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Multiple positive solutions for boundary value problems of second order delay differential equations with one-dimensional p-Laplacian ☆

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Abstract

We consider the boundary value problems: $(\phi_p(x'(t)))' + q(t)f(t, x(t), x(t-1), x'(t)) = 0$, $\phi_p(s) = |s|^{p-2}s$, $p > 1$, $t \in (0, 1)$, subject to some boundary conditions. By using a generalization of the Leggett–Williams fixed-point theorem due to Avery and Peterson, we provide sufficient conditions for the existence of at least three positive solutions to the above problems.

Keywords: Multiple positive solutions; Delay differential equations; Boundary value problems; One-dimensional p-Laplacian

1. Introduction

In this paper we consider the existence of triple positive solutions for the delay differential equation with one-dimensional p-Laplacian

$$\big(\phi_p\big(x'(t)\big)\big)' + q(t)f\big(t, x(t), x(t-1), x'(t)\big) = 0, \quad t \in (0, 1), \tag{1.1}$$

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subject to one of the following two pairs of boundary conditions:

$$\begin{cases} x(t)=\xi(t), & -1\leqslant t\leqslant 0,\\ x(1)=0, \end{cases}\tag{1.2}$$

$$\begin{cases} x(t)=\xi(t), & -1\leqslant t\leqslant 0,\\ x'(1)=0, \end{cases}\tag{1.3}$$

where $\phi_p(s)=|s|^{p-2}s$, $p>1$. By positive solution to the above problems, we mean a function $x(t)$ that is positive on $0<t<1$ and satisfies the differential equation (1.1) and the boundary conditions (1.2) or (1.3), respectively.

In recent years, there has been much attention focused on the existence and multiplicity of positive solutions for nonlinear ordinary differential equations and functional differential equations, see [5–17] and references therein. The Guo–Krasnosel'skii fixed point theorem [1,2], Leggett–Williams fixed point theorem [3], and a generalization of Leggett–Williams fixed point theorem due to Avery and Peterson [4] play an important role in the above studies.

In [12], Bai et al. studied the two-point boundary value problems by using a generalization of Leggett–Williams fixed point theorem [4]

$$\begin{cases} x''(t)+q(t)f\big(t,x(t),x'(t)\big)=0, & t\in(0,1),\\ x(0)=0=x(1),\\ x(0)=0=x'(1). \end{cases}$$

Very recently, with the same method, Shu and Xu in [13] investigated the problem

$$\begin{cases} x''(t)+q(t)f\big(t,x(t),x'(t-1)\big)=0, & t\in(0,1),\\ x(t)=\xi(t), & -1\leqslant t\leqslant 0,\\ x(1)=0. \end{cases}$$

Jiang [14] used a fixed point index theorem in cones to study the existence of at least one positive solution for the problem

$$\begin{cases} x''(t)+f\big(t,x(t-\tau)\big)=0, & t\in(0,1),\ \tau>0,\\ x(t)=0, & -\tau\leqslant t\leqslant 0,\\ x(1)=0. \end{cases}$$

In [15], Bai et al. considered the existence of multiple positive solutions for the one-dimensional p-Laplacian boundary value problems

$$\begin{cases} \big(\phi_p\big(x'(t)\big)\big)'+q(t)f\big(t,x(t),x'(t)\big)=0, & t\in(0,1),\\ \alpha\phi_p\big(x(0)\big)-\beta\phi_p\big(x'(0)\big)=0, & \gamma\phi_p\big(x(1)\big)+\delta\phi_p\big(x'(1)\big)=0, \end{cases}$$

and

$$\begin{cases} \big(\phi_p\big(x'(t)\big)\big)'+q(t)f\big(t,x(t),x'(t)\big)=0, & t\in(0,1),\\ x(0)-g_1\big(x'(0)\big)=0, & x(1)+g_2\big(x'(1)\big)=0. \end{cases}$$

The author [16] studied the boundary value problems

$$\begin{cases} \big(\phi_p\big(x'(t)\big)\big)'+q(t)f\big(t,x(t),x'(t)\big)=0, & t\in(0,1),\\ x(0)=0=x(1),\\ x(0)=0=x'(1). \end{cases}$$

By using a Leary–Schauder alternative and the Guo–Krasnosel'skii fixed point theorem, Jiang et al. [17] established the existence of single and multiple nonnegative solutions to the problem

$$\begin{cases} \big(\phi_p\big(x'(t)\big)\big)' + q(t) f\big(t, x(t-\tau)\big) = 0, & t \in (0,1) \setminus \{\tau\}, \\ x(t) = \xi(t), \quad -\tau \leqslant t \leqslant 0, \\ x(1) = 0. \end{cases}$$

Motivated by the above works, we investigate the problems (1.1)–(1.2) and (1.1)–(1.3). Our main results will depend on an application of a fixed-point theorem due to Avery and Peterson [4].

For the convenience of the reader, we present here the necessary definitions from the theory of cones in Banach spaces.

Definition 1.1. Let E be a real Banach space. A nonempty convex closed set $P \subset E$ is said to be a cone provided that

(i) $au \in P$ for all $u \in P$ and all $a \geqslant 0$ and
(ii) $u, -u \in P$ implies $u = 0$.

Note that every cone $P \subset E$ induces an ordering in E given by $x \leqslant y$ if $y - x \in P$.

Definition 1.2. The map α is said to be a nonnegative continuous *concave* functional on a cone P of a real Banach space E provided that $\alpha : P \to [0, \infty)$ is continuous and

$$\alpha\big(tx + (1-t)y\big) \geqslant t\alpha(x) + (1-t)\alpha(y)$$

for all $x, y \in P$ and $0 \leqslant t \leqslant 1$.

Similarly, we say the map β is a nonnegative continuous *convex* functional on a cone P of a real Banach space E provided that $\beta : P \to [0, \infty)$ is continuous and

$$\beta\big(tx + (1-t)y\big) \leqslant t\beta(x) + (1-t)\beta(y)$$

for all $x, y \in P$ and $0 \leqslant t \leqslant 1$.

Definition 1.3. An operator is called completely continuous if it is continuous and maps bounded sets into pre-compact sets.

Let γ and θ be nonnegative continuous convex functionals on P, α be a nonnegative continuous concave functional on P, and ψ be a nonnegative continuous functional on P. Then for positive real numbers a, b, c, and d, we define the following convex sets:

$$\begin{aligned} &P(\gamma, d) = \big\{x \in P \mid \gamma(x) < d\big\}, \\ &P(\gamma, \alpha, b, d) = \big\{x \in P \mid b \leqslant \alpha(x),\ \gamma(x) \leqslant d\big\}, \\ &P(\gamma, \theta, \alpha, b, c, d) = \big\{x \in P \mid b \leqslant \alpha(x),\ \theta(x) \leqslant c,\ \gamma(x) \leqslant d\big\} \end{aligned}$$

and a closed set

$$R(\gamma, \psi, a, d) = \big\{x \in P \mid a \leqslant \psi(x),\ \gamma(x) \leqslant d\big\}.$$

The following fixed-point theorem due to Avery and Peterson is fundamental in the proofs of our main results.

Theorem 1.1. [4] *Let P be a cone in a real Banach space E. Let γ and θ be nonnegative continuous convex functionals on P, α be a nonnegative continuous concave functional on P, and ψ be a nonnegative continuous functional on P satisfying $\psi(\lambda x) \leqslant \lambda\psi(x)$ for $0 \leqslant \lambda \leqslant 1$, such that for some positive numbers M and d,*

$$\alpha(x) \leqslant \psi(x) \quad \text{and} \quad \|x\| \leqslant M\gamma(x), \tag{1.4}$$

for all $x \in \overline{P(\gamma, d)}$. Suppose $T : \overline{P(\gamma, d)} \to \overline{P(\gamma, d)}$ is completely continuous and there exist positive numbers a, b, and c with $a < b$ such that

(S1) $\{x \in P(\gamma, \theta, \alpha, b, c, d) \mid \alpha(x) > b\} \neq \emptyset$ *and* $\alpha(Tx) > b$ *for* $x \in P(\gamma, \theta, \alpha, b, c, d)$;
(S2) $\alpha(Tx) > b$ *for* $x \in P(\gamma, \alpha, b, d)$ *with* $\theta(Tx) > c$;
(S3) $0 \notin R(\gamma, \psi, a, d)$ *and* $\psi(Tx) < a$ *for* $x \in R(\gamma, \psi, a, d)$ *with* $\psi(x) = a$.

Then T has at least three fixed points $x_1, x_2, x_3 \in \overline{P(\gamma, d)}$, such that

$$\gamma(x_i) \leqslant d \quad \text{for } i = 1, 2, 3;$$
$$b < \alpha(x_1);$$
$$a < \psi(x_2) \quad \text{with } \alpha(x_2) < b;$$
$$\psi(x_3) < a.$$

In this article, it is assumed that:

(C1) $f \in C([0, 1] \times [0, \infty) \times [0, \infty) \times R, [0, \infty))$;
(C2) $q(t)$ is nonnegative measurable function defined on $(0, 1)$ and there exist a natural number $k \geqslant 3$ and $t_0 \in (1/k, (k-1)/k)$ such that $q(t_0) > 0$. Furthermore, $q(t)$ satisfies $0 < \int_0^1 q(t)\,dt < \infty$;
(C3) $\xi(t) \in C[-1, 0]$, $\xi(t) > 0$ on $[-1, 0)$ and $\xi(0) = 0$.

Denote $M = \int_0^1 q(t)\,dt$, $Q = \max_{-1 \leqslant t \leqslant 0} \xi(t)$.
In Section 2, we assume that

$$\text{(H1)} \qquad \int_0^{1/2} \phi_p^{-1}\left(\int_s^{1/2} q(r)\,dr\right) ds + \int_{1/2}^1 \phi_p^{-1}\left(\int_{1/2}^s q(r)\,dr\right) ds < \infty$$

holds. Then, we can let

$$C_1 = \max\left\{\int_0^{1/2} \phi_p^{-1}\left(\int_s^{1/2} q(r)\,dr\right) ds,\ \int_{1/2}^1 \phi_p^{-1}\left(\int_{1/2}^s q(r)\,dr\right) ds\right\},$$

and

$$C_2 = \min\left\{\int_{1/k}^{1/2} \phi_p^{-1}\left(\int_s^{1/2} q(r)\,dr\right) ds,\ \int_{1/2}^{(k-1)/k} \phi_p^{-1}\left(\int_{1/2}^s q(r)\,dr\right) ds\right\}.$$

In Section 3, we assume that

(H2) $$\int_0^1 \phi_p^{-1}\left(\int_s^1 q(r)\,dr\right)ds < \infty$$

holds. Then we let

$$C_3 = \int_0^1 \phi_p^{-1}\left(\int_s^1 q(r)\,dr\right)ds, \qquad C_4 = \int_{1/k}^1 \phi_p^{-1}\left(\int_s^{(k-1)/k} q(r)\,dr\right)ds.$$

Here $\phi_p^{-1}(s) = |s|^{1/(p-1)}\operatorname{sgn}(s)$ is the inverse function to $\phi_p(s)$.

2. Triple positive solutions of (1.1)–(1.2)

Throughout this section, suppose condition (H1) holds. Let $X = (C^1[0,1],\ \|\cdot\|)$ be our Banach space with the maximum norm

$$\|x\| = \max\left\{\max_{0\leqslant t\leqslant 1}\left|x(t)\right|,\ \max_{0\leqslant t\leqslant 1}\left|x'(t)\right|\right\}.$$

Let

$$P = \left\{x \in X \mid x(t) \geqslant 0,\ x(0) = x(1) = 0,\ x \text{ is concave on } [0,1]\right\} \subset X.$$

Let the nonnegative continuous concave functional α, the nonnegative continuous convex functionals θ, γ, and the nonnegative continuous functional ψ be defined on the cone P by

$$\gamma(x) = \max_{0\leqslant t\leqslant 1}\left|x'(t)\right|, \qquad \psi(x) = \theta(x) = \max_{0\leqslant t\leqslant 1}\left|x(t)\right|, \qquad \alpha(x) = \min_{\frac{1}{k}\leqslant t\leqslant 1-\frac{1}{k}}\left|x(t)\right|.$$

Lemma 2.1. [12] *If $x \in P$, then* $\max_{0\leqslant t\leqslant 1}|x(t)| \leqslant \frac{1}{2}\max_{0\leqslant t\leqslant 1}|x'(t)|$.

Lemma 2.2. [6] *If $x \in P$, then $x(t) \geqslant t(1-t)\max_{0\leqslant t\leqslant 1}|x(t)|$.*

Lemma 2.3. *If $x(t)$ is a positive solution to the problem*

$$\left(\phi_p\left(x'(t)\right)\right)' + q(t)f\left(t, x(t), x(t-1) + w(t-1), x'(t)\right) = 0, \quad t \in (0,1), \tag{2.1}$$

$$\begin{cases} x(t) = 0, & -1 \leqslant t \leqslant 0, \\ x(1) = 0, \end{cases} \tag{2.2}$$

where

$$w(t) = \begin{cases} \xi(t), & -1 \leqslant t \leqslant 0, \\ 0, & 0 \leqslant t \leqslant 1, \end{cases}$$

then $\tilde{x}(t) = x(t) + w(t)$, $-1 \leqslant t \leqslant 1$, is a positive solution to BVP (1.1)–(1.2).

Proof. It is easy to check that $\tilde{x}(t)$ satisfies (1.1) and (1.2).

So we focus on BVP (2.1)–(2.2).

By Lemmas 2.1 and 2.2, their definitions, and the concavity of x, the functionals defined above satisfy:

$$\frac{k-1}{k^2}\theta(x) \leqslant \alpha(x) \leqslant \theta(x) = \psi(x), \qquad \|x\| = \max\{\theta(x), \gamma(x)\} = \gamma(x), \tag{2.3}$$

for all $x \in \overline{P(\gamma, d)} \subset P$. Therefore, condition (1.4) is satisfied.

For $x \in P$, we define

$$u(t) := \int_0^t \phi_p^{-1}\left(\int_s^t q(r) f\big(r, x(r), x(r-1) + w(r-1), x'(r)\big)\, dr\right) ds$$
$$- \int_t^1 \phi_p^{-1}\left(\int_t^s q(r) f\big(r, x(r), x(r-1) + w(r-1), x'(r)\big)\, dr\right) ds,$$

for $0 < t < 1$. Clearly, $u(t)$ is continuous and strictly increasing in $(0, 1)$ and $u(0+) < 0 < u(1-)$. Thus, $u(t)$ has zeros in $(0, 1)$. Let σ be a zero of $u(t)$ in $(0, 1)$. Then

$$\int_0^\sigma \phi_p^{-1}\left(\int_s^\sigma q(r) f\big(r, x(r), x(r-1) + w(r-1), x'(r)\big)\, dr\right) ds$$
$$= \int_\sigma^1 \phi_p^{-1}\left(\int_\sigma^s q(r) f\big(r, x(r), x(r-1) + w(r-1), x'(r)\big)\, dr\right) ds. \qquad \square \tag{2.4}$$

Theorem 2.1. *Suppose* (H1) *holds. In addition, assume that there exist numbers a, b, d with* $0 < a < b \leqslant (k-1)d/(2k^2)$ *such that the following conditions are satisfied*:

(H3) $$f(t, u, v, w) \leqslant \frac{1}{M}\phi_p(d) \quad \textit{for } (t, u, v, w) \in [0, 1] \times [0, d/2] \times [0, Q] \times [-d, d],$$

(H4) $$f(t, u, v, w) \geqslant \phi_p\left(\frac{k^2 b}{(k-1)C_2}\right)$$
$$\textit{for } (t, u, v, w) \in \left[\frac{1}{k}, \frac{k-1}{k}\right] \times \left[b, \frac{k^2}{k-1}b\right] \times [0, Q] \times [-d, d],$$

(H5) $$f(t, u, v, w) < \phi_p\left(\frac{a}{C_1}\right) \quad \textit{for } (t, u, v, w) \in [0, 1] \times [0, a] \times [0, Q] \times [-d, d].$$

Then the boundary-value problem (2.1)–(2.2) *has at least three positive solutions* x_1, x_2, *and* x_3 *satisfying*

$$\begin{aligned}
&\max_{0\leqslant t\leqslant 1} |x_i'(t)| \leqslant d, \quad \textit{for } i = 1, 2, 3,\\
&b < \min_{\frac{1}{k}\leqslant t\leqslant 1-\frac{1}{k}} |x_1(t)|,\\
&a < \max_{0\leqslant t\leqslant 1} |x_2(t)|, \quad \textit{with} \min_{\frac{1}{k}\leqslant t\leqslant 1-\frac{1}{k}} |x_2(t)| < b,\\
&\max_{0\leqslant t\leqslant 1} |x_3(t)| < a.
\end{aligned} \tag{2.5}$$

Proof. Problem (2.1)–(2.2) has a solution $x = x(t)$ if and only if x solves the operator equation

$$x(t) = (Tx)(t) := \begin{cases} \int_0^t \phi_p^{-1}\big(\int_s^{\sigma_x} q(r) f(r, x(r), x(r-1) + w(r-1), x'(r))\, dr\big)\, ds, & 0 \leqslant t \leqslant \sigma_x, \\ \int_t^1 \phi_p^{-1}\big(\int_{\sigma_x}^s q(r) f(r, x(r), x(r-1) + w(r-1), x'(r))\, dr\big)\, ds, & \sigma_x \leqslant t \leqslant 1, \\ 0, & -1 \leqslant t \leqslant 0, \end{cases}$$

where σ_x is defined in (2.4).

From the definition of T and above discussion, we deduce that for each $x \in P$, $Tx \in P$. Moreover, $(Tx)(\sigma_x)$ is the maximum value of T on $[0, 1]$.

We can prove that this operator, $T : P \to P$, is completely continuous. The proof is similar to that given for Lemma 2 in [11], so we omit it here. We now show that all the conditions of Theorem 1.1 are satisfied.

If $x \in \overline{P(\gamma, d)}$, then $\gamma(x) = \max_{0 \leqslant t \leqslant 1} |x'(t)| \leqslant d$. From Lemma 2.1 we have $\max_{0 \leqslant t \leqslant 1} |x(t)| \leqslant \frac{d}{2}$ and

$$\begin{aligned} \max_{0 \leqslant t \leqslant 1} \big|x(t-1) + w(t-1)\big| &\leqslant \max_{0 \leqslant t \leqslant 1} \big|x(t-1)\big| + \max_{0 \leqslant t \leqslant 1} \big|w(t-1)\big| \\ &= \max_{-1 \leqslant t \leqslant 0} \big|x(t)\big| + \max_{-1 \leqslant t \leqslant 0} \big|w(t)\big| \\ &= \max_{-1 \leqslant t \leqslant 0} \big|\xi(t)\big| \\ &= Q. \end{aligned}$$

Then assumption (H3) implies $f(t, x(t), x(t-1) + w(t-1), x'(t)) \leqslant \frac{1}{M}\phi_p(d)$. On the other hand, for $x \in P$, $Tx \in P$, Tx is concave on $[0, 1]$, and $\max_{t \in [0,1]} |(Tx)'(t)| = \max\{|(Tx)'(0)|, |(Tx)'(1)|\}$, so

$$\begin{aligned} \gamma(Tx) &= \max_{t \in [0,1]} \big|(Tx)'(t)\big| \\ &= \max_{t \in [0,1]} \big\{\big|(Tx)'(0)\big|,\ \big|(Tx)'(1)\big|\big\} \\ &= \max\left\{\phi_p^{-1}\left(\int_0^{\sigma_x} q(r) f\big(r, x(r), x(r-1) + w(r-1), x'(r)\big)\, dr\right),\right. \\ &\qquad \left.\phi_p^{-1}\left(\int_{\sigma_x}^1 q(r) f\big(r, x(r), x(r-1) + w(r-1), x'(r)\big)\, dr\right)\right\} \\ &\leqslant d \cdot \phi_p^{-1}\left(\frac{1}{M}\right) \max\left\{\phi_p^{-1}\left(\int_0^{\sigma_x} q(r)\, dr\right),\ \phi_p^{-1}\left(\int_{\sigma_x}^1 q(r)\, dr\right)\right\} \\ &\leqslant d \cdot \phi_p^{-1}\left(\frac{1}{M}\right) \cdot \phi_p^{-1}(M) = d. \end{aligned}$$

Hence, $T : \overline{P(\gamma, d)} \to \overline{P(\gamma, d)}$.

To check condition (S1) of Theorem 1.1, we choose

$$x_0(t) = -\frac{4k^2}{k-1} b\left(t - \frac{1}{2}\right)^2 + \frac{k^2}{k-1} b, \quad 0 \leqslant t \leqslant 1.$$

It is easy to see that $x_0 \in P(\gamma, \theta, \alpha, b, \frac{k^2}{k-1}b, d)$ and $\alpha(x_0) = \min_{\frac{1}{k} \leqslant t \leqslant 1-\frac{1}{k}} |x_0(t)| = x_0(\frac{1}{k}) = 4b > b$, and so $\{x \in P(\gamma, \theta, \alpha, b, \frac{k^2}{k-1}b, d) \mid \alpha(x) > b\} \neq \emptyset$. Hence, if $x \in P(\gamma, \theta, \alpha, b, \frac{k^2}{k-1}b, d)$, then $b \leqslant x(t) \leqslant \frac{k^2}{k-1}b$, $|x'(t)| \leqslant d$ for $1/k \leqslant t \leqslant (k-1)/k$, and

$$\begin{aligned}
\max_{\frac{1}{k} \leqslant t \leqslant 1-\frac{1}{k}} \bigl|x(t-1) + w(t-1)\bigr| &\leqslant \max_{\frac{1}{k} \leqslant t \leqslant 1-\frac{1}{k}} \bigl|x(t-1)\bigr| + \max_{\frac{1}{k} \leqslant t \leqslant 1-\frac{1}{k}} \bigl|w(t-1)\bigr| \\
&= \max_{-(1-\frac{1}{k}) \leqslant t \leqslant -\frac{1}{k}} \bigl|x(t)\bigr| + \max_{-(1-\frac{1}{k}) \leqslant t \leqslant -\frac{1}{k}} \bigl|w(t)\bigr| \\
&= \max_{-(1-\frac{1}{k}) \leqslant t \leqslant -\frac{1}{k}} \bigl|\xi(t)\bigr| \\
&\leqslant Q.
\end{aligned}$$

From assumption (H4) and Lemma 2.2, we have

$$\begin{aligned}
\alpha(Tx) &= \min_{\frac{1}{k} \leqslant t \leqslant 1-\frac{1}{k}} \bigl|(Tx)(t)\bigr| \\
&\geqslant \frac{k-1}{k^2} \max_{0 \leqslant t \leqslant 1} \bigl|(Tx)(t)\bigr| \\
&= \frac{k-1}{k^2} \int_0^{\sigma_x} \phi_p^{-1}\left(\int_s^{\sigma_x} q(r) f\bigl(r, x(r), x(r-1) + w(r-1), x'(r)\bigr)\, dr \right) ds \\
&= \frac{k-1}{k^2} \int_{\sigma_x}^{1} \phi_p^{-1}\left(\int_{\sigma_x}^{s} q(r) f\bigl(r, x(r), x(r-1) + w(r-1), x'(r)\bigr)\, dr \right) ds \\
&\geqslant \frac{k-1}{k^2} \min\left\{ \int_0^{1/2} \phi_p^{-1}\left(\int_s^{1/2} q(r) f\bigl(r, x(r), x(r-1) + w(r-1), x'(r)\bigr)\, dr \right) ds, \right. \\
&\qquad \left. \int_{1/2}^{1} \phi_p^{-1}\left(\int_{1/2}^{s} q(r) f\bigl(r, x(r), x(r-1) + w(r-1), x'(r)\bigr)\, dr \right) ds \right\} \\
&\geqslant \frac{k-1}{k^2} \min\left\{ \int_{1/k}^{1/2} \phi_p^{-1}\left(\int_s^{1/2} q(r) f\bigl(r, x(r), x(r-1) + w(r-1), x'(r)\bigr)\, dr \right) ds, \right. \\
&\qquad \left. \int_{1/2}^{(k-1)/k} \phi_p^{-1}\left(\int_{1/2}^{s} q(r) f\bigl(r, x(r), x(r-1) + w(r-1), x'(r)\bigr)\, dr \right) ds \right\} \\
&\geqslant \frac{k-1}{k^2} \frac{k^2 b}{(k-1)C_2} \min\left\{ \int_{1/k}^{1/2} \phi_p^{-1}\left(\int_s^{1/2} q(r)\, dr \right) ds, \int_{1/2}^{(k-1)/k} \phi_p^{-1}\left(\int_{1/2}^{s} q(r)\, dr \right) ds \right\} \\
&= \frac{b}{C_2} C_2 = b.
\end{aligned}$$

This shows that condition (S1) of Theorem 1.1 is satisfied.

Secondly, from (2.3) and $b \leqslant \frac{k-1}{2k^2}d$, we have

$$\alpha(Tx) \geqslant \frac{k-1}{k^2}\theta(Tx) > \frac{k-1}{k^2}\frac{k^2}{k-1}b = b,$$

for all $x \in P(\gamma, \alpha, b, d)$ with $\theta(Tx) > \frac{k^2}{k-1}b$. Thus, condition (S2) of Theorem 1.1 is satisfied.

We finally show that (S3) of Theorem 1.1 also holds. Clearly, as $\psi(0) = 0 < a$, we have $0 \notin R(\gamma, \psi, a, d)$. Suppose that $x \in R(\gamma, \psi, a, d)$ with $\psi(x) = a$. Then, by (H5), we have

$$\begin{aligned}
\psi(Tx) &= \max_{0\leqslant t\leqslant 1}\bigl|(Tx)(t)\bigr| \\
&= \int_0^{\sigma_x} \phi_p^{-1}\left(\int_s^{\sigma_x} q(r) f\bigl(r, x(r), x(r-1) + w(r-1), x'(r)\bigr)\, dr\right) ds \\
&= \int_{\sigma_x}^{1} \phi_p^{-1}\left(\int_{\sigma_x}^{s} q(r) f\bigl(r, x(r), x(r-1) + w(r-1), x'(r)\bigr)\, dr\right) ds \\
&\leqslant \max\left\{\int_0^{1/2} \phi_p^{-1}\left(\int_s^{1/2} q(r) f\bigl(r, x(r), x(r-1) + w(r-1), x'(r)\bigr)\, dr\right) ds,\right. \\
&\qquad \left.\int_{1/2}^{1} \phi_p^{-1}\left(\int_{1/2}^{s} q(r) f\bigl(r, x(r), x(r-1) + w(r-1), x'(r)\bigr)\, dr\right) ds\right\}.
\end{aligned}$$

In fact, if $\sigma_x = 1/2$, the above inequality holds; if $\sigma_x < 1/2$, we have

$$\begin{aligned}
\psi(Tx) &= \int_0^{\sigma_x} \phi_p^{-1}\left(\int_s^{\sigma_x} q(r) f\bigl(r, x(r), x(r-1) + w(r-1), x'(r)\bigr)\, dr\right) ds \\
&= \int_0^{\sigma_x} \phi_p^{-1}\left(\int_s^{1/2} q(r) f\bigl(r, x(r), x(r-1) + w(r-1), x'(r)\bigr)\, dr\right. \\
&\qquad \left. + \int_{1/2}^{\sigma_x} q(r) f\bigl(r, x(r), x(r-1) + w(r-1), x'(r)\bigr)\, dr\right) ds \\
&\leqslant \int_0^{\sigma_x} \phi_p^{-1}\left(\int_s^{1/2} q(r) f\bigl(r, x(r), x(r-1) + w(r-1), x'(r)\bigr)\, dr\right) ds \\
&\leqslant \int_0^{1/2} \phi_p^{-1}\left(\int_s^{1/2} q(r) f\bigl(r, x(r), x(r-1) + w(r-1), x'(r)\bigr)\, dr\right) ds,
\end{aligned}$$

if $\sigma_x > 1/2$, we have

$$\begin{aligned}\psi(Tx) &= \int_{\sigma_x}^{1} \phi_p^{-1}\left(\int_{\sigma_x}^{s} q(r) f\big(r, x(r), x(r-1)+w(r-1), x'(r)\big)\,dr\right) ds \\ &= \int_{\sigma_x}^{1} \phi_p^{-1}\left(\int_{\sigma_x}^{1/2} q(r) f\big(r, x(r), x(r-1)+w(r-1), x'(r)\big)\,dr\right. \\ &\quad \left. + \int_{1/2}^{s} q(r) f\big(r, x(r), x(r-1)+w(r-1), x'(r)\big)\,dr\right) ds \\ &\leqslant \int_{\sigma_x}^{1} \phi_p^{-1}\left(\int_{1/2}^{s} q(r) f\big(r, x(r), x(r-1)+w(r-1), x'(r)\big)\,dr\right) ds \\ &\leqslant \int_{1/2}^{1} \phi_p^{-1}\left(\int_{1/2}^{s} q(r) f\big(r, x(r), x(r-1)+w(r-1), x'(r)\big)\,dr\right) ds.\end{aligned}$$

So, the above inequality holds. Thus,

$$\begin{aligned}\psi(Tx) &= \max_{0\leqslant t\leqslant 1}\big|(Tx)(t)\big| \\ &= \int_{0}^{\sigma_x} \phi_p^{-1}\left(\int_{s}^{\sigma_x} q(r) f\big(r, x(r), x(r-1)+w(r-1), x'(r)\big)\,dr\right) ds \\ &= \int_{\sigma_x}^{1} \phi_p^{-1}\left(\int_{\sigma_x}^{s} q(r) f\big(r, x(r), x(r-1)+w(r-1), x'(r)\big)\,dr\right) ds \\ &\leqslant \max\left\{\int_{0}^{1/2} \phi_p^{-1}\left(\int_{s}^{1/2} q(r) f\big(r, x(r), x(r-1)+w(r-1), x'(r)\big)\,dr\right) ds,\right. \\ &\qquad \left.\int_{1/2}^{1} \phi_p^{-1}\left(\int_{1/2}^{s} q(r) f\big(r, x(r), x(r-1)+w(r-1), x'(r)\big)\,dr\right) ds\right\} \\ &< \frac{a}{C_1}\max\left\{\int_{0}^{1/2} \phi_p^{-1}\left(\int_{s}^{1/2} q(r)\,dr\right) ds, \int_{1/2}^{1} \phi_p^{-1}\left(\int_{1/2}^{s} q(r)\,dr\right) ds\right\} \\ &= \frac{a}{C_1}\cdot C_1 = a.\end{aligned}$$

So, condition (S3) of Theorem 1.1 is satisfied. Therefore, an application of Theorem 1.1 implies the boundary-value problem (2.1)–(2.2) has at least three positive solutions x_1, x_2, and x_3 satisfying (2.5). The proof is complete. □

So, in this section, we have the main result.

Corollary 2.1. *Suppose the assumptions of Theorem* 2.1 *hold. Then the boundary value problem* (1.1)–(1.2) *has at least three positive solutions* $\tilde{x}_i(t) = x_i(t) + w(t)$, $i = 1, 2, 3$, *where* $x_i(t)$ *satisfies* (2.5).

3. Triple positive solutions of (1.1)–(1.3)

Now we deal with problem (1.1)–(1.3). The method is similar to what we have done above. Throughout this section, suppose condition (H2) holds. Define the cone $P_1 \subset X$ by

$$P_1 = \{x \in X \mid x(t) \geqslant 0,\ x(0) = x'(1) = 0,\ x \text{ is concave and increasing on } [0,1]\}.$$

Let the nonnegative continuous concave functional α_1, the nonnegative continuous convex functional θ_1, γ_1, and the nonnegative continuous functional ψ_1 be defined on the cone P_1 by

$$\gamma_1(x) = \max_{t\in[0,1]} |x'(t)| = x'(0), \qquad \psi_1(x) = \theta_1(x) = \max_{t\in[0,1]} |x(t)| = x(1),$$

$$\alpha_1(x) = \min_{t\in[\frac{1}{k}, 1-\frac{1}{k}]} |x(t)| = x\left(\frac{1}{k}\right), \quad \text{for } x \in P_1.$$

Lemma 3.1. [12] *If* $x \in P_1$, *then* $x(1) \leqslant x'(0)$.

Lemma 3.2. *If* $x(t)$ *is a positive solution to the problem*

$$\big(\phi_p(x'(t))\big)' + q(t) f\big(t, x(t), x(t-1) + w(t-1), x'(t)\big) = 0, \quad t \in (0,1), \tag{3.1}$$

$$\begin{cases} x(t) = 0, & -1 \leqslant t \leqslant 0, \\ x'(1) = 0, \end{cases} \tag{3.2}$$

where

$$w(t) = \begin{cases} \xi(t), & -1 \leqslant t \leqslant 0, \\ 0, & 0 \leqslant t \leqslant 1, \end{cases}$$

then $\tilde{x}(t) = x(t) + w(t)$, $-1 \leqslant t \leqslant 1$, *is a positive solution to BVP* (1.1)–(1.3).

So we focus on BVP (3.1)–(3.2).

With Lemma 3.1 and the concavity of x, the functionals defined above satisfy

$$\frac{1}{k}\theta_1(x) \leqslant \alpha_1(x) \leqslant \theta_1(x) = \psi_1(x), \qquad \|x\| = \max\{\theta_1(x),\ \gamma_1(x)\} = \gamma_1(x), \tag{3.3}$$

for all $x \in \overline{P_1(\gamma, d)} \subset P_1$.

Theorem 3.1. *Suppose* (H2) *holds. In addition, assume that there exist numbers* a, b, d *with* $0 < a < b \leqslant (k-1)d/k^2$ *such that the following conditions are satisfied*:

(H6) $$f(t,u,v,w) \leqslant \frac{1}{M}\phi_p(d), \quad \textit{for } (t,u,v,w) \in [0,1] \times [0,d] \times [0,Q] \times [-d,d],$$

(H7) $$f(t,u,v,w) \geqslant \phi_p\left(\frac{kb}{C_4}\right),$$

$$\textit{for } (t,u,v,w) \in \left[\frac{1}{k}, \frac{k-1}{k}\right] \times \left[b, \frac{k^2}{k-1}b\right] \times [0,Q] \times [-d,d],$$

(H8) $\quad f(t,u,v,w) < \phi_p\left(\frac{a}{C_3}\right)$, *for* $(t,u,v,w) \in [0,1] \times [0,a] \times [0,Q] \times [-d,d]$.

Then the boundary-value problem (3.1)–(3.2) *has at least three positive solutions* x_1, x_2, *and* x_3 *satisfying*

$$\begin{aligned}
&\max_{0\leqslant t\leqslant 1}\left|x_i'(t)\right| \leqslant d, \quad \textit{for } i = 1,2,3,\\
&b < \min_{\frac{1}{2}\leqslant t\leqslant 1}\left|x_1(t)\right|,\\
&a < \max_{0\leqslant t\leqslant 1}\left|x_2(t)\right|, \quad \textit{with} \min_{\frac{1}{2}\leqslant t\leqslant 1}\left|x_2(t)\right| < b,\\
&\max_{0\leqslant t\leqslant 1}\left|x_3(t)\right| < a.
\end{aligned} \tag{3.4}$$

Proof. Problem (3.1)–(3.2) has a solution $x = x(t)$ if and only if x solves the operator equation

$$\begin{aligned}
x(t) &= (Tx)(t)\\
&:= \begin{cases} \int_0^t \phi_p^{-1}\left(\int_s^1 q(r) f(r, x(r), x(r-1) + w(r-1), x'(r))\,dr\right) ds, & 0 \leqslant t \leqslant 1,\\ 0, & -1 \leqslant t \leqslant 0. \end{cases}
\end{aligned}$$

We can prove that this operator, $T : P_1 \to P_1$, is completely continuous. We now show that all the conditions of Theorem 1.1 are satisfied.

From the definition of T, we deduce that for each $x \in P_1$, $Tx \in P_1$. Moreover, $(Tx)(1)$ is the maximum value of T on $[0, 1]$.

If $x \in \overline{P(\gamma_1, d)}$, then $\gamma_1(x) = \max_{0\leqslant t\leqslant 1} |x'(t)| \leqslant d$. From Lemma 3.1, we have $\max_{0\leqslant t\leqslant 1} |x(t)| \leqslant d$, and

$$\begin{aligned}
\left[x(t-1) + w(t-1)\right]\big|_{0\leqslant t\leqslant 1} &= \left[x(t) + w(t)\right]\big|_{-1\leqslant t\leqslant 0} = w(t)|_{-1\leqslant t\leqslant 0}\\
&= \xi(t)|_{-1\leqslant t\leqslant 0} \leqslant Q.
\end{aligned}$$

Then assumption (H6) implies $f(t, x(t), x(t-1) + w(t-1), x'(t)) \leqslant \frac{1}{M}\phi_p(d)$. On the other hand, for $x \in P_1$, $Tx \in P_1$, Tx is concave on $[0, 1]$, and $\max_{t\in[0,1]} |(Tx)'(t)| = |(Tx)'(0)|$, so

$$\begin{aligned}
\gamma_1(Tx) &= \max_{t\in[0,1]}\left|(Tx)'(t)\right|\\
&= \left|(Tx)'(0)\right|\\
&= \phi_p^{-1}\left(\int_0^1 q(r) f\big(r, x(r), x(r-1) + w(r-1), x'(r)\big)\,dr\right)\\
&\leqslant d \cdot \phi_p^{-1}\left(\frac{1}{M}\right)\phi_p^{-1}\left(\int_0^1 q(r)\,dr\right)\\
&= d \cdot \phi_p^{-1}\left(\frac{1}{M}\right) \cdot \phi_p^{-1}(M) = d.
\end{aligned}$$

Hence, $T : \overline{P(\gamma_1, d)} \to \overline{P(\gamma_1, d)}$.

To check condition (S1) of Theorem 1.1, we choose

$$x_0(t) = -\frac{k^2}{k-1} b(t-1)^2 + \frac{k^2}{k-1} b, \quad 0 \leqslant t \leqslant 1.$$

It is easy to see that $x_0 \in P(\gamma_1, \theta_1, \alpha_1, b, \frac{k^2}{k-1}b, d)$ and $\alpha_1(x_0) = \min_{t\in[\frac{1}{k}, 1-\frac{1}{k}]} |x_0(t)| = x_0(\frac{1}{k}) > b$, and so $\{x \in P(\gamma_1, \theta_1, \alpha_1, b, \frac{k^2}{k-1}b, d) \mid \alpha_1(x) > b\} \neq \emptyset$. Hence, if $x \in P(\gamma_1, \theta_1, \alpha_1, b, \frac{k^2}{k-1}b, d)$, then $b \leqslant x(t) \leqslant \frac{k^2}{k-1}b$, $|x'(t)| \leqslant d$ for $1/k \leqslant t \leqslant 1 - 1/k$. From assumption (H7), we have

$$\begin{aligned}
\alpha_1(Tx) &= \min_{\frac{1}{k}\leqslant t\leqslant 1-\frac{1}{k}} \left|(Tx)(t)\right| \geqslant \frac{1}{k} \max_{0\leqslant t\leqslant 1} \left|(Tx)(t)\right| = \frac{1}{k}(Tx)(1) \\
&= \frac{1}{k} \int_0^1 \phi_p^{-1}\left(\int_s^1 q(r) f\big(r, x(r), x(r-1) + w(r-1), x'(r)\big)\, dr \right) ds \\
&\geqslant \frac{1}{k} \int_{1/k}^1 \phi_p^{-1}\left(\int_s^{(k-1)/k} q(r) f\big(r, x(r), x(r-1) + w(r-1), x'(r)\big)\, dr \right) ds \\
&\geqslant \frac{b}{C_4} \int_{1/k}^1 \phi_p^{-1}\left(\int_s^{(k-1)/k} q(r)\, dr \right) ds \\
&= \frac{b}{C_4} \cdot C_4 = b.
\end{aligned}$$

This shows that condition (S1) of Theorem 1.1 is satisfied.

Secondly, from (3.3), we have

$$\alpha_1(Tx) \geqslant \frac{1}{k}\theta_1(Tx) > \frac{k}{k-1}b > b,$$

for all $x \in P(\gamma_1, \alpha_1, b, d)$ with $\theta_1(Tx) > \frac{k^2}{k-1}b$. Thus, condition (S2) of Theorem 1.1 is satisfied.

We finally show that (S3) of Theorem 1.1 also holds. Clearly, since $\psi_1(0) = 0 < a$, $0 \notin R(\gamma_1, \psi_1, a, d)$. Suppose that $x \in R(\gamma_1, \psi_1, a, d)$ with $\psi_1(x) = a$. Then, by assumption (H8), we have

$$\begin{aligned}
\psi_1(Tx) &= \max_{0\leqslant t\leqslant 1} \left|(Tx)(t)\right| \\
&= \int_0^1 \phi_p^{-1}\left(\int_s^1 q(r) f\big(r, x(r), x(r-1) + w(r-1), x'(r)\big)\, dr \right) ds \\
&\leqslant \frac{a}{C_3} \int_0^1 \phi_p^{-1}\left(\int_s^1 q(r)\, dr \right) ds \\
&< \frac{a}{C_3} \cdot C_3 = a.
\end{aligned}$$

So, condition (S3) of Theorem 1.1 is satisfied. Therefore, the boundary-value problem (3.1)–(3.2) has at least three positive solutions x_1, x_2, and x_3 satisfying (3.4). The proof is complete. □

So, in this section, we have the main result.

Corollary 3.1. *Suppose the assumptions of Theorem* 3.1 *hold. Then the boundary value problem* (1.1)–(1.3) *has at least three positive solutions* $\tilde{x}_i(t) = x_i(t) + w(t)$, $i = 1, 2, 3$, *where* $x_i(t)$ *satisfies* (3.4).

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