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On the complexity of Putinar's Positivstellensatz

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Abstract

Let $S=\{x\in\mathbb{R}^n \mid g_1(x)\geqslant 0,\ldots,g_m(x)\geqslant 0\}$ be a basic closed semialgebraic set defined by real polynomials g_i. Putinar's Positivstellensatz says that, under a certain condition stronger than compactness of S, every real polynomial f positive on S possesses a representation $f=\sum_{i=0}^m \sigma_i g_i$ where $g_0:=1$ and each σ_i is a sum of squares of polynomials. Such a representation is a certificate for the nonnegativity of f on S. We give a bound on the degrees of the terms $\sigma_i g_i$ in this representation which depends on the description of S, the degree of f and a measure of how close f is to having a zero on S. As a consequence, we get information about the convergence rate of Lasserre's procedure for optimization of a polynomial subject to polynomial constraints.

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1. Introduction

Always write $\mathbb{N}:=\{0,1,2,\ldots\}$ and $\mathbb{R}$ for the sets of nonnegative integers and real numbers, respectively. Denote by $\mathbb{R}[\bar{X}]$ the ring of polynomials in $n\geqslant 1$ indeterminates $\bar{X}:=(X_1,\ldots,X_n)$. We use suggestive notation like $\mathbb{R}[\bar{X}]^2:=\{p^2 \mid p\in\mathbb{R}[\bar{X}]\}$ for the set of squares and $\sum\mathbb{R}[\bar{X}]^2$ for the set of sums of squares of polynomials in $\mathbb{R}[\bar{X}]$. A subset $M\subseteq\mathbb{R}[\bar{X}]$ is called a *quadratic*

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module if it contains 1 and it is closed under addition and under multiplication with squares, i.e.,

$$1 \in M, \quad M + M \subseteq M \text{ and } \mathbb{R}[\bar{X}]^2 M \subseteq M.$$

A subset $T \subseteq \mathbb{R}[\bar{X}]$ is called a *preordering* if it contains all squares in $\mathbb{R}[\bar{X}]$ and it is closed under addition and multiplication, i.e.,

$$\mathbb{R}[\bar{X}]^2 \subseteq T, \quad T + T \subseteq T \text{ and } TT \subseteq T.$$

In other words, the preorderings are exactly the multiplicatively closed quadratic modules.

Throughout the article, we fix $m \in \mathbb{N}$ and a tuple $\bar{g} := (g_1, \dots, g_m)$ of polynomials $g_i \in \mathbb{R}[\bar{X}]$. It will be convenient to set $g_0 := 1 \in \mathbb{R}[\bar{X}]$. The quadratic module $M(\bar{g})$ generated by $\bar{g}$ (i.e., the smallest quadratic module containing each g_i) is

$$M(\bar{g}) = \sum_{i=0}^{m} \sum \mathbb{R}[\bar{X}]^2 g_i := \left\{ \sum_{i=0}^{m} \sigma_i g_i \mid \sigma_i \in \sum \mathbb{R}[\bar{X}]^2 \right\}. \tag{1}$$

Using the notation

$$\bar{g}^\delta := g_1^{\delta_1}, \dots, g_m^{\delta_m},$$

the preordering $T(\bar{g})$ generated by $\bar{g}$ can be written as

$$T(\bar{g}) = \sum_{\delta \in \{0,1\}^m} \sum \mathbb{R}[\bar{X}]^2 \bar{g}^\delta := \left\{ \sum_{\delta \in \{0,1\}^m} \sigma_\delta \bar{g}^\delta \mid \sigma_\delta \in \sum \mathbb{R}[\bar{X}]^2 \right\}, \tag{2}$$

i.e., $T(\bar{g})$ is the quadratic module generated by the 2^m products of g_i. It is obvious that all polynomials lying in $T(\bar{g}) \supseteq M(\bar{g})$ are nonnegative on the set

$$S(\bar{g}) := \{x \in \mathbb{R}^n \mid g_1(x) \geqslant 0, \dots, g_m(x) \geqslant 0\}.$$

Sets of this form are important in semialgebraic geometry (see [BCR]) and are called *basic closed semialgebraic sets*. In 1991, Schmüdgen [Smn] proved the following "Positivstellensatz" (a commonly used German term explained by the analogy with Hilbert's Nullstellensatz).

Theorem 1 (*Schmüdgen*). *Suppose the basic closed semialgebraic set $S(\bar{g})$ is compact. Then for every polynomial $f \in \mathbb{R}[\bar{X}]$,*

$$f > 0 \quad \text{on } S(\bar{g}) \Longrightarrow f \in T(\bar{g}).$$

Under a certain extra property on $M(\bar{g})$ which we will define now, this theorem remains true with $T(\bar{g})$ replaced by its subset $M(\bar{g})$. We introduce the notation

$$\|\bar{X}\|^2 := \sum_{i=1}^{n} X_i^2 \in \mathbb{R}[\bar{X}].$$

Definition 2. A quadratic module $M \subseteq \mathbb{R}[\bar{X}]$ is called *archimedean* if

$$N - \|\bar{X}\|^2 \in M \quad \text{for some } N \in \mathbb{N}.$$

Note that this definition applies also to preorderings since every preordering is a quadratic module. As a corollary from Schmüdgen's Theorem, we get the following well-known characterization of archimedean quadratic modules.

Corollary 3. *For a quadratic module $M \subseteq \mathbb{R}[\bar{X}]$, the following are equivalent*:

(i) *M is archimedean.*
(ii) *There is a polynomial $p \in M$ such that $S(p) = \{p \geqslant 0\} \subseteq \mathbb{R}^n$ is compact.*
(iii) *There is a tuple $\bar{g}$ of polynomials such that $S(\bar{g})$ is compact and M contains the preordering $T(\bar{g})$.*
(iv) *For all $p \in \mathbb{R}[\bar{X}]$, there is $N \in \mathbb{N}$ such that $N - p \in M$.*

Proof. Observe that (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) $\Longrightarrow$ (iv) $\Longrightarrow$ (i). All of these implications are trivial except (iii)$\Longrightarrow$ (iv) which follows from Theorem 1. □

In particular, we see that $S(\bar{g})$ is compact if and only if $T(\bar{g})$ is archimedean. Unfortunately, $S(\bar{g})$ might be compact without $M(\bar{g})$ being archimedean (see [PD, Example 6.3.1]). What has to be added to compactness of $S(\bar{g})$ in order to ensure that $M(\bar{g})$ is archimedean has been extensively investigated by Jacobi and Prestel [JP,PD]. Now we can state the Positivstellensatz proved by Putinar [Put] in 1993.

Theorem 4 (*Putinar*). *Suppose the quadratic module $M(\bar{g})$ is archimedean. Then for every $f \in \mathbb{R}[\bar{X}]$,*

$$f > 0 \quad \text{on } S(\bar{g}) \Longrightarrow f \in M(\bar{g}).$$

Both the proofs of Schmüdgen and Putinar use functional analysis and real algebraic geometry. They do not give information how to construct a representation of f showing that f lies in the preordering (an expression like in (2) involving 2^m sums of squares) or the quadratic module (a representation like in (1) with $m + 1$ sums of squares).

Based on an old theorem of Pólya [Pól], new proofs of both Schmüdgen's and Putinar's Positivstellensatz have been given in [Sw1,Sw3] which are to some extent constructive. By carefully analyzing a tame version of [Sw3] and using an effective version of Pólya's theorem [PR], upper bounds on the degrees of the sums of squares appearing in Schmüdgen's preordering representation have been obtained in [Sw2]. The aim of this article is to prove bounds on Putinar's quadratic module representation. They will depend on the same data but will be worse than the ones known for Schmüdgen's theorem.

Since it will appear in our bound, we will need a convenient measure of the size of the coefficients of a polynomial. For $\alpha \in \mathbb{N}^n$, we introduce the notation

$$|\alpha| := \alpha_1 + \cdots + \alpha_n \quad \text{and} \quad \bar{X}^\alpha := X_1^{\alpha_1} \cdots X_n^{\alpha_n},$$

as well as the multinomial coefficient

$$\binom{|\alpha|}{\alpha} := \frac{|\alpha|!}{\alpha_1! \cdots \alpha_n!}.$$

For a polynomial $f = \sum_\alpha a_\alpha \bar{X}^\alpha \in \mathbb{R}[\bar{X}]$ with coefficients $a_\alpha \in \mathbb{R}$, we set

$$\|f\| := \max_\alpha \frac{|a_\alpha|}{\binom{|\alpha|}{\alpha}}.$$

This defines a norm on the real vector space $\mathbb{R}[\bar{X}]$ with convenient properties illustrated by Proposition 14 below. For any $k \in \mathbb{R}_{\geqslant 0}$, we now define convex cones $T(\bar{g}, k)$ and $M(\bar{g}, k)$ in the finite-dimensional vector space $\mathbb{R}[\bar{X}]_{\leqslant k}$ of polynomials of degree at most k (i.e., at most $\lfloor k \rfloor$) by setting

$$T(\bar{g}, k) = \left\{ \sum_{\delta \in \{0,1\}^m} \sigma_\delta \bar{g}^\delta \mid \sigma_\delta \in \sum \mathbb{R}[\bar{X}]^2, \deg(\sigma_\delta \bar{g}^\delta) \leqslant k \right\} \subseteq T(\bar{g}) \cap \mathbb{R}[\bar{X}]_{\leqslant k},$$

$$M(\bar{g}, k) = \left\{ \sum_{i=0}^{m} \sigma_\delta \bar{g}^\delta \mid \sigma_\delta \in \sum \mathbb{R}[\bar{X}]^2, \deg(\sigma_\delta \bar{g}^\delta) \leqslant k \right\} \subseteq M(\bar{g}) \cap \mathbb{R}[\bar{X}]_{\leqslant k}.$$

We now recall the previously proved bound for Schmüdgen's theorem.

Theorem 5 (*Schweighofer [Sw2]*). *For all $\bar{g}$ defining a basic closed semialgebraic set $S(\bar{g})$ which is nonempty and contained in the open hypercube $(-1, 1)^n$, there is some $c \geqslant 1$ (depending on $\bar{g}$) such that for all $f \in \mathbb{R}[\bar{X}]$ of degree d with*

$$f^* := \min\{f(x) \mid x \in S(\bar{g})\} > 0,$$

we have

$$f \in T\left(\bar{g}, cd^2\left(1 + \left(d^2 n^d \frac{\|f\|}{f^*}\right)^c\right)\right).$$

In this article, we will prove the following bound for Putinar's theorem.

Theorem 6. *For all $\bar{g}$ defining an archimedean quadratic module $M(\bar{g})$ and a set $\emptyset \neq S(\bar{g}) \subseteq (-1, 1)^n$, there is some $c \in \mathbb{R}_{>0}$ (depending on $\bar{g}$) such that for all $f \in \mathbb{R}[\bar{X}]$ of degree d with*

$$f^* := \min\{f(x) \mid x \in S(\bar{g})\} > 0,$$

we have

$$f \in M\left(\bar{g}, c \exp\left(\left(d^2 n^d \frac{\|f\|}{f^*}\right)^c\right)\right).$$

In both theorems above, there have been made additional assumptions compared to Schmüdgen's and Putinar's original results. But these are not very serious and have only been made to simplify the statements: For example, if $S(\bar{g}) = \emptyset$, then $-1 \in T(\bar{g}, k)$ for some $k \in \mathbb{N}$ by Schmüdgen's theorem. Therefore $4f = (f + 1)^2 + (f - 1)^2(-1) \in T(\bar{g}, 2d + k)$ for each

$f \in \mathbb{R}[\bar{X}]$ of degree $d \geqslant 0$. The other hypothesis that $S(\bar{g})$ be contained in the open hypercube $(-1, 1)^n$ is only a matter of rescaling by a linear (or affine linear) transformation on $\mathbb{R}^n$. For example, if $r > 0$ is such that $S(\bar{g}) \subseteq (-r, r)^n$, then Theorem 5 remains true with $\|f\|$ replaced by $\|f(r\bar{X})\|$. Here it is important to note that the property that $M(\bar{g})$ be archimedean is preserved under affine linear coordinate changes. This is clear from Corollary 3. Confer also the proof of Proposition 9 below.

In both Theorems 5 and 6, the bound depends on three parameters:

- the description $\bar{g}$ of the basic closed semialgebraic set,
- the degree d of f and
- a measure of how close f comes to have a zero on $S(\bar{g})$, namely $\|f\|/f^*$.

The main difference between the two bounds is the exponential function appearing in the degree bound for the quadratic module representation. It is an open research problem whether this exponential function can be avoided. It could even be possible that the same bound than for Schmüdgen's theorem holds also for Putinar's theorem. In view of the impact on the convergence rate of Lasserre's optimization procedure (see Section 2 below), this question seems very interesting for applications. Whereas the bound for the preordering representation cannot be improved significantly (see [Ste]), this seems possible for the quadratic module representation.

The dependance on the third parameter $\|f\|/f^*$ is consistent with the fact that the condition $f^* > 0$ cannot be weakened to $f^* \geqslant 0$ in neither Schmüdgen's nor Putinar's theorem. Under certain conditions (e.g., on the derivatives of f), both theorems can however be extended to nonnegative polynomials (see [Sch,Mr2]). With the partially constructive approach from [Sw4] to representation of nonnegative polynomials with zeros, one might perhaps in the future gain bounds even for the case of nonnegative polynomials which depend however on further data (for example the norm of the Hessian at the zeros).

In special cases, Prestel had already proved the mere existence of a degree bound for Putinar's Theorem depending on the three parameters described above (see [PD, Section 8.4] and [Pre]). He used model theory and valuation theory to get the existence of such a bound. But the only information about the bound he gets (using Gödel's theorem on the completeness of first order logic) is that the bound is computable.

In contrast to this, our more constructive approach yields information in what way the above bound depends on the two parameters d and $\|f\|/f^*$. The constant c depends on the description $\bar{g}$ of the semialgebraic set, but no explicit formula is given. For a concretely given $\bar{g}$, one could possibly determine a constant c like in Theorems 5 and 6 by a very (probably too) tedious analysis of the proofs (cf. [Sw2, Remark 10]).

We conclude this introduction by considering the one variable case, i.e., $n = 1$. Scheiderer showed in [Sch, Corollary 3.4] that, in this case, compactness of $S(\bar{g})$ implies that $M(\bar{g}) = T(\bar{g})$ (and therefore $M(\bar{g})$ is archimedean). Now the equality $M(\bar{g}) = T(\bar{g})$ implies in particular that $\bar{g}^\delta \in M(\bar{g})$ for all $\delta \in \{0, 1\}^m$. As an easy consequence, we get that Theorem 5 remains valid with T replaced by M in the case of univariate polynomials. The bound in Theorem 6 is thus far from being sharp in the one variable case. As said above, in the multivariate case it is not known if the bound can be improved considerably.

The rest of the paper is organized as follows. In the next section, we use our result to investigate the accuracy of Lasserre's "sums of squares relaxations" for optimization of polynomials. In Section 3, we give the proof of Theorem 6.

2. Convergence rate of Lasserre's procedure

Consider the problem to compute (by a numerical procedure, i.e., up to some prescribable error) the minimum

$$f^* := \min\{f(x) \mid x \in S(\bar{g})\} \tag{3}$$

of a polynomial $f \in \mathbb{R}[\bar{X}]$ on a nonempty basic closed semialgebraic set $S(\bar{g})$. In other words, you want to minimize a polynomial under polynomial inequality constraints. When all the polynomials involved are linear, i.e., of degree $\leqslant 1$, this is a linear optimization problem (a linear program) and there are very efficient algorithms to solve this problem. For general polynomials this problem gets very hard. It is therefore a common approach to solve a much easier related problem, a so-called relaxation, namely to compute for $k \in \mathbb{N}$,

$$f_k^* := \sup\{a \in \mathbb{R} \mid f - a \in M(\bar{g}, k)\} \in \mathbb{R} \cup \{-\infty\}, \tag{4}$$

which is clearly a lower bound of f^*. The problem of finding f_k^* can be written as a semidefinite program whose size gets bigger when k grows (see the references below). Semidefinite programming is a well-known generalization of linear programming for which very efficient algorithms exist (see for example [Tod]). One can now solve a sequence of larger and larger semidefinite programs in order to get tighter and tighter lower bounds for f^*. Lasserre [Las] was the first to interpret Putinar's theorem as a convergence result.

Indeed, it is easy to see that Putinar's theorem just says that the ascending sequence $(f_k^*)_{k\in\mathbb{N}}$ converges to f^* under the condition that $M(\bar{g})$ be archimedean. In this section, we will interpret our bound for Putinar's Positivstellensatz as a result about the speed of convergence of this sequence.

For an introduction to the interplay of semidefinite programming, sums of squares, optimization of polynomials and results about positive polynomials, we refer to [Las,Mr1,Sw1] (with special regard to Putinar's Positivstellensatz) and [JL,DNP,NDS,PS]. There are several software tools which translate the problem of computing f_k^* into a semidefinite program and call a semidefinite programming solver. See [HL,KKW, Löf,PPP].

The following technical lemma will also be needed in Section 3.

Lemma 7. *For any polynomial $f \in \mathbb{R}[\bar{X}]$ of degree $d \geqslant 1$ and all $x \in [-1, 1]^n$,*

$$|f(x)| \leqslant 2\, dn^d \|f\|.$$

Proof. Writing $f = \sum_\alpha a_\alpha \binom{|\alpha|}{\alpha} \bar{X}^\alpha$ ($a_\alpha \in \mathbb{R}$), we have $\|f\| = \max_\alpha |a_\alpha|$ and

$$|f(x)| = \left|\sum_\alpha a_\alpha \binom{|\alpha|}{\alpha} x_1^{\alpha_1} \cdots x_n^{\alpha_n}\right| \leqslant \sum_\alpha |a_\alpha| \binom{|\alpha|}{\alpha} |x_1|^{\alpha_1} \cdots |x_n|^{\alpha_n}$$

for all $x \in [-1, 1]^n$. Using that $|a_\alpha| \leqslant \|f\|$ and $|x_i| \leqslant 1$, the multinomial identity now shows that $|f(x)| \leqslant \|f\| \sum_{k=0}^d n^k \leqslant (d+1)n^d \|f\| \leqslant 2\, dn^d \|f\|$. $\square$

Now we are ready to prove the main theorem of this section.

Theorem 8. *For all polynomials $\bar{g}$ defining an archimedean quadratic module $M(\bar{g})$ and a set $\emptyset \neq S(\bar{g}) \subseteq (-1,1)^n$, there is some $c > 0$ (depending on $\bar{g}$) such that for all $f \in \mathbb{R}[\bar{X}]$ of degree d with minimum f^* on S and for all integers $k > c\exp((2d^2n^d)^c)$, we have*

$$(f - f^*) + \frac{6d^3n^{2d}\|f\|}{\sqrt[c]{\log\frac{k}{c}}} \in M(\bar{g}, k),$$

and hence

$$0 \leqslant f^* - f_k^* \leqslant \frac{6d^3n^{2d}\|f\|}{\sqrt[c]{\log\frac{k}{c}}},$$

where f_k^* is defined as in (4).

Proof. Given $\bar{g}$, we choose $c > 0$ like in Theorem 6. Now let $f \in \mathbb{R}[\bar{X}]$ be of degree d with minimum f^* on S and

$$k > c\exp((2d^2n^d)^c) \tag{5}$$

be an integer. The case $d = 0$ is trivial. We assume therefore $d \geqslant 1$. Note that $k > c$ and hence $\log(k/c) > 0$. Setting

$$a := \frac{6d^3n^{2d}\|f\|}{\sqrt[c]{\log\frac{k}{c}}}, \tag{6}$$

all we have to prove is $h := f - f^* + a \in M(\bar{g}, k)$ because the second claim follows from this. By our choice of c and the observation $\deg h = \deg f = d$, it is enough to show that

$$c\exp\left(\left(d^2n^d\frac{\|h\|}{a}\right)^c\right) \leqslant k,$$

or equivalently

$$d^2n^d\|h\| \leqslant a\sqrt[c]{\log\frac{k}{c}} = 6d^3n^{2d}\|f\|.$$

Observing that $\|h\| \leqslant \|f\| + |f^*| + a$, it suffices to show that

$$\|f\| + |f^*| + a \leqslant 6\,dn^d\|f\|.$$

Lemma 7 tells us that $|f^*| \leqslant 2dn^d\|f\|$ and we are thus reduced to verify that

$$a \leqslant (4dn^d - 1)\|f\|,$$

which is by (6) equivalent to

$$6d^3n^{2d} \leqslant (4dn^d - 1)\sqrt[c]{\log\frac{k}{c}}.$$

By (5), it is finally enough to check that $6d^3n^{2d} \leqslant (4\,dn^d - 1)(2d^2n^d)$. $\square$

As already said in the introduction, the hypothesis that $S(\bar{g})$ is contained in the open unit hypercube is just a technicality to avoid that the bound gets even more complicated. In fact, if one does not insist on all the information given in Theorem 8, one gets a corollary which is easy to remember and still gives the most important part of information.

Corollary 9. *Suppose $M(\bar{g})$ is archimedean, $S(\bar{g}) \neq \emptyset$ and $f \in \mathbb{R}[\bar{X}]$. There is*

- *a constant $c > 0$ depending only on $\bar{g}$ and*
- *a constant $c' > 0$ depending on $\bar{g}$ and f*

such that for f^* and f_k^* as defined in (3) *and* (4),

$$0 \leqslant f^* - f_k^* \leqslant \frac{c'}{\sqrt[c]{\log \frac{k}{c}}} \quad \textit{for all large } k \in \mathbb{N}.$$

Proof. Without loss of generality, assume $f \neq 0$. Set $d := \deg f$. Since $M(\bar{g})$ is archimedean, $S(\bar{g})$ is compact. We can hence choose a rescaling factor $r > 0$ depending only on $\bar{g}$ such that $S(\bar{g}(r\bar{X})) \subseteq (-1, 1)^n$. Here $\bar{g}(r\bar{X})$ denotes the tuple of rescaled polynomials $g_i(r\bar{X})$. Now Theorem 8 applied to $g(r\bar{X})$ instead of $\bar{g}$ yields $c > 0$ that will together with $c' := 6d^3n^{2d}\|f(rX)\|$ have the desired properties by simple scaling arguments. $\square$

Remark 10. The bound on the difference $f^* - f_k^*$ presented in this section is much worse than the corresponding one presented in [Sw2, Section 2] which is based on preordering representations (i.e., where f_k^* would be defined using $T(\bar{g})$ instead of $M(\bar{g})$). This raises the question whether it is after all not such a bad thing to use preordering (instead of quadratic module) representations for optimization though they involve the 2^m products $\bar{g}^\delta$ letting the semidefinite programs get huge when m is not small. However, it is not known if Theorem 8 holds perhaps even with the bound from [Sw2, Theorem 4]. Compare also [Sw2, Remark 5].

3. The proof

In this section, we give the Proof of Theorem 6. The three main ingredients are

- the bound for Schmüdgen's theorem presented in Theorem 5 above,
- ideas from the (to some extent constructive) proof of Putinar's theorem in [Sw3, Section 2] and
- the Łojasiewicz inequality from semialgebraic geometry.

We start with some simple facts from calculus.

Lemma 11. *If $0 \neq f \in \mathbb{R}[\bar{X}]$ has degree d, then*

$$|f(x) - f(y)| \leqslant \|x - y\| d^2 n^{d-1} \sqrt{n} \|f\|$$

for all $x, y \in [-1, 1]^n$.

Proof. Denoting by *Df* the derivative of *f*, by the mean value theorem, it is enough to show that

$$|Df(x)(e)| \leqslant d^2 n^{d-1} \sqrt{n} \|f\| \tag{7}$$

for all $x \in [-1, 1]^n$ and $e \in \mathbb{R}^n$ with $\|e\| = 1$. A small computation (compare the Proof of Lemma 7) shows that

$$\left|\frac{\partial f(x)}{\partial x_i}\right| \leqslant \|f\| \sum_{k=1}^{d} k(|x_1| + \cdots + |x_n|)^{k-1} \leqslant \|f\| \sum_{k=1}^{d} kn^{k-1} \leqslant \|f\| d^2 n^{d-1},$$

from which we conclude for all $x \in [-1, 1]^n$ and $e \in \mathbb{R}^n$ with $\|e\| = 1$,

$$|Df(x)(e)| = \left|\sum_{i=1}^{n} \frac{\partial f(x)}{\partial x_i} e_i\right| \leqslant \sum_{i=1}^{n} \left|\frac{\partial f(x)}{\partial x_i}\right| \cdot |e_i| \leqslant \|f\| d^2 n^{d-1} \sum_{i=1}^{n} |e_i|.$$

Because for a vector e on the unit sphere in $\mathbb{R}^n$, $\sum_{i=1}^{n} |e_i|$ can reach at most $\sqrt{n}$, this implies (7). $\square$

Remark 12. For all $k \in \mathbb{N}$ and $y \in [0, 1]$, $(y-1)^{2k} y \leqslant 1/(2k+1)$.

The next lemma is a version of [Sw3, Lemma 2.3] caring about complexity issues. In [Sw3, Lemma 2.3], it is shown that, if $C \subseteq \mathbb{R}^n$ is any compact set, $g_i \leqslant 1$ on C for all i and $f \in \mathbb{R}[\bar{X}]$ is a polynomial with $f > 0$ on $S(\bar{g})$, then there exists $\lambda \geqslant 0$ such that for all sufficiently large $k \in \mathbb{N}$,

$$f - \lambda \sum_{i=1}^{m} (g_i - 1)^{2k} g_i > 0 \quad \text{on } C. \tag{8}$$

The idea is that, if you want to show that $f \in M(\bar{g})$, you first subtract another polynomial from f which lies obviously in $M(\bar{g})$ such that the difference can be proved to lie in $M(\bar{g})$ as well. This other polynomial must necessarily be nonnegative on $S(\bar{g})$ but it should take on only very small values on $S(\bar{g})$ so that the difference is still positive on $S(\bar{g})$. On the region where you are outside and not too far away from $S(\bar{g})$, the polynomial you subtract should take large negative values so that the difference gets positive on this region outside of $S(\bar{g})$ (where f itself might be negative). The hope is that the difference satisfies an improved positivity condition which will help us to show that it lies in $M(\bar{g})$. To understand the lemma, it is helpful to observe that the pointwise limit for $k \to \infty$ of this difference, which is the left hand side of (11), is f on $S(\bar{g})$ and ∞ outside of $S(\bar{g})$.

Lemma 13. *For all $\bar{g}$ such that $S := S(\bar{g}) \cap [-1, 1]^n \neq \emptyset$ and $g_i \leqslant 1$ on $[-1, 1]^n$, there are $c_0, c_1, c_2 > 0$ with the following property:*

For all polynomials $f \in \mathbb{R}[\bar{X}]$ of degree d with minimum $f^* > 0$ on S, if we set

$$L := d^2 n^{d-1} \frac{\|f\|}{f^*}, \quad \lambda := c_1 d^2 n^{d-1} \|f\| L^{c_2}, \tag{9}$$

and if $k \in \mathbb{N}$ satisfies

$$2k + 1 \geqslant c_0(1 + L^{c_0}), \tag{10}$$

then the inequality

$$f - \lambda \sum_{i=1}^{m} (g_i - 1)^{2k} g_i \geqslant \frac{f^*}{2} \tag{11}$$

holds on $[-1, 1]^n$.

Proof. By the Łojasiewicz inequality for semialgebraic functions [BCR, Corollary 2.6.7], we can choose $c_2, c_3 > 0$ such that

$$\operatorname{dist}(x, S)^{c_2} \leqslant -c_3 \min\{g_1(x), \dots, g_m(x), 0\} \tag{12}$$

for all $x \in [-1, 1]^n$ where $\operatorname{dist}(x, S)$ denotes the distance of x to S. Set

$$c_4 := c_3(4n)^{c_2}, \tag{13}$$

$$c_1 := 4nc_4, \tag{14}$$

and choose $c_0 \in \mathbb{N}$ big enough to guarantee that

$$c_0(1 + r^{c_0}) \geqslant 2(m-1)c_4 r^{c_2}, \quad \text{and} \tag{15}$$

$$c_0(1 + r^{c_0}) \geqslant 4mc_1 r^{c_2+1} \tag{16}$$

for all $r \geqslant 0$. Now suppose $f \in \mathbb{R}[\bar{X}]$ is of degree d with minimum $f^* > 0$ on S and consider the set

$$A := \left\{ x \in [-1, 1]^n \mid f(x) \leqslant \frac{3}{4} f^* \right\}.$$

By Lemma 11, we get for all $x \in A$ and $y \in S$

$$\frac{f^*}{4} \leqslant f(y) - f(x) \leqslant \|x - y\| d^2 n^{d-1} \sqrt{n} \|f\| \leqslant \|x - y\| d^2 n^d \|f\|.$$

Since this is valid for arbitrary $y \in S$, it holds that

$$\frac{f^*}{4d^2 n^d \|f\|} \leqslant \operatorname{dist}(x, S)$$

for all $x \in A$. We combine this now with (12) and get

$$\min\{g_1(x), \dots, g_m(x)\} \leqslant -\frac{1}{c_3} \left(\frac{f^*}{4d^2 n^d \|f\|} \right)^{c_2}$$

for $x \in A$. We have omitted the argument 0 in the minimum which is here redundant because of $A \cap S = \emptyset$. By setting

$$\delta := \frac{1}{c_4 L^{c_2}} > 0, \tag{17}$$

where we define L like in (9), and having a look at (13), we can rewrite this as

$$\min\{g_1(x), \dots, g_m(x)\} \leqslant -\delta. \tag{18}$$

Define λ and k like in (9) and (10). For later use, we note

$$\lambda = c_1 L^{c_2+1} f^*. \tag{19}$$

We claim now that

$$f + \frac{\lambda\delta}{2} \geqslant \frac{f^*}{2} \text{ on } [-1,1]^n, \tag{20}$$

$$\frac{\delta}{2} \geqslant \frac{m-1}{2k+1} \quad \text{and} \tag{21}$$

$$\frac{f^*}{4} \geqslant \frac{\lambda m}{2k+1}. \tag{22}$$

Let us prove these claims. If we choose in Lemma 11 for y a minimizer of f on S, we obtain

$$|f(x) - f^*| \leqslant \operatorname{diam}([-1,1]^n) d^2 n^{d-1}\sqrt{n}\|f\| = 2\sqrt{n} d^2 n^{d-1}\sqrt{n}\|f\| = 2d^2 n^d \|f\|$$

for all $x \in [-1,1]^n$, noting that the diameter of $[-1,1]^n$ is $2\sqrt{n}$. In particular, we observe

$$f \geqslant f^* - 2d^2 n^d \|f\| \geqslant \frac{f^*}{2} - 2d^2 n^d \|f\| \quad \text{on } [-1,1]^n.$$

Together with the equation

$$\frac{\lambda\delta}{2} = 2d^2 n^d \|f\|,$$

which is clear from (9), (14) and (17), this yields (20). Using (10), (15) and (17), we see that

$$(2k+1)\delta \geqslant c_0(1 + L^{c_0})\delta \geqslant 2(m-1)c_4 L^{c_2}\delta = 2(m-1),$$

which is nothing else than (21). Finally, we exploit (10), (16) and (19), to see that

$$(2k+1)f^* \geqslant c_0(1+L^{c_0})f^* \geqslant 4mc_1 L^{c_2+1} f^* = 4m\lambda,$$

i.e., (22) holds.

Now (20), (21) and (22) will enable us to show our claim (11). If $x \in A$, then in the sum

$$\sum_{i=1}^{m} (g_i(x) - 1)^{2k} g_i(x) \tag{23}$$

at most $m-1$ summands are nonnegative. By Remark 12, these nonnegative summands add up to at most $(m-1)/(2k+1)$. At least one summand is negative, even $\leqslant -\delta$ by (18). All in all, if we evaluate the left hand side of our claim (11) in a point $x \in A$, then it is

$$\geqslant f(x) - \lambda\frac{m-1}{2k+1} + \lambda\delta \geqslant \underbrace{f(x) + \frac{\lambda\delta}{2}}_{\geqslant \frac{f^*}{2} \text{ by (20)}} + \lambda\underbrace{\left(\frac{\delta}{2} - \frac{m-1}{2k+1}\right)}_{\geqslant 0 \text{ by (21)}} \geqslant \frac{f^*}{2}.$$

When we evaluate it in a point $x \in [-1,1]^n \setminus A$, all summands of the sum (23) might happen to be nonnegative. Again by Remark 12, they add up to at most $m/(2k+1)$. But at the same time, the definition of A gives us a good lower bound on $f(x)$ so that the result is

$$\geqslant \frac{3}{4} f^* - \lambda \frac{m}{2k+1} \geqslant \frac{f^*}{2} + \underbrace{\frac{f^*}{4} - \frac{\lambda m}{2k+1}}_{\geqslant 0 \text{ by } (22)} \geqslant \frac{f^*}{2}. \qquad \square$$

Proposition 14. *If $p, q \in \mathbb{R}[\bar{X}]$ are both homogeneous (i.e., all of their respective monomials have the same degree), then $\|pq\| \leqslant \|p\|\|q\|$. For arbitrary $s \in \mathbb{N}$ and polynomials $0 \neq p_1, \ldots, p_s \in \mathbb{R}[\bar{X}]$, we have*

$$\|p_1 \cdots p_s\| \leqslant (1 + \deg p_1) \cdots (1 + \deg p_s) \|p_1\| \cdots \|p_s\|.$$

Proof. The statement for homogeneous p and q can be found in [Sw2, Lemma 8]. The second claim follows from this by writing each p_i as a sum $p_i = \sum_k p_{ik}$ of homogeneous degree k polynomials p_{ik}. Multiply the p_i by distributing out all such sums and apply the triangle inequality to the sum which arises in this way. Then use

$$\|p_{1k_1} \cdots p_{sk_s}\| \leqslant \|p_{1k_1}\| \cdots \|p_{sk_s}\| \leqslant \|p_1\| \cdots \|p_s\|.$$

Now factor out $\|p_1\| \cdots \|p_s\|$ and recombine the terms of the sum which now are all constant 1. $\square$

Lemma 15. *For all $c_1, c_2, c_3 > 0$, there is $c > 0$ such that*

$$c_1 \exp(c_2 r^{c_3}) \leqslant c \exp(r^c) \quad \text{for all } r \geqslant 0.$$

Proof. Choose any $c \geqslant c_1 \exp(c_2 2^{c_3})$ such that $c_3 \leqslant c/2$ and $c_2 \leqslant 2^{c/2}$. Then for $r \in [0, 2]$,

$$c_1 \exp(c_2 r^{c_3}) \leqslant c_1 \exp(c_2 2^{c_3}) \leqslant c \leqslant c \exp(r^c)$$

and for $r \geqslant 2$ (observing that $c_1 \leqslant c$),

$$c_1 \exp(c_2 r^{c_3}) \leqslant c \exp(2^{c/2} r^{c/2}) \leqslant c \exp(r^c). \qquad \square$$

We resume the discussion before Lemma 13. With regard to (11), we can for the moment concentrate on polynomials positive on the hypercube $[-1, 1]^n$. If this hypercube could be described by a single polynomial inequality, i.e., if we had $[-1, 1]^n = S(p)$ for some $p \in \mathbb{R}[\bar{X}]$, then the idea would be to apply the bound for Schmüdgen's Positivstellensatz now. The clue is here that p is a single polynomial and hence preordering and quadratic module representations are the same, i.e., $T(p) = M(p)$. The following lemma works around the fact that $[-1, 1]^n = S(p)$ can only happen when $n = 1$. We round the edges of the hypercube.

Lemma 16. *Let $S \subseteq (-1, 1)^n$ be compact. Then $1 - \frac{1}{d} - (X_1^{2d} + \cdots + X_n^{2d}) > 0$ on S for all sufficiently large $d \in \mathbb{N}$.*

Proof. Consider for each $1 \leqslant d \in \mathbb{N}$ the set

$$A_d := \left\{ x \in S \mid x_1^{2d} + \cdots + x_n^{2d} \geqslant 1 - \frac{1}{d} \right\}.$$

This gives a decreasing sequence $A_1 \supseteq A_2 \supseteq A_3 \supseteq \cdots$ of compact sets whose intersection $\bigcap_{d=1}^{\infty} A_d$ is empty by calculus. By compactness, a finite subintersection is empty, i.e., $A_d = \emptyset$ for all large $d \in \mathbb{N}$. $\square$

Note that in the proof of Putinar's theorem in [Sw3, Section 2] where we were not interested in complexity, a different approach has been taken. Condition (8) has been established for a polyhedron C which is even bigger than the hypercube, so big that preordering representations certifying nonnegativity on C can be turned into quadratic module representations certifying nonnegativity on the hypercube. The advantage was that we could use Pólya's theorem [Pól] which is much more elementary than Schmüdgen's theorem. Despite the existence of the effective version [PR] of that theorem of Pólya, it seems that establishing positivity on such a big polyhedron C is too expensive from the complexity point of view. Though it is not so nice, we therefore work here with a rounded hypercube and Theorem 5 instead.

We finally attack the proof of Theorem 6.

Proof of Theorem 6. By a simple scaling argument, we may assume that $\|g_i\| \leqslant 1$ and $g_i \leqslant 1$ on $[-1, 1]$ for all i. According to Lemma 16, we can choose $d_0 \in \mathbb{N}$ such that

$$p := 1 - \frac{1}{d_0} - (X_1^{2d} + \cdots + X_n^{2d}) > 0 \quad \text{on } S(\bar{g}).$$

By Putinar's Theorem 4, we have $p \in M(\bar{g})$ and therefore

$$p \in M(\bar{g}, d_1) \tag{24}$$

for some $d_1 \in \mathbb{N}$. Choose $d_2 \in \mathbb{N}$ such that

$$1 + \deg g_i \leqslant d_2 \quad \text{for all } i \in \{1, \ldots, m\}. \tag{25}$$

Now we choose c_0, c_1, c_2 like in Lemma 13, define L and λ like in (9) and choose the smallest $k \in \mathbb{N}$ satisfying (10). Then

$$2k + 1 \leqslant c_0(1 + L^{c_0}) + 2. \tag{26}$$

Let $c_3 \geqslant 1$ denote the constant existing by Theorem 5 (which is there called c and gives the bound for preordering representations of polynomials positive on $S(\bar{g})$). Using Lemma 15, it is easy to see that we can choose $c_4, c_5, c_6, c_7, c \geqslant 0$ satisfying

$$c_3 2^{c_3} r^{2+2c_3} n^{c_3 r} \leqslant c_4(\exp(c_4 r)), \tag{27}$$

$$2r + 2c_1 r^{c_2+1} d_2^{r(1+r^{c_0})+1} \leqslant c_5 \exp(r^{c_5}), \tag{28}$$

$$c_4 \exp(2c_4 d_2 r(1 + r^{c_0} + 3)) \leqslant c_6 \exp(r^{c_6}), \tag{29}$$

$$c_5^{c_3} c_6 \exp(c_3 r^{c_5} + r^{c_6}) \leqslant c_7 \exp(r^{c_7}), \tag{30}$$

$$c_7 \exp(r^{c_7}) + d_1 \leqslant c \exp(r^c) \tag{31}$$

for all $r \geqslant 0$. Now let $f \in \mathbb{R}[\bar{X}]$ be a polynomial of degree $d \geqslant 1$ with

$$f^* := \min\{f(x) \mid x \in S(\bar{g})\} > 0.$$

We are going to apply Theorem 5 to

$$h := f - \lambda \sum_{i=1}^{m} (g_i - 1)^{2k} g_i.$$

By Lemma 13, (11) holds for this polynomial, in particular

$$h^* := \min\{h(x) \mid x \in S(p)\} \geqslant \frac{f^*}{2}. \tag{32}$$

By Proposition 14 and the definition of d_2 in (25),

$$\|h\| \leqslant \|f\| + \lambda d_2^{2k+1} \tag{33}$$

$$\deg h \leqslant \max\{d, (2k+1)d_2, 1\} =: d_h. \tag{34}$$

By Theorem 5 (respectively the above choice of $c_3 \geqslant 1$), we get

$$h \in T(p, k_h) \quad \text{where } k_h := c_3 d_h^2 \left(1 + d_h^2 n^{d_h} \frac{\|h\|}{h^*}\right)^{c_3}. \tag{35}$$

Note that $\|h\|/h^* \geqslant 1$ since $0 < h^* \leqslant h(0) \leqslant \|h\|$. We use this to simplify the degree bound in (35). Obviously

$$\begin{aligned} k_h &\leqslant c_3 d_h^2 \left(2 d_h^2 n^{d_h} \frac{\|h\|}{h^*}\right)^{c_3} \\ &\leqslant c_3 2^{c_3} d_h^{2+2c_3} n^{c_3 d_h} \left(\frac{\|h\|}{h^*}\right)^{c_3} \leqslant c_4 \exp(c_4 d_h) \left(\frac{\|h\|}{h^*}\right)^{c_3}, \end{aligned} \tag{36}$$

by choice of c_4 in (27). Moreover, we have

$$\begin{aligned} \frac{\|h\|}{h^*} &\leqslant \frac{2}{f^*}(\|f\| + \lambda d_2^{2k+1}) = 2\frac{\|f\|}{f^*} + 2c_1 d_2^{2k+1} L^{c_2+1} \\ &\leqslant 2L + 2c_1 d_2^{2k+1} L^{c_2+1} = 2L + 2c_1 L^{c_2+1} d_2^{c_0(1+L^{c_0})+1} \leqslant c_5 \exp(L^{c_5}), \end{aligned} \tag{37}$$

by (33), (32), (26), (19) and by the choice of c_5 in (28). It follows that

$$\begin{aligned} d_h &\leqslant d(2k+2)d_2 && \text{(by (34))} \\ &\leqslant d(c_0(1 + L^{c_0}) + 3)d_2 && \text{(by (26))} \\ &\leqslant 2d_2 d^2 n^d \tfrac{\|f\|}{2dn^d\|f\|}(c_0(1 + L^{c_0}) + 3) && \\ &\leqslant 2d_2 d^2 n^d \tfrac{\|f\|}{f^*}(c_0(1 + L^{c_0}) + 3) && \text{(by Lemma 7)} \\ &\leqslant 2d_2 nL(c_0(1 + (nL)^{c_0} + 3)) && \text{(by (9))}, \end{aligned}$$

and therefore

$$c_4 \exp(c_4 d_h) \leqslant c_6 \exp((nL)^{c_6}) \tag{38}$$

for the constant c_6 chosen in (29). We now get

$$\begin{aligned} k_h &\leqslant c_4 \exp(c_4 d_h) \left(\tfrac{\|h\|}{h^*}\right)^{c_3} && \text{(by (36))} \\ &\leqslant c_6 \exp((nL)^{c_6})(c_5 \exp(L^{c_5}))^{c_3} && \text{(by (38) and (37))} \\ &= c_5^{c_3} c_6 \exp(c_3 (nL)^{c_5} + (nL)^{c_6}) \\ &\leqslant c_7 \exp((nL)^{c_7}) && \text{(by choice of } c_7 \text{ in (30)).} \end{aligned}$$

Combining this with (35) and (24), i.e.,

$$h \in T(p, c_7 \exp((nL)^{c_7})) \quad \text{and} \quad p \in M(\bar{g}, d_1),$$

yields (by composing corresponding representations)

$$h \in M(\bar{g}, c \exp((nL)^c))$$

according to the choice of c in (31). Finally, we have that

$$f = h + \lambda \sum_{i=1}^{m} (g_i - 1)^{2k} g_i \in M(\bar{g}, c \exp((nL)^c)),$$

since

$$\deg((g_i - 1)^{2k} g_i) \leqslant d_h \leqslant k_h \leqslant c_7 \exp((nL)^{c_7}) \leqslant c \exp((nL)^c),$$

by choice of d_2 in (25), d_h in (34), k_h in (35) and c in (31). □

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