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Tensor products of some special rings ☆

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Abstract

In this paper we solve a problem, originally raised by Grothendieck, on the properties, i.e., complete intersection, Gorenstein, Cohen–Macaulay, that are conserved under tensor product of algebras over a field k.

Keywords: Regular; Complete intersection; Gorenstein; Cohen–Macaulay; Flat homomorphism of rings

Introduction

Throughout this note all rings and algebras considered in this paper are commutative with identity elements, and all ring homomorphisms are unital. Throughout, k stands for a field.

Among local rings there is a well-known chain

$$\text{Regular} \Rightarrow \text{Complete intersection} \Rightarrow \text{Gorenstein} \Rightarrow \text{Cohen–Macaulay}.$$

These concepts are extended to non-local rings: for example, a ring is regular if for all prime ideal $\mathfrak{p}$ of R, $R_{\mathfrak{p}}$ is a regular local ring.

In this paper, we shall investigate if these properties are conserved under tensor product operations. It is well-known that the tensor product $R \otimes_A S$ of regular rings is not regular in general, even if we assume R and S are A-algebra and A is a field, see Remark 7. In [5],

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Watanabe, Ishikawa, Tachibana, and Otsuka, showed that under a suitable condition tensor products of regular rings are complete intersections. It is proved in [3], that the tensor product $R \otimes_A S$ of Cohen–Macaulay rings are again Cohen–Macaulay if we assume R is flat A-module and S is a finitely generated A-module, and in [5], it is shown that the same is true for Gorenstein rings. Recently, in [1], Bouchiba and Kabbaj showed that if R and S are k-algebras such that $R \otimes_k S$ is Noetherian then $R \otimes_k S$ is a Cohen–Macaulay ring if and only if R and S are Cohen–Macaulay rings.

In this paper we shall show that the same is true for complete intersection and Gorenstein rings. Also it is shown that $R \otimes_k S$ satisfies Serre's condition (S_n) if and only if R and S satisfy (S_n).

Main results

A Noetherian local ring R is a complete intersection (ring) if its completion $\hat{R}$ is a residue class ring of a regular local ring S with respect to an ideal generated by an S-sequence. We say that a Noetherian ring is locally a complete intersection if all its localizations are complete intersections.

A Noetherian ring R satisfies Serre's condition (S_n) if depth $R_\mathfrak{p} \geqslant \operatorname{Min}\{n, \dim R_\mathfrak{p}\}$ for all prime ideal $\mathfrak{p}$ of R. Also, a Noetherian ring R satisfies Serre's normality condition (R_n) if $R_\mathfrak{p}$ is a regular local ring for all prime ideal $\mathfrak{p}$ with $\dim R_\mathfrak{p} \leqslant n$.

The following theorem is collected from [2, Remark 2.3.5, Corollary 3.3.15, Theorem 2.1.7, and Theorem 2.2.12].

Theorem 1. *Let φ: $(R, \mathfrak{m}) \to (S, \mathfrak{n})$ be a flat local homomorphism of Noetherian local rings. Then the following hold*:

(a) *S is a complete intersection* (*resp. Gorenstein, Cohen–Macaulay*) $\Leftrightarrow$ *R and $S/\mathfrak{m}S$ are complete intersections* (*resp. Gorenstein, Cohen–Macaulay*).
(bl) *If S is regular then R is regular.*
(b2) *If R and $S/\mathfrak{m}S$ are regular then S is regular.*

Corollary 2. *Let φ: $R \to S$ be a flat homomorphism of Noetherian rings. Then the following hold*:

(a) *If R and the fibers $R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p} \otimes_R S$, $\mathfrak{p} \in \operatorname{Spec}(R)$, are regular* (*resp. locally complete intersections, Gorenstein, Cohen–Macaulay*) *then S is regular* (*resp. locally complete intersection, Gorenstein, Cohen–Macaulay*).
(b) *If S is locally complete intersection* (*resp. Gorenstein, Cohen–Macaulay*) *then the fibres $R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p} \otimes_R S$, $\mathfrak{p} \in \operatorname{Spec}(R)$, are locally complete intersections* (*resp. Gorenstein, Cohen–Macaulay*).

Proof. (a) Let $\mathfrak{q} \in \operatorname{Spec}(S)$. Set $\mathfrak{p} = \mathfrak{q} \cap R \in \operatorname{Spec}(R)$. The induced homomorphism $\tilde{\varphi}$: $R_\mathfrak{p} \to S_\mathfrak{q}$ is flat and local. It is clear that $S_\mathfrak{q}/\mathfrak{p}R_\mathfrak{p}S_\mathfrak{q}$ is a localization of $R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p} \otimes_R S$. Now the assertion follows from Theorem 1.

(b) Let $\mathfrak{p} \in \operatorname{Spec}(R)$. Then $R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p} \otimes_R S \cong S_\mathfrak{p}/\mathfrak{p}S_\mathfrak{p}$, where $S_\mathfrak{p} = T^{-1}S$ and $T = R - \mathfrak{p}$, and we have

$$\operatorname{Spec}(S_\mathfrak{p}/\mathfrak{p}S_\mathfrak{p}) = \{\mathfrak{q}S_\mathfrak{p}/\mathfrak{p}S_\mathfrak{p} \mid \mathfrak{q} \in \operatorname{Spec}(S), \mathfrak{q} \supseteq \mathfrak{p}S, \mathfrak{q} \cap (R - \mathfrak{p}) = \varnothing\}.$$

For $\mathfrak{q}S_\mathfrak{p}/\mathfrak{p}S_\mathfrak{p} \in \operatorname{Spec}(S_\mathfrak{p}/\mathfrak{p}S_\mathfrak{p})$ we have to show that $(S_\mathfrak{p}/\mathfrak{p}S_\mathfrak{p})_{\mathfrak{q}S_\mathfrak{p}/\mathfrak{p}S_\mathfrak{p}} \cong S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q}$ is complete intersection *(*resp. Gorenstein, Cohen–Macaulay*)*. Consider the induced flat local homomorphism $\tilde{\varphi}\colon R_\mathfrak{p} \to S_\mathfrak{q}$. Now the assertion follows from Theorem 1. □

Theorem 3 (see [2], Propositions 2.1.16 and 2.2.21). *Let $\varphi\colon R \to S$ be a flat homomorphism of Noetherian rings. Then the following hold*:

(a) *Let $\mathfrak{q} \in \operatorname{Spec}(S)$ and $\mathfrak{p} = \mathfrak{q} \cap R$. If $S_\mathfrak{q}$ satisfies (S_n) (resp. (R_n)) then $R_\mathfrak{p}$ satisfies (S_n) (resp. (R_n)).*
(b) *If R and the fibers $R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p} \otimes_R S$, $\mathfrak{p} \in \operatorname{Spec}(R)$, satisfy (S_n) (resp. (R_n)) then S satisfies (S_n) (resp. (R_n)).*

Corollary 4. *Let $\varphi\colon R \to S$ be a faithfully flat homomorphism of Noetherian rings. Then the following hold*:

(a) *If S is regular (resp. locally complete intersection, Gorenstein, Cohen–Macaulay), then so is R.*
(b) *If S satisfies (S_n) (resp. (R_n)), then so does R.*

Proof. Let $\mathfrak{p} \in \operatorname{Spec}(R)$. Since φ is faithfully flat there exists $\mathfrak{q} \in \operatorname{Spec}(S)$ such that $\mathfrak{p} = \mathfrak{q} \cap R$. Consider the flat local homomorphism $\tilde{\varphi}\colon R_\mathfrak{p} \to S_\mathfrak{q}$ where $\tilde{\varphi}(r/s) = \varphi(r)/\varphi(s)$. Now the assertion follows from Theorems 1 and 3. □

Proposition 5. *Let k be a field, L and K be two extension fields of k. Suppose that $L \otimes_k K$ is Noetherian. Then the following hold*:

(a) *$L \otimes_k K$ is locally complete intersection.*
(b) *If k is perfect then $L \otimes_k K$ is regular.*

Proof. (a) With the same method in the proof of [4, Theorem 2.2], we can assume that K is a finitely generated extension field of k (note that, in view of Theorem 1, [4, Lemma 2.1] is true with "Gorenstein ring" replaced by "complete intersection"). Now using [2, Proposition 2.1.11] we have that $L \otimes_k K$ is isomorphic to

$$A = T^{-1}(L[x_1, x_2, \ldots, x_n])/(f_1, f_2, \ldots, f_m)T^{-1}(L[x_1, x_2, \ldots, x_n]),$$

where T is a multiplicatively closed subset of $L[x_1, x_2, \ldots, x_n]$ and $f_1, f_2, \ldots, f_m$ is a $T^{-1}(L[x_1, x_2, \ldots, x_n])$-sequence. Therefore A is locally complete intersection, cf. [2, Theorem 2.3.3(c)].

(b) The assertion follows from the note on page 49 of [4]. □

Theorem 6. *Let R and S be non-zero k-algebras such that $R \otimes_k S$ is Noetherian. Then the following hold*:

(a) *$R \otimes_k S$ is locally complete intersection* (*resp. Gorenstein*, *Cohen–Macaulay*) *if and only if R and S are locally complete intersections* (*resp. Gorenstein*, *Cohen–Macaulay*).
(b) *$R \otimes_k S$ satisfies (S_n) if and only if R and S satisfy (S_n).*
(c) *If $R \otimes_k S$ is regular then R and S are regular.*
(d) *If $R \otimes_k S$ satisfies (R_n) then R and S satisfy (R_n).*
(e) *The converse of parts* (c) *and* (d) *hold if* $\operatorname{char}(k) = 0$ *or* $\operatorname{char}(k) = p$ *such that* $k = \{a^p \mid a \in k\}$.

Proof. Consider two faithfully flat homomorphisms

$$\varphi : R \to R \otimes_k S \quad \text{and} \quad \psi : S \to R \otimes_k S$$

of Noetherian rings.

If $R \otimes_k S$ is regular (resp. locally complete intersection, Gorenstein, Cohen–Macaulay) then by Corollary 4 we have R and S are regular (resp. locally complete intersections, Gorenstein, Cohen–Macaulay). Also if $R \otimes_k S$ satisfies (S_n) (resp. (R_n)) then by Corollary 4, R and S satisfy (S_n) (resp. (R_n)).

Now let R and S be locally complete intersection (resp. Gorenstein, Cohen–Macaulay). By Corollary 2 it is enough to show that the fibres $(R \otimes_k S) \otimes_R R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p} \cong R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p} \otimes_k S$ over every prime ideal $\mathfrak{p}$ of R is locally complete intersection (resp. Gorenstein, Cohen–Macaulay). Consider the flat homomorphism $\gamma : S \to R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p} \otimes_k S$. Using Corollary 2, it is enough to show that the fibres $(R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p} \otimes_k S) \otimes_S S_\mathfrak{q}/\mathfrak{q}S_\mathfrak{q} \cong R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p} \otimes_k S_\mathfrak{q}/\mathfrak{q}S_\mathfrak{q}$ over every prime $\mathfrak{q}$ of S is locally complete intersection (resp. Gorenstein, Cohen–Macaulay). But it is clear to see that $R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p} \otimes_k S_\mathfrak{q}/\mathfrak{q}S_\mathfrak{q}$ is Noetherian, since it is a localization of $R/\mathfrak{p} \otimes_k S/\mathfrak{q} \cong R \otimes_k S/(\mathfrak{p} \otimes_k S + R \otimes_k \mathfrak{q})$, which is Noetherian. Now the assertion follows from Proposition 5.

If R and S satisfy (S_n), with the same proof $R \otimes_k S$ satisfies (S_n). By using the Proposition 5 the proof of part (e) is the same. □

Remark 7. The converse of part (c) in Theorem 6 is not true. For example, let k be an imperfect field of characteristic 3, let $a \in k$ be an element with no cube root in k. Then $K = k[x]/(x^3 - a)k[x]$ is a splitting field of $x^3 - a$ over k. Thus $K \otimes_k K \cong K[x]/(x^3 - a)K[x]$, which is not regular.

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