ADVANCES IN APPLIED MATHEMATICS 8, 434–445 (1987)

MSS Sequences, Colorings of Necklaces, and Periodic Points of $f(z) = z^2 - 2$

K. M. BRUCKS

Department of Mathematics, North Texas State University, Denton, Texas 76203

1. INTRODUCTION

Consider the various colorings of a necklace consisting of n beads, where each bead can be either red or black. Informally, two colorings will be considered to be the same if it is possible to get from one to the other by moving the clasp, interchanging colors, or both. A coloring is considered primitive if it has no proper subpattern. Gilbert and Riordan [4] give a formula which counts the number of such primitive colorings. Metropolis, Stein, and Stein [5] noted that for all $n \leq 15$, Gilbert and Riordan's formula also gives the number of MSS sequences of length n. In Section 5 we show that these two quantities agree for all n by means of an algorithm which for each n produces a bijection. We also have, but will not give here, a number theoretic argument.

In Section 4 we consider the periodic points of $f(z) = z^2 - 2$. Let f^0 be the identity map, and for n a positive integer define f^n inductively by $f^1 = f$ and $f^n = f(f^{n-1})$. If for some $n > 0$ $f^n(w) = w$, then we call w a periodic point of f and the minimum $\{n > 0 | f^n(w) = w\}$ is the period of w. For any z, the orbit of z is $\{f^n(z) | n \geq 0\}$. Given a periodic point x of f with period m, we call the orbit of x negative if $(f^m)'(x) < 0$. Of course if $(f^m)'(x)$ is negative, then $(f^m)'(y)$ is negative for all y in the orbit of x. For a given positive integer n, Myrberg [9] attempts to count the values of p for which zero is a periodic point of $h(z) = z^2 - p$ of period n. In his analysis he describes sequences which would later be called MSS sequences. Based on a hypothesis which he is unable to prove, he finds the number of such p to be the number of distinct negative orbits of order, or size, n using the function $f(z) = z^2 - 2$. The number of such p for $n \leq 16$, calculated by Myrberg, is equal to the number of MSS sequences of length n. In Section 4 we give two proofs that the number of MSS sequences of length

0196-8858/87 $7.50

n, for all $n \in N$, is the same as the number of distinct negative orbits of order n using the function $f(z) = z^2 - 2$.

For completeness, in Section 2 we briefly discuss MSS or shift maximal sequences. In Section 3 we follow the work of Sun Lichiang and G. Helmberg [10] to give a formula which, for each positive integer n, counts the number of MSS sequences of length n. We use the results of Section 3 throughout the paper. To our knowledge their work has not yet been published. We let N denote the positive integers.

2. MSS Sequences

In 1973 Metropolis, Stein, and Stein [5] developed a universal theory for a certain class of maps of $[0, 1]$ into itself. Given such a map f and a value λ between zero and one they formed a finite or possibly infinite sequence of R's and L's, $\{b_i\}$, by considering the iterates of the map λf at $\frac{1}{2}$. For $i \geq 1$, set

$$b_i = \begin{cases} R, & \text{if} \quad (\lambda f)^i\left(\frac{1}{2}\right) > \frac{1}{2}, \\ L, & \text{if} \quad (\lambda f)^i\left(\frac{1}{2}\right) < \frac{1}{2}, \\ C, & \text{if} \quad (\lambda f)^i\left(\frac{1}{2}\right) = \frac{1}{2}. \end{cases}$$

If $b_i = C$ for some i, then the sequence stops. Finite sequences of R's and L's obtained in this manner are called MSS sequences. In particular, for each $n \in N$, they were interested in the set of λ, and their associated sequence of R's and L's, for which $\frac{1}{2}$ was a periodic point of λf of minimal period n. For example, RLLRC is an MSS sequence of length five. For each positive integer n we will denote the set of possible MSS sequences of length n by MSS_n, and the cardinality of a set A will be denoted by $|A|$.

Collet and Eckmann [2] define shift maximal sequences. Briefly, a sequence w of symbols L, R, C is said to be admissible if w is an infinite sequence of L's and R's or if w is a finite (or empty) sequence of L's and R's followed by a C. We will refer to such sequences as words. The parity-lexicographic order is put on the set of admissible words: Set $L < C < R$. Let $w = \{w_i\}$ and $v = \{v_i\}$ be two distinct admissible words. Let k be the first index where they differ. If they differ in the first position, i.e., $k = 1$, then $w < v$ iff $w_1 < v_1$. Assume $k > 1$. If $w_1 \cdots w_{k-1} = v_1 \cdots v_{k-1}$ has an even number of R's, i.e., has even parity, then $w < v$ iff $w_k < v_k$. If there are an odd number of R's then $w < v$ iff $v_k < w_k$. A word is shift maximal if it is greater than or equal to all of its right shifts.

For example, RLLRC is shift maximal where as LLLC is not. Beyer, Mauldin, and Stein [1] show that a finite word is shift maximal iff it is a MSS sequence.

3. $|MSS_n|$

Sun Lichiang and G. Helmberg [10] expand the set of admissible words to include all finite sequences of R's and L's. They also extend the parity-lexicographic order as follows. If w and v are admissible words such that there is a $k \geq 1$ with $w = w_1 \cdots w_k w_{k+1} \cdots$ and $v = w_1 \cdots w_k$, then $w > v$ if the parity of v is odd, otherwise $v > w$. Given w, a finite sequence of R's and L's, we call w shift maximal in the extended parity-lexicographic order if w is greater than or equal to all of its right shifts. For simplicity we will simply call w shift maximal. A word of length n is said to be primitive provided its smallest subperiod is also of length n. Notice that if $w = w_1 \cdots w_n$ is primitive, then for each j, $2 \leq j \leq n$, $w \neq w_j \cdots w_n w_1 \cdots w_{j-1}$.

Only those results of Sun Lichiang and G. Helmberg which are used to establish their formula for $|MSS_n|$ are listed. We provide our own proofs for all except Theorem 3.2. in order to avoid their notation. Again we note that these results will be used in Sections 4 and 5.

LEMMA 3.1. *If $w = w_1 \cdots w_n \in \{R, L\}^n$ is shift maximal, then $w_1 \cdots w_{n-1}C \in MSS_n$ and if $b_1 \cdots b_{n-1}C$ is in MSS_n then both $b_1 \cdots b_{n-1}L$ and $b_1 \cdots b_{n-1}R$ are shift maximal.*

Proof. Lemma 3.1 follows from elementary observations.

THEOREM 3.2. *Let $w = w_1 \cdots w_n \in \{R, L\}^n$. Then the following are equivalent*:

(1) *w is shift maximal.*

(2) *w^∞ is shift maximal and either w is primitive or $w = v^2$,*

where v has odd parity and v is primitive.

Proof. We will first show that (1) implies (2). Observe that w^∞ is shift maximal iff for all i, $1 \leq i < n$,

$$w_{i+1} \cdots w_n w_1 \cdots w_i \leq w_1 \cdots w_{n-i} w_{n-i+1} \cdots w_n.$$

Let i be such that $1 \leq i < n$. First, w shift maximal implies that $w_{i+1} \cdots w_n \leq w_1 \cdots w_{n-i}$. We need only consider,

$$w_{i+1} \cdots w_n = w_1 \cdots w_{n-i}. \qquad (*)$$

So, assume that $(*)$ holds. Now, since w is shift maximal, the parity of $w_1 \cdots w_{n-i}$ is odd and $w_{n-i+1} \cdots w_n \leq w_1 \cdots w_i$. Thus,

$$w_{i+1} \cdots w_n w_1 \cdots w_i \leq w_1 \cdots w_{n-i} w_{n-i+1} \cdots w_n = w,$$

and therefore w^∞ is shift maximal.

Next, suppose w is not primitive. Express w as

$$w = w_1 \cdots w_k \cdots w_1 \cdots w_k,$$

where $n/k \geq 2$. The parity of $w_1 \cdots w_k$ is odd, since w is shift maximal. Thus $n/k = 2$, since otherwise w would be less than its right shift $w_1 \cdots w_k w_1 \cdots w_k$. So, (1) implies (2).

Assume that (2) holds and let i be such that $1 \leq i < n$. We will show that $w_{i+1} \cdots w_n \leq w$. We have $w_{i+1} \cdots w_n \leq w_1 \cdots w_{n-i}$, since w^∞ is shift maximal. So it suffices to only consider,

$$w_{i+1} \cdots w_n = w_1 \cdots w_{n-i}. \tag{$**$}$$

Thus we assume $(**)$ holds. Again, since w^∞ is shift maximal, we have

$$w_{n-i+1} \cdots w_n \leq w_1 \cdots w_i.$$

First, if $w_{n-i+1} \cdots w_n < w_1 \cdots w_i$, then the parity of $w_{i+1} \cdots w_n$ is odd, since $w_{i+1} \cdots w_n w_1 \cdots w_i \leq w_1 \cdots w_{n-i} w_{n-i+1} \cdots w_n = w$. Therefore, $w_{i+1} \cdots w_n < w$.

Next, if $w_{n-i+1} \cdots w_n = w_1 \cdots w_i$, then $w = w_1 \cdots w_{n-i} w_{n-i+1} \cdots w_n = w_{i+1} \cdots w_n w_1 \cdots w_i$. Thus, w is not primitive and therefore $w = v^2$ with v odd and primitive. Hence, $v = w_{i+1} \cdots w_n$ and therefore $w_{i+1} \cdots w_n < w$.

THEOREM 3.3. *Let $m \in N$ and $w = w_1 \cdots w_m \in \{R, L\}^m$ be shift maximal and primitive. Then for each j, $2 \leq j \leq m$, $w_j \cdots w_m w_1 \cdots w_{j-1}$ is not shift maximal.*

Proof. Suppose there exists a j, $2 \leq j \leq m$, so that $v = w_j \cdots w_m w_1 \cdots w_{j-1}$ is shift maximal. It suffices to assume $2(j-1) \leq m$. Let $k = [m/(j-1)]$; then $k \geq 2$.

First, w and v both shift maximal imply the following:

(1) $w_1 = w_j, w_2 = w_{j+1}, \ldots, w_{j-1} = w_{2(j-1)}$.

(2) The parity of $w_1 \cdots w_{j-1}$ is odd.

(3) $w_1 = w_{t(j-1)+1}, \ldots, w_{j-1} = w_{(t+1)(j-1)}$, for $t = 1, 2, \ldots, k-1$, if $k \geq 3$.

(4) $w_{k(j-1)+1} = w_1, \ldots, w_m = w_{m-k(j-1)}$.

The parity of $w_{k(j-1)+1} \cdots w_m$ is odd, since w is shift maximal. Note that

$k(j-1) < m$, since w is primitive. However, now

$$w < w_1 \cdots w_{j-1} w_1 \cdots w_{m-k(j-1)},$$

with $w_1 \cdots w_{j-1} w_1 \cdots w_{m-k(j-1)}$ a right shift of w. Hence v can not be shift maximal.

THEOREM 3.4. *Let $m \in N$, and $w \in \{R, L\}^m$ be primitive. Let* $\mathbf{C}$ *be the set of all cyclic permutations of w. Then there exists exactly one word in* $\mathbf{C}$ *which is shift maximal.*

Proof. Theorem 3.3 implies that there is at most one shift maximal word in $\mathbf{C}$. We will assume $m > 1$. Now, $m > 1$ and w being primitive imply that both R and L appear in w. Let

$$q = \max\{n \in N | \exists v \in \mathbf{C} \text{ so that } v \text{ begins with } RL^n\}.$$

Note that $q \geq 1$. Let $\mathbf{C}_1$ be the set of v in $\mathbf{C}$ which begin with RL^q. If $|\mathbf{C}_1| = 1$, then that one element is shift maximal.

Assume that $|\mathbf{C}_1| > 1$, and that each v in $\mathbf{C}_1$ is not shift maximal. Let $v = v_1 \cdots v_m \in \mathbf{C}_1$. Then Theorem 3.2 implies that v^∞ is not shift maximal and so there is some j, $2 \leq j \leq m$, with $v_j \cdots v_m v_1 \cdots v_{j-1} > v$. Let $v^1 = v_j \cdots v_m v_1 \cdots v_{j-1}$. Then $v^1 > v$ implies that $v^1 \in \mathbf{C}_1$. Similarly, there exists $v^2 \in \mathbf{C}_1$ so that $v^2 > v^1$. However, $\mathbf{C}_1$ is a finite set. Hence there is exactly one word in $\mathbf{C}$ which is shift maximal.

Now, for each $n \in N$ let

$$p(n) = |\{w | w \text{ is shift maximal, } w \in \{R, L\}^n, \text{and } w \text{ is primitive}\}|.$$

Then for $n \in N$,

$$p(n) = \frac{1}{n} \sum_{d|n} \mu(d) 2^{n/d}, \qquad (*)$$

where $\mu(m)$ is the Möbius function: $\mu(1) = 1$, $\mu(m) = (-1)^r$ if m is a product of r distinct primes, and $\mu(m) = 0$ otherwise. To see this, for each $s \in N$ let $\mathbf{P}(s)$ be the number of words of length s which are primitive. Then $2^n = \sum_{d|n} \mathbf{P}(d)$, and by möbius inversion,

$$\mathbf{P}(n) = \sum_{d|n} \mu(d) 2^{n/d}.$$

Theorem 3.4 now gives us $(*)$.

THEOREM 3.5. *For $n \in N$ we have that,*

$$p(n) = \begin{cases} 2|\mathrm{MSS}_n| - |\mathrm{MSS}_{n/2}|, & \text{if } n \text{ is even}, \\ 2|\mathrm{MSS}_n|, & \text{if } n \text{ is odd}. \end{cases}$$

Proof. First, Lemma 3.1 implies that $p(n) \le 2|\mathrm{MSS}_n|$ for all $n \in N$. For n odd, Theorem 3.2 immediately gives equality. So assume n is even.

The case $n = 2$ can be simply checked, so assume n is even and greater than two. We now let,

$$\mathbf{B} = \{B = b_1 \cdots b_{n-1}C | B \in \mathrm{MSS}_n,$$
$$\text{and either } b_1 \cdots b_{n-1}L \text{ or } b_1 \cdots b_{n-1}R \text{ is } \textbf{not} \text{ primitive}\}.$$

It suffices to show that $|\mathbf{B}| = |\mathrm{MSS}_{n/2}|$.

Let $b_1 \cdots b_{n-1}C \in \mathbf{B}$. Then $b_1 \cdots b_{n/2}b_1 \cdots b_{n/2}$ is shift maximal and therefore $b_1 \cdots b_{n/2}$ is shift maximal. Thus, $b_1 \cdots b_{n/2-1}C \in \mathrm{MSS}_{n/2}$ and so $|\mathbf{B}| \le |\mathrm{MSS}_{n/2}|$.

Let $D = d_1 \cdots d_{n/2-1}C \in \mathrm{MSS}_{n/2}$. Choose $d_{n/2}$ from $\{R, L\}$ so that $d_1 \cdots d_{n/2}$ is primitive. Since $D \in \mathrm{MSS}_{n/2}$, $d_1 \cdots d_{n/2}$ is shift maximal. We claim that $d_1 \cdots d_{n/2}d_1 \cdots d_{n/2}$ is also shift maximal. For if not, there is some j, $2 \le j \le n/2$, so that

$$d_j \cdots d_{n/2}d_1 \cdots d_{n/2} > d_1 \cdots d_{n/2}d_1 \cdots d_{n/2}.$$

However, $d_1 \cdots d_{n/2}$ shift maximal implies that $(d_1 \cdots d_{n/2})^\infty$ is shift maximal and therefore

$$d_j \cdots d_{n/2}d_1 \cdots d_{n/2} = d_1 \cdots d_{n/2}d_1 \cdots d_{n/2-j+1}.$$

In particular, $d_j \cdots d_{n/2}d_1 \cdots d_{j-1} = d_1 \cdots d_{n/2}$ which contradicts $d_1 \cdots d_{n/2}$ being primitive. So our claim holds. Thus, $d_1 \cdots d_{n/2}d_1 \cdots d_{n/2-1}C \in \mathbf{B}$ and $|\mathbf{B}| = |\mathrm{MSS}_{n/2}|$.

Theorem 3.5 makes the proof of Theorem 3.6 a simple induction argument.

THEOREM 3.6. *If $n = 2^k(2m - 1)$, then*

$$|\mathrm{MSS}_n| = \sum_{i=1}^{k+1} 2^{-i}p(n/2^{i-1}).$$

4. PERIODIC POINTS OF $f(z) = z^2 - 2$

We want to consider the periodic points of $f(z) = z^2 - 2$ and relate these to MSS sequences.

THEOREM 4.1. *f^n has exactly 2^n distinct real fixed points each of which is between -2 and 2.*

Proof. Fix n and let z be a real valued fixed point of f^n. Then, $|z| \le 2$ implies that there is a $u \in [0, \pi]$ so that $z = 2\cos u$. Hence, $f^k(z) =$

$2\cos 2^k u$, $k \geq 1$, and therefore we need only solve $\cos 2^n t = \cos t$ for $t \in [0, \pi]$. We find,

$$t = \begin{cases} 2\pi m/(2^n - 1), & m = 0,1,2,\ldots,2^{n-1} - 1, \\ 2\pi m'/(2^n + 1), & m' = 1,2,\ldots,2^{n-1}. \end{cases}$$

We now classify the periodic points of $f(z) = z^2 - 2$. First notice that for any real valued z and positive integer n we have that

$$(f^n)'(z) = 2^n \prod_{i=0}^{n-1} f^i(z).$$

So, for w a periodic point of $f(z) = z^2 - 2$ of period m, we call the orbit of w positive if $(f^m)'(w) > 0$ and negative if $(f^m)'(w) < 0$.

Now, to each periodic point of $f(z) = z^2 - 2$ associate its period. Then for each $n \in N$, partition periodic points of period n into orbits and orbits according to sign of derivative. So, for each $n \in N$, we have positive and negative orbits of order n. Let $q(n)$ be the number of such negative orbits of order n, $n \in N$.

Fix $n \in N$ and let z be a fixed point of f^n. One can shows that $(f^n)'(z) < 0$ iff $z = 2\cos t$ for t of the form $2\pi m'/(2^n + 1)$. So, $q(n) \leq 2^{n-1}/n$. More accurately,

$$q(n) = \left(2^{n-1} - \epsilon_n\right)/n,$$

where

$$\epsilon_n = \begin{cases} 1, & n \text{ an odd prime}, \\ 0, & n = 2^m,\ m = 0,1,2,\ldots \\ n \sum\limits_{\substack{d|n \\ d \text{ odd} \\ d>1}} q(n/d)/d, & \text{otherwise}. \end{cases}$$

We remark that we have derived $q(n)$ in much the same way as Myrberg [9], and include the derivation to more clearly present our own work which follows.

We will first show that $q(n) = |\mathrm{MSS}_n|$ for all n, using a number theoretic argument. For n an odd prime, Theorem 3.5 gives that $|\mathrm{MSS}_n| = \frac{1}{2}p(n)$. Thus,

$$|\mathrm{MSS}_n| = \frac{1}{2}\left[\frac{1}{n}(2^n - 2)\right] = \frac{1}{n}[2^{n-1} - 1] = q(n).$$

Similarly for n a power of 2, the argument requires no induction. We now consider the "otherwise" case.

THEOREM 4.2. *For every $s \in N$ the following holds. If $k \geq 0$ and $n = 2^k p_1 \cdots p_s$, where the p_i are odd primes not necessarily distinct, then* $|MSS_n| = q(n)$.

Proof. We will induct on s. First assume n is of the form $n = 2^k p_1$, $k \geq 0$. We only need consider $k \geq 1$. Now,

$$|\mathrm{MSS}_n| = \sum_{i=1}^{k+1} 2^{-i} p\left(2^{k-i+1} p_1\right) = (1/2n) \sum_{i=1}^{k+1} \mathbf{P}\left(2^{k-i+1} p_1\right).$$

So, we need

$$\sum_{i=1}^{k+1} \mathbf{P}\left(2^{k-i+1} p_1\right) = \left[\sum_{s|n} \mathbf{P}(s)\right] - 2\epsilon_n. \qquad (*)$$

However,

$$\epsilon_n = n \sum_{\substack{d|n \\ d \text{ odd} \\ d>1}} q(n/d)/d = (n/p_1)|\mathrm{MSS}_{2^k}| = \frac{1}{2} \sum_{i=1}^{k+1} \mathbf{P}(2^{k-i+1}).$$

It now follows that $(*)$ holds.

The induction step is straightforward and patterned after the case $s = 1$. Hence we delete it.

Next we give a bijection from MSS_n onto the negative orbits of order n. We first state the following theorem in order to help clarify how we are going to represent the negative orbits.

THEOREM 4.3. *For each $n \in N$, define φ_n from the fixed points of f^n into $\{1, -1\}^n$ as follows*:

$$\varphi_n(z) = \langle e_0, e_1, \ldots, e_{n-1} \rangle,$$

where

$$e_i = \begin{cases} 1, & \text{if } f^i(z) > 0 \\ -1, & \text{if } f^i(z) < 0. \end{cases}$$

Then φ_n is a bijection.

The following two facts follow from Theorem 4.3:

(1) If w is a periodic point of f of period m, then $\varphi_m(w)$ is primitive.

(2) If α, $\beta \in \{1, -1\}^m$ are primitive and β is a cyclic permutation of α, then $\varphi_m^{-1}(\beta)$ is in the orbit of $\varphi_m^{-1}(\alpha)$.

Thus if $e = \langle e_0, \ldots, e_{m-1}\rangle \in \{1, -1\}^m$ is primitive and $\prod_{i=0}^{m-1} e_i < 0$, then e or any cyclic permutation of e can be used to represent the negative orbit of $\varphi_m^{-1}(e)$. For example, if $m = 3$, we can use $\langle 1, -1, 1\rangle$, $\langle -1, 1, 1\rangle$, or $\langle 1, 1, -1\rangle$ to represent the negative orbit of $\varphi_3^{-1}(\langle 1, 1, -1\rangle)$.

THEOREM 4.4. *Let $n \in N$, $n > 1$. Define g from MSS_n into the negative orbits of order n as follows.*

$$g(b_1 \cdots b_{n-1}C) = \begin{cases} \langle e_1, \ldots, e_{n-1}, -1\rangle, & \text{if } \prod_{i=1}^{n-1} e_i > 0, \\ \langle e_1, \ldots, e_{n-1}, 1\rangle, & \text{if } \prod_{i=1}^{n-1} e_i < 0, \end{cases}$$

where

$$e_i = \begin{cases} 1, & b_i = L \\ -1, & b_i = R. \end{cases}$$

Then g is a bijection.

Proof. Let $B = b_1 \cdots b_{n-1}C \in \mathrm{MSS}_n$. Then both $b_1 \cdots b_{n-1}L$ and $b_1 \cdots b_{n-1}R$ are shift maximal. Thus, using Theorem 3.2. and the fact that $\prod_{i=1}^{n}[g(b_1 \cdots b_{n-1}C)]_i < 0$, it follows that $g(b_1 \cdots b_{n-1}c)$ is primitive. Hence, the range of g is as claimed.

Suppose $D \in \mathrm{MSS}_n$ with $D \neq B$. Then Lemma 3.1 and Theorem 3.3 imply that $g(D)$ is not a cyclic permutation of $g(B)$ and therefore that g is 1–1. That g is onto follows from Theorem 3.4 and Lemma 3.1. For $n = 1$ one simply sends C to $\langle -1\rangle$.

5. COLORINGS OF NECKLACES

For each $n \in N$ we partition the elements of $\{1, -1\}^n$ into equivalence classes, where equivalence is determined by $C_n \times S_2$. Here C_n is a cyclic group of n elements, namely cyclic permutations, and S_2 is the permutation group on two elements. So, if w and v are two elements of $\{1, -1\}^n$, we say w and v are the same iff there exists some $\gamma \in C_n \times S_2$ so that $\gamma(w) = v$. Each equivalence class containing primitive elements gives a distinct coloring for a necklace consisting of n beads, where there are two possible colors for each bead. We let CL_n denote the collection of such equivalence classes and we will use arbitrary members of a class to represent it. For example, we can express CL_4 as, $\mathrm{CL}_4 = \{\langle -1, 1, 1, 1\rangle, \langle -1, 1, 1, -1\rangle\}$. Of course, among others, $\langle 1, 1, 1, -1\rangle$ and $\langle -1, -1, -1, 1\rangle$ are equivalent to $\langle -1, 1, 1, 1\rangle$. The coloring $\langle 1, -1, 1, -1\rangle$ is not primitive. Gilbert and

Riordan [4] give a formula which yields $|CL_n|$ for each $n \in N$, and they computed $|CL_n|$ for $1 \leq n \leq 20$. Their values match those in Metropolis, Stein, and Steins' Table 2 [5] and the table given by Myrberg [9]. We proceed with our algorithm.

DEFINITION 5.1. *Let $n \geq 2$ and $B = b_1 \cdots b_{n-1}C \in MSS_n$. Then define $h(B)$ as*

$$h(B) = \langle e_1, e_2, \dots, e_n \rangle,$$

where $e_1 = -1$, $e_2 = 1$, and for $3 \leq i \leq n$

$$e_i = \begin{cases} e_{i-1}, & \text{if } b_{i-1} = L, \\ -e_{i-1}, & \text{if } b_{i-1} = R. \end{cases}$$

Thus $h\colon \bigcup_{n \geq 2} MSS_n \to \bigcup_{n \geq 2}\{-1, 1\}^n$.

THEOREM 5.2. *For every $n \geq 2$, $h|_{MSS_n}$ is a bijection onto CL_n.*

COROLLARY 5.3. *$|MSS_n| = |CL_n|$, for each $n \in N$.*

Corollary 5.3 follows immediately from Theorem 5.2. We prove Theorem 5.2 with the next four lemmas.

LEMMA 5.4. *Let $n \geq 2$ and $B = b_1 \cdots b_{n-1}C \in MSS_n$. Then $h(B)$ is primitive.*

Proof. Suppose $h(B)$ is not primitive. Then we can express $h(B)$ as

$$h(B) = \langle \alpha_1, \dots, \alpha_p, \dots, \alpha_1, \dots, \alpha_p \rangle,$$

where $n/p \geq 2$. Thus $B = b_1 \cdots b_p \cdots b_1 \cdots b_{p-1}C$ and therefore, using Theorem 3.2, n/p is either one or two. Assume $n/p = 2$. We have two cases.

Case 1. Assume the parity of $b_1 \cdots b_{p-1}$ is odd, i.e., there are an odd number of R's appearing in $b_1 \cdots b_{p-1}$. Then $B = b_1 \cdots b_p b_1 \cdots b_{p-1}C \in \text{MSS}_n$ implies that $b_p = L$, for otherwise $b_1 \cdots b_p b_1 \cdots b_p$ is less than its right shift $b_1 \cdots b_p$. Thus $\alpha_p = -1$. This implies that $\alpha_1 = 1$, since the parity of $b_1 \cdots b_{p-1}$ is odd. However, $\alpha_1 = -1$ by the definition of h. Thus $n/p = 1$ and $h(B)$ is primitive.

Case 2. Assume the parity of $b_1 \cdots b_{p-1}$ is even. The argument is similar to Case 1.

LEMMA 5.5. *Let $n \geq 2$, and $B \neq D$ in MSS_n. Then $h(B)$ and $h(D)$ are inequivalent colorings.*

Proof. Let $B = b_1 \cdots b_{n-1}C$ and $D = d_1 \cdots d_{n-1}C$. Suppose that $h(B)$ and $h(D)$ are equivalent colorings. Express $h(B)$ as $h(B) =$

$\langle e_1,\ldots,e_n\rangle$. Then $\exists\, j$, $2 \le j \le n$, so that,

$h(D) = \langle e_j,\ldots,e_n,e_1,\ldots,e_{j-1}\rangle$ or $h(D) = -\langle e_j,\ldots,e_n,e_1,\ldots,e_{j-1}\rangle$

In either case,

$$D = b_j \cdots b_{n-1}bb_1 \cdots b_{j-2}C,$$

where b is L if $e_n = e_1$ and R if $e_n \neq e_1$. Observe that both $w = b_1 \cdots b_{n-1}b$ and $v = b_j \cdots b_{n-1}bb_1 \cdots b_{j-2}b_{j-1}$ are shift maximal. Now, w is primitive or has minimal subperiod $n/2$. If w is primitive we contradict Theorem 3.3. If w is not primitive, then j must be $(n/2)+1$ and therefore $B = D$. Hence the result holds.

DEFINITION 5.6. *Let $n \ge 2$, and $\langle e_1,\ldots,e_n\rangle \in \{1,-1\}^n$. Then we define $\psi(\langle e_1,\ldots,e_n\rangle)$ as follows.*

$$\psi(\langle e_1,\ldots,e_n\rangle) = b_1 \cdots b_{n-1},$$

where for $1 \le i \le n-1$

$$b_i = \begin{cases} R, & \text{if } e_i \neq e_{i+1}, \\ L, & \text{if } e_i = e_{i+1}. \end{cases}$$

LEMMA 5.7. *Let $n \ge 2$, and $e = \langle e_1,\ldots,e_n\rangle \in \{-1,1\}^n$ be primitive. Then,*

(i) *$\psi(\langle e_1,\ldots,e_n,e_1\rangle)$ is either primitive or has minimal subperiod $n/2$.*

(ii) *If $\psi(\langle e_1,\ldots,e_n,e_1\rangle) = b_1 \cdots b_{n/2}b_1 \cdots b_{n/2}$, then the parity of $b_1 \cdots b_{n/2}$ is odd.*

Proof. Let $w = b_1 \cdots b_n = \psi(\langle e_1,\ldots,e_n,e_1\rangle)$. Suppose (i) does not hold. Express w as,

$$w = b_1 \cdots b_p b_1 \cdots b_p \cdots b_1 \cdots b_p.$$

Then $e_{p+1} = -e_1$, since otherwise e would not be primitive. So, e now looks like

$$e = \langle e_1,\ldots,e_p,-e_1,\ldots,-e_p,e_1,\ldots,e_p,\ldots\rangle,$$

depending on the size of n/p, and therefore e_n is either e_p or $-e_p$. Recall that $b_n = b_p$ is obtained by comparing e_n to e_1, or e_p to e_{p+1}. Thus, $e_{p+1} = -e_1$ implies that $e_n = -e_p$. However, this implies that e is not primitive. Thus (i) holds.

We will now show (ii). Assume the parity of $b_1 \cdots b_{n/2}$ is even. Then e primitive implies that

$$e = \langle e_1,\ldots,e_{n/2},-e_1,\ldots,-e_{n/2}\rangle.$$

However, the parity of $b_1 \cdots b_{n/2}$ being even implies that there must be an even number of "sign changes" in

$$e_1, \ldots, e_{n/2}, e_{n/2+1}.$$

Thus, $e_1 = e_{n/2+1}$. Now we have $e_1 = e_{n/2+1} = -e_1$. So, (ii) holds.

LEMMA 5.8. *Let $n \geq 2$. Then $h|_{MSS_n}$ is onto CL_n.*

Proof. Let $e = \langle e_1, \ldots, e_n \rangle \in \{-1, 1\}^n$ be primitive. We have two cases.

Case 1. Assume that $\psi(\langle e_1, \ldots, e_n, e_1 \rangle) = w = b_1 \cdots b_n$ is primitive. If w is shift maximal, then $B = b_1 \cdots b_{n-1}C \in \mathrm{MSS}_n$ and $h(B) = e$ or $h(B) = -e$. If w is not shift maximal, then, by Theorem 3.4, there exists some j, $2 \leq j \leq n$, such that $b_j \cdots b_n b_1 \cdots b_{j-1}$ is shift maximal. Then $D = b_j \cdots b_n b_1 \cdots b_{j-2}C$ is in MSS_n and $h(D)$ is equivalent to e.

Case 2. Assume that $\psi(\langle e_1, \ldots, e_n, e_1 \rangle)$ has minimal period $n/2$. The argument is similar using Lemma 5.7 (ii) and Theorem 3.4 applied to $b_1 \cdots b_{n/2}$.

ACKNOWLEDGMENTS

I thank Dr. Beyer and Dr. P. Stein of Los Alamos Laboratory. In particular, the idea for the algorithm given in Section 5 was shown to me by Dr. Beyer.

REFERENCES

1. W. A. BEYER, R. D. MAULDIN, AND P. R. STEIN, Shift-maximal sequences in function iteration: Existence, uniqueness, and multiplicity, *J. Math. Anal. Appl.* **115** (1986), 305–362.
2. P. COLLET AND J. -P. ECKMANN, "Iterated Maps on the Interval as Dynamical Systems," Birkhauser, Basel, 1980.
3. B. DERRIDA, A. GERVOIS, AND Y. POMEAU, Iteration of endomorphisms on the real axis and representations of numbers, *Ann. Inst. H. Poincaré* **29** (1978), 305–356.
4. E. N. GILBERT AND J. RIORDAN, Symmetry types of periodic sequences, *Illinois J. Math.* **5** (1961), 657.
5. N. METROPOLIS, M. L. STEIN, AND P. R. STEIN, On finite limit sets for transformations on the unit interval, *J. Combin. Theory* **15** (1973), 25–44.
6. P. J. MYRBERG, Iteration der reellen Polynome zweiten Grades, *Ann. Acad. Sci. Fenn. Ser. A I Math.* **256** (1958).
7. P. J. MYRBERG, Iteration von Quadratwurzeloperationen, *Ann. Acad. Sci. Fenn. Ser. A I Math.* **259** (1958).
8. P. J. MYRBERG, Iteration der reellen Polynome zweiten Grades II, *Ann. Acad. Sci. Fenn. Ser. A I Math.*, **268** (1959).
9. P. J. MYRBERG, Iteration der reellen Polynome zweiten Grades III, *Ann. Acad. Sci. Fenn. Ser. A I Math.*, **336**, No. 3, (1963).
10. SUN LICHIANG AND G. HELMBERG, Maximal words connected with unimodal maps, reprint.