,b)|m^b = d_a$ by the Möbius inversion formula (2.4). By (1), k^a is a weighting for $\mathcal{F}$ if $d_\bullet$ is G-invariant. Dually, If $m_b = \sum_a t^a\mu(a,b)$ then $\sum_a m_a|\mathcal{S}(a,b)| = t^b$ by the Möbius inversion formula (2.4). By (2), k_b is a coweighting for $\mathcal{F}$ if $t^\bullet$ is G-invariant. □

3. The Möbius function of a finite group

In this section we introduce the Möbius function of a finite group and show how it can be used to express weightings and coweightings of p-subgroup categories.

Definition 3.1. Let G be any finite group.

- $\overline{\mathcal{S}}_G$ is the poset of *all* subgroups of G.
- $\mu\colon \mathrm{Ob}(\overline{\mathcal{S}}_G)\times\mathrm{Ob}(\overline{\mathcal{S}}_G)\to\mathbf{Q}$ is the Möbius function for $\overline{\mathcal{S}}_G$.

We noted in Section 2.3 that the value of μ on a pair (H,K) of subgroups of G, only depends on H and K and not on G. This is the reason for writing just μ, rather than μ_G, for the Möbius function of $\overline{\mathcal{S}}_G$. In particular, for any subgroup $K\le G$, $\mu(1,K)$ only depends on K, not on the whole group G, and it is customary to write $\mu(K)$ for $\mu(1,K)$ [19].
The Möbius function μ for the full subgroup poset $\overline{\mathcal{S}}_G$ restricts to Möbius functions for the right ideal $\mathcal{S}_G$ of $\overline{\mathcal{S}}_G$ and for the left ideal $\mathcal{S}_G^*$ of $\mathcal{S}_G$ (Remark 2.6). The next lemma gives μ on pairs of p-subgroups.

Lemma 3.2 *([18], [19, Corollary 3.5]). Let H and K be p-subgroups of G. Then*

$$\mu(H,K) = \begin{cases} (-1)^n p^{\binom{n}{2}} & \Phi(K)\le H\le K, \quad p^n = |K:H| \\ 0 & \text{otherwise.}\end{cases}$$

In particular, $\mu(K)=\mu(1,K)=0$ unless K is elementary abelian where

$$\mu(K) = (-1)^n p^{\binom{n}{2}}, \quad p^n = |K|.$$

Proof. If $\mu(H,K)\ne 0$ then $H\triangleleft K$ with $H\backslash K$ elementary abelian and $\mu(H,K)=\mu(H\backslash K)$ [21, Proposition 2.4], [22, Lemme 4.1]. Burnside's basis theorem [30, 5.3.2], [15, Lemma 3.15], $\Phi(K) = [K,K]K^p$, shows that $H\triangleleft K$ with $H\backslash K$ elementary abelian if and only if $\Phi(K)\le H$. □

Theorem 3.3. *Let $\mathcal{C}$ be one of the five categories $\mathcal{S}_G$, $\mathcal{T}_G$, $\mathcal{L}_G$, $\mathcal{F}_G$, or $\mathcal{O}_G$. Weightings $k^\bullet_{\mathcal{C}}$ for $\mathcal{C}$, coweightings $k_\bullet^{\mathcal{C}^*}$ for $\mathcal{C}^*$, and Euler characteristics for $\mathcal{C}^*$ are as in Table 1.*

Table 1
Weightings, coweightings, and Euler characteristics for categories of nonidentity p-subgroups.

$\mathcal{C}$	$k_{\mathcal{C}}^{H}$	$k_K^{\mathcal{C}^*}$	$\chi(\mathcal{C}^*)$
$\mathcal{S}_G$	$\sum_K \mu(H,K)$	$-\mu(K)$	$\sum_{K\gneq 1} -\mu(K)$
$\mathcal{T}_G$	$\|G\|^{-1}\sum_K \mu(H,K)$	$-\|G\|^{-1}\mu(K)$	$\sum_{[K]\neq[1]} \frac{-\mu(K)}{\|\mathcal{T}_G^*(K)\|}$
$\mathcal{L}_G$	$\|G\|^{-1}\sum_K \mu(H,K)\|O^pC_G(K)\|$	$-\|G\|^{-1}\mu(K)\|O^pC_G(K)\|$	$\sum_{[K]\neq[1]} \frac{-\mu(K)}{\|\mathcal{L}_G^*(K)\|}$
$\mathcal{F}_G$	$\|G\|^{-1}\sum_K \mu(H,K)\|C_G(K)\|$	$-\|G\|^{-1}\mu(K)\|C_G(K)\|$	$\sum_{[K]\neq[1]} \frac{-\mu(K)}{\|\mathcal{F}_G^*(K)\|}$
$\mathcal{O}_G$	$\|G\|^{-1}\|H\|\sum_K \mu(H,K)$	$\|G\|^{-1}\sum_H \|H\|\mu(H,K)$	$\|G\|^{-1}\sum_{H,K\gneq 1}\|H\|\mu(H,K)$

Proof. $\mathcal{S}_G$ is a poset with Möbius function μ as in Definition 3.1. By Example 2.10,

$$k_{\mathcal{S}}^{H} = \sum_{1\leq H\leq K} \mu(H,K), \qquad k_K^{\mathcal{S}^*} = \sum_{1\lneq H\leq K} \mu(H,K) = -\mu(K)$$

are, respectively, a weighting for $\mathcal{S}_G$ and a coweighting for $\mathcal{S}_G^*$.

Next note that the homotopy orbit category for the conjugation action of G on the poset $\mathcal{S}_G$ is the transporter category $\mathcal{T}_G = (\mathcal{S}_G)_{hG}$. Since also

$$|\mathcal{T}_G(H,K)| = |O^pC_G(H)||\mathcal{L}_G(H,K)| = |C_G(H)||\mathcal{F}_G(H,K)| = |\mathcal{O}_G(H,K)||K| \tag{3.4}$$

we are in a position to apply Theorem 2.18. For example, in case of $\mathcal{F}_G^*$, Theorem 2.18.(3) shows that

$$k_{\mathcal{F}}^{H} = |G|^{-1}\sum_{H\leq K} \mu(H,K)|C_G(K)|, \qquad k_K^{\mathcal{F}^*} = |G|^{-1}\sum_{1\lneq H\leq K} \mu(H,K)|C_G(K)| = -|G|^{-1}\mu(K)|C_G(K)|$$

are, respectively, a weighting for $\mathcal{F}_G$ and a coweighting for $\mathcal{F}_G^*$. Note that the coweighting is constant over the conjugacy class $[K]$ of K of size $|[K]| = |G\colon N_G(K)|$. The function

$$k_{[K]}^{[\mathcal{F}^*]} = |G\colon N_G(H)|k_K^{\mathcal{F}^*} = \frac{-\mu(K)}{|\mathcal{F}_G^*(K)|} \tag{3.5}$$

is thus a coweighting for the set $[\mathcal{F}_G^*]$ of isomorphism classes of objects (Section 2.4). The Euler characteristic of $\mathcal{F}_G^*$,

$$\chi(\mathcal{F}_G^*) = \sum_{[K]} \frac{-\mu(K)}{|\mathcal{F}_G^*(K)|}$$

is the sum of the values for the coweighting for $[\mathcal{F}_G^*]$. □

The quotient category $\widetilde{\mathcal{F}}_G^*$ is missing from Table 1 because Theorem 2.18 does not directly apply. We shall later see that $\widetilde{\mathcal{F}}_G^*$ and $\mathcal{F}_G^*$ have identical coweightings and Euler characteristics (Theorem 7.7).

Lemma 2.9 implies that $\chi(\mathcal{C}^*) = \chi(\mathcal{C}^{\mathrm{ea}})$ for $\mathcal{C} = \mathcal{S}, \mathcal{T}, \mathcal{L}, \mathcal{F}$ because the coweightings for these categories are concentrated on the elementary abelian p-subgroups of G (Lemma 3.2). Quillen shows in [29, Proposition 2.1] the much stronger result that the posets $\mathcal{S}_G^*$ and $\mathcal{S}_G^{\mathrm{ea}}$ (the 'Quillen poset') are homotopy equivalent.

Example 3.6. Let D_{2pn} be the dihedral group of order $2pn$, $n \geq 1$, A_p the alternating group of index $p > 2$, and $\mathrm{SL}_n(\mathbf{F}_q)$ the special linear group where q is a power of p and $n \geq 2$. Then

$$\chi(\mathcal{S}_{D_{2pn}}^*) = 1, \qquad \chi(\mathcal{S}_{A_p}^*) = (p-2)!, \qquad \widetilde{\chi}(\mathcal{S}_{\mathrm{SL}_n(\mathbf{F}_q)}^*) = (-1)^n q^{\binom{n}{2}}.$$

See [29, Example 2.7] for the Euler characteristic of $\mathcal{S}_{A_p}^*$. Let $V_n(q)$ be an n-dimensional vector space over $\mathbf{F}_q$ and $L_n(q)$ the poset of $\mathbf{F}_q$-subspaces of $V_n(q)$. The poset $\mathcal{S}_{\mathrm{SL}_n(\mathbf{F}_q)}^{\mathrm{ea}}$ of nonidentity elementary abelian p-subgroups is homotopy equivalent to the open interval $(0, V_n(q))$ [29, Theorem 3.1], the building for $\mathrm{SL}_n(\mathbf{F}_q)$ [1, Example 6.5]. Therefore

$$\widetilde{\chi}(\mathcal{S}_{\mathrm{SL}_n(\mathbf{F}_q)}^{\mathrm{ea}}) = \widetilde{\chi}((0, V_n(q))) = \mu_{L_n(q)}(0, V_n(q)) = (-1)^n q^{\binom{n}{2}}$$

by the computation of the Möbius function $\mu_{L_n(q)}$ in $L_n(q)$ [34, Example 3.10.2] [22, Proposition 3.6]. In this example we may replace $\mathrm{SL}_n(\mathbf{F}_q)$ by any of the groups $\mathrm{GL}_n(\mathbf{F}_q)$, $\mathrm{PSL}_n(\mathbf{F}_q)$, or $\mathrm{PGL}_n(\mathbf{F}_q)$ since they all have identical p-subgroup posets. The computer-generated Table 3 displays Euler characteristics of poset categories at $p = 2$ of small alternating groups.

Example 3.7. If P is a nonidentity p-group we immediately have that

$$\chi(\mathcal{S}_P^*) = 1, \qquad \chi(\mathcal{T}_P^*) = |P|^{-1}, \qquad \chi(\mathcal{L}_P^*) = |P|^{-1}, \qquad \chi(\mathcal{F}_P^*) = 1, \qquad \chi(\mathcal{O}_P^*) = 1, \qquad \chi(\widetilde{\mathcal{F}}_P^*) = 1 \tag{3.8}$$

because P is terminal in $\mathcal{S}_P^*$ and $\mathcal{O}_P^*$, $\mathcal{T}_P^* = \mathcal{L}_P^*$, and $\chi(\mathcal{T}_P^*) = |P|^{-1}\chi(\mathcal{S}_G^*) = |P|^{-1}$ by Theorem 2.18, Proposition 5.1 applies to $\mathcal{F}_P^*$, and Theorem 7.7 to $\widetilde{\mathcal{F}}_P^*$. More generally, if G has a normal p-complement, then $\chi(\mathcal{F}_G^*) = 1$ because $\mathcal{F}_G^* = \mathcal{F}_P^*$ according to the Frobenius normal p-complement theorem [16, Proposition 16.10][30, 10.3.2].

Example 3.9. The Euler characteristics for the subgroup categories generated by *all* p-subgroups of G (including the identity subgroup) are

$$\chi(\mathcal{S}_G) = 1, \qquad \chi(\mathcal{T}_G) = |G|^{-1}, \qquad \chi(\mathcal{L}_G) = |G\colon O^pG|^{-1}, \qquad \chi(\mathcal{F}_G) = 1,$$
$$\chi(\mathcal{O}_G) = \frac{1}{p|G|} + \frac{p-1}{p}\sum_{[C]}\frac{1}{|\mathcal{O}_G(C)|}, \qquad \chi(\widetilde{\mathcal{F}}_G) = 1.$$

Observe that $\mathcal{S}_G$, $\mathcal{F}_G$, and $\widetilde{\mathcal{F}}_G$ have initial objects and that $\mathcal{T}_G$ deformation retracts onto $\mathcal{T}_G(1) = G$ and $\mathcal{L}_G$ deformation retracts onto $\mathcal{L}_G(1) = O^pG\backslash G$. (See the proof of Proposition 5.1 for the definition of deformation retracts.) In the formula for $\chi(\mathcal{O}_G)$, $[C]$ runs through the set of conjugacy classes of cyclic p-subgroups C of G, see Corollary 4.2.(3).

3.1. Alternative weightings and coweightings

In this subsection we shall determine the Möbius functions for $[\mathcal{T}_G]$, $[\mathcal{L}_G]$, $[\mathcal{F}_G]$, and $[\mathcal{O}_G]$ (Corollary 2.15).

Let $H, K \leq G$. The rational number

$$[\mu](H,K) = \frac{1}{|N_G(H)|}\sum_{B\in[K]}\mu(H,B) \tag{3.10}$$

only depends on the conjugacy classes of H and K[1], and

$$\begin{aligned}[\mu](H,K) &= \frac{1}{|N_G(H)|}\sum_{B\in[K]}\mu(H,B) = \frac{1}{|N_G(H)||N_G(K)|}\sum_{g\in G}\mu(H,K^g)\\ &= \frac{1}{|N_G(H)||N_G(K)|}\sum_{g\in G}\mu(H^g,K) = \frac{1}{|N_G(K)|}\sum_{A\in[H]}\mu(A,K).\end{aligned}$$

In particular, $[\mu](K) = |N_G(K)|^{-1}\mu(K)$, where $[\mu](K)$ is short for $[\mu](1,K)$. As in Equation (2.11),

$$\sum_{[1]\neq[H]}[\mu]([H],[K]) = \frac{1}{|N_G(K)|}\sum_{1\neq H}\mu(H,K) = \frac{-\mu(K)}{|N_G(K)|} = -[\mu]([K])$$

for any subgroup K of G.

Proposition 3.11. *The above function $[\mu]([H],[K])$, derived from Equation (3.10), is the Möbius function for $[\mathcal{T}_G]$.*

Proof. We claim that $([\mu]([H],[K]))_{[H],[K]\in[\mathcal{T}_G]}$ is the inverse of the matrix $(|N_G(H,K)|)_{[H],[K]\in[\mathcal{T}_G]}$. If H and L are (p-)subgroups of G then

$$\begin{aligned}|N_G(H)|\sum_{[K]}[\mu]([H],[K])|N_G(K,L)| &= \sum_K \mu(H,K)|N_G(K,L)| = \sum_K\mu(H,K)\sum_{g\in G}\mathcal{S}_G(K^g,L)\\ &= \sum_{g\in G}\sum_K\mu(H,K)\mathcal{S}_G(K,L^{g^{-1}}) = \sum_{g\in G}\delta(H,L^{g^{-1}}) = \sum_{g\in G}\delta(H^g,L).\end{aligned}$$

This last sum is 0 if H and L are not conjugate, and it is $|N_G(H)|$ if they are conjugate. □

Theorem 3.12. *Möbius functions and weightings for $[\mathcal{C}]$, and coweightings and Euler characteristics for $[\mathcal{C}^*]$ are as in Table 2 when $\mathcal{C}$ is one of the p-subgroup categories $\mathcal{S}_G$, $\mathcal{T}_G$, $\mathcal{L}_G$, $\mathcal{F}_G$, $\mathcal{O}_G$.*

Proof. The relations (3.4) between sizes of homomorphism sets allow us to determine the Möbius functions for $\mathcal{C} = \mathcal{L}_G, \mathcal{F}_G, \mathcal{O}_G$. The row sums for these $[\mu]$-matrices are the weightings shown in the third column of Table 2. The $[\mu]$-matrices restrict to Möbius functions on the left ideal $[\mathcal{C}^*]$ (Remark 2.6). The column sums of these restricted matrices are the coweightings in the fourth column. This explains all entries of Table 2 but the ones of the first row.

[1] The Möbius function $[\mu]$ for the homotopy quotient $[\mathcal{T}_G] = [(\mathcal{S}_G)_{hG}]$ is not the same as the Möbius function λ_G [19, Section 7] [27] for the quotient $\mathcal{S}_G/G$, the poset of p-subgroup classes ordered by subconjugation.

Table 2
Möbius functions, weightings, coweightings, and Euler characteristics for categories of p-subgroups.

$\mathcal{C}$	$[\mu(\mathcal{C})]([H],[K])$	$k^{[H]}_{[\mathcal{C}]}$	$k^{[\mathcal{C}*]}_{[K]}$	$\chi(\mathcal{C}^*)$
$\mathcal{S}_G$	$\mu(H,K)$	$\lvert N_G(H)\rvert k^{[H]}_{[\mathcal{T}_G]}$	$k^{[\mathcal{T}^*_G]}_{[K]}\lvert N_G(K)\rvert$	$\lvert G\rvert\chi(\mathcal{T}^*_G)$
$\mathcal{T}_G$	$[\mu]([H],[K])$	$\sum_{[K]}[\mu]([H],[K])$	$-[\mu]([K])$	$\sum_{[K]}-[\mu]([K])$
$\mathcal{L}_G$	$[\mu]([H],[K])\lvert O^pC_G(K)\rvert$	$\sum_{[K]}[\mu]([H],[K])\lvert O^pC_G(K)\rvert$	$k^{[\mathcal{T}^*_G]}_{[K]}\lvert O^pC_G(K)\rvert$	$\sum_{[K]}-[\mu]([K])\lvert O^pC_G(K)\rvert$
$\mathcal{F}_G$	$[\mu]([H],[K])\lvert C_G(K)\rvert$	$\sum_{[K]}[\mu]([H],[K])\lvert C_G(K)\rvert$	$k^{[\mathcal{T}^*_G]}_{[K]}\lvert C_G(K)\rvert$	$\sum_{[K]}-[\mu]([K])\lvert C_G(K)\rvert$
$\mathcal{O}_G$	$\lvert H\rvert[\mu]([H],[K])$	$\lvert H\rvert\sum_{[K]}[\mu]([H],[K])$	$\sum_{[H]}\lvert H\rvert[\mu]([H],[K])$	$\sum_{[H],[K]}\lvert H\rvert[\mu]([H],[K])$

We now focus on the first row. Note that $[\mathcal{S}_G]=\mathcal{S}_G$ as there are no nonidentity isomorphisms in $\mathcal{S}_G$. The weighting for $\mathcal{S}_G$ and the coweighting for $\mathcal{S}^*_G$ are

$$k^H_{\mathcal{S}}=\sum_{1\le K}\mu(H,K)=\sum_{[K]}\sum_{B\in[K]}\mu(H,B)=|N_G(H)|\sum_{[K]}[\mu]([H],[K])=|N_G(H)|k^{[H]}_{[\mathcal{T}]}$$
$$k^{\mathcal{S}^*}_K=\sum_{1<H}\mu(H,K)=\sum_{[1]\ne[H]}\sum_{A\in[H]}\mu(A,K)=|N_G(K)|\sum_{[1]\ne[H]}\mu([H],[K])=-|N_G(K)|[\mu]([K])=k^{[\mathcal{T}^*]}_{[K]}|N_G(K)|.$$

The Euler characteristic of $\mathcal{S}^*_G$ is

$$\chi(\mathcal{S}^*_G)=\sum_{1\ne K}k^{\mathcal{S}^*}_K=\sum_{[1]\ne[K]}|G:N_G(K)|k^{\mathcal{S}^*}_K=|G|\sum_{[1]\ne[K]}k^{[\mathcal{T}^*]}_{[K]}=|G|\chi(\mathcal{T}^*_G).$$

This explains the first row of Table 2. □

The Möbius functions of the second column of Table 2 restrict to Möbius functions on the (left or right) ideals $\mathcal{C}^*$, $\mathcal{C}^{\mathrm{ea}}$, and $\mathcal{C}^{\mathrm{sc}}$ of $\mathcal{C}$. The weightings of the third column of Table 2 restrict to weightings on the left ideals $\mathcal{C}^*$ and $\mathcal{C}^{\mathrm{sc}}$ (Remark 2.6).

Remark 3.13. Let $H,K\le G$. Define the μ-transporter from H to K to be the set

$$N^{\mu}_G(H,K)=\{g\in G\mid \Phi(K)\le H^g\le K\},\quad H,K\in\mathrm{Ob}(\mathcal{S}^*_G)$$

of group elements g that conjugate H into K such that $\mu(H^g,K)\ne 0$.

The map $g\to K^{g^{-1}}$ is a bijection between $N^{\mu}_G(H,K)/N_G(K)$ and the set $\{L\in[K]\mid H\le L,\ \mu(H,L)\ne 0\}$ of subgroups L of G conjugate to K and containing H with $\mu(H,L)\ne 0$. Therefore

$$[\mu]([H],[K])=(-1)^np^{\binom{n}{2}}\frac{|N^{\mu}_G(H,K)|}{|N_G(H)||N_G(K)|},\quad H,K\in\mathrm{Ob}(\mathcal{S}^*_G),\ |K|=p^n|H|$$

can be computed from these transporter sets.

Next we note that the values of the weightings for the p-subgroup categories $\mathcal{S}_G$, $\mathcal{T}_G$, $\mathcal{O}_G$, and $\mathcal{F}_G$ can be computed locally.

Proposition 3.14. *The functions*

$$k^H_{\mathcal{S}}=-\widetilde{\chi}(\mathcal{S}^*_{\mathcal{O}_G(H)}),\qquad k^H_{\mathcal{T}}=\frac{-\widetilde{\chi}(\mathcal{S}^*_{\mathcal{O}_G(H)})}{|G|},\qquad k^H_{\mathcal{O}}=\frac{-\widetilde{\chi}(\mathcal{S}^*_{\mathcal{O}_G(H)})}{|G\colon H|}$$

are weightings for $\mathcal{S}_G$, $\mathcal{T}_G$, *and* $\mathcal{O}_G$, *respectively. The function*

$$k^H_{\mathcal{F}}=\sum_{x\in C_{N_G(H)}(H)}\frac{-\widetilde{\chi}(\mathcal{S}^*_{C_{N_G(H)}(x)/H})}{|G|}$$

is a weighting for $\mathcal{F}_G$.

Proof. Let H be a p-subgroup of G. Consider the projection $N_G(H)\to N_G(H)/H=\mathcal{O}_G(H)$ of the p-local subgroup $N_G(H)$ onto its quotient $N_G(H)/H$. Following the bar convention [14, pp. 18, 139], we write $\overline{N_G(H)}$ for $N_G(H)/H$ and $\overline{K}$ for the image in $\overline{N_G(H)}$ of any subgroup K of $N_G(H)$.

We see from Table 1 that the weighting for the poset $\mathcal{S}_G$ is

$$k^H_{\mathcal{S}}=\sum_K\mu(H,K)=\sum_{K\in[H,N_G(H)]}\mu(H,K)=\sum_{\overline{K}\le\mathcal{O}_G(H)}\mu(\overline{K})\overset{(2.12)}{=}-\widetilde{\chi}(\mathcal{S}^*_{\mathcal{O}_G(H)})$$

as $\mu(H,K)=0$ unless H is normalized by K (Lemma 3.2). (Indeed, the subposets $(H,G]$ and $(H,N_G(H)]$ of $\mathcal{S}_G$ are homotopy equivalent [29, Proposition 6.1].) The formulas for the weightings for $\mathcal{T}_G$ and $\mathcal{O}_G$ now follow from Table 1.

In the formula from Table 1 for the weighting $k^H_{\mathcal{F}}$ for $\mathcal{F}_G$ we may replace $C_G(K)$ by $C_{N_G(H)}(K)$ because $\mu(H,K)=0$ unless $H\le K\le N_G(H)$ (Lemma 3.2) and then $C_G(K)\le C_G(H)\le N_G(H)$. Thus we see from Table 1 that

$$|G|k^H_{\mathcal{F}}=\sum_{H\le K\le N_G(H)}\mu(H,K)|C_G(K)|=\sum_{\overline{K}\le\overline{N_G(H)}}\mu(\overline{K})|C_{N_G(H)}(K)|.$$

The order of the group $C_{N_G(H)}(K)$ is

$$\begin{aligned}|C_{N_G(H)}(K)| &= |\{x\in N_G(H)\mid x\in C_G(K)\}|=|\{x\in C_{N_G(H)}(H)\mid x\in C_G(K)\}|\\ &=|\{x\in C_{N_G(H)}(H)\mid K\le C_{N_G(H)}(x)\}|=\sum_{x\in C_{N_G(H)}(H)}|\overline{\mathcal{S}}_{\overline{N_G(H)}}\big(\overline{K},\overline{C_{N_G(H)}(x)}\big)|\end{aligned}$$

where we use the poset $\overline{\mathcal{S}}_{\overline{N_G(H)}}$ of all subgroups of $\overline{N_G(H)}$ (Definition 3.1). We conclude that

$$\begin{aligned}|G|k^H_{\mathcal{F}} &= \sum_{x\in C_{N_G(H)}(H)}\sum_{\overline{K}\le\overline{N_G(H)}}\mu(\overline{K})|\overline{\mathcal{S}}_{\overline{N_G(H)}}\big(\overline{K},\overline{C_{N_G(H)}(x)}\big)|=\sum_{x\in C_{N_G(H)}(H)}\sum_{\overline{K}\le\overline{C_{N_G(H)}(x)}}\mu(\overline{K})\\ &=\sum_{x\in C_{N_G(H)}(H)}-\widetilde{\chi}(\mathcal{S}^*_{\overline{C_{N_G(H)}(x)}})\end{aligned}$$

using Eq. (2.12) to get the last identity. □

Using the expressions from Proposition 3.14, we find the following weightings $k^{[H]}_{[\mathcal{C}]}=|G:N_G(H)|k^H_{\mathcal{C}}$ for $[\mathcal{C}]$ any of $[\mathcal{T}_G]$, $[\mathcal{O}_G]$, and $[\mathcal{F}_G]$:

$$k^{[H]}_{[\mathcal{T}]}=\frac{-\widetilde{\chi}(\mathcal{S}^*_{\mathcal{O}_G(H)})}{|\mathcal{T}_G(H)|},\qquad k^{[H]}_{[\mathcal{O}]}=\frac{-\widetilde{\chi}(\mathcal{S}^*_{\mathcal{O}_G(H)})}{|\mathcal{O}_G(H)|},\qquad k^{[H]}_{[\mathcal{F}]}=\sum_{x\in C_{N_G(H)}(H)}\frac{-\widetilde{\chi}(\mathcal{S}^*_{C_{N_G(H)}(x)/H})}{|N_G(H)|}. \tag{3.15}$$

Thus the Euler characteristics of $\mathcal{S}^*_G$, $\mathcal{T}^*_G$, and $\mathcal{O}^*_G$ are

$$\chi(\mathcal{S}^*_G)=\sum_H-\widetilde{\chi}(\mathcal{S}^*_{\mathcal{O}_G(H)}),\qquad \chi(\mathcal{T}^*_G)=\sum_{[H]}\frac{-\widetilde{\chi}(\mathcal{S}^*_{\mathcal{O}_G(H)})}{|\mathcal{T}_G(H)|},\qquad \chi(\mathcal{O}^*_G)=\sum_{[H]}\frac{-\widetilde{\chi}(\mathcal{S}^*_{\mathcal{O}_G(H)})}{|\mathcal{O}_G(H)|}. \tag{3.16}$$

The Euler characteristic of $\mathcal{F}^*_G$

$$\chi(\mathcal{F}^*_G)=\sum_{[H]}\sum_{x\in C_{N_G(H)}(H)}\frac{-\widetilde{\chi}(\mathcal{S}^*_{C_{N_G(H)}(x)/H})}{|N_G(H)|}=\chi(\mathcal{F}_G)-k^1_{\mathcal{F}}=1+\frac{1}{|G|}\sum_{x\in G}\widetilde{\chi}(\mathcal{S}^*_{C_G(x)})=\frac{1}{|G|}\sum_{x\in G}\chi(\mathcal{S}^*_{C_G(x)}) \tag{3.17}$$

is the average of the Euler characteristics for nonidentity p-subgroup posets of the element-centralizers in G.

Definition 3.18. The p-subgroup H of G is

- p-radical if $O_p\mathcal{O}_G(H)=1$ [4, Proposition 4]
- $\mathcal{F}_G$-radical if $O_p\widetilde{\mathcal{F}}_G(H)=1$ [7, Definition A.9]

Corollary 3.19. *The weightings for $\mathcal{S}_G$, $\mathcal{T}_G$, and $\mathcal{O}_G$ are supported on the p-radical subgroups of G.*

Proof. If H is not p-radical then the weightings for $\mathcal{S}_G$, $\mathcal{T}_G$, and $\mathcal{O}_G$ from Proposition 3.14 vanish on H because $\widetilde{\chi}(\mathcal{S}^*_{\mathcal{O}_G(H)})=0$ by Quillen [29, Proposition 2.4]. □

For $\mathcal{C} = \mathcal{S}_G, \mathcal{T}_G, \mathcal{O}_G$, let $\mathcal{C}^{\mathrm{ra}}$ denote the full subcategory of $\mathcal{C}^*$ generated by all nonidentity p-radical p-subgroups. The category $\mathcal{C}^{\mathrm{ra}}$ has Euler characteristic (Corollary 2.15), and

$$\chi(\mathcal{C}^{\mathrm{ra}}) = \chi(\mathcal{C}^*)$$

by Corollary 3.19 and Lemma 2.9.(1). Bouc [5, Corollaire] shows the stronger result that $\mathcal{S}_G^{\mathrm{ra}}$ (the 'Bouc poset') and $\mathcal{S}_G^*$ are homotopy equivalent posets. Thévenaz and Webb [35, Theorem 2.3] describe $\mathcal{S}_G^{\mathrm{ra}}$ when G is simple group of Lie type in defining characteristic p.

Remark 3.20. We suspect that the support for the weightings for $\mathcal{S}_G$, $\mathcal{T}_G$, and $\mathcal{O}_G$ is the set of p-radical p-subgroups of G. This would be the case if the strong Quillen conjecture (1.4) turned out to be true.

The weighting for $\mathcal{F}_G$ is supported on the set of p-subgroups $H \leq G$ for which $H = O_pC_{N_G(H)}(x)$ for some x in $C_G(H)$. There are examples (the symmetric group Σ_8 at $p = 2$) where the support of the weighting $k^{\bullet}_{\mathcal{F}_G^*}$ is *strictly* contained in this set of p-subgroups. The Frobenius category $\mathcal{F}_G^*$ is not able to detect p-radical p-subgroups of G: the dihedral groups D_8 and $D_{24} = C_3 \rtimes D_8$ of order 8 and 24 have equivalent Frobenius categories at $p = 2$ but distinct sets of 2-radical subgroups.

There is no simple general relation between the two concepts of radical subgroups from Definition 3.18 [7, Appendix A]. If P is an abelian nonidentity p-group, then all subgroups of P are $\mathcal{F}_p$-radical but only P itself is p-radical. However, if H is a p-selfcentralizing p-subgroup of G (Definition 8.1) then $O^pC_G(H)$ is a p'-group (Lemma 8.2.(1)) and the short exact sequence

$$1 \to O^pC_G(H) \to \mathcal{O}_G^{\mathrm{sc}}(H) \to \widetilde{\mathcal{F}}_G^c(H) \to 1$$

can be used to verify the implication

$$H \text{ is } p\text{-selfcentralizing and } \mathcal{F}_G\text{-radical} \Longrightarrow H \text{ is } p\text{-selfcentralizing and } p\text{-radical}.$$

The converse implication does not hold in general: let $p = 2$. The normal cyclic subgroup $H = O_pG$ of order 4 in the dihedral group $G = D_{24}$ of order 24 is a p-selfcentralizing subgroup that is p-radical ($\mathcal{O}_G^{\mathrm{sc}}(H) = \Sigma_3$) but not $\mathcal{F}_G$-radical ($\widetilde{\mathcal{F}}_G^c(H) = C_2$).

4. Orbit categories

We shall now derive a more concise expression than the ones given in Table 1 or Table 2 for the Euler characteristic of $\mathcal{O}_G$ and $\mathcal{O}_G^*$.

Theorem 4.1. *The Euler characteristics of the orbit categories $\mathcal{O}_G$ and $\mathcal{O}_G^*$ are*

$$\chi(\mathcal{O}_G) = \frac{p + (p-1)\sum_{C>1}|C|}{p|G|}, \qquad \chi(\mathcal{O}_G^*) = \frac{p\chi(\mathcal{S}_G^*) + (p-1)\sum_{C>1}|C|}{p|G|}$$

where C runs through the set of nonidentity cyclic p-subgroups of G.

Proof. Observe that $|\mathcal{O}_G(H,K)||K| = |\mathcal{T}_G(H,K)|$ for all p-subgroups $H, K \leq G$. Therefore

$$k_K^{\mathcal{O}} = \frac{1}{|G|}\sum_{H\geq 1}|H|\mu(H,K), \quad K \in \mathrm{Ob}(\mathcal{O}_G),$$

is a coweighting for $\mathcal{O}_G$ according to Theorem 2.18.(3). The value of this coweighting is

$$k_K^{\mathcal{O}} = \begin{cases} \frac{1}{|G|} & K = 1 \\ \frac{p-1}{p}\frac{|K|}{|G|} & K > 1, K \text{ cyclic} \\ 0 & K > 1, K \text{ noncyclic} \end{cases}$$

by Corollary 4.5 below. The sum of these values is the Euler characteristic of $\mathcal{O}_G$.

The coweighting for $\mathcal{O}_G^*$ is

$$k_K^{\mathcal{O}^*} = \frac{1}{|G|}\sum_{H>1}|H|\mu(H,K) = \frac{-\mu(K)}{|G|} + k_K^{\mathcal{O}}, \quad K \in \mathrm{Ob}(\mathcal{O}_G^*)$$

according to Table 1. The formula

$$\chi(\mathcal{O}_G) - \chi(\mathcal{O}_G^*) = \sum_{K>1}\frac{\mu(K)}{|G|} + k_1^{\mathcal{O}} = \frac{-\chi(\mathcal{S}_G^*)}{|G|} + \frac{1}{|G|} = \frac{-\widetilde{\chi}(\mathcal{S}_G^*)}{|G|}$$

relates the Euler characteristics of $\mathcal{O}_G^*$ and $\mathcal{O}_G$. □

Corollary 4.2. *Let G be any finite group and p any prime number.*

(1) $|G|_{p'}k_{[\mathcal{O}]}^{[H]}$ *is an integer for every p-subgroup $H \leq G$*

(2) *$|G|_{p'}\chi(\mathcal{O}_G)$ and $|G|_{p'}\chi(\mathcal{O}_G^*)$ are integers*
(3) *The Euler characteristic of $\mathcal{O}_G$ is*

$$\frac{1+(p-1)\sum_C |C|}{p|G|} = \chi(\mathcal{O}_G) = \sum_{[H]} \frac{-\widetilde{\chi}(\mathcal{S}^*_{\mathcal{O}_G}(H))}{|\mathcal{O}_G(H)|}$$

where C runs through the set of cyclic p-subgroups of G and $[H]$ through the set of conjugacy classes of p-radical p-subgroups of G.

Proof. (1) The weighting for $[\mathcal{O}_G]$ is given in Equation (3.15). We see that

$$|G|_{p'}k^{[H]}_{[\mathcal{O}]} = \frac{|G|_{p'}}{|\mathcal{O}_G(H)|_{p'}} \frac{-\widetilde{\chi}(\mathcal{S}^*_{\mathcal{O}_G(H)})}{|\mathcal{O}_G(H)|_p}$$

is an integer by Brown's theorem [8, Corollary 2] [29, Corollary 4.2] [36, Corollary 3.3] [19, Corollary 3.9].
(2) The rational number $|G|_{p'}\chi(\mathcal{O}_G)$ is an integer because it is the sum of the integers $|G|_{p'}k^{[H]}_{\mathcal{O}}$ where $[H]$ runs through the set of conjugacy classes of p-subgroups of G. The difference

$$|G|_{p'}(\chi(\mathcal{O}_G) - \chi(\mathcal{O}_G^*)) = |G|_{p'}\frac{-\widetilde{\chi}(\mathcal{S}^*_G)}{|G|} = \frac{-\widetilde{\chi}(\mathcal{S}^*_G)}{|G|_p}$$

is also an integer by Brown's theorem again.
(3) The expression to the left is the Euler characteristic of $\mathcal{O}_G$ computed from a coweighting for $\mathcal{O}_G$ and the expression to the right is the Euler characteristic of $\mathcal{O}_G$ computed from a coweighting of $[\mathcal{O}_G]$. □

Corollary 4.3. *Let G be any finite group and p any prime number. Then*

$$(1-p)\sum_{1\le C\le G}|C| \equiv 1 \bmod p|G|_p$$

where the sum ranges over the set of all cyclic p-subgroups C of G.

Proof. From Corollary 4.2.(3) we have that

$$1+(p-1)\sum_{1\le C\le G}|C| = p|G|\chi(\mathcal{O}_G) = p|G|_p|G|_{p'}\chi(\mathcal{O}_G)$$

and $|G|_{p'}\chi(\mathcal{O}_G)$ is an integer by Corollary 4.2.(2). □

The congruence relation of Corollary 4.3 reduces to the familiar relation

$$(1-p)(1+p+\cdots+p^n) \equiv 1 \bmod p^{n+1}$$

when G is cyclic of order p^n.

We shall now prove the combinatorial identities used in Theorem 4.1. The Gaussian p-binomial coefficient

$$\binom{[n]}{[d]} = \frac{\prod_{j=1}^{d}(p^n - p^{j-1})}{\prod_{j=1}^{d}(p^d - p^{j-1})} = \frac{\prod_{j=1}^{d}(p^{n+1-j}-1)}{\prod_{j=1}^{d}(p^j - 1)}$$

counts the number of d-dimensional subspaces of the n-dimensional $\mathbf{F}_p$-vector space $\mathbf{F}_p^n$ [34, 1.3.18].

Lemma 4.4. *For any $n \ge 1$,*

$$\sum_{d=0}^{n}(-1)^d\binom{[n]}{[d]}p^{\binom{d}{2}}p^{n-d} = \begin{cases} p-1 & n=1 \\ 0 & n>1. \end{cases}$$

Proof. Note first the formulas [34, p. 26]

$$\binom{[n]}{[d]} = \binom{[n-1]}{[d]} + p^{n-d}\binom{[n-1]}{[d-1]}, \qquad \binom{[n]}{[0]} = 1 = \binom{[n]}{[n]}, \qquad \binom{[2]}{[1]} = 1+p,$$

for the Gaussian p-binomial coefficients.

For $n=1$ and $n=2$, the sums we are evaluating are the polynomials

$$\binom{[1]}{[0]}p - \binom{[1]}{[1]} = p-1, \qquad \binom{[2]}{[0]}p^2 - \binom{[2]}{[1]}p + \binom{[2]}{[2]}p = p^2-(1+p)p+p = 0.$$

For $n > 2$ the sum has the value

$$\begin{aligned}\sum_{d=0}^{n}(-1)^d\binom{[n]}{[d]}p^{\binom{d}{2}}p^{n-d} &= p^n+\sum_{d=1}^{n-1}(-1)^d\binom{[n]}{[d]}p^{\binom{d}{2}}p^{n-d}+(-1)^np^{\binom{n}{2}}\\ &= p^n+\sum_{d=1}^{n-1}(-1)^d\left(\binom{[n-1]}{[d]}+p^{n-d}\binom{[n-1]}{[d-1]}\right)p^{\binom{d}{2}}p^{n-d}+(-1)^np^{\binom{n}{2}}\\ &= \left(p^n+\sum_{d=1}^{n-1}(-1)^d\binom{[n-1]}{[d]}p^{\binom{d}{2}}p^{n-d}\right)+\left(\sum_{d=1}^{n-1}(-1)^d\binom{[n-1]}{[d-1]}p^{\binom{d}{2}}p^{2(n-d)}+(-1)^np^{\binom{n}{2}}\right)\\ &= \sum_{d=0}^{n-1}(-1)^d\binom{[n-1]}{[d]}p^{\binom{d}{2}}p^{n-d}+\sum_{d=1}^{n}(-1)^d\binom{[n-1]}{[d-1]}p^{\binom{d}{2}}p^{2(n-d)}.\end{aligned}$$

The first term is

$$\sum_{d=0}^{n-1}(-1)^d\binom{[n-1]}{[d]}p^{\binom{d}{2}}p^{n-d}=p\sum_{d=0}^{n-1}(-1)^d\binom{[n-1]}{[d]}p^{\binom{d}{2}}p^{n-1-d}$$

and the second term is

$$\begin{aligned}\sum_{d=1}^{n}(-1)^d\binom{[n-1]}{[d-1]}p^{\binom{d}{2}}p^{2(n-d)} &= -\sum_{d=0}^{n-1}(-1)^d\binom{[n-1]}{[d]}p^{\binom{d+1}{2}}p^{2(n-d-1)}=-\sum_{d=0}^{n-1}(-1)^d\binom{[n-1]}{[d]}p^{\binom{d}{2}}p^dp^{2(n-d-1)}\\ &= -\sum_{d=0}^{n-1}(-1)^d\binom{[n-1]}{[d]}p^{\binom{d}{2}}p^{2n-d-2}=-p^{n-1}\sum_{d=0}^{n-1}(-1)^d\binom{[n-1]}{[d]}p^{\binom{d}{2}}p^{n-1-d}.\end{aligned}$$

We have now proved the recursive relation

$$\sum_{d=0}^{n}(-1)^d\binom{[n]}{[d]}p^{\binom{d}{2}}p^{n-d}=p(1-p^{n-2})\sum_{d=0}^{n-1}(-1)^d\binom{[n-1]}{[d]}p^{\binom{d}{2}}p^{n-1-d}$$

for $n > 2$. Since the sum equals 0 for $n = 2$, it equals 0 for all $n \geq 2$. □

Corollary 4.5. *For any $K \in \mathrm{Ob}(\mathcal{S}_G^*)$*

$$\frac{1}{|\Phi(K)|}\sum_{1\leq H\leq K}|H|\mu(H,K)=\begin{cases}p-1 & K \text{ is cyclic}\\ 0 & K \text{ is not cyclic.}\end{cases}$$

Proof. Suppose that the Frattini quotient $K/\Phi(K)$ is elementary abelian of order p^n for some $n > 0$. Recall that $n = 1$, ie $K/\Phi(K)$ is cyclic, if and only if K is cyclic [13, Chp 5, Corollary 1.2]. The sum of this corollary,

$$\sum_{\substack{H\\ \Phi(K)\leq H\leq K}}|H/\Phi(K)|\mu(K/H)=\sum_{d=0}^{n}(-1)^{n-d}\binom{[n]}{[d]}p^{\binom{n-d}{2}}p^d=\sum_{d=0}^{n}(-1)^d\binom{[n]}{[d]}p^{\binom{d}{2}}p^{n-d},$$

is evaluated in Lemma 4.4. It is nontrivial only if $n = 1$ where it has value $p - 1$. □

It is also possible to derive Corollary 4.5 from relations in the Burnside ring of K [21, Proposition 2.8].

5. The range of $\chi(\mathcal{F}_G^*)$

We shall first identify a class of finite groups G with $\chi(\mathcal{F}_G^*) = 1$ and $\chi(\mathcal{L}_G^*) = |G\colon O^pG|^{-1}$.

Proposition 5.1. *If G contains a nontrivial central p-subgroup, then $\chi(\mathcal{C}^*) = \chi(\mathcal{C})$ for $\mathcal{C} = \mathcal{F}_G, \mathcal{L}_G$.*

Proof. Let Z be a nonidentity central p-subgroup of G and Z^+ the full subcategory of $\mathcal{F}_G^*$ generated by p-subgroups containing Z. The category Z^+ has Euler characteristic by Corollary 2.15. Z^+ is a deformation retract of $\mathcal{F}_G^*$ in the sense that there are functors and natural transformations

$$Z^+\underset{L}{\overset{R}{\rightleftarrows}}\mathcal{F}_G^*,\qquad 1_{Z^+}=LR,\quad 1_{\mathcal{F}_G^*}\Longrightarrow RL,$$

where R is the inclusion functor. Its left adjoint is the functor L that takes $Q \leq G$ to $LQ = QZ$ and the $\mathcal{F}_G^*$-morphism $c_g : P \to Q$ to $c_g : LP \to LQ$ (where $c_g : x \mapsto x^g = g^{-1}xg$ is conjugation by $g \in G$). If P and Q are nonidentity p-subgroups of G with $Q \geq Z$ then

$$Z^+(LP, Q) = \mathcal{F}_G^*(PZ, Q) = C_G(PZ)\backslash N_G(PZ, Q) = C_G(P)\backslash N_G(P, Q) = \mathcal{F}_G^*(P, Q) = \mathcal{F}_G^*(P, RQ)$$

showing that L and R are adjoint functors with $L \dashv R$. By Lemma 2.8, the two categories Z^+ and $\mathcal{F}_G^*$ have the same Euler characteristics. But $\chi(Z^+) = 1$ as Z^+ has initial object Z (Example 2.7). Thus $\chi(\mathcal{F}_G^*) = \chi(Z^+) = 1 \stackrel{\text{Example 3.9}}{=} \chi(\mathcal{F}_G)$.

Similarly, let now Z^+ be the full subcategory of $\mathcal{L}_G^*$ generated by p-subgroups containing Z. Again, Z^+ has Euler characteristic. The two categories Z^+ and $\mathcal{L}_G^*$ have identical Euler characteristics because there is an adjunction between them. By definition of $\mathcal{L}_G^*(Z, G)$ we have that $|Z^+(Z, Q)| = |G : O^pG|$ for any object Q of Z^+. Thus Z^+ has a coweighting $k_\bullet^{Z^+} = |G : O^pG|^{-1}\delta(Z, \bullet)$ concentrated at the object Z (as in Example 2.7). Thus $\chi(\mathcal{L}_G^*) = \chi(Z^+) = |G : O^pG|^{-1} \stackrel{\text{Example 3.9}}{=} \chi(\mathcal{L}_G)$. □

The converse of Proposition 5.1 is not true as $\chi(\mathcal{F}_G^*) = 1$ and $Z(G) = 1$ for $G = \Sigma_3$ and $p = 2$. We do not know how to characterize the finite groups G with $\chi(\mathcal{F}_G^*) = 1$.

Note also that $O_2(\Sigma_4) > 1$, so that $\mathcal{S}_{\Sigma_4}^*$ is contractible and $\chi(\mathcal{S}_{\Sigma_4}^*) = 1$, but $\chi(\mathcal{F}_{\Sigma_4}^*) \neq 1$ at $p = 2$.

It is immediate that the Euler characteristic $\chi(\mathcal{F}_G^*) = |G|^{-1}\sum k_K^{\mathcal{F}^*} = -|G|^{-1}\sum_K \mu(K)|C_G(K)|$ is a rational number and that $|G|\chi(\mathcal{F}_G^*)$ is an integer. We now improve this observation using the coweighting $k_\bullet^{[\mathcal{F}^*]}$ (3.5) for the set $[\mathcal{F}_G^*]$ of isomorphism classes of objects of $\mathcal{F}_G^*$.

Corollary 5.2. *$|G|_{p'}\chi(\mathcal{F}_G^*)$ is an integer; in fact, $|G|_{p'}k_{[K]}^{[\mathcal{F}^*]}$ is an integer for any nonidentity p-subgroup conjugacy class $[K]$, $K \leq G$.*

Proof. Recall from Lemma 3.2 that $\mu(K) = 0$ unless the nonidentity p-subgroup $K \leq G$ is elementary abelian. Suppose that $K \leq G$ is elementary abelian of order $|K| = p^n$. Then

$$|G|_{p'}k_{[K]}^{[\mathcal{F}^*]} = |G|_{p'}\frac{\mu(K)}{|\mathcal{F}_G^*(K)|} = \frac{\mu(K)}{|\mathcal{F}_G^*(K)|_p} \cdot \frac{|G|_{p'}}{|\mathcal{F}_G^*(K)|_{p'}}$$

is an integer because $|\mathcal{F}_G^*(K)|_p$ divides $|\operatorname{Aut}(K)|_p = p^{\binom{n}{2}} = \mu(K)$ as $|\mathcal{F}_G^*(K)|$ divides $|\operatorname{Aut}(K)|$, and $|\mathcal{F}_G^*(K)|_{p'}$ divides $|G|_{p'}$ as $|\mathcal{F}_G^*(K)| = |N_G(K) : C_G(K)|$ divides $|G|$. □

Corollary 5.3. *Suppose that G has a normal Sylow p-subgroup, P, (so that $G = P \rtimes G/P$) and let $k_{\mathcal{F}^*}^H$, $H \in \mathrm{Ob}(\mathcal{S}_G^*)$, be the weighting for $\mathcal{F}_G^*$ from Table 1.*

(1) $|G|k_{\mathcal{F}^*}^H = |\{x \in G \mid C_P(x) = H\}|$

(2) $k_{\mathcal{F}^*}^H \geq 0$ *with* $k_{\mathcal{F}}^H > 0$ *if and only if* $H = C_P(x)$ *for some* $x \in G$

(3) $|G|\chi(\mathcal{F}_G^*) = |\{x \in G \mid C_P(x) > 1\}|$.

Proof. For any nonidentity p-subgroup K and any element $x \in G$, since $K \leq P$,

$$x \in C_G(K) \iff K \leq C_G(x) \iff K \leq P \cap C_G(x) = C_P(x)$$

so that

$$|G|k_{\mathcal{F}^*}^H = \sum_K \mu(H, K)|C_G(K)| = \sum_{x\in G}\sum_K \mu(H, K)|\mathcal{S}_G^*(K, C_P(x))| = \sum_{x\in G}\delta(H, C_P(x))$$

by the Möbius inversion formula (2.4). This proves (1) which immediately implies (2) and (3). □

Corollary 5.4. *Suppose that G has an abelian Sylow p-subgroup, P. Then*

$$\chi(\mathcal{F}_G^*) = \frac{|\{\varphi \in \mathcal{F}_G(P) \mid C_P(\varphi) > 1\}|}{|\mathcal{F}_G(P)|}.$$

Proof. When P is abelian, $\mathcal{F}_G^*(P)$ has order prime to p and

$$\mathcal{F}_G = \mathcal{F}_{N_G(P)} = \mathcal{F}_{P \rtimes \mathcal{F}_G(P)}$$

where the first identity is Burnside's Fusion Theorem [15, Lemma 16.9] which says that $N_G(P)$ controls p-fusion in G. For the second equality, observe that all morphisms in the Frobenius category of $N_G(P)$ extend to automorphisms of P. Now apply Corollary 5.3.(3) to $P \rtimes \mathcal{F}_G(P)$. □

Example 5.5. Let $p = 2$ and $G = P \rtimes C_3$ where the cyclic group C_3 cyclically permutes the three factors of $P = C_2^3$. Then $\chi(\mathcal{F}_G^*) = 1$ by Proposition 5.1; indeed, $k_{\mathcal{F}}^{Z(G)} = 2/3$, $k_{\mathcal{F}}^P = 1/3$, and $k_{\mathcal{F}}^H = 0$ for all other nonidentity 2-subgroups $H \leq G$ by Corollary 5.3.

Table 3
Euler characteristics of nonidentity 2-subgroup posets and categories of alternating groups computed by Magma [3].

n	4	5	6	7	8	9	10	11	12	13	14
$\chi(\mathcal{S}^*_{A_n})$	1	5	−15	−175	65	5121	15105	55935	−288255	1626625	23664641
$\chi(\mathcal{L}^*_{A_n})$	1/12		−1/24		1081/2016		971/6720		90035/145152		406699/1451520
$\chi(\mathcal{F}^*_{A_n})$	1/3		1/3		41/63		18/35		389/567		233/405
$\chi(\mathcal{O}^*_{A_n})$	1/3	1/3	1/3	2/9	13/63	44/315	178/945	46/315	397/2835	160/1701	2101/42525

Example 5.6. Let D_{2pn} be the dihedral group of order $2pn$, $n \geq 1$, A_p the alternating group of index $p > 2$, and $\mathrm{SL}_2(\mathbf{F}_q)$ the special linear group where q is a power of p. Then

$$\chi(\mathcal{F}^*_{D_{2pn}}) = \frac{1}{(2, p-1)}, \qquad \chi(\mathcal{F}^*_{A_p}) = \frac{2}{p-1}, \qquad \chi(\mathcal{F}^*_{\mathrm{SL}_2(\mathbf{F}_q)}) = \frac{(2, q-1)}{q-1}$$

where (m, n) stands for the greatest common divisor of the natural numbers m and n.

Example 5.7. The computer-generated Table 3 displays Euler characteristics of Frobenius categories at $p = 2$ of small alternating groups. (The Frobenius categories for A_{2n} and A_{2n+1} at $p = 2$ are equivalent.) We do not know if the sequence $\chi(\mathcal{F}^*_{A_n})$ converges. See [31] for information about the Euler characteristic $\chi(1, \Sigma_n)$ of the intervals $(1, \Sigma_n)$ in $\overline{\mathcal{S}}_{\Sigma_n}$.

Example 5.8. The group $H = (C_3 \times C_3) \rtimes C_2$, where C_2 swaps the two copies of C_3, has an irreducible 4-dimensional representation over $\mathbf{F}_2$. Let $G = C_2^4 \rtimes H$ be the associated semi-direct product. Then $|G| = 288$ and $\chi(\mathcal{F}^*_G) = 10/9$ at $p = 2$.

We have not been able to settle these two questions:

- Is $\chi(\mathcal{F}^*_G)$ always positive when p divides the order of G?
- Can $\chi(\mathcal{F}^*_G)$ get arbitrarily large?

In Corollaries 5.3 and 5.4, and in all the concrete examples that we have checked, $\chi(\mathcal{F}^*_G)$ is positive when p divides the order of G. For some time we suspected that also $\chi(\mathcal{F}^*_G) \leq 1$ for all G but Example 5.8 shows that $\chi(\mathcal{F}^*_G)$ can be greater than 1.

6. Product formulas

We present product formulas for the Euler characteristics of the subgroup poset $\mathcal{S}^*_{G_1 \times G_2}$ and the Frobenius category $\mathcal{F}^*_{G_1 \times G_2}$ for the product of two finite groups G_1 and G_2. According to Table 1,

$$-\widetilde{\chi}(\mathcal{S}^*_G) = \sum_{K \in \mathrm{Ob}(\mathcal{S}_G)} \mu(K), \qquad -\widetilde{\chi}(\mathcal{F}^*_G) = \frac{1}{|G|} \sum_{K \in \mathrm{Ob}(\mathcal{S}_G)} \mu(K)|C_G(K)|$$

with summation over *all* p-subgroups K of G. We shall use these expressions to derive formulas for the Euler characteristic of the subgroup poset and Frobenius category of a direct product of groups.

Theorem 6.1. *Let $G_1, \ldots, G_n$ be finite groups. Then*

$$-\widetilde{\chi}(\mathcal{S}^*_{\prod_{i=1}^n G_i}) = \prod_{i=1}^n -\widetilde{\chi}(\mathcal{S}^*_{G_i}), \qquad -\widetilde{\chi}(\mathcal{F}^*_{\prod_{i=1}^n G_i}) = \prod_{i=1}^n -\widetilde{\chi}(\mathcal{F}^*_{G_i}).$$

Proof. By induction over n it is enough to prove the two formulas for a product of two groups, G_1 and G_2. The first formula then asserts that

$$\sum_{H \in \mathrm{Ob}(\mathcal{S}_{G_1 \times G_2})} \mu(H) = \sum_{H_1 \in \mathrm{Ob}(\mathcal{S}_{G_1})} \mu(H_1) \quad \cdot \sum_{H_2 \in \mathrm{Ob}(\mathcal{S}_{G_2})} \mu(H_2). \tag{6.2}$$

Let $\pi_1 : G_1 \times G_2 \to G_1$ and $\pi_2 : G_1 \times G_2 \to G_2$ be the projections. The product poset $\mathcal{S}_{G_1} \times \mathcal{S}_{G_2}$ [34, Chp 3.2] is a deformation retract of $\mathcal{S}_{G_1 \times G_2}$ in the sense that there are poset morphisms and natural transformations

$$\mathcal{S}_{G_1 \times G_2} \underset{R}{\overset{L}{\rightleftarrows}} \mathcal{S}_{G_1} \times \mathcal{S}_{G_2}, \qquad 1_{\mathcal{S}_{G_1} \times \mathcal{S}_{G_2}} = LR, \qquad 1_{\mathcal{S}_{G_1 \times G_2}} \Longrightarrow RL,$$

where $LH = (\pi_1(H), \pi_2(H))$, $H \leq G_1 \times G_2$, $R(H_1, H_2) = H_1 \times H_2$, $H_1 \leq G_1$, $H_2 \leq G_2$. We have

$$H \leq R(H_1, H_2) \iff LH \leq (H_1, H_2)$$

so that L and R are adjoint functors. For any two p-subgroups, $H_1 \leq G_1$ and $H_2 \leq G_2$,

$$\sum_{\substack{H \leq G_1 \times G_2 \\ LH=(H_1,H_2)}} \mu(H) = \mu_{\mathcal{S}_{G_1}\times\mathcal{S}_{G_2}}((1,1),(H_1,H_2)) = \mu(H_1)\mu(H_2)$$

where the sum is taken over all p-subgroups $H \leq G_1 \times G_2$ with $\pi_1(H) = H_1$ and $\pi_2(H) = H_2$. The first equality is a special case of [23, Proposition 4.4] and the second equality is a special case of the formula for the Möbius function $\mu_{\mathcal{S}_{G_1}\times\mathcal{S}_{G_2}}$ of the product poset $\mathcal{S}_{G_1} \times \mathcal{S}_{G_2}$ [34, Proposition 3.8.2]. Eq. (6.2) now easily follows.

For a product, $G_1 \times G_2$, of two groups, the second formula asserts that

$$\sum_{H\in \mathrm{Ob}(\mathcal{S}_{G_1\times G_2})} \mu(H)|C_{G_1\times G_2}(H)| = \sum_{H_1\in \mathrm{Ob}(\mathcal{S}_{G_1})} \mu(H_1)|C_{G_1}(H_1)| \cdot \sum_{H_2\in \mathrm{Ob}(\mathcal{S}_{G_2})} \mu(H_2)|C_{G_2}(H_2)|.$$

But again this follows from Eq. (6.2) because $C_{G_1\times G_2}(H) = C_{G_1}(H_1)\times C_{G_2}(H_2)$ when $H \leq G_1\times G_2$ and $H_1 = \pi_1(H), H_2 = \pi_2(H)$ are the projections of H. □

The first part of Theorem 6.1 also follows Quillen's work. According to [29, Proposition 2.6], $\mathcal{S}^{ea}_{G_1\times G_2}$ is homotopy equivalent to the join $\mathcal{S}^{ea}_{G_1} * \mathcal{S}^{ea}_{G_2}$ and therefore

$$1-\chi(\mathcal{S}^{ea}_{G_1\times G_2}) = 1-\chi(\mathcal{S}^{ea}_{G_1} * \mathcal{S}^{ea}_{G_2}) = \big(1-\chi(\mathcal{S}^{ea}_{G_1})\big)\big(1-\chi(\mathcal{S}^{ea}_{G_2})\big)$$

as $1-\chi(X*Y) = \big(1-\chi(X)\big)\big(1-\chi(Y)\big)$ for any two finite abstract simplicial complexes, X and Y.

Example 6.3. The permutation group $\Sigma_n = A_n \rtimes C_2$ is the semi-direct product of the alternating group A_n with a group of order two. Assume that $p = 2$. When $n = 6$, $-\widetilde{\chi}(\mathcal{S}^*_{\Sigma_6}) = 16$ and $-\widetilde{\chi}(\mathcal{S}^*_{C_2}) = 0$. When $n = 4$, $-\widetilde{\chi}(\mathcal{F}^*_{\Sigma_4}) = 1/3$ and $-\widetilde{\chi}(\mathcal{F}^*_{C_2}) = 0$.

These two examples show that neither the first nor the second part of Theorem 6.1 generalize to semi-direct products.

7. Abstract Frobenius categories

We shall first formulate an alternative expression for the coweighting of a Frobenius category for a finite group G. We shall next see that this new expression easily extends to all abstract Frobenius categories.

For any two p-subgroups, H and K, of G, let

$$\mathcal{S}_G([H],K) = \sum_{A\in[H]} |\mathcal{S}_G(A,K)| = |\{A\in[H] \mid A\leq K\}| \tag{7.1}$$

denote the number of subgroups of K that are G-conjugate to H. Quite similarly, for arbitrary subgroups, H and K, of G, let

$$\overline{\mathcal{S}}_G([H],K) = \sum_{A\in[H]} |\overline{\mathcal{S}}_G(A,K)| = |\{A\in[H] \mid A\leq K\}| \tag{7.2}$$

denote the number of subgroups of K that are G-conjugate to H. (Recall from Definition 3.1 that $\overline{\mathcal{S}}_G$ is the poset of all subgroups of G.) In particular, $\overline{\mathcal{S}}_G([H],G) = |[H]| = |G\colon N_G(H)|$ is the number of conjugates of H in G.

Let P be a subgroup of G of index prime to p, for instance, a Sylow p-subgroup of G. Write $P\cap\mathcal{F}^*_G$ for the full subcategory of $\mathcal{F}^*_G$ generated by all nonidentity p-subgroups of P. Then $P\cap\mathcal{F}^*_G$ and $\mathcal{F}^*_G$ are equivalent so they have identical Euler characteristics (Lemma 2.8).

Corollary 7.3. *The function*

$$k_K^{P\cap\mathcal{F}^*_G} = -\frac{\mu(K)}{|\mathcal{F}^*_G(K,P)|}, \quad K\in \mathrm{Ob}(P\cap\mathcal{F}^*_G),$$

is a coweighting for $P\cap\mathcal{F}^*_G$ and the Euler characteristic of $P\cap\mathcal{F}^*_G$ is

$$\chi(P\cap\mathcal{F}^*_G) = \sum_K -\frac{\mu(K)}{|\mathcal{F}^*_G(K,P)|} = \chi(\mathcal{F}^*_G)$$

with summation over the nonidentity elementary abelian p-subgroups K of P.

Proof. The set of isomorphism classes of objects $[P\cap\mathcal{F}^*_G] = [\mathcal{F}^*_G]$ has Möbius inversion and unique coweighting $k_{[K]}^{[\mathcal{F}^*_G]} = -\mu(K)/|\mathcal{F}^*_G(K)|$ (3.5). The isomorphism class of any p-subgroup $K\leq P$ contains $\overline{S}_G([K],P)$ objects in $P\cap\mathcal{F}^*_G$. According to Proposition 2.14

$$k_K^{P\cap\mathcal{F}^*_G} = \frac{1}{\overline{S}_G([K],P)}k_{[K]}^{[\mathcal{F}^*_G]} = \frac{-\mu(K)}{\overline{S}_G([K],P)|\mathcal{F}^*_G(K)|} = \frac{-\mu(K)}{|\mathcal{F}^*_G(K,P)|}$$

is a coweighting for the category $P\cap\mathcal{F}^*_G$. Here, we used that $\overline{\mathcal{S}}_G([K],P)|\mathcal{F}^*_G(K)| = |\mathcal{F}^*_G(K,P)|$. □

Let P be a finite p-group and $\mathcal{F}$ an abstract Frobenius category over P [28, Chp 2] [7]. The objects of $\mathcal{F}$ are all subgroups H, K of P. Write $[H]$ for the isomorphism class in $\mathcal{F}$ of the object H, and

$$S_P([H], K) = \sum_{A \in [H]} |S_P(A, K)|$$

for the number of objects of $[H]$ contained in K. Thus $|[H]| = \mathcal{S}_p([H], P)$. The Divisibility Axiom [28, 2.3.1] for the Frobenius category $\mathcal{F}$ implies that

$$|\mathcal{F}(H, K)| = |\mathcal{F}(H)|\mathcal{S}_p([H], K). \tag{7.4}$$

Define $\mathcal{F}^*$ to be the full subcategory of $\mathcal{F}$ generated by all *nonidentity* subgroups of P. It is clear that $[\mathcal{F}^*]$ has Möbius inversion and that $[\mathcal{F}^*]$ and $\mathcal{F}^*$ have Euler characteristics (Proposition 2.14).

Theorem 7.5. *The functions*

$$k_K^{\mathcal{F}^*} = \frac{-\mu(K)}{|\mathcal{F}^*(K, P)|}, \qquad k_{[K]}^{[\mathcal{F}^*]} = \frac{-\mu(K)}{|\mathcal{F}^*(K)|}, \quad 1 \lneqq K \leq P,$$

are coweightings for $\mathcal{F}^*$ and $[\mathcal{F}^*]$, respectively. The Euler characteristic of $\mathcal{F}^*$ is

$$\chi(\mathcal{F}^*) = \sum_{[K] \in [\mathcal{F}^*]} \frac{-\mu(K)}{|\mathcal{F}^*(K)|}.$$

All values of the coweighting for $[\mathcal{F}^*]$ and the Euler characteristic of $\mathcal{F}^*$ are p-local integers.

Proof. We verify that $k_\bullet^{\mathcal{F}^*}$ is a coweighting for $\mathcal{F}^*$ (2.5). For any nonidentity subgroup $K \leq P$,

$$\sum_{1 \lneqq H \leq P} \frac{-\mu(H)}{|\mathcal{F}^*(H, P)|}|\mathcal{F}^*(H, K)| = \sum_{1 \lneqq H \leq P} -\mu(H)\frac{\mathcal{S}_p([H], K)}{\mathcal{S}_p([H], P)} = \sum_{[H] \in [\mathcal{F}^*]} -\mu(H)\mathcal{S}_p([H], K) =$$
$$\sum_{1 \lneqq H \leq P} -\mu(H)\mathcal{S}_p(H, K) = \sum_{1 \lneqq H \leq K} -\mu(H) \overset{\text{Table 1}}{=} \chi(\mathcal{S}_K^*) \overset{(3.8)}{=} 1.$$

Because the coweighting $k_K^{\mathcal{F}^*}$ is constant over the isomorphism class of K,

$$k_{[K]}^{[\mathcal{F}^*]} = \sum_{B \in [K]} k_K^{\mathcal{F}^*} = \mathcal{S}_p([K], P)k_K^{\mathcal{F}^*} = \mathcal{S}_p([K], P)\frac{-\mu(K)}{|\mathcal{F}^*(K)|\mathcal{S}_p([K], P)} = \frac{-\mu(K)}{|\mathcal{F}^*(K)|}$$

is a coweighting for $[\mathcal{F}^*]$. The formula for the Euler characteristic of $\mathcal{F}^*$ or $[\mathcal{F}^*]$ follows (Proposition 2.14).

Recall from Lemma 3.2 that the Möbius function vanishes on all p-groups but the elementary abelian ones. Since $|\mathcal{F}^*(K)|_p$ divides $|\operatorname{Aut}(K)|_p = \mu(K)$ when K is elementary abelian (see proof of Corollary 5.2), all values of the coweighting $k_{[K]}^{[\mathcal{F}^*]}$ lie in the ring $\mathbf{Z}_{(p)}$ of p-local integers. (Following standard notation we write $\mathbf{Z}_{(p)}$ for the localization of the integer ring $\mathbf{Z}$ at the prime ideal (p) generated by p.) □

The exterior quotient $\widetilde{\mathcal{F}}$ of the Frobenius category $\mathcal{F}$ has the same objects as $\mathcal{F}_G$, all subgroups H, K of P. The morphism set $\widetilde{\mathcal{F}}(H, K) = \mathcal{F}(H, K)/K$ is the set of K-conjugacy classes of $\mathcal{F}$-morphisms. Composition of morphisms in $\mathcal{F}$ induces morphism composition in $\widetilde{\mathcal{F}}$.

Define $\widetilde{\mathcal{S}}_p$ to be the category with the same objects as $\mathcal{F}$, all subgroups H, K of P. The morphism sets are

$$\widetilde{\mathcal{S}}_p(H, K) = \begin{cases} C_K(H) & H \leq K \\ \emptyset & H \not\leq K. \end{cases}$$

Group multiplication in P induces composition of morphisms in $\widetilde{\mathcal{S}}_p$. For any objects H and K in $\mathcal{F}$, let

$$\widetilde{\mathcal{S}}_p([H], K) = \sum_{A \in [H]} |\widetilde{\mathcal{S}}_p(A, K)|$$

be the sum of the orders of the centralizers $C_K(A)$ over all subgroups A of P that are isomorphic to H and contained in K. Then we have

$$|\widetilde{\mathcal{F}}(H, K)||K| = |\mathcal{F}(H)|\widetilde{\mathcal{S}}_p([H], K) \tag{7.6}$$

from the The Divisibility Axiom [28, 2.3.1] for $\mathcal{F}$ combined with Burnside's counting lemma (Lemma 7.9).

Recall that $\widetilde{\mathcal{F}}^*$ denotes the full subcategory of $\widetilde{\mathcal{F}}$ generated by all *nonidentity* subgroups of P; $[\widetilde{\mathcal{F}}^*]$ has Möbius inversion and $[\widetilde{\mathcal{F}}^*]$ and $\widetilde{\mathcal{F}}^*$ have Euler characteristics (Corollary 2.15).

Theorem 7.7. *The coweighting $k_\bullet^{\mathcal{F}^*}$ for $\mathcal{F}^*$ from Theorem 7.5 is also a coweighting for $\widetilde{\mathcal{F}}^*$. The category $\mathcal{F}^*$ has the same Euler characteristic as its exterior quotient $\widetilde{\mathcal{F}}^*$.*

Proof. We verify that $k_\bullet^{\mathcal{F}^*}$ from Theorem 7.5 is a coweighting for $\widetilde{\mathcal{F}}^*$ (2.5). For any nonidentity subgroup $K \le P$,

$$\sum_{1 \lneq H \le P} \frac{-\mu(H)}{|\mathcal{F}^*(H,P)|} |\widetilde{\mathcal{F}}^*(H,K)| = \frac{1}{|K|} \sum_{1 \lneq H \le P} -\mu(H) \frac{\widetilde{\mathcal{S}}_P([H],K)}{\mathcal{S}_P([H],P)} = \frac{1}{|K|} \sum_{[H] \in [\mathcal{F}^*]} -\mu(H) \widetilde{\mathcal{S}}_P([H],K) =$$

$$\frac{1}{|K|} \sum_{1 \lneq H \le P} -\mu(H) \widetilde{\mathcal{S}}_P(H,K) = \frac{1}{|K|} \sum_{1 \lneq H \le K} -\mu(H) |C_K(H)| \overset{\text{Table 1}}{=} \chi(\mathcal{F}_K^*) \overset{(3.8)}{=} 1.$$

Since $\mathcal{F}^*$ and $\widetilde{\mathcal{F}}^*$ have identical coweightings, they also have identical Euler characteristics. □

Remark 7.8. Our original proof of Theorem 7.7, only valid for Frobenius categories of finite groups, was extended to abstract Frobenius categories in Eske Sparsø's Master's thesis [33, Theorem 21]. The thesis also contains the computation

$$\chi(\mathcal{F}^*_{\mathrm{Sol}(q)}) = \frac{209}{315}$$

of the Euler characteristic of the Frobenius category for the Solomon 2-local finite group $\mathrm{Sol}(q)$ [24,25] defined for any odd prime power q.

The two categories $\mathcal{F}_G^*$ and $\widetilde{\mathcal{F}}_G^*$, even their left ideals $\mathcal{F}_G^{\mathrm{sc}}$ and $\widetilde{\mathcal{F}}_G^{\mathrm{sc}}$, do not in general have identical weightings.

Lemma 7.9 (*Burnside's Counting Lemma [26]*)**.** *If X is a finite right G-set then*

$$\sum_{g \in G} |X^g| = |X/G||G| = \sum_{x \in X} |{}^xG|$$

where $X^g = \{x \in X \mid xg = x\}$ is the fixed set for $g \in G$ and ${}^xG = \{g \in G \mid xg = x\}$ is the isotropy subgroup for $x \in X$.

8. Self-centralizing subgroups

This section deals with the subcategories $\mathcal{C}^{\mathrm{sc}}$ of the p-subgroup categories generated $\mathcal{C}$ by the p-selfcentralizing subgroups. We mention here some facts to justify our interest in these subcategories of p-selfcentralizing subgroups:

- The category $\mathcal{L}_G^{\mathrm{sc}}$ is a complete algebraic invariant of the p-completed classifying space of G [6, Theorem A]
- The Frobenius category $\mathcal{F}_G^*$ is completely determined by $\mathcal{F}_G^{\mathrm{sc}}$ [28, Chp 4–5]
- All morphisms in the category $\widetilde{\mathcal{F}}_G^{\mathrm{sc}}$ are epimorphisms [28, Corollary 4.9]
- All morphisms in the category $\widetilde{\mathcal{F}}_G^{\mathrm{sc}}$ have unique maximal extensions [28].

Now follow the definition and a few standard properties of p-selfcentralizing p-subgroups.

Definition 8.1 (*[28, 4.8.1], [7, Definition A.3]*)**.** The p-subgroup H of G is p-selfcentralizing if the center of H is a Sylow p-subgroup of $C_G(H)$.

Lemma 8.2 (*[28, Chp 4], [7, Appendix A]*)**.** *Let H be a p-subgroup of G and let P be a Sylow p-subgroup of G.*

(1) *H is p-selfcentralizing if and only if $C_G(H) \cong Z(H) \times O^pC_G(H)$ and $O^pC_G(H)$ is a p'-group.*
(2) *H is p-selfcentralizing if and only if $C_P(H^g) \le H^g$ for every $g \in N_G(H,P)$.*
(3) *If H is p-selfcentralizing and $H^g \le K$ for some $g \in G$ and some p-subgroup K of G, then K is p-selfcentralizing, $Z(H^g) = C_K(H^g)$, and $Z(H^g) \ge Z(K)$.*
(4) *If $Q \le P$ and $C_P(Q)$ is a Sylow p-subgroup of $C_G(Q)$, then $QC_P(Q)$ is p-selfcentralizing.*

We shall determine weightings for the left ideals $\mathcal{C}^{\mathrm{sc}}$ of $\mathcal{C} = \mathcal{T}_G, \mathcal{L}_G, \mathcal{F}_G, \mathcal{O}_G, \widetilde{\mathcal{F}}_G$ generated by the p-self-centralizing p-subgroups of G. The first four cases are very easy. The weightings for $[\mathcal{T}_G^{\mathrm{sc}}], [\mathcal{L}_G^{\mathrm{sc}}], [\mathcal{F}_G^{\mathrm{sc}}], [\mathcal{O}_G^{\mathrm{sc}}]$ are simply the restrictions of the weightings from Table 1 for $[\mathcal{T}_G], [\mathcal{L}_G], [\mathcal{F}_G], [\mathcal{O}_G]$ (Remark 2.6).

We now determine the weighting for $[\widetilde{\mathcal{F}}_G^{\mathrm{sc}}]$. If H is p-selfcentralizing and $H \le K$, then $C_K(H) = Z(H)$ (Lemma 8.2.(3)). Identity (7.6) for sizes of morphism sets in exterior quotients simplifies to

$$|C_G(H)|_{p'} |\widetilde{\mathcal{F}}_G^{\mathrm{sc}}(H,K)||K| = |\mathcal{T}_G^{\mathrm{sc}}(H,K)|$$

which immediately gives us the Möbius function

$$[\mu(\widetilde{\mathcal{F}}_G^{\mathrm{sc}})]([H],[K]) = |H| [\mu]([H],[K]) |C_G(K)|_{p'} \tag{8.3}$$

for $[\widetilde{\mathcal{F}}_G^{\mathrm{sc}}]$ expressed by means of the Möbius function $[\mu]$ for $[\mathcal{T}_G]$ (Proposition 3.11).

Table 4
Weightings, coweightings and Euler characteristics for categories of p-selfcentralizing p-subgroups.

$\mathcal{C}$	$k^{[H]}_{[\mathcal{C}]}$	$k_{[K]}^{[\mathcal{C}]}$	$\chi(\mathcal{C})$
$\mathcal{T}_G^{sc}$	$\sum_{[K]}[\mu]([H],[K])$	$\sum_{[H]}[\mu]([H],[K])$	$\sum_{[H],[K]}[\mu]([H],[K])$
$\mathcal{L}_G^{sc}$	$\sum_{[K]}[\mu]([H],[K])\lvert C_G(K)\rvert_{p'}$	$k_{[K]}^{[\mathcal{T}_G^{sc}]}\lvert C_G(K)\rvert_{p'}$	$\sum_{[H],[K]}[\mu]([H],[K])\lvert C_G(K)\rvert_{p'}$
$\mathcal{F}_G^{sc}$	$\sum_{[K]}[\mu]([H],[K])\lvert C_G(K)\rvert$	$k_{[K]}^{[\mathcal{T}_G^{sc}]}\lvert C_G(K)\rvert$	$\sum_{[H],[K]}[\mu]([H],[K])\lvert C_G(K)\rvert$
$\mathcal{O}_G^{sc}$	$\lvert H\rvert k^{[H]}_{[\mathcal{T}_G^{sc}]}$	$\sum_{[H]}\lvert H\rvert[\mu]([H],[K])$	$\sum_{[H],[K]}\lvert H\rvert[\mu]([H],[K])$
$\widetilde{\mathcal{F}}_G^{sc}$	$\lvert H\rvert k^{[H]}_{[\mathcal{L}_G^{sc}]}$	$k_{[K]}^{[\mathcal{O}_G^{sc}]}\lvert C_G(K)\rvert_{p'}$	$\sum_{[H],[K]}\lvert H\rvert[\mu]([H],[K])\lvert C_G(K)\rvert_{p'}$

Theorem 8.4. *Weightings $k^\bullet$, coweightings $k_\bullet$, and Euler characteristics for the finite categories $\mathcal{T}_G^{sc}$, $\mathcal{L}_G^{sc}$, $\mathcal{F}_G^{sc}$, $\mathcal{O}_G^{sc}$, and $\widetilde{\mathcal{F}}_G^{sc}$ of p-selfcentralizing p-subgroups of G are as in Table 4.*

The sums that express the weightings of Table 4 run over all $[K] \geq_G [H]$ for a fixed p-selfcentralizing p-subgroup H. The sums that express the coweighting run over all *p-selfcentralizing* $[H] \leq_G [K]$ for a fixed p-selfcentralizing p-subgroup K.

We next note that the weightings for $\mathcal{L}_G^{sc}$ and $\widetilde{\mathcal{F}}_G^{sc}$ can be computed p-locally in analogy with the situation of Proposition 3.14.

Proposition 8.5. *The values of the weightings for $[\mathcal{L}_G^{sc}]$ and $[\widetilde{\mathcal{F}}_G^{sc}]$ at the conjugacy class of the p-selfcentralizing p-subgroup $H \leq G$ are*

$$k^{[H]}_{[\mathcal{L}_G^{sc}]} = \frac{-\widetilde{\chi}(\mathcal{S}^*_{\widetilde{\mathcal{F}}_G^{sc}(H)})}{|\mathcal{L}_G^{sc}(H)|}, \qquad k^{[H]}_{[\widetilde{\mathcal{F}}_G^{sc}]} = \frac{-\widetilde{\chi}(\mathcal{S}^*_{\widetilde{\mathcal{F}}_G^{sc}(H)})}{|\widetilde{\mathcal{F}}_G^{sc}(H)|}.$$

All values of the weighting for $[\widetilde{\mathcal{F}}_G^{sc}]$ and the Euler characteristic $\chi(\widetilde{\mathcal{F}}_G^{sc})$ are p-local integers.

Proof. We only need to prove the second of the above two formulas because $|\mathcal{L}_G^{sc}(H)| = |H||\widetilde{\mathcal{F}}_G^{sc}(H)|$ and $k^{[H]}_{[\widetilde{\mathcal{F}}_G^{sc}]} = |H|k^{[H]}_{[\mathcal{L}_G^{sc}]}$ (Table 4). For any nonidentity p-subgroup $H \leq G$, there is a commutative diagram

$$\begin{array}{ccccc}
Z(H) & \longrightarrow & H & \longrightarrow & H/Z(H) \\
\downarrow & & \downarrow & & \downarrow \\
C_G(H) & \longrightarrow & \mathcal{T}_G^{sc}(H) & \longrightarrow & \mathcal{F}_G^{sc}(H) \\
\downarrow & & \downarrow & & \downarrow \\
C_G(H)/Z(H) & \longrightarrow & \mathcal{O}_G^{sc}(H) & \longrightarrow & \widetilde{\mathcal{F}}_G^{sc}(H)
\end{array}$$

with exact rows and columns. Let K be a p-subgroup such that $H \leq K \leq N_G(H)$. Then $C_G(K) \leq C_G(H) \leq N_G(H)$ so that $C_G(K) = C_{N_G(H)}(K)$. In case H is a p-selfcentralizing subgroup of G, the chain of inclusions, obtained using Lemma 8.2.(3),

$$Z(K) \leq Z(H) \cap C_G(K) \leq H \cap C_G(K) \leq K \cap C_G(K) = Z(K)$$

is, in fact, a chain of identities so that $Z(K) = Z(H) \cap C_G(K) = H \cap C_G(K) = C_H(K)$. The projection $\mathcal{T}_G^{sc}(H) = N_G(H) \to \mathcal{O}_G^{sc}(H) = N_G(H)/H$ takes $C_G(K)$ to $\overline{C_G(K)}$ with kernel $C_H(K)$, the Sylow p-subgroup of $C_G(K)$. Thus $\overline{C_G(K)} = C_G(K)/C_H(K)$ and $|\overline{C_G(K)}| = |C_G(K)|_{p'}$.

According to Tables 1, 4, and Eq. (3.10)

$$|\widetilde{\mathcal{F}}_G^{sc}(H)|k^{[H]}_{\widetilde{\mathcal{F}}_G^{sc}} = \sum_{[\overline{K}]} \mu(\overline{K})|\mathcal{O}_G^{sc}(H) : N_{\mathcal{O}_G^{sc}(H)}(\overline{K})| \frac{|C_G(K)|_{p'}}{|C_G(H)|_{p'}}, \qquad -\widetilde{\chi}(\mathcal{S}^*_{\widetilde{\mathcal{F}}_G^{sc}(H)}) = \sum_{[\overline{K}]} \mu(\overline{K})|\widetilde{\mathcal{F}}_G^{sc}(H) : N_{\widetilde{\mathcal{F}}_G^{sc}(H)}(\overline{K})|$$

where the first sum runs over the set of conjugacy classes of (elementary abelian) p-subgroups $\overline{K}$ of $\mathcal{O}_G^{sc}(H)$ and the second sum over the set of conjugacy classes of (elementary abelian) p-subgroups $\overline{K}$ of $\widetilde{\mathcal{F}}_G^{sc}(H)$. By Lemma 8.9 the projection $\mathcal{O}_G^{sc}(H) \to \widetilde{\mathcal{F}}_G^{sc}(H)$ has kernel of order prime to p, and therefore induces a bijection between these two sets. It suffices to prove that

$$\frac{|\mathcal{O}_G^{sc}(H)|}{|N_{\mathcal{O}_G^{sc}(H)}(\overline{K})|} \frac{|C_G(K)|_{p'}}{|C_G(H)|_{p'}} = \frac{|\widetilde{\mathcal{F}}_G^{sc}(H)|}{|N_{\widetilde{\mathcal{F}}_G^{sc}(H)}(\overline{K})|}$$

Table 5
Euler characteristics at $p = 2$ of p-selfcentralizing p-subgroup posets and categories of alternating groups computed by Magma [3].

n	4–5	6	7	8–9	10	11	12–13
$\chi(\mathcal{O}^{sc}_{A_n})$	1/3	1/3	2/9	13/63	19/105	106/945	388/2835
$\chi(\mathcal{T}^{sc}_{A_n})$	1/12	−1/24	−5/72	13/4032	29/4480	653/120960	1133/1451520
$\chi(\mathcal{L}^{sc}_{A_n})$	1/12	−1/24		13/4032	29/4480		1133/1451520
$\chi(\mathcal{F}^{sc}_{A_n})$	1/3	1/3		13/63	19/105		388/2835
$\chi(\widetilde{\mathcal{F}}^{sc}_{A_n})$	1/3	1/3		13/63	19/105		388/2835

or, equivalently, that $|N_{\mathcal{O}^{sc}_G(H)}(\overline{K})| = |N_{\widetilde{\mathcal{F}}^{sc}_G(H)}(\overline{K})||\overline{C_G(K)}|$ for all p-subgroups $\overline{K}$ of $\mathcal{O}^{sc}_G(H)$. The surjection $\mathcal{O}^{sc}_G(H) \to \widetilde{\mathcal{F}}^{sc}_G(H)$ restricts to a surjection [15, Lemma 4.2.(ii)] $N_{\mathcal{O}^{sc}_G(H)}(\overline{K}) \to N_{\widetilde{\mathcal{F}}^{sc}_G(H)}(\overline{K})$ and we claim that the kernel is $\overline{C_G(K)}$. Using Lemma 8.8 below we see that the kernel is

$$\overline{C_G(H)} \cap N_{N_G(H)/H}(\overline{K}) = \overline{C_G(H)} \cap \overline{N_{N_G(H)}(K)} = \overline{C_G(H)H \cap N_{N_G(H)}(K)}$$

so we need to prove that $C_G(H)H \cap N_{N_G(H)}(K) = C_G(K)H$ where the group on the left hand side is $N_{C_G(H)}(K)H$ (Dedekind's modular law). Thus the claim follows from the basic identity $N_{C_G(H)}(K) = C_G(K)Z(H)$ [7, Proposition A.8], [28, Corollary 4.7].

As the numbers $|\widetilde{\mathcal{F}}^{sc}_G(H)|_{p'} k^{[H]}_{[\widetilde{\mathcal{F}}^{sc}_G]}$ are integers by Brown's theorem [8, Corollary 2], all values of the weighting for $[\widetilde{\mathcal{F}}^{sc}_G]$ are p-local integers. □

Parallel to the situation of Corollary 3.19 we can now narrow down the support for the weightings for $\mathcal{L}^{sc}_G$ and $\widetilde{\mathcal{F}}^{sc}_G$.

Corollary 8.6. *The weightings for $\mathcal{L}^{sc}_G$ and $\widetilde{\mathcal{F}}^{sc}_G$ are supported on the p-selfcentralizing $\mathcal{F}_G$-radical p-subgroups of G.*

According to Proposition 8.5 and Corollary 8.6 the Euler characteristics of $\mathcal{L}^{sc}_G$ and $\widetilde{\mathcal{F}}^{sc}_G$ are

$$\chi(\mathcal{L}^{sc}_G) = \sum_{[H]} \frac{-\widetilde{\chi}(\mathcal{S}^*_{\widetilde{\mathcal{F}}^{sc}_G(H)})}{|\mathcal{L}^{sc}_G(H)|}, \qquad \chi(\widetilde{\mathcal{F}}^{sc}_G) = \sum_{[H]} \frac{-\widetilde{\chi}(\mathcal{S}^*_{\widetilde{\mathcal{F}}^{sc}_G(H)})}{|\widetilde{\mathcal{F}}^{sc}_G(H)|} \tag{8.7}$$

with summations running over the set of conjugacy classes of p-selfcentralizing $\mathcal{F}_G$-radical p-subgroups of G.

The following elementary two lemmas were used in the proof of Proposition 8.5.

Lemma 8.8. *Let $N \lhd G$ and consider the projection $G \to G/N$ of G onto its factor group by N. Write $\overline{X}$ for the image of a subgroup $X \le G$ in the factor group G/N. Then $\overline{K_1 \cap K_2} = \overline{K_1} \cap \overline{K_2}$ if at least one of $K_1, K_2 \le G$ contains N.*

Lemma 8.9. *If the normal subgroup $N \lhd G$ has order prime to p, then the projection $G \to G/N$ induces a bijection $[\mathcal{T}_G] \to [\mathcal{T}_{G/N}]$ between the sets of conjugacy classes of p-subgroups of G and G/N.*

Proof. Let $\pi : G \to G/N$ be the projection. If $K \le G$ is a p-group, also $\pi(K)$ is a p-group and $K \cong \pi(K)$. Thus $K \to \pi(K)$ induces a map $[\mathcal{T}_G] \to [\mathcal{T}_{G/N}]$ from the set of conjugacy classes of p-subgroups of G to the set of conjugacy classes of p-subgroups of G/N. We first show that this map is injective: Suppose that $K_1 \le G, K_2 \le G$ and that $\pi(K_1)$ and $\pi(K_2)$ are conjugate in G/N. Then K_1N and K_2N are conjugate in G. Thus $K_1N = (K_2N)^g = K_2^g N^g$ for some $g \in G$. Then the Sylow p-subgroups K_1 and K_2^g of K_1N are conjugate in K_1N. Thus K_1 and K_2 are conjugate subgroups of G. Next, we show surjectivity: If H is a p-subgroup in G/N, let K be a Sylow p-subgroup of $\pi^{-1}H$, its preimage in G. Then $\pi(K) = H$ by order considerations for $|H| = |\pi^{-1}H|_p = |K| = |\pi(K)|$. This shows that the induced map $[\mathcal{T}_G] \to [\mathcal{T}_{G/N}]$ is bijective. □

Remark 8.10. We suspect that the support for the weightings for $[\mathcal{C}] = [\mathcal{L}^{sc}_G], [\widetilde{\mathcal{F}}^{sc}_G]$ is the set of p-selfcentralizing $\mathcal{F}_G$-radical p-subgroups of G. This would be the case if the strong Quillen conjecture (1.4) turned out to be true.

Remark 8.11. Based on explicit computations we suspect that $\chi(\mathcal{F}^{sc}_G) = \chi(\widetilde{\mathcal{F}}^{sc}_G)$. We saw in Theorem 7.7 that $\mathcal{F}^*_G$ and $\widetilde{\mathcal{F}}^*_G$ have identical coweightings and Euler characteristics. However, it is not generally true that $\mathcal{F}^{sc}_G$ and $\widetilde{\mathcal{F}}^{sc}_G$ have identical coweightings or weightings. Nevertheless, it happens that $\chi(\mathcal{F}^{sc}_G) = \chi(\widetilde{\mathcal{F}}^{sc}_G)$ at $p = 2$ if $|G| \le 500$ or if G is one of the alternating groups A_n, $4 \le n \le 13$ of Table 5.

Example 8.12. If P is a nonidentity p-group then

$$\chi(\mathcal{S}^{sc}_P) = 1, \qquad \chi(\mathcal{T}^{sc}_P) = |P|^{-1}, \qquad \chi(\mathcal{L}^{sc}_P) = |P|^{-1}, \qquad \chi(\mathcal{F}^{sc}_P) = 1, \qquad \chi(\mathcal{O}^{sc}_P) = 1, \qquad \chi(\widetilde{\mathcal{F}}^{sc}_P) = 1$$

because $\mathcal{S}^{sc}_P$, $\mathcal{O}^{sc}_P$, and $\widetilde{\mathcal{F}}^{sc}_P$ have P as terminal object and $\mathcal{L}^{sc}_P = \mathcal{T}^{sc}_P$ is the Grothendieck construction for the P-action on $\mathcal{S}^{sc}_P$. Corollary 5.3.(2) shows that the weighting for $\mathcal{F}^*_P$ is supported on the subgroups of the form $C_P(x)$, $x \in P$, so that $\chi(\mathcal{F}^{sc}_P) = \chi(\mathcal{F}^*_P) = 1$ as these subgroups are p-selfcentralizing by Lemma 8.2.(4).

By Example 5.5 it is not true for general groups G that the weighting for $\mathcal{F}_G^*$ is supported on the p-selfcentralizing subgroups. Also, Tables 3 and 5 contain several examples of alternating groups where the nonidentity and the centric Frobenius categories have different Euler characteristics.

9. Möbius algebras of finite categories

In this section we introduce Möbius algebras of square matrices and finite categories.

9.1. The Möbius algebra of a square matrix

Let S be a finite set, R an integral domain with field of fractions k, and $\zeta = \big(\zeta(a,b)\big)_{a,b\in S}$ a square matrix indexed by S with entries in R. Assume that the determinant of ζ is nonzero. The matrix ζ can also be perceived as a function $\zeta : S\times S \to R$ with values in R. For instance, if $\delta\colon S\times S\to R$ is Kronecker's δ-function, then $\delta = \big(\delta(a,b)\big)_{a,b\in S}$ is the identity matrix.

Let R^S denote the free right R-module with basis S. The map $\zeta : S\times S \to R$ extends to a R-bilinear form $\zeta : R^S\times R^S\to R$ and to a R-linear homomorphism $M(\zeta)\colon R^S\to R^S$, the table of marks for ζ, given by

$$\zeta\left(\sum_a ar_a, \sum_b bs_b\right) = \sum_{a,b}\zeta(a,b)r_as_b, \qquad M(\zeta)(b) = \sum_{a\in S} a\zeta(a,b), \quad a,b\in S \tag{9.1}$$

$M(\zeta)$ is an injective homomorphism between free R-modules as ζ has nonzero determinant.

Similarly, let k^S denote the right k-vector space with basis S. Viewing ζ as a square matrix with entries in k, the map $\zeta : S\times S\to k$ extends to a k-bilinear form $k^S\times k^S\to k$ and to a k-linear homomorphism $M(\zeta)\colon k^S\to k^S$ given by the same expressions as in (9.1). This $M(\zeta)$ over k is an isomorphism of k-vector spaces.

Equip $R^S = \prod_{a\in S} R$ and $k^S = \prod_{a\in S} k$ with the product algebra structures. The product in R^S and k^S is given by

$$a\bullet b = \begin{cases} a & a=b \\ 0 & a\neq b\end{cases}$$

for all $a, b\in S$. The unit element in R^S and k^S is the sum $1 = \sum_{a\in S} a$ of the basis elements.

Definition 9.2. The Möbius k-algebra of ζ, $M(\zeta; k)$, is the k-algebra with underlying k-vectorspace k^S and with the commutative product $\cdot$ defined by

$$\forall x,y\in k^S\colon M(\zeta)(x\cdot y) = M(\zeta)(x)\bullet M(\zeta)(y).$$

The algebra $(k^S,\bullet)$ is the Möbius algebra $M(\delta; k)$ of the δ-function. The product in the Möbius k-algebra $M(\zeta; k)$ makes the table of marks k-vector space isomorphism $M(\zeta)\colon M(\zeta;k)\to M(\delta;k)$ an isomorphism of k-algebras. Equivalently, the product in $M(\zeta;k)$ satisfies $\zeta(a, x\cdot y) = \zeta(a,x)\zeta(a,y)$ for all $x,y\in M(\zeta;k)$ and all $a\in S$.

Let $\mu = \big(\mu(a,b)\big)_{a,b\in S}$ be the inverse (over k) of ζ. Then $k_\zeta^\bullet = \sum_{b\in S}\mu(\bullet,b)\colon S\to k$ is the weighting, $k_\bullet^\zeta = \sum_{a\in S}\mu(a,\bullet)\colon S\to k$ the coweighting for ζ (Section 2.1), and $M(\mu)\colon M(\delta;k)\to M(\zeta;k)$ is an isomorphism of Möbius algebras.

The sum

$$1 = \sum_{b\in S} b$$

is a decomposition of the unit of the Möbius k-algebra $M(\delta;k)$ into a sum of orthogonal idempotents, and

$$1 = M(\mu)(1) = \sum_{b\in S} M(\mu)b = \sum_{b\in S} e_b$$

is a decomposition of the unit of the Möbius k-algebra $M(\zeta;k)$ into a sum of the orthogonal idempotents

$$e_b = M(\mu)b = \sum_{a\in S} a\mu(a,b), \quad b\in S \tag{9.3}$$

of $M(\zeta;k)$. Thus the unit of $M(\zeta;k)$,

$$1 = \sum_{b\in S} e_b = \sum_{b\in S}\sum_{a\in S} a\mu(a,b) = \sum_{a\in S} a\sum_{b\in S}\mu(a,b) = \sum_{a\in S} ak_\zeta^a \tag{9.4}$$

is the weighted sum of basis elements. We collect these observations in the following proposition. (Remember that $\delta\colon k^S\times k^S\to k$ (also) is the bilinear form $\delta(\sum_{a\in S} ar_a, \sum_{b\in S} bs_b) = \sum_{a\in S} r_as_a$.)

Proposition 9.5. *The Möbius function, weighting, coweighting, and Euler characteristic of ζ are*

$$\mu(a,b)=\delta(a,e_b),\qquad k_\zeta^\bullet=\delta(\bullet,1),\qquad k_\bullet^\zeta=\sum_{a\in S}\delta(a,e_\bullet),\qquad \chi(\zeta)=\sum_{a,b\in S}\delta(a,e_b)=\sum_{a\in S}\delta(a,1).$$

The Möbius inversion formula

$$x=M(\mu)M(\zeta)x=M(\mu)\Big(\sum_{b\in S}b\zeta(b,x)\Big)=\sum_{b\in S}e_b\zeta(b,x)=\sum_{a\in S}a\sum_{b\in S}\mu(a,b)\zeta(b,x) \tag{9.6}$$

holds for any $x\in M(\zeta;k)$.

Let x and y be elements of the Möbius k-algebra $M(\zeta;k)$. The product of x and y in $M(\zeta;k)$ is

$$x\cdot y=\Big(\sum_{b\in S}e_b\zeta(b,x)\Big)\cdot\Big(\sum_{b\in S}e_b\zeta(b,y)\Big)=\sum_{b\in S}e_b\zeta(b,x)\zeta(b,y)=\sum_{a\in S}a\sum_{b\in S}\mu(a,b)\zeta(b,x)\zeta(b,y) \tag{9.7}$$

as the elements e_b, $b\in S$, are orthogonal idempotents in $M(\zeta,k)$.

Definition 9.8. Suppose that the sub-R-module R^S of the Möbius k-algebra $M(\zeta;k)$ is stable under multiplication and contains the unit element. The Möbius R-algebra of ζ, $M(\zeta;R)$, is the R-algebra R^S with product and unit inherited from $M(\zeta;k)$.

The Möbius R-algebra of ζ is defined if and only if the conditions of Definition 9.8 are satisfied.

Corollary 9.9. *The Möbius R-algebra for ζ is defined if and only if*

(1) $\sum_{b\in S}\mu(a,b)\zeta(b,c)\zeta(b,d)\in R$ *for all* $a,c,d\in S$, *and*
(2) $k_\zeta^a\in R$ *for all* $a\in S$.

Proof. Let c and d be elements of the basis S. According to the product formula (9.7)

$$c\cdot d\in M(\zeta;R)\iff \forall a\in S\colon\ \sum_{b\in S}\mu(a,b)\zeta(b,c)\zeta(b,d)\in R.$$

Thus the first condition is equivalent to the condition that $c\cdot d$ lies in $M(\zeta;R)$ for any two $c,d\in S$. The second condition is equivalent to the condition that the unit element of $M(\zeta;k)$ (9.4) lies in $M(\zeta;R)$. □

Remark 9.10. The sum of the coefficients in the product $x\cdot y$ (9.7) is

$$\sum_{a\in S}\delta(a,x\cdot y)=\sum_{a\in S}\sum_{b\in S}\mu(a,b)\zeta(b,x)\zeta(b,y)=\sum_{b\in S}\sum_{a\in S}\mu(a,b)\zeta(b,x)\zeta(b,y)=\sum_{b\in S}k_b^\zeta\zeta(b,x)\zeta(b,y).$$

In particular, if ζ has a constant row, ie if there are $a\in S$ and $r\in R$ so that $\zeta(a,b)=r$ for all $b\in S$, then the $k_b^\zeta=r^{-1}\delta(a,b)$ (Example 2.7), and the sum of coefficients equals r for all $x,y\in S$.

Suppose that $\zeta_1=\big(\zeta_1(a,b)\big)_{a,b\in S}$ and $\zeta_2=\big(\zeta_2(a,b)\big)_{a,b\in S}$ are two square matrices indexed by S with entries in R and with nonzero determinants. The k-algebras $M(\zeta_1;k)$ and $M(\zeta_2;k)$ are of course isomorphic as abstract k-algebras. By the very construction, there is a commutative diagram

$$\begin{array}{ccc} M(\zeta_1;k) & \xrightarrow[\cong]{\varphi(\zeta_1,\zeta_2)=M(\mu_2)M(\zeta_1)} & M(\zeta_2;k)\\ & {\scriptstyle M(\zeta_1)}\searrow\cong \quad \cong\swarrow{\scriptstyle M(\zeta_2)} & \\ & M(\delta;k) & \end{array} \tag{9.11}$$

of k-algebra isomorphisms. However, if $M(\zeta_1;R)$ and $M(\zeta_2;R)$ are defined, it is not always the case that the k-algebra isomorphism $\varphi(\zeta_1,\zeta_2)=M(\mu_2)M(\zeta_1)$ restricts to an R-algebra isomorphism between $M(\zeta_1;R)$ and $M(\zeta_2;R)$.

Corollary 9.12. *Suppose that $M(\zeta_1;R)$ and $M(\zeta_2;R)$ are defined. If all entries of the matrix $M(\mu_2)M(\zeta_1)$ lie in R and the determinant in $R^\times$, then $M(\zeta_1;R)$ and $M(\zeta_2;R)$ are isomorphic R-algebras.*

As a special, and obvious case, we note that if ζ is invertible over R then the Möbius R-algebra for ζ is defined, and $M(\zeta;R)$ and $M(\delta;R)$ are isomorphic R-algebras.

9.2. The Möbius algebra of a finite category $\mathcal{C}$ and of $[\mathcal{C}]$

Let $\mathcal{C}$ be a finite category and $[\mathcal{C}]$ its finite set of isomorphism classes of objects as in Section 2.4.

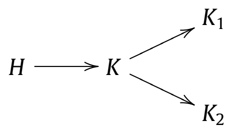
Definition 9.13 (*[17, Definition 4.1, Theorem 4.5], [32, Theorem 1]*)**.**

- If $\mathcal{C}$ has Möbius inversion then its rational Möbius algebra $M(\mathcal{C};\mathbf{Q})$ is the rational Möbius algebra of $\zeta(\mathcal{C})$.
- If $[\mathcal{C}]$ has Möbius inversion then its rational Möbius algebra $M([\mathcal{C}];\mathbf{Q})$ is the rational Möbius algebra of $[\zeta(\mathcal{C})]$.

Assume that $[\mathcal{C}]$ has Möbius inversion. The product in $M([\mathcal{C}];\mathbf{Q})$ is $\mathbf{Q}$-bilinear. The product of two basis elements

$$\forall K_1, K_2 \in [\mathcal{C}]: K_1 \cdot K_2 = \sum_{H\in[\mathcal{C}]} H \sum_{K\in[\mathcal{C}]} \mu(H,K)\zeta(K,K_1)\zeta(K,K_2) \tag{9.14}$$

is given by Eq. (9.7). Since $[\mu](H,K) \neq 0 \Longrightarrow [\zeta](H,K) \neq 0$ [23, Theorem 4.1], the inner sum runs over all $K \in [\mathcal{C}]$ that admit morphisms



in $\mathcal{C}$.

Proposition 9.15. *Let $\mathcal{I}$ be a left ideal and $\mathcal{J}$ a right ideal in $\mathcal{C}$ (Remark 2.6).*

(1) *The projection map $M([\mathcal{C}];\mathbf{Q}) \to M([\mathcal{I}];\mathbf{Q})$ is a homomorphism of $\mathbf{Q}$-algebras.*
(2) *$M([\mathcal{J}];\mathbf{Q})$ is an ideal in the Möbius algebra $M([\mathcal{C}];\mathbf{Q})$.*
(3) *$M([\mathcal{I}];\mathbf{Q}) \cong M([\mathcal{C}];\mathbf{Q})/M([\mathcal{C}]-[\mathcal{I}];\mathbf{Q})$.*

Proof. Since $[\mathcal{C}]$ has Möbius inversion, also $[\mathcal{I}]$ and $[\mathcal{J}]$ have Möbius inversion and their Möbius functions are restrictions of the Möbius function for $[\mathcal{C}]$ (Remark 2.6).
(1) It follows from (9.14) that the projection map preserves products. Since the weighting for $[\mathcal{C}]$ restricts to a weighting for $[\mathcal{I}]$, the projection also preserves units (9.4).
(2) Suppose that $K_1 \in [\mathcal{C}]$ and $K_2 \in [\mathcal{J}]$. If $H \in [\mathcal{C}]$ occurs in the product $K_1 \cdot K_2$ (9.14) with a nonzero rational coefficient, then $[\zeta](H,K_2) \neq 0$. As $\mathcal{J}$ is a right ideal in $\mathcal{C}$, H is in $[\mathcal{J}]$. This means that $K_1 \cdot K_2$ lies in the vector space $M([\mathcal{J}];\mathbf{Q})$ spanned by $[\mathcal{J}]$.
(3) The kernel of the projection of algebras $M([\mathcal{C}];\mathbf{Q}) \to M([\mathcal{I}];\mathbf{Q})$ is the ideal $M([\mathcal{C}]-[\mathcal{I}];\mathbf{Q})$. □

10. Möbius algebras of *p*-subgroup categories

In this section we discuss Möbius algebras of the p-subgroup categories $\mathcal{C} = \mathcal{S}_G, \mathcal{T}_G, \mathcal{L}_G, \mathcal{F}_G, \mathcal{O}_G, \widetilde{\mathcal{F}}_G$. We also consider sub- and supercategories of $\mathcal{C}$.

Definition 10.1. Let $\mathcal{C}$ be one of the four p-subgroup categories $\mathcal{S}_G$, $\mathcal{T}_G$, $\mathcal{F}_G$ or $\mathcal{O}_G$. Then $\overline{\mathcal{C}}$ denotes the corresponding category whose objects are *all* subgroups of G and whose morphism sets are defined by the same coset formulas as in $\mathcal{C}$ (see Section 1.1).

In all cases $[\mathcal{C}]$ has Möbius inversion (Corollary 2.15). Thus we may consider the Möbius algebras $M([\mathcal{C}];\mathbf{Q})$. From Proposition 9.15 we have that

- $M([\mathcal{C}];\mathbf{Q})$ is an ideal in $M([\overline{\mathcal{C}}];\mathbf{Q})$ for $\mathcal{C} = \mathcal{S}_G, \mathcal{T}_G, \mathcal{F}_G, \mathcal{O}_G$.
- $M([\mathcal{C}^*];\mathbf{Q}) \cong M([\mathcal{C}];\mathbf{Q})/M([1];\mathbf{Q})$
- $M([\mathcal{C}^{sc}];\mathbf{Q}) \cong M([\mathcal{C}^*];\mathbf{Q})/M([\mathcal{C}^*]-[\mathcal{C}^{sc}]);\mathbf{Q})$.

Example 10.2. Möbius algebras of finite categories include as a special case the classical Burnside algebra of G. If X is a left G-set and H a subgroup of G, the mark of $[H]$ in X is the number $|{}^HX|$ of H-fixed points in X. Left G-sets are determined up to isomorphism by the marks they put on the conjugacy classes of subgroups of G [9, Chapter XII, Section 180] [11, Lemma 1]. If also K is a subgroup of G then the mark of H in the transitive left G-set G/K is

$$m(K,H) = |{}^H(G/K)| = |\{gK \in G/K \mid H^g \leq K\}| = |N_G(H,K)/K| = |\overline{\mathcal{O}}_G(H,K)|.$$

Thus Burnside's table of marks matrix for G, $(m(K,H))_{H,K}$, is the transpose of the ζ-matrix for $[\overline{\mathcal{O}}_G]$. Obviously the mark of H in the product G-set $G/K_1 \times G/K_2$ is the product $|\overline{\mathcal{O}}_G(H,K_1)||\overline{\mathcal{O}}_G(H,K_2)|$ of the marks of H in G/K_1 and G/K_2. This shows that the rational Burnside algebra of G is the rational Möbius algebra $M([\overline{\mathcal{O}}_G];\mathbf{Q})$ of the orbit category of G.

Table 6
Möbius functions and idempotents in rational Möbius algebras $M([\mathcal{C}];\mathbf{Q})$.

$\mathcal{C}$	$[\mu(\mathcal{C})]([H],[K])$	$[\mu(\mathcal{C})]([H],[K])$	$e_{[K]}^{[\mathcal{C}]}$
$\mathcal{S}_G$	$\mu(H,K)$	$\mu(H,K)$	$\sum_{1\le H\le K} H\mu(H,K)$
$\mathcal{T}_G$	$[\mu]([H],[K])$	$\frac{1}{\lvert N_G(H)\rvert}\sum_{B\in[K]}\mu(H,B)$	$\frac{1}{\lvert \mathcal{T}_G(K)\rvert}\sum_{1\le H\le K}[H]\mu(H,K)$
$\mathcal{L}_G$	$[\mu]([H],[K])\lvert O^pC_G(K)\rvert$	$\frac{\lvert O^pC_G(K)\rvert}{\lvert N_G(H)\rvert}\sum_{B\in[K]}\mu(H,B)$	$\frac{1}{\lvert \mathcal{L}_G(K)\rvert}\sum_{1\le H\le K}[H]\mu(H,K)$
$\mathcal{F}_G$	$[\mu]([H],[K])\lvert C_G(K)\rvert$	$\frac{\lvert C_G(K)\rvert}{\lvert N_G(H)\rvert}\sum_{B\in[K]}\mu(H,B)$	$\frac{1}{\lvert \mathcal{F}_G(K)\rvert}\sum_{1\le H\le K}[H]\mu(H,K)$
$\mathcal{O}_G$	$\lvert H\rvert[\mu]([H],[K])$	$\frac{1}{\lvert \mathcal{O}_G(H)\rvert}\sum_{B\in[K]}\mu(H,B)$	$\frac{1}{\lvert N_G(K)\rvert}\sum_{1\le H\le K}[H]\lvert H\rvert\mu(H,K)$
$\widetilde{\mathcal{F}}_G^{sc}$	$\lvert H\rvert[\mu]([H],[K])\lvert C_G(K)\rvert_{p'}$	$\frac{\lvert C_G(K)\rvert_{p'}}{\lvert \mathcal{O}_G^{sc}(H)\rvert}\sum_{B\in[K]}\mu(H,B)$	$\frac{\lvert C_G(K)\rvert_{p'}}{\lvert N_G(K)\rvert}\sum_{H\le K}[H]\lvert H\rvert\mu(H,K)$

Example 10.3 (*Burnside's Example*)**.** Let $G = A_4$ be the alternating group on four elements. We shall consider the orbit categories $[\overline{O}_G]$, $[\mathcal{O}_G]$, $[\mathcal{O}_G^*]$, $[\mathcal{O}_G^{sc}]$ of all subgroups, all 2-subgroup, all nonidentity 2-subgroups, and all 2-selfcentralizing 2-subgroups of G.

The subgroup conjugacy classes of G are H_1, H_2, H_3, H_4, H_5 of orders 1, 2, 3, 4, 12 where $H_1 = 1$ and $H_5 = G$. Of the 2-subgroups, H_1, H_2, H_4, only H_4 is 2-self-centralizing. The table of marks $[\zeta(\overline{\mathcal{O}}_G)]$, that can be found in [9, Section 185], produces the multiplication table for the Möbius algebra $M([\overline{\mathcal{O}}_G];\mathbf{Q})$ of all subgroups of G and its ideal $M([\mathcal{O}_G];\mathbf{Q}) = (H_1, H_2, H_4)$.

$M([\overline{\mathcal{O}}_G];\mathbf{Q})$	H_1	H_2	H_3	H_4	H_5
H_1	$12H_1$	$6H_1$	$4H_1$	$3H_1$	H_1
H_2	·	$2H_1+2H_2$	$2H_1$	$3H_2$	H_2
H_3	·	·	H_1+H_3	H_1	H_3
H_4	·	·	·	$3H_4$	H_4
H_5	·	·	·	·	H_5

$M([\mathcal{O}_G];\mathbf{Q})$	H_1	H_2	H_4
H_1	$12H_1$	$6H_1$	$3H_1$
H_2	·	$2H_1+2H_2$	$3H_2$
H_4	·	·	$3H_4$

The Möbius algebras $M([\mathcal{O}_G^*];\mathbf{Q}) = (H_1,H_2,H_4)/(H_1)$ and $M([\mathcal{O}_G^{sc}];\mathbf{Q}) = (H_1,H_2,H_4)/(H_1,H_2)$ are quotients of the ideal (H_1, H_2, H_4). Their multiplication tables are

$M([\mathcal{O}_G^*];\mathbf{Q})$	H_2	H_4
H_2	$2H_2$	$3H_2$
H_4	·	$3H_4$

$M([\mathcal{O}_G^{sc}];Q)$	H_4
H_4	$3H_4$

The unit of $M([\overline{\mathcal{O}}_G];\mathbf{Q})$ is $H_5 = G$ while $\frac{1}{3}H_4$ is the unit of $M([\mathcal{O}_G];\mathbf{Q})$, $M([\mathcal{O}_G^*];\mathbf{Q})$, and $M([\mathcal{O}_G^{sc}];Q)$. We read off that $\chi(\overline{\mathcal{O}}_G) = 1$, $\chi(\mathcal{O}_G) = \chi(\mathcal{O}_G^*) = \chi(\mathcal{O}_G^{sc}) = \frac{1}{3}$ from the expressions for the units (9.4).

We first calculate the Möbius function for $[\mathcal{C}]$ and the idempotents $e_{[K]} \in M([\mathcal{C}];\mathbf{Q})$, $[K] \in [\mathcal{C}]$, in case $\mathcal{C}$ is a p-subgroup category.

Proposition 10.4. *The Möbius function $[\mu]$ for $[\mathcal{C}]$ and the idempotents $e_{[K]}$ (9.3) of the Möbius algebra $M([\mathcal{C}];\mathbf{Q})$ are as in Table 6 for the categories $\mathcal{C} = \mathcal{S}_G, \mathcal{L}_G, \mathcal{F}_G, \mathcal{O}_G, \widetilde{\mathcal{F}}_G^{sc}$.*

Proof. The second column of Table 6 displays the Möbius functions already determined in Table 2 and Eq. (8.3). The third column displays rewritings, more suitable for our purpose here, using Eq. (3.10). The fourth column shows the idempotents (9.3) in $M([\mathcal{C}];\mathbf{Q})$. The idempotents are calculated in a similar fashion, for example, for $[\mathcal{T}_G]$ we find that

$$
\begin{aligned}
e_{[K]}^{[\mathcal{T}_G]} &= \sum_{[H]}[H][\mu]([H],[K]) = \frac{1}{\lvert N_G(K)\rvert}\sum_{[H]}[H]\sum_{A\in[H]}\mu(A,K) = \frac{1}{\lvert N_G(K)\rvert}\sum_{H}[H]\mu(H,K) \\
&= \frac{1}{\lvert \mathcal{T}_G(K)\rvert}\sum_{H}[H]\mu(H,K)
\end{aligned}
$$

using (the equation just below) Eq. (3.10). □

Note that we have recovered, in a uniform way, the formulas [12, Proposition] [36] for the idempotents in $M([\mathcal{O}_G];\mathbf{Q})$ and (the nonexotic version of) the formula [10, Theorem 3.3] for the idempotents in $M([\widetilde{\mathcal{F}}_G^{sc}];\mathbf{Q})$.

We shall now determine the products in the six Möbius algebras of Table 6 using the product formula of Eq. (9.7).

The product in the Möbius algebra $M(\mathcal{S}_G;\mathbf{Q})$ of p-subgroups K_1 and K_2 of G is

$$
\begin{aligned}
K_1\cdot K_2 &= \sum_H H\sum_K \mu(H,K)\mathcal{S}_G(K,K_1)\mathcal{S}_G(K,K_2) = \sum_H H\sum_K \mu(H,K)\mathcal{S}_G(K,K_1\cap K_2) \\
&= \sum_H H\delta(H,K_1\cap K_2) = K_1\cap K_2.
\end{aligned}
$$

The product of K_1 and K_2 in the Möbius algebra $M([\mathcal{T}_G]; \mathbf{Q})$ is

$$\begin{aligned}
K_1 \cdot K_2 &= \sum_{H \in [\mathcal{T}_G]} H \sum_{K \in [\mathcal{T}_G]} [\mu](H,K)[\zeta(\mathcal{T}_G)](K,K_1)[\zeta(\mathcal{T}_G)](K,K_2) \\
&\overset{\text{Table 6}}{=} \sum_{H \in [\mathcal{T}_G]} \frac{H}{|N_G(H)|} \sum_{K \in \mathrm{Ob}(\mathcal{T}_G)} \mu(H,K)\mathcal{T}_G(K,K_1)\mathcal{T}_G(K,K_2) \\
&= \frac{1}{|G|} \sum_{H \in [\mathcal{T}_G]} |G : N_G(H)| H \sum_{K \in \mathrm{Ob}(\mathcal{T}_G)} \mu(H,K)\mathcal{T}_G(K,K_1)\mathcal{T}_G(K,K_2) \\
&= \frac{1}{|G|} \sum_{H \in \mathrm{Ob}(\mathcal{T}_G)} [H] \sum_{K \in \mathrm{Ob}(\mathcal{T}_G)} \sum_{(g_1,g_2) \in G \times G} \mu(H,K)\mathcal{S}_G(K,K_1^{g_1})\mathcal{S}_G(K,K_2^{g_2}) \\
&= \frac{1}{|G|} \sum_{H \in \mathrm{Ob}(\mathcal{T}_G)} [H] \sum_{(g_1,g_2) \in G \times G} \sum_{K \in \mathrm{Ob}(\mathcal{T}_G)} \mu(H,K)\mathcal{S}_G(K,K_1^{g_1} \cap K_2^{g_2}) \\
&= \frac{1}{|G|} \sum_{H \in \mathrm{Ob}(\mathcal{T}_G)} [H] \sum_{(g_1,g_2) \in G \times G} \delta(H, K_1^{g_1} \cap K_2^{g_2}) \\
&= \frac{1}{|G|} \sum_{(g_1,g_2) \in G \times G} [K_1^{g_1} \cap K_2^{g_2}] = \frac{1}{|G|} \sum_{(g_1,g_2) \in G \times G} [K_1 \cap K_2^{g_2 g_1^{-1}}] = \sum_{g \in G} [K_1^g \cap K_2].
\end{aligned}$$

The product of K_1 and K_2 in the Möbius algebra $M([\mathcal{F}_G]; \mathbf{Q})$ is

$$\begin{aligned}
K_1 \cdot K_2 &= \sum_{H \in [\mathcal{F}_G]} H \sum_{K \in [\mathcal{F}_G]} [\mu(\mathcal{F}_G)](H,K)[\zeta(\mathcal{F}_G)](K,K_1)[\zeta(\mathcal{F}_G)](K,K_2) \\
&\overset{\text{Table 6}}{=} \sum_{H \in [\mathcal{F}_G]} H \sum_{K \in [\mathcal{F}_G]} |C_G(K)|[\mu](H,K)|\mathcal{F}_G(K,K_1)||\mathcal{F}_G(K,K_2)| \\
&\overset{(7.1)}{=} \sum_{H \in [\mathcal{F}_G]} H \sum_{K \in [\mathcal{F}_G]} |C_G(K)|[\mu](H,K)|\mathcal{F}_G(K)|\mathcal{S}_G([K],K_1)|\mathcal{F}_G(K)|\mathcal{S}_G([K],K_2) \\
&= \sum_{H \in [\mathcal{F}_G]} H \sum_{K \in [\mathcal{F}_G]} |\mathcal{F}_G(K)|\mathcal{S}_G([K],K_1)\mathcal{S}_G([K],K_2)|N_G(K)|[\mu](H,K)
\end{aligned}$$

where all terms in the inner sum are integers. Alternatively,

$$\begin{aligned}
K_1 \cdot K_2 &= \sum_{H \in [\mathcal{F}_G]} H \sum_{K \in [\mathcal{F}_G]} [\mu(\mathcal{F}_G)](H,K)[\zeta(\mathcal{F}_G)](K,K_1)[\zeta(\mathcal{F}_G)](K,K_2) \\
&\overset{\text{Table 6}}{=} \sum_{H \in [\mathcal{F}_G]} \frac{H}{|N_G(H)|} \sum_{K \in \mathrm{Ob}(\mathcal{F}_G)} |C_G(K)|\mu(H,K)|\mathcal{F}_G(K,K_1)||\mathcal{F}_G(K,K_2)| \\
&= \frac{1}{|G|} \sum_{H \in \mathrm{Ob}(\mathcal{F}_G)} [H] \sum_{K \in \mathrm{Ob}(\mathcal{F}_G)} |C_G(K)|\mu(H,K)|\mathcal{F}_G(K,K_1)||\mathcal{F}_G(K,K_2)| \\
&= \frac{1}{|G|} \sum_{H \in \mathrm{Ob}(\mathcal{F}_G)} [H] \sum_{(g_1,g_2) \in G \times G} \sum_{K \in \mathrm{Ob}(\mathcal{F}_G)} \frac{\mu(H,K)}{|C_G(K)|} \mathcal{S}_G(K,K_1^{g_1})\mathcal{S}_G(K,K_2^{g_2}) \\
&= \frac{1}{|G|} \sum_{H \in \mathrm{Ob}(\mathcal{F}_G)} [H] \sum_{(g_1,g_2) \in G \times G} \sum_{K \in [H, K_1^{g_1} \cap K_2^{g_2}]} \frac{\mu(H,K)}{|C_G(K)|}.
\end{aligned}$$

Similarly, the product in $M([\mathcal{L}_G]; \mathbf{Q})$ is

$$\begin{aligned}
K_1 \cdot K_2 &= \sum_{H \in [\mathcal{L}_G]} H \sum_{K \in [\mathcal{L}_G]} |\mathcal{L}_G(K)|\mathcal{S}_G([K],K_1)\mathcal{S}_G([K],K_2)|N_G(K)|[\mu](H,K) \\
&= \frac{1}{|G|} \sum_{H \in \mathrm{Ob}(\mathcal{L}_G)} [H] \sum_{(g_1,g_2) \in G \times G} \sum_{K \in [H, K_1^{g_1} \cap K_2^{g_2}]} \frac{\mu(H,K)}{|O^p C_G(K)|}.
\end{aligned}$$

Table 7
Coefficient rings in Möbius algebras.

$\mathcal{C}$	$\mathcal{S}_G$	$\mathcal{T}_G$	$\mathcal{L}_G$	$\mathcal{F}_G$	$\mathcal{O}_G$	$\mathcal{O}_G^{sc}$	$\widetilde{\mathcal{F}}_G^{sc}$
R	$\mathbf{Z}$	$\mathbf{Z}[1/\|G\|]$	$\mathbf{Z}[1/\|G\|]$	$\mathbf{Z}[1/\|G\|]$	$\mathbf{Z}_{(p)}$	$\mathbf{Z}_{(p)}$	$\mathbf{Z}_{(p)}$

The product of K_1 and K_2 in the classical Burnside algebra $M([\mathcal{O}_G];\mathbf{Q})$ is

$$\begin{aligned}
K_1\cdot K_2 &= \sum_{H\in[\mathcal{O}_G]} H \sum_{K\in[\mathcal{O}_G]} [\mu(\mathcal{O}_G)](H,K)[\zeta(\mathcal{O}_G)](K,K_1)[\zeta(\mathcal{O}_G)](K,K_2)\\
&\overset{\text{Table 6}}{=} \sum_{H\in[\mathcal{O}_G]} \frac{H}{|\mathcal{O}_G(H)|}\sum_{K\in \mathrm{Ob}(\mathcal{O}_G)} \mu(H,K)|\mathcal{O}_G(K,K_1)||\mathcal{O}_G(K,K_2)|\\
&= \frac{1}{|G|}\sum_{H\in[\mathcal{O}_G]} H|G:N_G(H)||H| \sum_{K\in \mathrm{Ob}(\mathcal{O}_G)} \mu(H,K)|\mathcal{O}_G(K,K_1)||\mathcal{O}_G(K,K_2)|\\
&= \frac{1}{|G|}\sum_{H\in \mathrm{Ob}(\mathcal{O}_G)} [H]\frac{|H|}{|K_1||K_2|}\sum_{K\in \mathrm{Ob}(\mathcal{O}_G)}\sum_{(g_1,g_2)\in G\times G} \mu(H,K)\mathcal{S}_G(K,K_1^{g_1})\mathcal{S}_G(K,K_2^{g_2})\\
&= \frac{1}{|G|}\sum_{H\in \mathrm{Ob}(\mathcal{O}_G)} [H]\frac{|H|}{|K_1||K_2|}\sum_{(g_1,g_2)\in G\times G}\sum_{K\in \mathrm{Ob}(\mathcal{O}_G)} \mu(H,K)\mathcal{S}_G(K,K_1^{g_1})\mathcal{S}_G(K,K_2^{g_2})\\
&= \frac{1}{|G|}\sum_{H\in \mathrm{Ob}(\mathcal{O}_G)} [H]\frac{|H|}{|K_1||K_2|}\sum_{(g_1,g_2)\in G\times G} \delta(H,K_1^{g_1}\cap K_2^{g_2}) = \frac{1}{|G|}\sum_{(g_1,g_2)\in G\times G}[K_1^{g_1}\cap K_2^{g_2}]\frac{|K_1^{g_1}\cap K_2^{g_2}|}{|K_1||K_2|}\\
&= \sum_{g\in G}[K_1^g\cap K_2]\frac{|K_1^g\cap K_2|}{|K_1||K_2|} = \sum_{g\in K_1\backslash G/K_2}[K_1^g\cap K_2].
\end{aligned}$$

The final equality uses the identity $|K_1gK_2||K_1^g\cap K_2| = |K_1||K_2|$ for $g\in G$. Similarly, the product in $M([\widetilde{\mathcal{F}}_G^{sc}];\mathbf{Q})$ of p-selfcentralizing p-subgroups K_1 and K_2 is

$$K_1\cdot K_2 = \sum_{g\in K_1O^pC_G(K_1)\backslash G/K_2O^pC_G(K_2)} [K_1^g\cap K_2]$$

where the conjugacy class $[K_1^g\cap K_2]=0$ in case $K_1^g\cap K_2$ is not p-selfcentralizing.

Proposition 10.5. *Multiplication in the rational Möbius algebra $M([\mathcal{C}];\mathbf{Q})$ restricts to a bilinear map*

$$M([\mathcal{C}];\mathbf{Z})\times M([\mathcal{C}];\mathbf{Z})\to M([\mathcal{C}];\mathbf{Z}),\quad \mathcal{C}=\mathcal{S}_G,\mathcal{L}_G,\mathcal{F}_G,\mathcal{O}_G,\mathcal{O}_G^{sc},\widetilde{\mathcal{F}}_G^{sc}.$$

Proof. This follows from the explicit expressions for the product that we just worked out. □

We already know that

- The weighting for $\mathcal{S}_G$ is a $\mathbf{Z}$-valued function.
- The weightings for $[\mathcal{T}_G]$, $[\mathcal{L}_G]$, and $[\mathcal{F}_G]$ are $\mathbf{Z}[1/|G|]$-valued functions.
- The weightings for $[\mathcal{O}_G]$, $[\mathcal{O}_G^{sc}]$ and $[\widetilde{\mathcal{F}}_G^{sc}]$ are $\mathbf{Z}_{(p)}$-valued functions.

The weighting for $[\mathcal{S}_G]=\mathcal{S}_G$ is a $\mathbf{Z}$-valued function (Proposition 3.14). The weightings for $[\mathcal{T}_G]$, $[\mathcal{L}_G]$, and $[\mathcal{F}_G]$ are $\mathbf{Z}[1/|G|]$-valued as their upper triangular ζ-matrices are invertible over this ring; see the proof of Corollary 2.15. The weightings for $[\mathcal{O}_G]$ (Corollary 4.2.(1)), its left ideal $[\mathcal{O}_G^{sc}]$ (Remark 2.6), and $[\widetilde{\mathcal{F}}_G^{sc}]$ (Proposition 8.5) are $\mathbf{Z}_{(p)}$-valued.

Corollary 10.6 (*Möbius R-Algebras of p-Subgroups*)**.** *If $(\mathcal{C},R)$ is as in Table 7 then the Möbius R-algebra for $[\mathcal{C}]$ is defined. $M([\mathcal{C}];R)$ is a commutative R-algebra with unit.*

Proof. This follows directly from Proposition 9.9 using Proposition 10.5 and the above information about values of weightings. □

In case $[\mathcal{C}]$ has a final element, i.e. an element b such that $[\zeta](a,b)=1$ for all $a\in[\mathcal{C}]$, then the unit actually lies in the lattice $M([\mathcal{C}];\mathbf{Z})$. For example, the classical Möbius ring for the orbit category of all subgroups of G, $M([\overline{\mathcal{O}}_G];\mathbf{Z})$, is a ring with unit $1=[G]$ given by (the conjugacy class of) G.

Let $\mathcal{C}$ be any of the four p-subgroup categories $\mathcal{T}_G$, $\mathcal{L}_G$, $\mathcal{F}_G$, $\mathcal{O}_G$. There are isomorphisms of $\mathbf{Z}[1/|G|]$-algebras

$$[\zeta(\mathcal{C})]\colon M([\mathcal{C}];\mathbf{Z}[1/|G|]) \xrightarrow{\cong} M(\delta;\mathbf{Z}[1/|G|])$$

represented by upper triangular matrices with entries in $\mathbf{N}$. The ζ-matrix for $\mathcal{C}$ is invertible over $\mathbf{Z}[1/|G|]$. Therefore the Möbius algebra for $[\mathcal{C}]$ is defined over $\mathbf{Z}[1/|G|]$ and it is isomorphic to $M(\delta;\mathbf{Z}[1/|G|])$. (The same argument immediately shows that the Möbius algebra over R for the poset of conjugacy classes of subgroups of G is isomorphic to the Burnside ring $M([\overline{\mathcal{O}}_G];R)$ over R when the order of G is invertible in R [32, Theorem 2].)

By a *nonnegative integral matrix* we understand a matrix with entries in the semiring $\mathbf{N} = \{0,1,2,\dots\}$ of natural numbers.

Proposition 10.7. *There are isomorphisms of* $\mathbf{Z}[1/|G|]$*-algebras*

$$\varphi([\mathcal{T}_G],[\mathcal{L}_G])\colon M([\mathcal{T}_G];\mathbf{Z}[1/|G|]) \xrightarrow{\cong} M([\mathcal{L}_G];\mathbf{Z}[1/|G|])$$
$$\varphi([\mathcal{T}_G],[\mathcal{F}_G])\colon M([\mathcal{T}_G];\mathbf{Z}[1/|G|]) \xrightarrow{\cong} M([\mathcal{F}_G];\mathbf{Z}[1/|G|])$$
$$\varphi([\mathcal{T}_G],[\mathcal{O}_G])\colon M([\mathcal{T}_G];\mathbf{Z}[1/|G|]) \xrightarrow{\cong} M([\mathcal{O}_G];\mathbf{Z}[1/|G|])$$

represented by upper triangular nonnegative integral matrices.

Proof. Let $\zeta = (\zeta_{ij})$ be an invertible upper triangular rational matrix with inverse μ. Write $\zeta_i = \zeta_{ii}$ for the diagonal entries of ζ. Also, let $\Delta = \Delta(c_i)$ be a diagonal matrix with the rational numbers c_i in the diagonal. Then $\mu\Delta\zeta$ is an upper triangular matrix with diagonal entries c_i and off-diagonal entries

$$(\mu\Delta\zeta)_{ij} = \sum_{k=1}^{j-i}(-1)^{k+1}\frac{\zeta_{i_0i_1}\zeta_{i_1i_2}\cdots\zeta_{i_{k-1}i_k}}{\zeta_{i_0}\zeta_{i_1}\cdots\zeta_{i_{k-1}}}(c_{i_{k-1}}-c_{i_k}), \quad i<j,$$

with summation over all k-simplices $i=i_0<i_1<\cdots<i_k=j$ from i to j. In particular we see that $\mu\Delta\zeta$ has nonnegative integer entries when ζ_i is a factor in ζ_{ij} for all $j \geq i$ and Δ is any diagonal matrix with decreasing entries in $\mathbf{N}$. This applies to $\zeta = [\zeta(\mathcal{T}_G)]$.

- Because $\Delta(|O^pC_G(H)|)[\zeta(\mathcal{L}_G)] = [\zeta(\mathcal{T}_G)]$, where H runs through the p-subgroups of G, the $\mathbf{Q}$-algebra isomorphism (9.11) $M([\mathcal{T}_G];\mathbf{Q}) \xrightarrow{\cong} M([\mathcal{L}_G];\mathbf{Q})$ is given by the integral matrix
$$\varphi([\mathcal{T}_G],[\mathcal{L}_G]) = [\mu(\mathcal{L}_G)][\zeta(\mathcal{T}_G)] = [\mu(\mathcal{T}_G)]\Delta(|O^pC_G(H)|)[\zeta(\mathcal{T}_G)]$$
with entries in $\mathbf{N}$.
- Because $\Delta(|C_G(H)|)[\zeta(\mathcal{F}_G)] = [\zeta(\mathcal{T}_G)]$, where H runs through the p-subgroups of G, the $\mathbf{Q}$-algebra isomorphism (9.11) $M([\mathcal{T}_G];\mathbf{Q}) \xrightarrow{\cong} M([\mathcal{F}_G];\mathbf{Q})$ is given by the integral matrix
$$\varphi([\mathcal{T}_G],[\mathcal{F}_G]) = [\mu(\mathcal{F}_G)][\zeta(\mathcal{T}_G)] = [\mu(\mathcal{T}_G)]\Delta(|C_G(H)|)[\zeta(\mathcal{T}_G)]$$
with entries in $\mathbf{N}$.
- Because $[\zeta(\mathcal{O}_G)]\Delta(|K|) = [\zeta(\mathcal{T}_G)]$, where K runs through the p-subgroups of G, the $\mathbf{Q}$-algebra isomorphism (9.11) $M([\mathcal{T}_G];\mathbf{Q}) \xrightarrow{\cong} M([\mathcal{O}_G];\mathbf{Q})$ is given by the diagonal matrix
$$\varphi([\mathcal{T}_G],[\mathcal{O}_G]) = [\mu(\mathcal{O}_G)][\zeta(\mathcal{T}_G)] = \Delta(|H|)$$
with entries in $\mathbf{N}$.

This finishes the proof. □

Because $[\zeta(\widetilde{\mathcal{F}}_G^{sc})]\Delta(|K|) = [\zeta(\mathcal{L}_G^{sc})]$, the $\mathbf{Q}$-algebra isomorphism (9.11) $M([\mathcal{T}_G^{sc}];\mathbf{Q}) \xrightarrow{\cong} M([\widetilde{\mathcal{F}}_G^{sc}];\mathbf{Q})$ is given by the upper triangular integral matrix

$$\varphi([\widetilde{\mathcal{F}}_G^{sc})][\mathcal{T}_G^{sc}]) = [\mu(\widetilde{\mathcal{F}}_G^{sc})][\zeta(\mathcal{T}_G^{sc})] = \Delta(|H|)[\mu(\mathcal{L}_G^{sc})][\zeta(\mathcal{T}_G^{sc})] = \Delta(|H|)\varphi(\mathcal{T}_G^{sc},\mathcal{L}_G^{sc}) \geq 0$$

where $\varphi([\mathcal{T}_G^{sc}],[\mathcal{L}_G^{sc}])$ is the submatrix of $\varphi([\mathcal{T}_G],[\mathcal{L}_G])$ of Proposition 10.7 determined by the p-selfcentralizing p-subgroups.

Lemma 10.8. *Let H and M be p-selfcentralizing p-subgroups of G with $H \leq M$. Then*

$$\sum_{K\in[H,M]}\mu(H,K)|C_G(K)|_{p'} = |\{x \in G\{p'\} \mid H = C_M(x)\}|$$

where $G\{p'\}$ denotes the set of elements of G of order prime to p.

Proof. We have

$$\sum_{K\in[H,M]} \mu(H,K)|O^pC_G(K)| = \sum_{(K,x)} \mu(H,K) = \sum_{x\in G\{p'\}} \sum_{K\in[H,C_M(x)]} \mu(H,K) = \sum_{x\in G\{p'\}} \delta(H, C_M(x))$$

where (K, x) runs through the set

$$\begin{aligned}\{(K,x) \mid K\in[H,M],\ x\in O^pC_G(K)\} &= \{(K,x) \mid x\in O^pC_G(H),\ K\in[H,C_M(x)]\}\\ &= \{(K,x) \mid x\in G\{p'\},\ K\in[H,C_M(x)]\}\end{aligned}$$

We use properties of p-selfcentralizing p-subgroups from Lemma 8.2 for these rewritings. □

The next result is a reformulation of [10, Theorem 3.11].

Theorem 10.9. *There is an isomorphism of algebras*

$$\varphi([\mathcal{O}_G^{sc}], [\widetilde{\mathcal{F}}_G^{sc}]) \colon M([\mathcal{O}_G^{sc}]; \mathbf{Z}_{(p)}) \xrightarrow{\cong} M([\widetilde{\mathcal{F}}_G^{sc}]; \mathbf{Z}_{(p)})$$

given by an upper triangular nonnegative integral matrix.

Proof. The Möbius $\mathbf{Z}_{(p)}$-algebras are defined for $[\mathcal{O}_G^{sc}]$ and $[\widetilde{\mathcal{F}}_G^{sc}]$ according to Corollary 10.6. Because $|\mathcal{O}_G^{sc}(H,K)| = |C_G(H)|_{p'}|\widetilde{\mathcal{F}}_G^{sc}(H,K)|$ for all p-selfcentralizing p-subgroups H, K, of G we get

$$\varphi([\mathcal{O}_G^{sc}], [\widetilde{\mathcal{F}}_G^{sc}]) = [\mu(\widetilde{\mathcal{F}}_G^{sc})][\zeta(\mathcal{O}_G^{sc})] = [\mu(\mathcal{O}_G^{sc})]\Delta(|C_G(K)|_{p'})[\zeta(\mathcal{O}_G^{sc})].$$

We must show that the $([H], [L])$-entry, for all $[H], [L] \in [\mathcal{O}_G^{sc}]$, of this matrix,

$$\sum_{[K]\in[\mathcal{O}_G^{sc}]} [\mu(\mathcal{O}_G^{sc})](H,K)|C_G(K)|_{p'}[\zeta(\mathcal{O}_G^{sc})](K,L), \tag{10.10}$$

is a nonnegative integer and that the determinant is a unit in $\mathbf{Z}_{(p)}$ (Corollary 9.12). The determinant is $\prod |C_G(K)|_{p'}$ which certainly is a unit in $\mathbf{Z}_{(p)}$. Using the expression for $[\mu(\mathcal{O}_G^{sc})]$ from Table 6, the sum (10.10) becomes

$$\begin{aligned}\sum_{[K]}[\mu(\mathcal{O}_G^{sc})](H,K)|C_G(K)|_{p'}[\zeta(\mathcal{O}_G^{sc})](K,L) &= \frac{1}{|\mathcal{O}_G^{sc}(H)|}\sum_{[K]}\sum_{M\in[K]} \mu(H,M)|C_G(K)|_{p'}|\mathcal{O}_G^{sc}(M,L)|\\ &= \frac{1}{|\mathcal{O}_G^{sc}(H)|}\sum_{K} \mu(H,K)|C_G(K)|_{p'}|\mathcal{O}_G^{sc}(K,L)|\\ &= \frac{1}{|\mathcal{O}_G^{sc}(H)||L|}\sum_{K} \mu(H,K)|C_G(K)|_{p'}|N_G(K,L)|\\ &= \frac{1}{|\mathcal{O}_G^{sc}(H)||L|}\sum_{x\in G}\sum_{K} \mu(H,K)|C_G(K)|_{p'}\mathcal{S}_G^{sc}(K,L^x)\\ &= \frac{|\mathcal{O}_G^{sc}(L)|}{|\mathcal{O}_G^{sc}(H)|}\sum_{M\in[L]}\sum_{K} \mu(H,K)|C_G(K)|_{p'}\mathcal{S}_G^{sc}(K,M)\\ &= \frac{|\mathcal{O}_G^{sc}(L)|}{|\mathcal{O}_G^{sc}(H)|}\sum_{M\in[L]}\sum_{K\in[H,M]} \mu(H,K)|C_G(K)|_{p'}.\end{aligned}$$

According to Lemma 10.8, this last expression equals

$$\begin{aligned}\frac{|\mathcal{O}_G^{sc}(L)|}{|\mathcal{O}_G^{sc}(H)|}|\{(M,x)\in[L]\times G\{p'\} \mid H = C_M(x)\}| &= \frac{1}{|L|}\frac{1}{|\mathcal{O}_G^{sc}(H)|}|\{(g,x)\in G\times G\{p'\} \mid H = C_{L^g}(x)\}|\\ &= \frac{1}{|L|}\frac{1}{|\mathcal{O}_G^{sc}(H)|}|\{(g,x)\in G\times G\{p'\} \mid H = C_{L^g}(x^g)\}|\\ &= \frac{1}{|L|}\frac{1}{|\mathcal{O}_G^{sc}(H)|}|\{(g,x)\in G\times G\{p'\} \mid H^g = C_L(x)\}|\\ &= |L:H|^{-1}|\{(K,x)\in[H]\times G\{p'\} \mid K = C_L(x)\}|.\end{aligned}$$

The group L acts by conjugation on the set $\{(K,x)\in[H]\times G\{p'\} \mid K = C_L(x)\}$. Observe that the stabilizer $L_{(K,x)}$ of (K, x) is a subgroup of K: if the element y of L satisfies $K^y = K$ and $x^y = x$ then $y \in C_L(x) = K$. The cardinality of the L-orbit containing (K, x),

$$\frac{|L|}{|L_{(K,x)}|} = |L:K|\frac{|K|}{|L_{(K,x)}|} = |L:H|\frac{|K|}{|L_{(K,x)}|}$$

is a nonnegative integral multiple of $|L : H|$ and we can now conclude that (10.10) is a nonnegative integer. □

It is now natural to define the rational (centric) Burnside ring of an abstract Frobenius category $\mathcal{F}$ to be the Möbius algebra $M([\widetilde{\mathcal{F}}^{sc}]; \mathbf{Q})$ [10, Definition 2.11].

Example 10.11 ($G = A_7$, $p = 2$)**.** The alternating group $G = A_7$ of order $2520 = 2^3 3^3 5^1 7^1$ contains five nonidentity 2-subgroup conjugacy classes H_1–H_5 of orders 2, 4, 4, 4, 8 and lengths (normalizer indices) 105, 35, 105, 315, 315. The four classes H_2, H_3, H_4, H_5 are p-selfcentralizing and $\Delta(|C_G(H)|_{2'}) = (3, 1, 1, 1)$. Here are the multiplication tables for the rational Möbius algebras

- $M([\mathcal{O}_G^{sc}]; \mathbf{Q})$ with unit $1 = -\frac{4}{9}H_2 - \frac{1}{3}H_3 + H_5$ and Euler characteristic $\chi = \frac{2}{9}$
- $M([\mathcal{F}_G^{sc}]; \mathbf{Q})$ with unit $1 = -\frac{1}{12}H_2 - \frac{1}{12}H_3 + \frac{1}{4}H_4 + \frac{1}{4}H_5$ and Euler characteristic $\chi = \frac{1}{3}$
- $M([\widetilde{\mathcal{F}}_G^{sc}]; \mathbf{Q})$ with unit $1 = -\frac{1}{3}H_2 - \frac{1}{3}H_3 + H_5$ and Euler characteristic $\chi = \frac{1}{3}$

$M([\mathcal{O}_G^{sc}]; \mathbf{Q})$	H_2	H_3	H_4	H_5
H_2	$18H_2$	0	0	$9H_2$
H_3	$\cdot$	$6H_3$	0	$3H_3$
H_4	$\cdot$	$\cdot$	$2H_4$	H_4
H_5	$\cdot$	$\cdot$	$\cdot$	$4H_2 + H_3 + H_5$

$M([\mathcal{F}_G^{sc}]; \mathbf{Q})$	H_2	H_3	H_4	H_5
H_2	$6H_2$	0	0	$6H_2$
H_3	$\cdot$	$6H_3$	0	$6H_3$
H_4	$\cdot$	$\cdot$	$2H_4$	$2H_4$
H_5	$\cdot$	$\cdot$	$\cdot$	$2H_2 + 2H_3 - 2H_4 + 4H_5$

$M([\widetilde{\mathcal{F}}_G^{sc}]; \mathbf{Q})$	H_2	H_3	H_4	H_5
H_2	$6H_2$	0	0	$3H_2$
H_3	$\cdot$	$6H_3$	0	$3H_3$
H_4	$\cdot$	$\cdot$	$2H_4$	H_4
H_5	$\cdot$	$\cdot$	$\cdot$	$H_2 + H_3 + H_5$

The upper triangular nonnegative integral matrix

$$\varphi([\mathcal{O}_G^{sc}], [\widetilde{\mathcal{F}}_G^{sc}]) = \begin{pmatrix} 3 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

is the $\mathbf{Z}_{(2)}$-algebra isomorphism $M([\mathcal{O}_G^{sc}]; \mathbf{Z}_{(2)}) \cong M([\widetilde{\mathcal{F}}_G^{sc}]; \mathbf{Z}_{(2)})$ given in Theorem 10.

Let $\mathcal{S}_G^{sc}/G$ denote the poset of conjugacy classes of p-selfcentralizing p-subgroups of G ordered by subconjugation.

Corollary 10.12. *Let H and K be objects of $\mathcal{S}_G^{sc}/G$ with $H < K$. Then*

$$\frac{|H|}{|K|}\sum_{k\geq 1}(-1)^{k+1}\frac{|N_G(H_{i_0}, H_{i_1})|\cdots|N_G(H_{i_{k-1}}, H_{i_k})|}{|N_G(H_{i_0})|\cdots|N_G(H_{i_{k-1}})|}(|C_G(H_{i_{k-1}})|_{p'} - |C_G(H_{i_k})|_{p'})$$

is a nonnegative integer. The summation runs over all $k \geq 1$ and all k-simplices in $\mathcal{S}_G^{sc}/G$

$$H = H_{i_0} < \cdots < H_{i_{k-1}} < H_{i_k} = K$$

from H to K.

Proof. The upper triangular matrix $\varphi([\mathcal{O}_G^{sc}], [\widetilde{\mathcal{F}}_G^{sc}]) = \mu([\mathcal{O}_G^{sc}])\Delta(|C_G(H)|_{p'})\zeta([\mathcal{O}_G^{sc}])$ has nonnegative integral entries by Theorem 10. According to the proof of Proposition 10.7, the off-diagonal entries are

$$\sum_{k=1}^{j-i}\frac{|\mathcal{O}_G^{sc}(H_{i_0}, H_{i_1})|\cdots|\mathcal{O}_G^{sc}(H_{i_{k-1}}, H_{i_k})|}{|\mathcal{O}_G^{sc}(H_{i_0})|\cdots|\mathcal{O}_G^{sc}(H_{i_{k-1}})|}(|C_G(H_{i_{k-1}})|_{p'} - |C_G(H_{i_k})|_{p'})$$
$$= \frac{|H_{i_0}|}{|H_{i_k}|}\sum_{k=1}^{j-i}\frac{|N_G(H_{i_0}, H_{i_1})|\cdots|N_G(H_{i_{k-1}}, H_{i_k})|}{|N_G(H_{i_0})|\cdots|N_G(H_{i_{k-1}})|}(|C_G(H_{i_{k-1}})|_{p'} - |C_G(H_{i_k})|_{p'}), \quad i < j,$$

where the summation runs over all k-simplices in $\mathcal{S}_G^{sc}/G$, $H_i = H_{i_0} < \cdots < H_{i_k} = H_j$, from H_i to H_j. □

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